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Longest increasing paths with gaps

Basdevant A.-L.* Gerin L.†

May 22, 2018

Abstract

We study a variant of the continuous and discrete Ulam-Hammersley problems. We obtain the limiting behaviour of the maximal length of an increasing path through a Poisson point process (or a Bernoulli point process) with the restriction that there must be minimal gaps between abscissae and ordinates of successive points of the path.

We also establish that, as in the classical Ulam-Hammersley problem, the fluctuations around the mean are given by the Tracy-Widom distribution.

MSC 2010 Classification: 60K35, 60F15.
Keywords: combinatorial probability, longest increasing subsequences, longest increasing paths, last-passage percolation, Hammersley's process, Ulam's problem, BLIP (Bernoulli Longest Increasing Paths)

1 Introduction

Motivated by the Ulam problem (which asks for the asymptotic behavior of the maximal length of an increasing subsequence in a uniform random permutation), Hammersley [8] studied the problem of the maximal length $L_{(x,t)}$ of an increasing path in a Poisson process with intensity one in $(0,x) \times (0,t)$. He used subadditivity to prove the existence of a constant $\pi/2 \leq c \leq e$ such that $L_{(t,t)}/c \to t$ in probability and conjectured $c = 2$. The first probabilistic proof of $c = 2$ was obtained by Aldous-Diaconis [1], by exploiting the geometric construction of Hammersley. (We refer to [15] for a nice and modern introduction to the Ulam-Hammersley problem.) In this article we obtain the limiting behaviour of the maximal length of an increasing path in a Poisson process, if we impose minimal *gaps* between abscissae and ordinates of successive points in the path.

Our proof uses a coupling with the original Ulam-Hammersley problem, and therefore we make a strong use of Aldous-Diaconis' result. This coupling also allows us to use the celebrated result by Baik-Deift-Johansson regarding the fluctuations of $L_{(x,t)}$ around its mean. We obtain that, with the proper rescaling, the fluctuations of our problem around the mean are also given by the Tracy-Widom distribution.

In two papers, Seppäläinen [17],[18] obtained analogous results for two discrete variants of the Ulam-Hammersley problem, we recall his results below. It turns out that our strategy also applies to the discrete settings: we obtain explicit asymptotic results for the length of the longest increasing path with *gaps* through Bernoulli random points on the square lattice. Furthermore,

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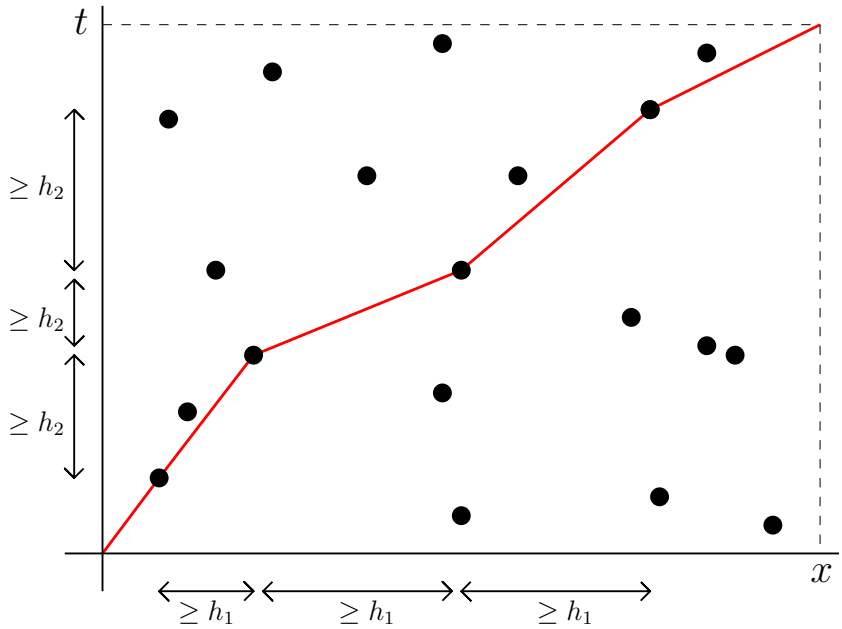


Figure 1: A realization of $L^{\mathbf{h}}_{(x,t)}$ (points of Ξ are represented with $\bullet$). Here we have $L^{\mathbf{h}}_{(x,t)} = 4$, one of the maximizing paths is drawn in red.

we also discuss the connection between longest increasing paths and last-passage percolation with geometric weights.

We now state our results in both settings.

Continuous settings

Let Ξ be a homogenous Poisson point process in $(0,+\infty)^2$ with intensity 1. We write $\Xi_{y,s} = 0/1$ for the absence/presence of a point of Ξ at (y,s), and we say that $(y,s) \in \Xi$ if $\Xi_{y,s} = 1$. Let $\mathbf{h} = (h_1, h_2)$ be a pair of non negative real numbers. We introduce the strict order on $(0,+\infty)^2$ defined by

$$(y,s) \overset{\mathbf{h}}{\prec} (y',s') \text{ if and only if } \begin{cases} y + h_1 \leq y', \\ s + h_2 \leq s'. \end{cases}$$

For $x, t > 0$, we consider the random variable $L^{\mathbf{h}}_{(x,t)}$ given by the length of the longest increasing path in $\Xi \cap [0,x] \times [0,t]$ with horizontal *gaps* h_1 and vertical *gaps* h_2. Namely,

$$L^{\mathbf{h}}_{(x,t)} = \max\Big\{ L;\ \text{there are } (y_1,s_1),\dots,(y_L,s_L) \in \Xi \text{ such that}$$
$$0 < y_1 < \cdots < y_L < x, \quad 0 < s_1 < \cdots < s_L < t, \quad (y_1,s_1) \overset{\mathbf{h}}{\prec} (y_2,s_2) \overset{\mathbf{h}}{\prec} \cdots \overset{\mathbf{h}}{\prec} (y_L,s_L). \Big\}$$

In the case $\mathbf{h} = \mathbf{0} := (0,0)$, the random variable $L_{(x,t)} := L^{\mathbf{0}}_{(x,t)}$ is just the length of the longest increasing path. The asymptotic behavior of $L_{(at,bt)}$ for every a, b was obtained by Aldous-Diaconis [1] (a different probabilistic proof can be found in [16]).

Proposition 1 ([1], Th.5 (a))**.** *Let $a, b > 0$. Then*

$$f(a,b) := \lim_{t \to +\infty} \frac{L_{(at,bt)}}{t} = 2\sqrt{ab}. \tag{1}$$

The convergence holds a.s. and in L^1.

Here we use a coupling between the cases with and without gaps (see Lemma 3 below) to extend the formula (1) to every pair of gaps :

Theorem 1. *For every $h_1, h_2 \geq 0$, we have the following limit:*

$$f^{\mathbf{h}}(a,b) := \lim_{t\to\infty} \frac{L^{\mathbf{h}}_{(at,bt)}}{t} = \begin{cases} \dfrac{2(ah_2+bh_1) - 2\sqrt{(ah_2-bh_1)^2+ab}}{4h_1h_2-1} & \text{if } h_1h_2 \neq 1/4, \\ \dfrac{ab}{h_1b+h_2a} & \text{if } h_1h_2 = 1/4. \end{cases} \tag{2}$$

The convergence holds a.s. and in L^1.

(We have no probabilistic interpretation of the case $h_1h_2 = 1/4$, but one can check that the right-hand side of (2) is continuous at every point of the line $h_1h_2 = 1/4$.)

In some cases the above formula for $f^{\mathbf{h}}(a,b)$ simplifies:

- If $h_1 = h_2 = h$, then
$$f^{(h,h)}(1,1) = \frac{2}{1+2h}.$$
- If $h_2 = 0$ then
$$f^{(h,0)}(1,1) = 2\sqrt{h^2+1} - 2h.$$

For $\mathbf{h} = (0,0)$, the fluctuations of $L_{(at,bt)}$ around its mean have been determined by Baik-Deift-Johansson [3].

Proposition 2 (Cube root fluctuations ([3])**.** *For every $a, b > 0$ and $x \in \mathbb{R}$,we have*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L_{(at,bt)} - 2\sqrt{ab}t}{(\sqrt{ab}t)^{1/3}} \leq x \right) = F_{TW}(x),$$

where F_{TW} is the distribution function of the Tracy-Widom distribution.

In fact the main result of [3] is stated for the longest increasing subsequence in a uniform permutation. Proposition 2 follows by elementary poissonization arguments. This proposition can also been extended for every pair of gaps :

Theorem 2. *For every $h_1, h_2, a, b \geq 0$ and $x \in \mathbb{R}$, we have*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{L^{\mathbf{h}}_{(at,bt)} - f^{\mathbf{h}}(a,b)t}{\sigma^{\mathbf{h}}(a,b)t^{1/3}} \leq x \right) = F_{TW}(x), \tag{3}$$

where F_{TW} is the distribution function of the Tracy-Widom distribution and

$$\sigma^{\mathbf{h}}(a,b) = \frac{f^{\mathbf{h}}(a,b)^{4/3}}{2^{1/3}} \frac{1}{2(bh_1+ah_2) + f^{\mathbf{h}}(a,b)(1-4h_1h_2)}.$$

(In some cases the expression for $\sigma^{\mathbf{h}}(a,b)$ simplifies, for instance $\sigma^{(h,h)}(1,1) = (1+2h)^{-4/3}$.)

The main strategy of this paper is to make a random dilation and map the rectangle $[0,x] \times [0,t]$ onto a rectangle $[0,x'] \times [0,t']$, for some $x' \geq x$, $t' \geq t$ which will be defined later. Then, we construct a coupling between $L_{(x,t)}$ (without gaps) and $L^{\mathbf{h}}_{(x',t')}$. Finally, Theorem 1 is obtained using the asymptotic for $L_{(at,bt)}$ given in Proposition 1 and Theorem 2 using Proposition 2.

Thanks to the scale-invariance property of the Poisson process we also easily obtain asymptotic results in the case where gaps and intensity of the Poisson process depend on t (see Section 2.4).

