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# Weakness of the topological twin building of a masure for the cone topology 

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#### Abstract

Masures are generalizations of Bruhat-Tits buildings introduced by Gaussent and Rousseau in order to study Kac-Moody groups over valued fields. A masure admits a building at infinity $\operatorname{Ch}(\partial \Delta)$, which is a twin building. Ciobotaru, Mühlherr and Rousseau equipped $\operatorname{Ch}(\partial \Delta)$ with a topology called the cone topology. They proved that this equips $\operatorname{Ch}(\partial \Delta)$ with a structure of weak topological twin building in the definition of Hartnick, Köhl and Mars. In this note, we prove however that unless $G$ is reductive, $\mathrm{Ch}(\partial \Delta)$ is not a topological twin building.


## 1 Introduction

Bruhat-Tits buildings are an important tool in the study of reductive groups over nonarchimedean local fields. Kac-Moody groups are interesting generalizations of reductive groups. In order to have an analogous tool in the Kac-Moody frameworks, Gaussent and Rousseau defined masures in [GR08] (where they are called hovels). Thanks to the works of Charignon and Rousseau, a masure is now associated to each almost-split Kac-Moody group over a non-archimedean valued field, see [Cha10], [Rou11], [Rou16] and [Rou17]. Rousseau gave an axiomatic definition of masures in [Rou11] and we simplified it in [Héb17].

Let $G$ be a split Kac-Moody group over a non-archimedean local field and $\Delta$ be its masure. As in the Bruhat-Tits theory, $\Delta$ is a union of apartments, all isomorphic to a standard one $\mathbb{A}$. The group $G$ acts on $\Delta$ and the apartments of $\Delta$ are the $g . \mathbb{A}$ for $g \in G$. Let us describe $\mathbb{A}$. There is still a Weyl group $W^{v}$ and a system $\Phi$ of roots but contrary to the reductive case, $W^{v}$ and $\Phi$ are infinite except if $G$ is reductive. One can consider $\Phi$ as a set of linear forms on a finite dimensional affine space $\mathbb{A}$ and we can define a fundamental chamber $C_{f}^{v}$. The Tits cone is the set $\mathcal{T}=\bigcup_{w \in W^{v}} \overline{w \cdot C_{f}^{v}}$. When $G$ is reductive, $\mathcal{T}=\mathbb{A}$ but when $G$ is not reductive, $\mathcal{T}$ is a proper convex cone of $\mathbb{A}$. If $\epsilon \in\{-,+\}$, one defines $\partial \Delta_{\epsilon}$ to be the set of $g . \epsilon \xi$ such that $g \in G$ and $\xi$ is the direction of a ray included in $\epsilon \mathcal{T}$. Then $\partial \Delta_{+}$and $\partial \Delta_{-}$ are buildings of type $W^{v}$. Moreover, $\partial \Delta_{+}$and $\partial \Delta_{-}$are naturally twinned and one obtains a twin building $\partial \Delta$ at infinity of $\Delta$. This twin building coincides with the twin building of $G$.

If $G$ is reductive, then $\Delta$ is a Bruhat-Tits building. It is thus equipped with a structure of $\operatorname{CAT}(0)$-space. This enables in particular to define a topology - the cone topology - on the set of chambers $\partial \Delta$ of the building at infinity $\partial \Delta$. When $G$ is no more reductive, no structure of CAT(0)-space on $\Delta$ is known. However, motivated by the study of actions of
groups on $\Delta$, Ciobotaru and Rousseau generalized the definition of the cone topology to the frameworks of masures, see [CR15].

It seems that topological twin buildings were first studied and axiomatized by Kramer in [Kra02]. In [HKM13], Hartnick, Köhl and Mars propose an (a priori non equivalent) axiomatization of topological twin buildings. For them a topological twin building is a twin building satisfying the axioms TTB1 to TTB4 of Section 3 of [HKM13]. In [CMR17], which contains the results of [CR15], Ciobotaru, Mühlherr and Rousseau study the properties of the cone topology. They prove that it satisfies the axioms TTB1, TTB2 and TTB4. Thus $\partial \Delta$ is a weak topological building in the terminology of [HKM13]. They also prove that it satisfies additional axioms and variations, see 3.4 of [CMR17]. The aim of this note is to prove that when $G$ is not reductive, then TTB3 is not satisfied and thus $\partial \Delta$ is not a topological twin building (see Proposition 3.1).

Actually we do not limit our study to masures associated to Kac-Moody groups: for us a masure is a set satisfying the axioms of [Héb17] (which are equivalent to the axioms of [Rou11]) and whose apartments are associated to a root generating system (and thus to a Kac-Moody matrix). We do not assume that there exists a group acting strongly transitively on it. In order to simplify the notation, we consider only semi-discrete masures, which means that if $M$ is a wall of $\mathbb{A}$, the set of walls parallel to it is discrete. When the masure is associated to a Kac-Moody group over a valued field, this is equivalent to assuming that the valuation is discrete. This hypothesis can be suppressed with minor changes.

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## 2 Masure and cone topology

In this section we give the definition of abstract masures as defined in [Rou11]. We use the axiomatic of [Héb17]. We only define semi-discrete masures, see [Rou11], [Rou17] or [Héb17] for a definition in a full generality.

### 2.1 Vectorial apartment

### 2.1.1 Root generating system

A Kac-Moody matrix (or generalized Cartan matrix) is a square matrix $A=\left(a_{i, j}\right)_{i, j \in I}$ with integers coefficients, indexed by a finite set $I$ and satisfying:

1. $\forall i \in I, a_{i, i}=2$
2. $\forall(i, j) \in I^{2} \mid i \neq j, a_{i, j} \leq 0$
3. $\forall(i, j) \in I^{2}, a_{i, j}=0 \Leftrightarrow a_{j, i}=0$.

A root generating system is a 5 -tuple $\mathcal{S}=\left(A, X, Y,\left(\alpha_{i}\right)_{i \in I},\left(\alpha_{i}^{\vee}\right)_{i \in I}\right)$ made of a KacMoody matrix $A$ indexed by $I$, of two dual free $\mathbb{Z}$-modules $X$ (of characters) and $Y$ (of cocharacters) of finite rank $\operatorname{rk}(X)$, a family $\left(\alpha_{i}\right)_{i \in I}$ (of simple roots) in $X$ and a family $\left(\alpha_{i}^{\vee}\right)_{i \in I}$ (of simple coroots) in $Y$. They have to satisfy the following compatibility condition:
$a_{i, j}=\alpha_{j}\left(\alpha_{i}^{\vee}\right)$ for all $i, j \in I$. We also suppose that the family $\left(\alpha_{i}\right)_{i \in I}$ is free in $X$ and that the family $\left(\alpha_{i}^{\vee}\right)_{i \in I}$ is free in $Y$.

We now fix a Kac-Moody matrix $A$ and a root generating system with matrix $A$.
Let $\mathbb{A}=Y \otimes \mathbb{R}$. Every element of $X$ induces a linear form on $\mathbb{A}$. We will consider $X$ as a subset of the dual $\mathbb{A}^{*}$ of $\mathbb{A}$ : the $\alpha_{i}$ 's, $i \in I$ are viewed as linear forms on $\mathbb{A}$. For $i \in I$, we define an involution $r_{i}$ of $\mathbb{A}$ by $r_{i}(v)=v-\alpha_{i}(v) \alpha_{i}^{\vee}$ for all $v \in \mathbb{A}$. Its space of fixed points is $\operatorname{ker} \alpha_{i}$. The subgroup of GL( $\left.\mathbb{A}\right)$ generated by the $\alpha_{i}$ 's for $i \in I$ is denoted by $W^{v}$ and is called the Weyl group of $\mathcal{S}$. The system ( $W^{v},\left\{r_{i} \mid i \in I\right\}$ ) is a Coxeter system.

Let $Q_{\mathbb{Z}}^{\vee}=\bigoplus_{i \in I} \mathbb{Z} \alpha_{i}^{\vee}$ be the coroot lattice of $\mathbb{A}$.
One defines an action of the group $W^{v}$ on $\mathbb{A}^{*}$ by the following way: if $x \in \mathbb{A}, w \in W^{v}$ and $\alpha \in \mathbb{A}^{*}$ then $(w . \alpha)(x)=\alpha\left(w^{-1} \cdot x\right)$. Let $\Phi=\left\{w \cdot \alpha_{i} \mid(w, i) \in W^{v} \times I\right\}, \Phi$ is the set of real roots. Then $\Phi \subset Q$, where $Q=\bigoplus_{i \in I} \mathbb{Z} \alpha_{i}$ is the root lattice of $\mathbb{A}$. Let $W=Q^{\vee} \rtimes W^{v} \subset \mathrm{GA}(\mathbb{A})$ be the affine Weyl group of $\mathcal{S}$, where $\mathrm{GA}(\mathbb{A})$ is the group of affine isomorphisms of $\mathbb{A}$.