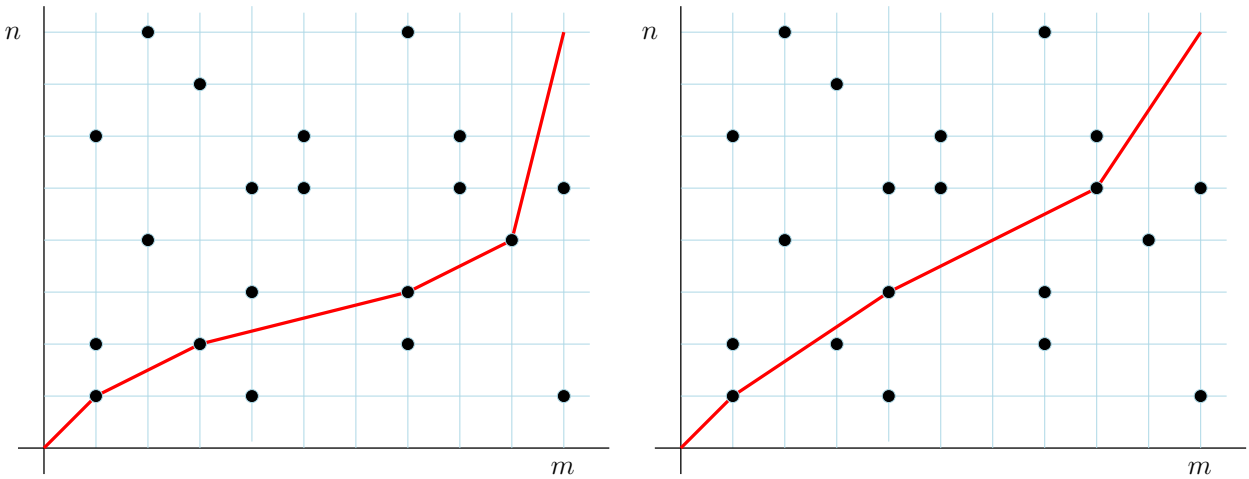


Figure 2: Realizations of $\mathcal{L}^{(2,1)}_{(m,n)}$ (left) and $\mathcal{L}^{(3,2)}_{(m,n)}$ (right) for $(m,n) = (10,8)$ and the same sampling of Ξ (points of Ξ are represented with $\bullet$). Here we have $\mathcal{L}^{(2,1)}_{(m,n)} = 4$, $\mathcal{L}^{(3,2)}_{(m,n)} = 3$. In both pictures one of the maximizing paths is drawn in red.

Discrete settings

The very same strategy allows us to obtain analogous results in the discrete settings. Let $\Xi = (\Xi_{i,j})_{i,j\in\mathbb{Z}_{>0}}$ be i.i.d. Bernoulli random variables with mean p. We also consider Ξ as a random set of integer points of the quarter-plane by saying that (i,j) is present in Ξ if $\Xi_{i,j} = 1$.

Let $\mathbf{h} = (h_1, h_2)$ be a pair of non-negative integers, we assume $\mathbf{h} \neq \mathbf{0} = (0,0)$. We introduce the strict order on $(\mathbb{Z}_{>0})^2$ defined by

$$(i,j) \overset{\mathbf{h}}{\prec} (i',j') \text{ if and only if } \begin{cases} i + h_1 \leq i', \\ j + h_2 \leq j'. \end{cases}$$

We consider the random variable given by the length of the longest non-decreasing path from $(1,1)$ to (m,n) in Ξ with horizontal gaps h_1 and vertical gaps h_2. Namely,

$$\mathcal{L}^{\mathbf{h}}_{(m,n)} = \max\Big\{L;\ \text{there are } (i_1,j_1),\dots,(i_L,j_L) \in \Xi \text{ such that}$$
$$1 \leq i_1 \leq \dots \leq i_L \leq m, \quad 1 \leq j_1 \leq \dots \leq j_L \leq m, \quad (i_1,j_1) \overset{\mathbf{h}}{\prec} (i_2,j_2) \overset{\mathbf{h}}{\prec} \dots \overset{\mathbf{h}}{\prec} (i_L,j_L)\Big\}.$$

This problem is close to what is sometimes called *slope-constrained longest increasing subsequence* (SCLIS) in the literature of algorithms [5]. In the case $\mathbf{h} = (1,1)$, the random variable $\mathcal{L}_{(m,n)} := \mathcal{L}^{(1,1)}_{(m,n)}$ is just the length of the longest increasing path and the asymptotic behavior of $\mathcal{L}_{(\lfloor an\rfloor,\lfloor bn\rfloor)}$ was obtained for every $a,b > 0$ by Seppäläinen in [17] using hydrodynamic limits of a given particle system (see also [12, Sec.III], [2, Th.1.1.] ,[4, Th.2.2] for different proofs).

Proposition 3 (Law of large numbers for $\mathcal{L}_{m,n}$ ([17],Th.1))**.** *Let $a,b > 0$, $p \in (0,1)$. Then*

$$\lim_{n\to\infty} \frac{\mathcal{L}_{(\lfloor an\rfloor,\lfloor bn\rfloor)}}{n} \overset{a.s.}{=} g(a,b) := \begin{cases} \dfrac{\sqrt{p}\left(2\sqrt{ab} - (a+b)\sqrt{p}\right)}{1-p} & \text{if } p < \min\{a/b, b/a\}, \\ \min\{a,b\} & \text{otherwise.} \end{cases} \tag{4}$$

The second case in (4) corresponds to a *flat edge* in the limiting shape. This differs from the continuous case.

We now can state our result in the discrete settings, the limiting constant is less explicit than in the continuous settings.

Theorem 3. *Let h_1, h_2 be two non-negative integers such that $\mathbf{h} = (h_1, h_2) \neq (0,0)$. For every $a, b > 0$, there exists a constant*

$$g^{\mathbf{h}}(a,b) := \lim_{n\to+\infty} \frac{1}{n}\mathcal{L}^{\mathbf{h}}_{(\lfloor an\rfloor,\lfloor bn\rfloor)},$$

where the convergence holds a.s. and in L^1. Moreover, $g^{\mathbf{h}}(a,b)$ is the unique solution of equation

$$g^{\mathbf{h}}(a,b) = g\left(a-(h_1-1)g^{\mathbf{h}}(a,b), b-(h_2-1)g^{\mathbf{h}}(a,b)\right), \tag{5}$$

where g is defined by (4).

We explicit here the solution of (5) in some cases:

- **Symmetric case.** If $(a,b) = (1,1)$ and $h_1 = h_2 = h$ we can easily solve (5) and we get

$$g^{(h,h)}(1,1) = \frac{2\sqrt{p}}{1+(2h-1)\sqrt{p}}.$$

- **Non-decreasing paths.** If $\mathbf{h} = (1,0)$, the random variable $\mathcal{L}^{(1,0)}_{(m,n)}$ represents the length of the longest non-decreasing path in Ξ. The above formula reduces to

$$g^{(1,0)}(a,b) = \begin{cases} 2\sqrt{abp(1-p)} + (a-b)p & \text{if } p < a/(a+b), \\ a & \text{otherwise.} \end{cases}$$

  This was also first proved by Seppäläinen in a second article ([18], Th.1) (see again [2] for a different proof). In the more general case $\mathbf{h} = (h,0)$ we obtain with (5) the following expression (we only write the formula for $a = b = 1$):

$$g^{(h,0)}(1,1) = \begin{cases} \dfrac{2(1+h)p(1-p) + 2\sqrt{p(1-p+h^2p)(1-p)}}{\left(h\sqrt{p} + \sqrt{(1-p+h^2p)(1-p)}\right)^2} & \text{if } p < 1/(h+1), \\ 1/h & \text{otherwise.} \end{cases}$$

- If $(a,b) = (1,1)$ and $\mathbf{h} = (h,1)$ for some $h \geq 0$

$$g^{(h,1)}(1,1) = \begin{cases} \dfrac{2hp(p-1) + 2\sqrt{p(1-p)(1+ph^2-2hp)}}{(hp-1)^2} & \text{if } p < 1/h, \\ 1/h & \text{otherwise.} \end{cases}$$

Coupling with last-passage percolation

In the discrete case our coupling is related to previous results in literature. If we apply our coupling to the case $\mathbf{h} = (0,0)$ we recover a coupling between random variables $\left\{\mathcal{L}_{(m,n)}\right\}_{m,n}$ and last passage percolation with geometric weights (which in turn is in correspondence with synchronous TASEP). We explain this formally in Section 3.3. This coupling was already introduced in [13],[7]. Note that another coupling with *asynchronous* TASEP (also called directed TASEP) was also given in [6, Sec.3],[14].

More precisely, let $\Xi' = \left(\Xi'_{i,j}\right)_{i,j\in\mathbb{Z}_{>0}}$ be i.i.d. geometric random variables with law

$$\mathbb{P}(\Xi'_{i,j} = k) = p^k(1-p) \text{ for } k \geq 0$$

and let

$$T_{(m,n)} = \max\{\sum_{(i,j)\in P} \Xi'_{i,j}\ ;\ P \in \mathcal{P}_{m,n}\},$$

where $\mathcal{P}_{m,n}$ denotes the set of paths from $(1,1)$ to (m,n) taking only North and East steps. If we formally take the parameter $\mathbf{h}$ equal to $(0,0)$ in our proof, we get the following result. For every $a, b > 0$,

$$\lim_{n\to+\infty} \frac{1}{n} T_{(\lfloor an\rfloor,\lfloor bn\rfloor)} = g^{\mathbf{0}}(a,b),$$

where $g^{\mathbf{0}}(a,b)$ is the unique solution of (5) with parameters $h_1 = h_2 = 0$. This can be made rigorous and thus we recover the known result for the limiting shape in last-passage percolation, originally due to Jockusch-Propp-Shor (in the context of synchronous TASEP).

Theorem 4 (([9],Th.2) see also ([19],Th.2.2))**.** *For every $a, b > 0$, $p \in (0,1)$*

$$\lim_{n\to+\infty} \frac{1}{n} T_{(\lfloor an\rfloor,\lfloor bn\rfloor)} = \frac{\sqrt{p}\left(2\sqrt{ab} + (a+b)\sqrt{p}\right)}{1-p}.$$

2 Proofs in the continuous settings

We fix a pair $\mathbf{h} = (h_1, h_2)$ of non-negative real numbers all along this section.

2.1 Preliminary results

We first justify that $\frac{1}{t}L^{\mathbf{h}}_{(at,bt)}$ converges almost surely and in L^1. Let us stress that for any $x, x', t, t' \geq 0$, we have the stochastic domination

$$L^{\mathbf{h}}_{(x+x',t+t')} \succcurlyeq L^{\mathbf{h}}_{(x,t)} + L'^{\mathbf{h}}_{(x',t')} - 1,$$

where $L'^{\mathbf{h}}_{(x',t')}$ has the same distribution as $L^{\mathbf{h}}_{(x',t')}$ but is independent of $L^{\mathbf{h}}_{(x,t)}$.

Indeed, if $(i_1, j_1), \ldots, (i_L, j_L)$ is a longest increasing path in Ξ with gaps $\mathbf{h}$ in $(0,x)\times(0,t)$ and $(i'_1, j'_1), \ldots, (i'_{L'}, j'_{L'})$ a longest increasing path in $(x, x+x')\times(t, t+t')$, then

$$(i_1, j_1), \ldots, (i_L, j_L), (i'_2, j'_2), \ldots, (i'_{L'}, j'_{L'})$$

is an increasing path with gaps h in $(0, x+x')\times(0, t+t')$. Thus, the family of random variables $\left\{L^{\mathbf{h}}_{(x,t)} - 1\right\}_{x>0,t>0}$ is superadditive. Hence, Kingman's subadditive theory (see for example [15, Th.A2-A3]) implies the existence of a constant

$$f^{\mathbf{h}}(a,b) := \lim_{t\to\infty} \frac{L^{\mathbf{h}}_{(at,bt)}}{t},$$

where the limit is a.s. and in L^1.

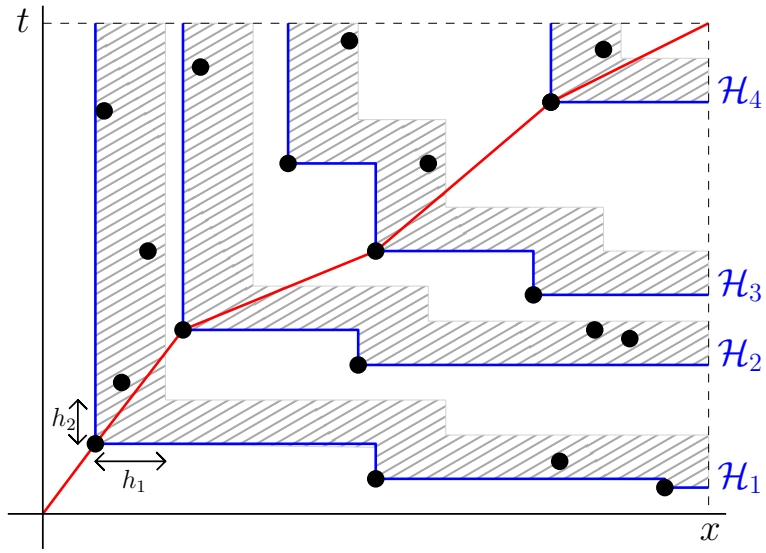


Figure 3: An example of Hammersley lines for the same realization of Ξ as that of Fig.1. The four Hammersley lines are drawn in blue.

2.2 Hammersley's lines and dilatation

A very useful way to handle the random variables $L^{\mathbf{h}}_{(x,t)}$ is the geometric interpretation of Hammersley's lines. In the classical case $\mathbf{h} = \mathbf{0}$ this construction was first implicitly introduced by Hammersley [8], a more explicit construction was given by Aldous-Diaconis in [1] (continuous settings) and by Seppäläinen [17] (discrete settings).

We now define Hammersley lines formally. These are a sequence $\mathcal{H}^{\mathbf{h}}_1, \mathcal{H}^{\mathbf{h}}_2, \dots$ of broken lines in $(0, +\infty)^2$ defined inductively as follows (an example is provided in Fig.3).

The broken line $\mathcal{H}^{\mathbf{h}}_1$ is the shortest path made of vertical and horizontal straight lines whose minimal points for $\overset{\mathbf{0}}{\prec}$ are exactly the minimal points of Ξ for $\overset{\mathbf{0}}{\prec}$.

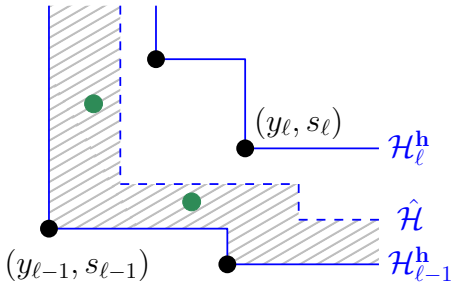
The line $\mathcal{H}^{\mathbf{h}}_2$ is defined as follows: we remove the points of $\mathcal{H}^{\mathbf{h}}_1 + [0, h_1] \times [0, h_2]$ (hatched in gray in Fig.3) and reiterate the procedure: $\mathcal{H}^{\mathbf{h}}_2$ is the shortest path made of vertical and horizontal straight lines whose minimal points for $\overset{\mathbf{0}}{\prec}$ are exactly the minimal points of $\Xi \setminus \big(\mathcal{H}^{\mathbf{h}}_1 + [0, h_1] \times [0, h_2]\big)$ for $\overset{\mathbf{0}}{\prec}$. Inductively we define $\mathcal{H}^{\mathbf{h}}_3, \mathcal{H}^{\mathbf{h}}_4, \dots$ in the same way.

Lemma 1. *For each $(x, t) \in (0, +\infty)^2$, there are exactly $L^{\mathbf{h}}_{(x,t)}$ distinct Hammersley lines which intersect $(0, x) \times (0, t)$.*

Proof of Lemma 1. Let denote by $\mathcal{N}_{(x,t)}$ the number of distinct Hammersley lines which intersect $(0, x) \times (0, t)$.

Proof of $L^{\mathbf{h}}_{(x,t)} \leq \mathcal{N}_{(x,t)}$. Let $P = (y_1, s_1) \prec \cdots \prec (y_{L_{(x,t)}}, s_{L_{(x,t)}})$ be a maximizing path in Ξ for $L_{(x,t)}$. For every $\ell \leq \mathcal{N}_{(x,t)}$, there is at most one point of P in the area $\mathcal{H}^{\mathbf{h}}_\ell + [0, h_1] \times [0, h_2]$. Therefore $L^{\mathbf{h}}_{(x,t)} \leq \mathcal{N}_{(x,t)}$.

Proof of $L^{\mathbf{h}}_{(x,t)} \geq \mathcal{N}_{(x,t)}$. Let $\mathcal{H}^{\mathbf{h}}_1, \mathcal{H}^{\mathbf{h}}_2, \dots, \mathcal{H}^{\mathbf{h}}_\ell$ be given, we will construct an admissible path with ℓ points of Ξ (from top-right to bottom-left). We first take any point (y_ℓ, s_ℓ) of $\mathcal{H}^{\mathbf{h}}_\ell$. Let $\hat{\mathcal{H}} = \mathcal{H}^{\mathbf{h}}_{\ell-1} + (h_1, h_2)$ be the translation of $\mathcal{H}^{\mathbf{h}}_{\ell-1}$ by the gaps. By construction of $\mathcal{H}^{\mathbf{h}}_\ell$ the broken line $\hat{\mathcal{H}}$ intersects $(0, y_\ell) \times (0, s_\ell)$. Since $\hat{\mathcal{H}}$ takes only directions North/West, necessarily there is a point $(y_{\ell-1} + h_1, s_{\ell-1} + h_2) \in \hat{\mathcal{H}} \cap (0, y_\ell) \times (0, s_\ell)$, with $(y_{\ell-1}, s_{\ell-1}) \in \Xi$.



Therefore, $(y_{\ell-1}, s_{\ell-1}) \overset{\mathbf{h}}{\prec} (y_\ell, s_\ell)$. By induction we construct an admissible path of ℓ points in Ξ.

□

The following Proposition was used in [1] in the case $\mathbf{h} = \mathbf{0}$, it extends plainly to the general case.

Proposition 4 (spatial Markov property for Hammersley's lines)**.** *Conditional on the ℓ-th Hammersley line $\mathcal{H}_\ell^{\mathbf{h}}$,*

$$\left\{\Xi_{y,s},\ (y,s) \overset{\mathbf{0}}{\succ} \mathcal{H}_\ell^{\mathbf{h}}\right\} \text{ is independent of } \left\{\Xi_{y,s},\ (y,s) \overset{\mathbf{0}}{\prec} \mathcal{H}_\ell^{\mathbf{h}}\right\},$$

and distributed as a homogeneous Poisson process with intensity one. Here $(y,s) \overset{\mathbf{0}}{\succ} \mathcal{H}_\ell^{\mathbf{h}}$ (resp. $\overset{\mathbf{0}}{\prec}$) means that $(y,s) \overset{\mathbf{0}}{\succ} (y',s')$ (resp. $\overset{\mathbf{0}}{\prec}$) for at least one point (y',s') in $\mathcal{H}_\ell^{\mathbf{h}}$.

In particular, conditional on $\mathcal{H}_\ell^{\mathbf{h}}$, the line $\mathcal{H}_{\ell+1}^{\mathbf{h}}$ is independent of $\mathcal{H}_1^{\mathbf{h}}, \mathcal{H}_2^{\mathbf{h}}, \ldots, \mathcal{H}_{\ell-1}^{\mathbf{h}}$.

Proof. Let $\mathcal{H}_1^{\mathbf{h}}, \mathcal{H}_2^{\mathbf{h}}, \ldots, \mathcal{H}_{\ell-1}^{\mathbf{h}}$ be given. By construction of Hammersley lines, the fact that (y,s) belongs to $\mathcal{H}_\ell^{\mathbf{h}}$ or not only depends on Ξ in the rectangle $[0,y] \times [0,s]$. □

We want to make a coupling between random variables $L_{(x,t)}$ and $L^{\mathbf{h}}_{(x',t')}$ for some $x' \geq x$, $t' \geq t$. We fix a realization of Ξ in the quarter-plane, and denote by $\left\{L_{(x,t)}(\Xi),\ (x,t) \in (0,+\infty)^2\right\}$ the lengths of the longest paths corresponding to this realization.

We introduce the (random) function

$$\begin{array}{rccc} \phi^{\mathbf{h}}: & (0,+\infty)^2 & \to & (0,+\infty)^2 \\ & (y,s) & \mapsto & (y + h_1 L_{(y,s)^-}, s + h_2 L_{(y,s)^-}), \end{array}$$

where $L_{(y,s)^-} = \lim_{\varepsilon \to 0} L_{(y-\varepsilon, s-\varepsilon)}$.

An example is drawn in Fig.4. By construction, the image by $\phi^{\mathbf{h}}$ of every Hammersley line $\mathcal{H}_\ell^{\mathbf{0}}$ is a translation of $\mathcal{H}_\ell^{\mathbf{0}}$ (and the area between two consecutive Hammersley lines $\mathcal{H}_\ell^{\mathbf{0}}, \mathcal{H}_{\ell+1}^{\mathbf{0}}$ is also translated by $\phi^{\mathbf{h}}$).

The main idea is that on the right picture of Figure 4 we re-sample new points in the regions which do not belong to the image of $\phi^{\mathbf{h}}$, according to an independent Poisson process.