### 2.2 Vectorial faces and Tits cone

Define $C_{f}^{v}=\left\{v \in \mathbb{A} \mid \alpha_{i}(v)>0, \forall i \in I\right\}$. We call it the fundamental chamber. For $J \subset I$, one sets $F^{v}(J)=\left\{v \in \mathbb{A} \mid \alpha_{i}(v)=0 \forall i \in J, \alpha_{i}(v)>0 \forall i \in J \backslash I\right\}$. Then the closure $\overline{C_{f}^{v}}$ of $C_{f}^{v}$ is the union of the $F^{v}(J)$ for $J \subset I$. The positive (resp. negative) vectorial faces are the sets $w \cdot F^{v}(J)$ (resp. $-w \cdot F^{v}(J)$ ) for $w \in W^{v}$ and $J \subset I$. A vectorial face is either a positive vectorial face or a negative vectorial face. We call positive chamber (resp. negative) every cone of the shape $w . C_{f}^{v}$ for some $w \in W^{v}$ (resp. $-w . C_{f}^{v}$ ). For all $x \in C_{f}^{v}$ and for all $w \in W^{v}, w \cdot x=x$ implies that $w=1$. In particular the action of $w$ on the positive chambers is simply transitive. The Tits cone $\mathcal{T}$ is defined by $\mathcal{T}=\bigcup_{w \in W^{v}} w \cdot \overline{C_{f}^{v}}$. We also consider the negative cone $-\mathcal{T}$. We define a $W^{v}$-invariant relation $\leq$ on $\mathbb{A}$ by: $\forall(x, y) \in \mathbb{A}^{2}$, $x \leq y \Leftrightarrow y-x \in \mathcal{T}$. This relation is called the Tits preorder.

A vectorial face is spherical if its fixer in $W^{v}$ is finite.

### 2.3 Masure

### 2.3.1 Filters

Definition 2.1. A filter in a set $E$ is a nonempty set $F$ of nonempty subsets of $E$ such that, for all subsets $S$, $S^{\prime}$ of $E$, if $S, S^{\prime} \in F$ then $S \cap S^{\prime} \in F$ and, if $S^{\prime} \subset S$, with $S^{\prime} \in F$ then $S \in F$.

If $F$ is a filter in a set $E$, and $E^{\prime}$ is a subset of $E$, one says that $F$ contains $E^{\prime}$ if every element of $F$ contains $E^{\prime}$. If $E^{\prime}$ is nonempty, the set $F_{E^{\prime}}$ of subsets of $E$ containing $E^{\prime}$ is a filter. By abuse of language, we will sometimes say that $E^{\prime}$ is a filter by identifying $F_{E^{\prime}}$ and $E^{\prime}$. If $F$ is a filter in $E$, its closure $\bar{F}$ (resp. its convex hull) is the filter of subsets of $E$ containing the closure (resp. the convex hull) of some element of $F$. A filter $F$ is said to be contained in an other filter $F^{\prime}: F \subset F^{\prime}$ (resp. in a subset $Z$ in $E: F \subset Z$ ) if and only if any set in $F^{\prime}($ resp. if $Z)$ is in $F$.

If $x \in \mathbb{A}$ and $\Omega$ is a subset of $\mathbb{A}$ containing $x$ in its closure, then the germ of $\Omega$ in $x$ is the filter $\operatorname{germ}_{x}(\Omega)$ of subsets of $\mathbb{A}$ containing a neighborhood in $\Omega$ of $x$.

A sector in $\mathbb{A}$ is a set of the shape $\mathfrak{s}=x+C^{v}$ with $C^{v}= \pm w \cdot C_{f}^{v}$ for some $x \in \mathbb{A}$ and $w \in W^{v}$. The point $x$ is its base point and $C^{v}$ is its direction. The intersection of two sectors of the same direction is a sector of the same direction.