Lemma 2 (Dilatation)**.** *Let $\tilde{\Xi}$ be the field of random points defined by*

$$\begin{cases} \tilde{\Xi}_{y',s'} & = \Xi_{y,s} \text{ if } (y',s') \in \mathrm{Image}(\phi^{\mathbf{h}}) \text{ and } \phi^{\mathbf{h}}(y,s) = (y',s'), \\ \tilde{\Xi}_{y',s'} & = \mathbf{Y}_{y',s'} \text{ if } (y',s') \notin \mathrm{Image}(\phi^{\mathbf{h}}), \end{cases}$$

where $\mathbf{Y}$ is a homogeneous Poisson process with intensity one, independent of Ξ. Then $\tilde{\Xi}$ is also a homogeneous Poisson process with intensity one.

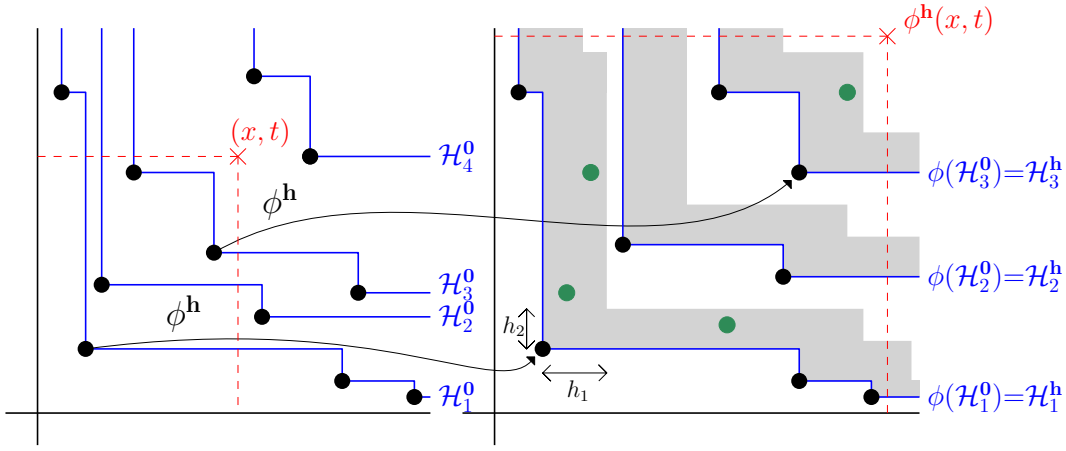


Figure 4: An example of the function $\phi^{\mathbf{h}}$. Left: A sample of Ξ. Right: The same realization after the dilatation $\phi^{\mathbf{h}}$. The gray areas correspond to regions which are not in the image of $\phi^{\mathbf{h}}$. New points of $\tilde{\Xi} \setminus \phi^{\mathbf{h}}(\Xi)$ are drawn in green. We have that $L_{(x,t)}(\Xi) = 3 = L^{\mathbf{h}}_{\phi^{\mathbf{h}}(x,t)}(\tilde{\Xi})$, as stated in Eq.(6).

Proof of Lemma 2. The point process $\tilde{\Xi}$ can be discovered by the following Markovian exploration of $(0,+\infty)^2$.

From bottom-left to top-right, both point processes Ξ and $\tilde{\Xi}$ coincide up to $\mathcal{H}_1^{\mathbf{0}}$.

Then, conditional to $\mathcal{H}_1^{\mathbf{0}}$, the points in the area $\phi^{\mathbf{h}}(\mathcal{H}_1^{\mathbf{0}}) + (0,h_1) \times (0,h_2)$ (this is the first gray region in Fig.4) are also distributed as an independent homogeneous Poisson process. This shows that, conditional to $\mathcal{H}_1^{\mathbf{0}}$, $\tilde{\Xi}$ is also a Poisson process up to $\phi^{\mathbf{h}}(\mathcal{H}_1^{\mathbf{0}}) + (0,h_1) \times (0,h_2)$. But now, thanks to the Markovian property of Hammersley lines applied to $\mathcal{H}_1^{\mathbf{0}}$ we can reiterate the argument to show that conditional to $\mathcal{H}_2^{\mathbf{0}}$, $\tilde{\Xi}$ is a Poisson process up to $\phi^{\mathbf{h}}(\mathcal{H}_2^{\mathbf{0}}) + (0,h_1) \times (0,h_2)$, and so on.

□

Lemma 3 (Removing gaps)**.** *For every $x,t > 0$, and every $k \in \mathbb{Z}_{\geq 0}$,*

$$\mathbb{P}(L^{\mathbf{h}}_{(x,t)} \leq k) = \mathbb{P}(L_{(x-h_1k,t-h_2k)} \leq k).$$

Proof of Lemma 3. We will first prove that almost surely, for every x,t we have

$$L_{(x,t)}(\Xi) = L^{\mathbf{h}}_{(x',t')}(\tilde{\Xi}), \tag{6}$$

where $(x',t') = \phi^{\mathbf{h}}(x,t)$ *i.e.*

$$(x',t') = (x + h_1 L_{(x,t)^-}(\Xi), t + h_2 L_{(x,t)^-}(\Xi)).$$

The quarter-plane $(0,+\infty)^2$ is divided in two types of regions:

1. The region $\mathcal{W}$ defined by the interior of $\phi^{\mathbf{h}}((0,+\infty)^2)$ (represented in white in Fig.4). There are no points of $\tilde{\Xi}$ in $\mathcal{W}$.

2. The regions $\mathcal{G}_\ell = \phi^{\mathbf{h}}(\mathcal{H}_\ell^{\mathbf{0}}) + [0,h_1] \times [0,h_2]$, for $\ell \geq 1$ (represented in gray in Fig.4).

Let $P = (x_1, s_1) \overset{\mathbf{0}}{\prec} \cdots \overset{\mathbf{0}}{\prec} (y_{L_{(x,t)}}, s_{L_{(x,t)}})$ be a maximizing path in Ξ. Because of the dilatation, the points of $\phi^{\mathbf{h}}(P)$ have horizontal gaps h_1 and vertical gaps h_2. Therefore the path $\phi^{\mathbf{h}}(P)$ satisfies the gaps constraints and

$$L_{(x,t)}(\Xi) \leq L^{\mathbf{h}}_{(x',t')}(\tilde{\Xi}).$$

For the reverse inequality, we observe that because of the gaps constraint, an admissible path for the order $\overset{\mathbf{h}}{\prec}$ takes at most one point in each $\mathcal{G}_\ell$. Since there are $L_{(x,t)}(\Xi)$ gray regions which intersect $(0, x') \times (0, t')$, this proves that $L^{\mathbf{h}}_{(x',t')}(\tilde{\Xi}) \leq L_{(x,t)}(\Xi)$. Finally we have proved (6).

We now conclude the proof of the lemma. Let $\gamma \geq 0$ be such that

$$\gamma = \sup \left\{ y \geq 0, L_{(x-yh_1, t-yh_2)} \geq k+1 \right\}$$

(with $\sup \varnothing = 0$). If $\gamma > 0$ we have

$$k+1 = L_{(x-\gamma h_1, t-\gamma h_2)} = L_{(x-\gamma h_1, t-\gamma h_2)^-} + 1.$$

From (6)

$$k+1 = L^{\mathbf{h}}_{(x-\gamma h_1 + h_1 k, t-\gamma h_2 + h_2 k)} = L^{\mathbf{h}}_{(x-\gamma h_1 + h_1 k, t-\gamma h_2 + h_2 k)^-} + 1.$$

By monotonicity of L we deduce that

$$\mathbb{P}(L^{\mathbf{h}}_{(x,t)} < k+1) = \mathbb{P}(\gamma < k).$$

On the other hand, by definition of γ,

$$\mathbb{P}(\gamma < k) = \mathbb{P}(L_{(x-h_1 k, t-h_2 k)} < k+1),$$

and the lemma is proved.

□

Remark 1. *There is a geometric interpretation of the coupling equality* (6)*: the image of a Hammersley line under mapping $\phi^{\mathbf{h}}$ is a Hammersley line as well. More precisely, for every ℓ,*

$$\phi^{\mathbf{h}}\left(\mathcal{H}^{\mathbf{0}}_\ell\right) = \mathcal{H}^{\mathbf{h}}_\ell,$$

where on the left-hand side $\mathcal{H}^{\mathbf{0}}_\ell$ is defined with the points of Ξ and on the right-hand side, $\mathcal{H}^{\mathbf{h}}_\ell$ is defined with the points of $\tilde{\Xi}$.

Proof of Theorem 1. Recall the asymptotics known for the length of the longest increasing path:

$$f(a,b) := \lim_{t \to +\infty} \frac{L_{(at,bt)}}{t} = 2\sqrt{ab}.$$

We fix $(a,b) \in (0,+\infty)^2$, let $\lambda > 0$ and t such that $\lambda t \in \mathbb{Z}_{\geq 0}$. Combining Lemma 3 and Proposition 1, we obtain

$$\mathbb{P}(L^{\mathbf{h}}_{(at,bt)} \leq \lambda t) = \mathbb{P}(L_{(at-h_1\lambda t, bt-h_2\lambda t)} \leq \lambda t) \overset{t \to +\infty}{\to} \begin{cases} 0 & \text{if } \lambda < f\left(a-h_1\lambda, b-h_2\lambda\right), \\ 1 & \text{if } \lambda > f\left(a-h_1\lambda, b-h_2\lambda\right). \end{cases}$$

Therefore, $\frac{1}{t} L^{\mathbf{h}}_{(at,bt)}$ converges in probability to the unique solution λ of the equation

$$\lambda = f\left(a-h_1\lambda, b-h_2\lambda\right) \qquad i.e. \qquad \lambda = 2\sqrt{(a-h_1\lambda)(b-h_2\lambda)}. \tag{7}$$

We easily check that if $h_1 h_2 \neq 1/4$ the solution of (7) is given by

$$\lambda = \frac{2(ah_2 + bh_1) - 2\sqrt{(ah_2 - bh_1)^2 + ab}}{4h_1h_2 - 1}.$$

If $h_1 h_2 = 1/4$, then (7) reduces to $\lambda = ab/(h_1 b + h_2 a)$. □

2.3 Fluctuations of $L^{\mathbf{h}}(at, bt)$

Let us now explain how the combination of Baik-Deift-Johansson's result (Proposition 2) and the equality given of Lemma 3 implies Theorem 2 for the fluctuations of $L^{\mathbf{h}}(at, bt)$.

Proof of Theorem 2. Using the scaling invariance of a Poisson point process under transformations which preserve the volume, in the case without gaps constraint, the distribution of $L_{(x,t)}$ only depends on the value of the product xt. Thus, we can define a family of random variables $(Z(s), s \geq 0)$ such that $Z(xt) \stackrel{d}{=} L_{(x,t)}$ for all $x, t \geq 0$. Proposition 2 yields that

$$\forall c \in \mathbb{R}, \quad \lim_{s\to\infty} \mathbb{P}(Z(s^2) \leq 2s + cs^{1/3}) = F_{TW}(c),$$

where F_{TW} is the distribution function of the Tracy-Widom distribution.

Fix now $a, b > 0$ and let $\lambda = f^{\mathbf{h}}(a, b)$. Using Lemma 3, we have that, for any $t \geq 0$ and $\beta \in \mathbb{R}$ such that $\lambda t + \beta t^{1/3} \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \mathbb{P}(L^{\mathbf{h}}_{(at,bt)} \leq \lambda t + \beta t^{1/3}) &= \mathbb{P}(L_{(t(a-h_1\lambda)-\beta h_1 t^{1/3}, t(b-h_2\lambda)-\beta h_2 t^{1/3})} \leq \lambda t + \beta t^{1/3}). \\ &= \mathbb{P}(Z(s^2) \leq \lambda t + \beta t^{1/3}) \end{aligned} \tag{8}$$

with $s \geq 0$ defined by

$$\begin{aligned} s^2 &:= (t(a - h_1\lambda) - \beta h_1 t^{1/3})(t(b - h_2\lambda) - \beta h_2 t^{1/3}) \\ &= t^2(a - \lambda h_1)(b - \lambda h_2) - t^{4/3}\beta(h_1 b + h_2 a - 2\lambda h_1 h_2) + \mathcal{O}(t) \\ &= t^2\frac{\lambda^2}{4} - t^{4/3}\beta(h_1 b + h_2 a - 2\lambda h_1 h_2) + \mathcal{O}(t), \end{aligned}$$

where we use (7) in the last line. Inverting this equality gives

$$t = \frac{2}{\lambda}s + \beta\delta s^{1/3} + \mathcal{O}(1) \quad \text{with} \quad \delta := 2^{4/3}\frac{h_1 b + h_2 a - 2\lambda h_1 h_2}{\lambda^{7/3}}.$$

Plugging this expression of t in (8), we get

$$\mathbb{P}(L^{\mathbf{h}}_{(at,bt)} \leq \lambda t + \beta t^{1/3}) = \mathbb{P}\left(Z(s^2) \leq 2s + \beta\Big(\lambda\delta + \frac{2^{1/3}}{\lambda^{1/3}}\Big)s^{1/3} + \mathcal{O}(1)\right).$$

If we set $\sigma^{\mathbf{h}}(a, b) = \big(\lambda\delta + \frac{2^{1/3}}{\lambda^{1/3}}\big)^{-1}$ and apply the above equation with $\beta = c\sigma^{\mathbf{h}}(a, b)$, we obtain

$$\lim_{t\to\infty} \mathbb{P}(L^{\mathbf{h}}_{(at,bt)} \leq \lambda t + c\sigma^{\mathbf{h}}(a, b)t^{1/3}) = \lim_{s\to\infty} \mathbb{P}(Z(s^2) \leq 2s + cs^{1/3}) = F_{TW}(c).$$

One can check that this definition of $\sigma^{\mathbf{h}}(a, b)$ coincides with the one given in Theorem 2. □

2.4 Case where h,λ depend on t

In this short section we show how to use the scale-invariance of the Poisson point process to derive asymptotics in the case where gaps and intensity of points depend on t. For the sake of simplicity we assume that vertical and horizontal gaps are identical.

For every t, let $\mathbf{h_t} = (h_t, h_t)$ be a pair of gaps, $\lambda_t > 0$ and denote by $L^{\mathbf{h_t},\lambda_t}_{(t,t)}$ be the length of the longest increasing path with gaps $\mathbf{h_t}$ when Ξ is a Poisson process with intensity λ_t.

Theorem 5. *Let $c_t = h_t\sqrt{\lambda_t}$ and assume that $c := \lim_{t\to\infty} c_t$ exists. Then,*

(i) If $c = 0$ and $\sqrt{\lambda_t}t \to +\infty$ then

$$\frac{1}{\sqrt{\lambda_t}t} L^{\mathbf{h_t},\lambda_t}_{(at,bt)} \overset{prob.}{\to} f^{(0,0)}(a,b).$$

(ii) If $c \in (0,+\infty)$ and $\sqrt{\lambda_t}t \to +\infty$ then

$$\frac{1}{\sqrt{\lambda_t}t} L^{\mathbf{h_t},\lambda_t}_{(at,bt)} \overset{prob.}{\to} f^{(c,c)}(a,b).$$

(iii) If $c = +\infty$ and $t/h_t \to +\infty$ then

$$\frac{h_t}{t} L^{\mathbf{h_t},\lambda_t}_{(at,bt)} \overset{prob.}{\to} \min\{a,b\}.$$

Proof. We observe that by the scaling invariance of the Poisson process we have

$$L^{\mathbf{h_t},\lambda_t}_{(at,bt)} \overset{(d)}{=} L^{\mathbf{h_t}\sqrt{\lambda_t},1}_{(at\sqrt{\lambda_t},bt\sqrt{\lambda_t})}.$$

Assume first that $c = \lim_{t\to\infty} h_t\sqrt{\lambda_t} < +\infty$. Let $\varepsilon \in (0,1)$ and T such that $c(1-\varepsilon) \le h_t\sqrt{\lambda_t} \le c(1+\varepsilon)$ for $t \ge T$. Since $L^{\mathbf{h}}_{(at,bt)}$ is a non-increasing function in $\mathbf{h}$, we get, for $t \ge T$, the stochastic domination:

$$\frac{1}{t\sqrt{\lambda_t}} L^{\mathbf{c}(\mathbf{1}+\varepsilon),1}_{(at\sqrt{\lambda_t},bt\sqrt{\lambda_t})} \preccurlyeq \frac{1}{t\sqrt{\lambda_t}} L^{\mathbf{h_t},\lambda_t}_{(at,bt)} \preccurlyeq \frac{1}{t\sqrt{\lambda_t}} L^{\mathbf{c}(\mathbf{1}-\varepsilon),1}_{(at\sqrt{\lambda_t},bt\sqrt{\lambda_t})}$$

Assuming that $t\sqrt{\lambda_t}$ tends to infinity, the left-hand side tends to $f^{(c(1+\varepsilon),c(1+\varepsilon))}(a,b)$ whereas the right-hand side tends to $f^{(c(1-\varepsilon),c(1-\varepsilon))}(a,b)$. We conclude by continuity in c of the expression of $f^{(c,c)}(a,b)$.

Assume now that $c = \lim_{t\to\infty} h_t\sqrt{\lambda_t} = +\infty$. First, by definition of gaps, $L^{\mathbf{h_t},\lambda}_{(at,bt)} \le \min\{a,b\} \times t/h_t$ a.s. This gives the upper bound in (iii).
For the lower bound, let $A > 0$ and T such that $h_t\sqrt{\lambda_t} \ge A$ for $t \ge T$. For $t \ge T$, $\lambda_t \ge (A/h_t)^2$, thus using the monotony of a Poisson point process with respect to its intensity, we have

$$\frac{h_t}{t} L^{\mathbf{h_t},\lambda_t}_{(at,bt)} \succcurlyeq \frac{h_t}{t} L^{\mathbf{h_t},(A/h_t)^2}_{(at,bt)}. \tag{9}$$

Using (ii) with $\tilde{\lambda}_t = (A/h_t)^2$, we get, if $t/h_t \to +\infty$, the following convergence in probability:

$$\lim_{t\to\infty} \frac{h_t}{At} L^{\mathbf{h_t},(A/h_t)^2}_{(t,t)} = f^{(A,A)}(a,b).$$

Observe that $\lim_{A\to+\infty} Af^{(A,A)}(a,b) = \frac{a+b-|a-b|}{2} = \min\{a,b\}$, so we obtain the lower bound by letting A tend to infinity in (9).

□

3 Proofs in the discrete settings

3.1 The case $h_1 > 0, h_2 > 0$

In the discrete settings and if $h_1 > 0, h_2 > 0$, the proof of Theorem 3 is almost identical to that of Theorem 1. We only explain how to change the definitions of the Hammersley lines and the function $\phi^{\mathbf{h}}$.

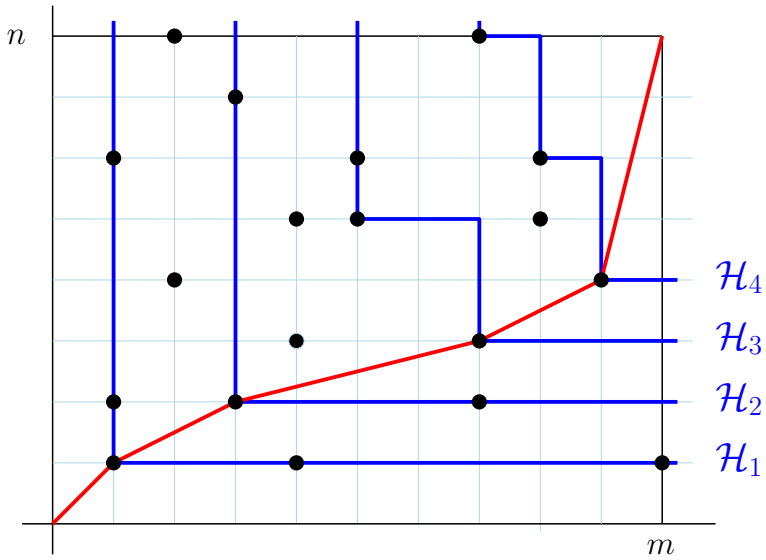


Figure 5: An example with $\mathbf{h} = (2,1)$ for the same realization of Ξ as that of Fig.2. In every unit square (i,j) we write the value of $\mathcal{L}^{(2,1)}_{(i,j)}$, the four Hammersley lines are drawn in blue.

The definition of the Hammersley lines is identical to the continuous case (an example is provided in Fig.5): the broken line $\mathcal{H}^{\mathbf{h}}_1$ is the shortest path made of vertical and horizontal straight lines whose minimal points for $\overset{\mathbf{0}}{\prec}$ are exactly the minimal points of Ξ for $\overset{\mathbf{0}}{\prec}$. The line $\mathcal{H}^{\mathbf{h}}_2$ is defined as follows: we remove the points of $\mathcal{H}^{\mathbf{h}}_1 + \{0,1,\dots,h_1-1\} \times \{0,1,\dots,h_2-1\}$ and reiterate the procedure: $\mathcal{H}^{\mathbf{h}}_2$ is the shortest path made of vertical and horizontal straight lines whose minimal points for $\overset{\mathbf{0}}{\prec}$ are exactly the minimal points of $\Xi \setminus \left(\mathcal{H}^{\mathbf{h}}_1 + \{0,1,\dots,h_1-1\} \times \{0,1,\dots,h_2-1\}\right)$ for $\overset{\mathbf{0}}{\prec}$. Inductively we define $\mathcal{H}^{\mathbf{h}}_3, \mathcal{H}^{\mathbf{h}}_4, \dots$ in the same way.

The function $\phi^{\mathbf{h}}$ has to be replaced by its discrete counterpart:

$$\begin{array}{rccc} \phi^{\mathbf{h}}: & (\mathbb{Z}_{>0})^2 & \to & (\mathbb{Z}_{>0})^2 \\ & (m,n) & \mapsto & (m + (h_1-1)\mathcal{L}_{(m-1,n-1)}, n + (h_2-1)\mathcal{L}_{(m-1,n-1)}). \end{array} \tag{10}$$

We define a new set of points $\tilde{\Xi}$ by

$$\begin{cases} \tilde{\Xi}_{i',j'} & = \Xi_{i,j} \text{ if } (i',j') \in \mathrm{Image}(\phi^{\mathbf{h}}) \text{ and } \phi^{\mathbf{h}}(i,j) = (i',j'), \\ \tilde{\Xi}_{i',j'} & = \mathbf{Y}_{i',j'} \text{ if } (i',j') \notin \mathrm{Image}(\phi^{\mathbf{h}}), \end{cases}$$

where $(\mathbf{Y}_{i',j'})_{i,j\geq 1}$ are independent Bernoulli random variables with mean p. In the same manner as in the continuous settings, we prove that $(\tilde{\Xi}_{i,j})_{i,j\geq 1}$ are i.i.d Bernoulli random variables with mean p and for every $m,n \geq 1$ we have

$$\mathcal{L}_{(m,n)}(\Xi) = \mathcal{L}^{\mathbf{h}}_{(m',n')}(\tilde{\Xi}),$$

where $(m',n') = \phi^{\mathbf{h}}(m,n) = (m + (h_1-1)\mathcal{L}_{(m-1,n-1)}, n + (h_2-1)\mathcal{L}_{(m-1,n-1)})$. We deduce then the discrete version of Lemma 3:

Lemma 4 (Removing gaps: the discrete case)**.** *For every $m,n \in \mathbb{Z}_{\geq 0}$, and every $k \in \mathbb{Z}_{\geq 0}$,*

$$\mathbb{P}(\mathcal{L}^{\mathbf{h}}_{(m,n)} \leq k) = \mathbb{P}(\mathcal{L}_{(m-(h_1-1)k, n-(h_2-1)k)} \leq k).$$

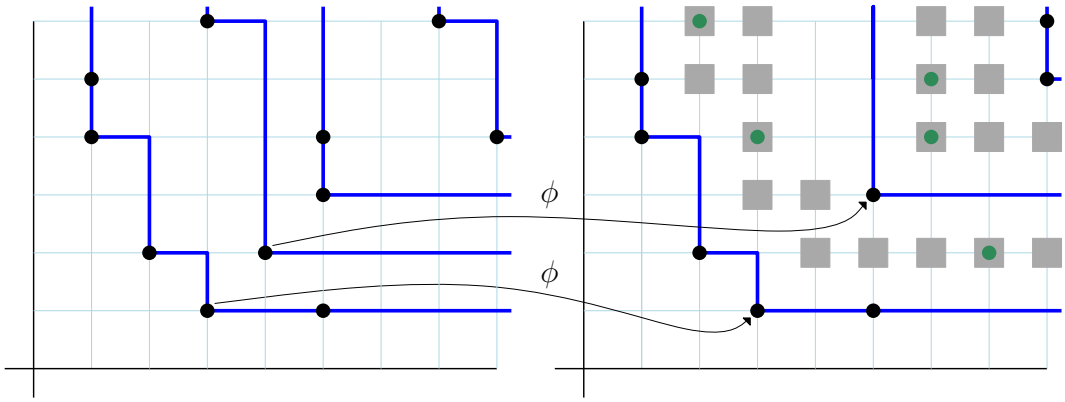


Figure 6: An example of the function $\phi^{\mathbf{h}}$ for $(m,n) = (8,6)$ and $\mathbf{h} = (3,2)$. The points of the quarter-plane indicated by small gray squares are not in the image of $\phi^{\mathbf{h}}$. The values of $\tilde{\Xi}$ at these points are independent of Ξ.

Proof of Theorem 3. We fix $(a,b) \in (0,+\infty)^2$, let $\lambda > 0$ and n such that $\lambda n \in \mathbb{Z}_{\geq 0}$. By Lemma 3 and Proposition 3,

$$\begin{aligned}\mathbb{P}(\mathcal{L}^{\mathbf{h}}_{(an,bn)} \leq \lambda n) &= \mathbb{P}(\mathcal{L}_{(an-(h_1-1)\lambda n, bn-(h_2-1)\lambda n)} \leq \lambda n) \\ &\overset{n\to+\infty}{\to} \begin{cases} 0 & \text{if } \lambda < g\left(a-(h_1-1)\lambda, b-(h_2-1)\lambda\right), \\ 1 & \text{if } \lambda > g\left(a-(h_1-1)\lambda, b-(h_2-1)\lambda\right). \end{cases}\end{aligned}$$

Therefore, $\frac{1}{n}\mathcal{L}^{\mathbf{h}}_{(an,bn)}$ converges in probability to the unique solution λ of the equation

$$\lambda = g\left(a-(h_1-1)\lambda, b-(h_2-1)\lambda\right), \tag{11}$$

where g is defined in (4). Let us note that Eq. (11) has indeed a unique solution since the right hand side of (11) decreases with respect to λ. □

3.2 The case $h_1 > 0, h_2 = 0$

As in the previous section, we can exhibit a coupling between $\mathcal{L}^{(h,0)}$ and $\mathcal{L}$ which shows that Lemma 4 also holds for $\mathbf{h} = (h_1, 0)$. However, some change must be made compared to the case $h_1 h_2 > 0$ since the function $\phi^{\mathbf{h}}$ defined in (10) is no more a dilation.

To make the exposition clearer, it is more convenient to explain the coupling between the model with gap $(h,0)$ with the one with gap $(h,1)$. Thus, let us consider a Bernoulli field Ξ on $(\mathbb{Z}_{>0})^2$ and construct the associated random variables $\mathcal{L}^{(h,1)}_{(m,n)}$. Define the function ψ^h by

$$\begin{array}{rccc} \psi^h: & (\mathbb{Z}_{>0})^2 & \to & (\mathbb{Z}_{>0})^2 \\ & (m,n) & \mapsto & (m, n - \mathcal{L}^{(h,1)}_{(m-1,n-1)}). \end{array}$$

Contrary to $\phi^{\mathbf{h}}$, the function ψ^h is surjective but no more injective. More precisely, for any $(m,n') \in (\mathbb{Z}_{>0})^2$, there exist $n \geq 1$ and $k \geq 0$ such that

$$(\psi^h)^{-1}(m,n') = \{(m,n), (m,n+1), \ldots, (m,n+k)\}.$$

We define now the new set of points $\tilde{\Xi}$ by

$$\tilde{\Xi}_{m,n'} = \Xi_{m,n+k} \text{ where } (\psi^h)^{-1}(m,n') = \{(m,n), (m,n+1), \ldots, (m,n+k)\}.$$

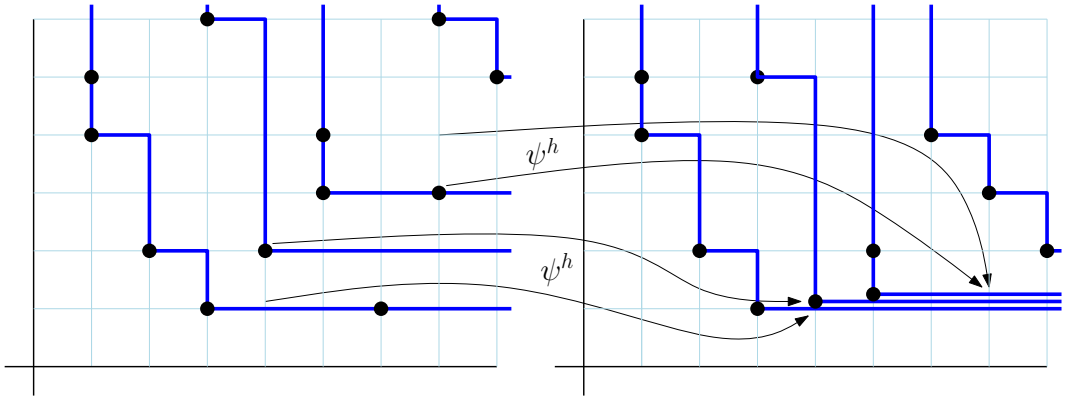


Figure 7: An example of the function ψ^h and the definition of set of points $\tilde{\Xi}$ for $\mathbf{h} = (1,0)$. There is a point in $\tilde{\Xi}$ at (i',j') i.f.f. there is one in Ξ at its highest antecedent by ψ^h.

Again, one can prove that the random variables $\tilde{\Xi} = (\tilde{\Xi}_{i,j})_{i,j\geq 1}$ are i.i.d. Bernoulli random variables with mean p and for every $m,n \geq 1$ we have

$$\mathcal{L}^{(h,1)}_{(m,n)}(\Xi) = \mathcal{L}^{(h,0)}_{(m',n')}(\tilde{\Xi}),$$

where $(m,n') = \psi^h(m,n) = (m, n - \mathcal{L}^{(h,1)}_{(m-1,n-1)})$. Then, with the same argument as in the proof of Lemma 3, we get that, for every $m,n \in \mathbb{Z}_{\geq 0}$, and every $k \in \mathbb{Z}_{\geq 0}$,

$$\mathbb{P}(\mathcal{L}^{(h,0)}_{(m,n)} \leq k) = \mathbb{P}(\mathcal{L}^{(h,1)}_{(m,n+k)} \leq k) = \mathbb{P}(\mathcal{L}_{(m-(h-1)k,n+k)} \leq k).$$

This in turn yields

$$\mathbb{P}(\mathcal{L}^{(h,0)}_{(an,bn)} \leq \lambda n) \overset{n\to+\infty}{\to} \begin{cases} 0 & \text{if } \lambda < g\left(a-(h_1-1)\lambda, b+\lambda\right), \\ 1 & \text{if } \lambda > g\left(a-(h_1-1)\lambda, b+\lambda\right). \end{cases}$$

Therefore, $\frac{1}{n}\mathcal{L}^{(h,0)}_{(an,bn)}$ converges in probability to the unique solution λ of the equation

$$\lambda = g\left(a-(h_1-1)\lambda, b+\lambda\right), \tag{12}$$

where g is defined in (4).

3.3 The case $h_1 = 0, h_2 = 0$: coupling with last-passage percolation

We now explain why taking the gap constraint $\mathbf{h} = (0,0)$ provides a coupling between the length of the longest increasing path in a Bernoulli random field and the model of last passage percolation with geometric weights. As this short section does not contain new results, we only sketch the arguments.

Thus, let us consider a Bernoulli field Ξ on $(\mathbb{Z}_{>0})^2$ and construct the associated random variables $\mathcal{L}_{(m,n)}$ associated to the gaps $(1,1)$. In the case $\mathbf{h} = (0,0)$, the function $\phi^{\mathbf{h}}$ defined in (10) becomes

$$\begin{array}{rccc} \phi^{\mathbf{0}}: & (\mathbb{Z}_{>0})^2 & \to & (\mathbb{Z}_{>0})^2 \\ & (m,n) & \mapsto & (m-\mathcal{L}_{(m-1,n-1)}, n-\mathcal{L}_{(m-1,n-1)}). \end{array} \tag{13}$$

As in the previous case $\mathbf{h} = (h,0)$, the function $\phi^{\mathbf{0}}$ is surjective but not injective. More precisely, for any $(m',n') \in (\mathbb{Z}_{>0})^2$, there exist $n,m \geq 1$ and $k \geq 0$ such that

$$(\phi^{\mathbf{0}})^{-1}(m',n') = \{(m,n),(m+1,n+1),\ldots,(m+k,n+k)\}.$$

We first define a new collection of random variables $\hat{\Xi} = \{\hat{\Xi}_{i,j}, i,j \geq 1\} \in \{0,1\}^{(\mathbb{Z}_{>0})^2}$ by

$$\begin{cases} \hat{\Xi}_{m,n} & = \Xi_{m,n} = 1 \text{ if } (m,n) \text{ is a minimal point of some } \mathcal{H}_\ell^{(1,1)} \\ \hat{\Xi}_{m,n} & = 0 \text{ otherwise,} \end{cases}$$

and we define now the family of random variables $\tilde{\Xi} = \{\tilde{\Xi}_{i,j}, i,j \geq 1\} \in (\mathbb{Z}_{\geq 0})^{(\mathbb{Z}_{>0})^2}$ by

$$\tilde{\Xi}_{m',n'} = \sum_{(m,n) \in (\phi^{\mathbf{0}})^{-1}(m',n')} \hat{\Xi}_{m,n}.$$

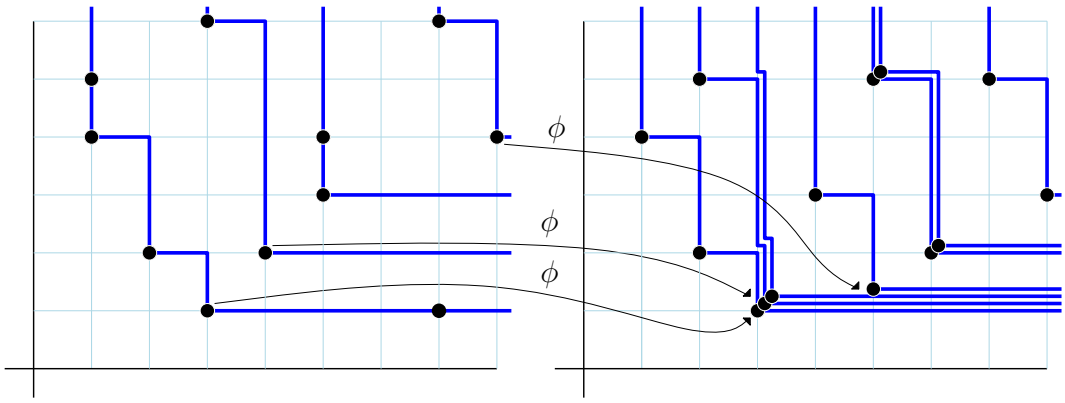


Figure 8: An example of the function $\phi^{\mathbf{0}}$. Left: a realization of Hammersley lines $\mathcal{H}_\ell^{(1,1)}$. Right: The associated realization of last-passage percolation with geometric weights. We have $\tilde{\Xi}_{3,1} = \hat{\Xi}_{3,1} + \hat{\Xi}_{4,2} + \hat{\Xi}_{5,3} = 1 + 1 + 1 = 3$.

For every m', n' we have $\tilde{\Xi}_{(m',n')} = k$ if

$$\hat{\Xi}_{(m,n)} = \hat{\Xi}_{(m+1,n+1)} = \cdots = \hat{\Xi}_{(m+k-1,n+k-1)} = 1, \quad \hat{\Xi}_{(m+k,n+k)} = 0,$$

which occurs with probability $p^k(1-p)$. Therefore we can show that $\tilde{\Xi}$ is a family of i.i.d. geometric random variables: $\mathbb{P}(\tilde{\Xi}_{m,n} = k) = p^k(1-p)$ for $k \geq 0$.

For $m, n \geq 1$, recall the notation

$$T_{(m,n)}(\tilde{\Xi}) = \max\{ \sum_{(i,j) \in P} \tilde{\Xi}_{i,j} \ ; \ P \in \mathcal{P}_{m,n}\},$$

where $\mathcal{P}_{m,n}$ denotes the set of paths from $(1,1)$ to (m,n) taking only North and East steps.

With the same arguments of the previous cases one can prove that for every $m, n \geq 1$ we have

$$\mathcal{L}^{(1,1)}_{(m,n)}(\Xi) = T_{(m',n')}(\tilde{\Xi}),$$

where $(m',n') = \phi^{\mathbf{0}}(m,n) = (m - \mathcal{L}^{(1,1)}_{(m-1,n-1)}, n - \mathcal{L}^{(1,1)}_{(m-1,n-1)})$. We deduce that Lemma 4 is also valid for $\mathbf{h} = (0,0)$: for every $m, n \in \mathbb{Z}_{\geq 0}$, and every $k \in \mathbb{Z}_{\geq 0}$,

$$\mathbb{P}(T_{(m,n)} \leq k) = \mathbb{P}(\mathcal{L}_{(m+k,n+k)} \leq k),$$

which is equivalent to Eq.(4.1) in [7]. All these arguments can be made rigorous and allow us to recover the expression given in Theorem 4.

3.4 Fluctuations of $\mathcal{L}^{\mathbf{h}}_{(an,bn)}$

Recall the limiting shape of last-passage percolation (Theorem 4):

$$\lim_{n\to\infty} \frac{1}{n} T_{(\lfloor an\rfloor,\lfloor bn\rfloor)} = g^{\mathbf{0}}(a,b) = \frac{\sqrt{p}\left(2\sqrt{ab}+(a+b)\sqrt{p}\right)}{1-p}.$$

Johansson [10] has computed the fluctuations of $T_{(\lfloor an\rfloor,\lfloor bn\rfloor)}$ around its mean:

Proposition 5 (Cube root fluctuations ([10], Theorem 1.2))**.** *For every $a,b>0$ and $x\in\mathbb{R}$, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{T_{(\lfloor an\rfloor,\lfloor bn\rfloor)} - ng^{\mathbf{0}}(a,b)}{\sigma(a,b)n^{1/3}} \le x\right) = F_{TW}(x),$$

where F_{TW} is the distribution function of the Tracy-Widom distribution, and

$$\sigma(a,b) = \frac{p^{1/6}}{1-p}(ab)^{-1/6}(\sqrt{a}+\sqrt{pb})^{2/3}(\sqrt{b}+\sqrt{pa})^{2/3}.$$

In another paper Johansson [11, Th.5.3] has also obtained Tracy-Widom fluctuations for longest increasing paths in the case $\mathbf{h}=(1,0)$. The authors of [13] state a close result for the fluctuations of $\mathcal{L}_{(an,bn)}$ around its mean (see also Section 4 in [14]). However, we have not been able to fill the gap between their result and the convergence of rescaled fluctuations.

From Proposition 5, it is not obvious to obtain a result as neat as Theorem 2 for every direction (a,b). The proof of Theorem 2 relies on the scaling invariance property of the Poisson process: the law of $L_{(x,t)}$ only depends on the value of xt. There is of course no analogous for fields of Bernoulli random points. However, one can still show that for any $a,b\ge 0$ and $\mathbf{h}$, the fluctuations of $\mathcal{L}^{\mathbf{h}}_{(\lfloor an\rfloor,\lfloor bn\rfloor)}$ are also of order $n^{1/3}$ (outside the flat edges of the limiting shape). Before stating our result about the fluctuations of $\mathcal{L}^{\mathbf{h}}_{(\lfloor an\rfloor,\lfloor bn\rfloor)}$, we must first prove a technical lemma.

Lemma 5. *Let $\mathbf{h}$ be a gap constraint. For all $a,b>0$ such that $g^{\mathbf{h}}(a,b)<\min\{a/h_1,b/h_2\}$, there exists a unique solution (α,β) of positive numbers to the system*

$$\begin{cases} \alpha + h_1 g^{\mathbf{0}}(\alpha,\beta) = a, \\ \beta + h_2 g^{\mathbf{0}}(\alpha,\beta) = b. \end{cases}$$

Moreover, $g^{\mathbf{0}}(\alpha,\beta)$ is the unique λ such that

$$\lambda = g^{\mathbf{0}}(a-h_1\lambda, b-h_2\lambda).$$

and we have $g^{\mathbf{h}}(a,b)=g^{\mathbf{0}}(\alpha,\beta)$.

Proof. By symmetry, we can assume that $a/h_1\le b/h_2$. Two applications of Lemma 4 yield

$$\mathbb{P}(\mathcal{L}^{\mathbf{h}}_{(m,n)}\le k) = \mathbb{P}(\mathcal{L}_{(m-(h_1-1)k,n-(h_2-1)k)}\le k) = \mathbb{P}(T_{(m-h_1k,n-h_2k)}\le k). \tag{14}$$

Hence, as in the proof of Theorem 3, we get

$$\mathbb{P}(\mathcal{L}^{\mathbf{h}}_{(an,bn)}\le \lambda n) \overset{n\to+\infty}{\to} \begin{cases} 0 & \text{if } \lambda < g^{\mathbf{0}}\left(a-h_1\lambda, b-h_2\lambda\right), \\ 1 & \text{if } \lambda > g^{\mathbf{0}}\left(a-h_1\lambda, b-h_2\lambda\right). \end{cases}$$

Note that $g^{\mathbf{0}}\left(a-h_1\lambda, b-h_2\lambda\right)$ is only defined for $\lambda\le a/h_1$ and the equation

$$\lambda = g^{\mathbf{0}}\left(a-h_1\lambda, b-h_2\lambda\right) \tag{15}$$

have a (unique) solution if and only if

$$\frac{a}{h_1} \geq g^{\mathbf{0}}(0, b - \frac{ah_2}{h_1}), \qquad \text{i.e.} \qquad p \leq a/(bh_1 - (h_2 - 1)a).$$

In this case, we deduce that $g^{\mathbf{h}}(a,b)$ is the solution of (15). Moreover, $g^{\mathbf{h}}(a,b) < a/h_1$ if and only if

$$p < a/(bh_1 - (h_2 - 1)a). \tag{16}$$

Let us now assume that $g^{\mathbf{h}}(a,b) < a/h_1$ and consider a solution (α, β) of the system

$$\alpha + h_1 g^{\mathbf{0}}(\alpha, \beta) = a, \qquad \beta + h_2 g^{\mathbf{0}}(\alpha, \beta) = b.$$

Putting $g^{\mathbf{0}}(\alpha, \beta)$ in (15), we see that $g^{\mathbf{0}}(\alpha, \beta)$ satisfies this equality and thus $g^{\mathbf{0}}(\alpha, \beta) = g^{\mathbf{h}}(a,b) < a/h_1$. In particular, we necessarily have $\alpha, \beta > 0$. Noticing that $g^{\mathbf{0}}(\alpha, \beta) = \beta g^{\mathbf{0}}(\alpha/\beta, 1)$ and setting $\gamma = \alpha/\beta$, we see now that the system is equivalent to

$$\beta(\gamma + h_1 g^{\mathbf{0}}(\gamma, 1)) = a, \qquad \beta(1 + h_2 g^{\mathbf{0}}(\gamma, 1)) = b. \tag{17}$$

In particular, we have

$$b(\gamma + h_1 g^{\mathbf{0}}(\gamma, 1)) = a(1 + h_2 g^{\mathbf{0}}(\gamma, 1)).$$

Hence, if $a/h_1 = b/h_2$, we get $\gamma = a/b$. In the other case : $a/h_1 < b/h_2$, we get

$$g^{\mathbf{0}}(\gamma, 1) = \frac{a - b\gamma}{bh_1 - ah_2}.$$

The left hand side is increasing with γ whereas the right hand side decreases. Thus, there exists a unique solution $\gamma > 0$ if and only if

$$\frac{p}{1-p} = g^{\mathbf{0}}(0, 1) < \frac{a}{bh_1 - ah_2} \qquad \text{i.e.} \qquad p < a/(bh_1 - (h_2 - 1)a)$$

and we recover the same condition as in (16). Finally, using (17), we see that the existence and unicity of γ implies the existence and unicity of (α, β). □

Proposition 6. *Let* $\mathbf{h} = (h_1, h_2)$ *be a gap constraint and let* $a, b > 0$ *be such that* $g^{\mathbf{h}}(a,b) < \min\{a/h_1, b/h_2\}$. *Let us define* (α, β) *as in Lemma 5 such that* $g^{\mathbf{h}}(a,b) = g^{\mathbf{0}}(\alpha, \beta)$. *Set*

$$\mathcal{W}^{\mathbf{h}}_{an,bn} = \frac{\mathcal{L}^{\mathbf{h}}_{(\lfloor an \rfloor, \lfloor bn \rfloor)} - g^{\mathbf{h}}(a,b)n}{\sigma(\alpha, \beta) n^{1/3}}.$$

Assuming for example that $a/h_1 \leq b/h_2$, *we have, for all* $x \geq 0$,

$$F_{TW}\left(\frac{bx}{b - h_2 g^{\mathbf{h}}(a,b)}\right) \leq \liminf_{n\to\infty} \mathbb{P}\left(\mathcal{W}^{\mathbf{h}}_{an,bn} \leq x\right) \leq \limsup_{n\to\infty} \mathbb{P}\left(\mathcal{W}^{\mathbf{h}}_{an,bn} \leq x\right) \leq F_{TW}\left(\frac{ax}{a - h_1 g^{\mathbf{h}}(a,b)}\right)$$

$$F_{TW}\left(\frac{-ax}{a - h_1 g^{\mathbf{h}}(a,b)}\right) \leq \liminf_{n\to\infty} \mathbb{P}\left(\mathcal{W}^{\mathbf{h}}_{an,bn} \leq -x\right) \leq \limsup_{n\to\infty} \mathbb{P}\left(\mathcal{W}^{\mathbf{h}}_{an,bn} \leq -x\right) \leq F_{TW}\left(\frac{-bx}{b - h_2 g^{\mathbf{h}}(a,b)}\right).$$

In one particular direction the LHS and RHS of above inequalities coincide:

Corollary 1. *Let* $\mathbf{h} = (h_1, h_2)$ *be a gap constraint and let* $a, b > 0$ *be such that* $a/h_1 = b/h_2$. *Then*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\mathcal{L}^{\mathbf{h}}_{(an,bn)} - g^{\mathbf{h}}(a,b)n}{\sigma^{\mathbf{h}}(a,b) n^{1/3}} \leq x\right) = F_{TW}(x), \tag{18}$$

where

$$\sigma^{\mathbf{h}}(a,b) = \sigma(\alpha, \beta)(1 - \frac{h_1}{a} g^{\mathbf{h}}(a,b)).$$

Again, we have a simple expression for $\mathbf{h} = (h, h)$ and $a = b = 1$:

$$\sigma^{(h,h)}(1,1) = \frac{(1-p)^{1/3}p^{1/6}}{(1+(2h-1)\sqrt{p})^{4/3}}.$$

Remark 2. *Proposition 6 states that, ouside the flat edge of the limiting shape, the fluctuations of $\mathcal{L}^{\mathbf{h}}_{(an,bn)}$ are of order $n^{1/3}$. Inside the flat edge (except in the critical direction), one can easily show that $\mathbb{P}(|\mathcal{L}^{\mathbf{h}}_{(\lfloor an\rfloor,\lfloor bn\rfloor)} - g^{\mathbf{h}}(a,b)n| \geq 1)$ tends to 0.*

Proof. A change of indexes in (14) yields

$$\mathbb{P}(T_{(m,n)} \leq k) = \mathbb{P}(\mathcal{L}^{\mathbf{h}}_{(m+h_1k,n+h_2k)} \leq k). \tag{19}$$

Let $a, b > 0$ such that $g^{\mathbf{h}}(a,b) < \min\{a/h_1, b/h_2\}$ and, according to Lemma 5, take $\alpha, \beta > 0$ solution of the system

$$\alpha + h_1 g^{\mathbf{0}}(\alpha,\beta) = a, \qquad \beta + h_1 g^{\mathbf{0}}(\alpha,\beta) = b$$

and such that $g^{\mathbf{h}}(a,b) = g^{\mathbf{0}}(\alpha,\beta) =: \lambda$. Using (19), we obtain

$$\begin{aligned}\mathbb{P}(T_{(\alpha n,\beta n)} \leq \lambda n + yn^{1/3}) &= \mathbb{P}(\mathcal{L}^{\mathbf{h}}_{(n(\alpha+h_1\lambda)+h_1yn^{1/3},n(\beta+h_2\lambda)+h_2yn^{1/3})} \leq \lambda n + yn^{1/3}).\\ &= \mathbb{P}(\mathcal{L}^{\mathbf{h}}_{(an+h_1yn^{1/3},bn+h_2yn^{1/3})} \leq \lambda n + yn^{1/3}).\end{aligned} \tag{20}$$

Assume now that $a/h_1 \leq b/h_2$. Set

$$N := n + \frac{h_1}{a}yn^{1/3} \text{ so that } n = N - \frac{h_1}{a}yN^{1/3} + o(N^{1/3})$$

and define the function γ such that

$$N\gamma(N) = n + \frac{h_2}{b}yn^{1/3},$$

observe that because of $a/h_1 \leq b/h_2$ we have

$$N\gamma(N) \leq n + \frac{h_1}{a}yn^{1/3} = N. \tag{21}$$

With this notation, (20) becomes

$$\mathbb{P}(\mathcal{L}^{\mathbf{h}}_{(aN,b\gamma(N)N)} \leq \lambda N + y(1-\frac{h_1\lambda}{a})N^{1/3} + o(N^{1/3})) = \mathbb{P}(T_{(\alpha n,\beta n)} \leq \lambda n + yn^{1/3}).$$

Let us notice that $1 - \frac{h_1\lambda}{a} > 0$ since we assume $g^{\mathbf{h}}(a,b) < \min\{a/h_1, b/h_2\}$. Using (21) we have that $\mathcal{L}^{\mathbf{h}}_{(aN,bN\gamma(N))} \leq \mathcal{L}^{\mathbf{h}}_{(aN,bN)}$ and therefore by putting $x = y(1 - h_1\lambda/a)$ we have for any $x \geq 0$,

$$\begin{aligned}\mathbb{P}(\mathcal{L}^{\mathbf{h}}_{(aN,bN)} \leq \lambda N + xN^{1/3} + o(N^{1/3})) &\leq \mathbb{P}(\mathcal{L}^{\mathbf{h}}_{(aN,bN\gamma(N))} \leq \lambda N + xN^{1/3} + o(N^{1/3}))\\ &\leq \mathbb{P}(T_{(\alpha n,\beta n)} \leq \lambda n + x(1-\frac{h_1\lambda}{a})^{-1}n^{1/3}).\end{aligned}$$

Using Johansson's result, we obtain, for $x \geq 0$,

$$\limsup_{N\to\infty} \mathbb{P}(\mathcal{W}^{\mathbf{h}}_{(aN,bN)} \leq x) \leq \lim_{n\to\infty} \mathbb{P}(T_{(\alpha n,\beta n)} \leq \lambda n + x\sigma(\alpha,\beta)(1-\frac{h_1\lambda}{a})^{-1}n^{1/3}) = F_{TW}(x(1-\frac{h_1\lambda}{a})^{-1}).$$

We obtain the lower bound in the same way, setting

$$\tilde{N} := n + \frac{h_2}{b} y n^{1/3}$$

and $\tilde{\gamma}$ the function such that

$$\tilde{N}\tilde{\gamma}(\tilde{N}) = n + \frac{h_1}{a} y n^{1/3}.$$

Due to the condition $a/h_1 \leq b/h_2$, we now have $\tilde{\gamma}(\tilde{N}) \geq 1$ for any $y \geq 0$. The case $x \leq 0$ is also obtained with similar arguments.

Finally, note that in the particular case $a/h_1 = b/h_2$, we have $\gamma(N) = 1$, and we obtain the corollary. □

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