The sector-germ of a sector $\mathfrak{s}=x+C^{v}$ is the filter $\mathfrak{S}$ of subsets of $\mathbb{A}$ containing an $\mathbb{A}$-translate of $\mathfrak{s}$. It only depends on the direction $C^{v}$.

A sector-face in $\mathbb{A}$ is a set of the shape $\mathfrak{s}=x+F^{v}$ with $F^{v}= \pm w \cdot F^{v}(J)$ for some $x \in \mathbb{A}$, $w \in W^{v}$ and some $J \subset I$. The the germ at infinity or simply the germ of a sector-face $\mathfrak{s}=x+F^{v}$ is the filter $\mathfrak{S}$ of subsets of $\mathbb{A}$ containing $u+x+F^{v}$ for some $u \in \overline{F^{v}}$.

A ray $\Xi$ with base point $x$ and containing $y \neq x$ (or the interval $] x, y]=[x, y] \backslash\{x\}$ or $[x, y])$ is called preordered if $x \leq y$ or $y \leq x$ and generic if $y-x \in \pm \mathcal{T}$, the interior of $\pm \mathcal{T}$.

Let $\Xi$ be a generic ray. The germ $\operatorname{germ}_{\infty}(\Xi)$ of $\Xi$ is the set of subsets $E$ of $\mathbb{A}$ such that $\Xi \backslash E$ is bounded.

### 2.3.2 Definitions of walls, enclosures, faces and related notions

Enclosure A hyperplane of the form $\alpha^{-1}(\{k\})$ with $\alpha \in \Phi$ and $k \in \mathbb{Z}$ is called a wall. A half-space of $\mathbb{A}$ delimited by a wall is a half-apartment. If $\alpha \in \Phi$ and $k \in \mathbb{Z}$, one sets $M(\alpha, k)=\{x \in \mathbb{A} \mid \alpha(x)+k=0\}, D(\alpha, k)=\{x \in \mathbb{A} \mid \alpha(x)+k \geq 0\}$ and $D^{\circ}(\alpha, k)=\stackrel{\circ}{D}(\alpha, k)$.

A set $P \subset \mathbb{A}$ is said to be enclosed if there exist $n \in \mathbb{N}$ and half-apartments $D_{1}, \ldots, D_{n}$ such that $P=\bigcap_{i=1}^{n} D_{i}$.

If $\mathcal{X}$ is a filter of $\mathbb{A}$, its enclosure is the filter $\operatorname{cl}(\mathcal{X})$ defined as follows. A set $E$ is in $\operatorname{cl}(\mathcal{X})$ if and only if there exists an enclosed set $E^{\prime} \subset E$ such that $E^{\prime}$ is enclosed and $E^{\prime} \in \mathcal{X}$.

Faces A local face $F^{\ell}$ in $\mathbb{A}$ is a filter associated to a point $x \in \mathbb{A}$, its vertex and a vectorial face $F^{v} \subset \mathbb{A}$, its direction. It is defined by $F^{\ell}=\operatorname{germ}_{x}\left(x+F^{v}\right)$ and we denote it by $F^{\ell}\left(x, F^{v}\right)$. A face $F$ in $\mathbb{A}$ is a filter associated to a point $x \in \mathbb{A}$ and a vectorial face $F^{v} \subset \mathbb{A}$. More precisely, a subset $S$ of $\mathbb{A}$ is an element of the face $F=F\left(x, F^{v}\right)$ if and only if it contains a finite intersection of half-apartments or open half-apartments containing $F^{\ell}\left(x, F^{v}\right)$.

There is an order on the (local) faces: if $F \subset \overline{F^{\prime}}$ we say that " $F$ is a face of $F^{\prime}$ " or " $F^{\prime}$ contains $F$ " or " $F^{\prime}$ dominates $F$ ". The dimension of a face $F$ is the smallest dimension of an affine space generated by some $S \in F$. Such an affine space is unique and is called its support.

As $W^{v}$ stabilizes $\Phi$, any element of $W^{v}$ permutes the sets of the shape $D(\alpha, k)$ where $\alpha$ runs over $\Phi$ and $k \in \mathbb{Z}$. Thus $W$ permutes the enclosures, faces, $\ldots$ of $\mathbb{A}$.

A chamber (or alcove) is a maximal face, i.e a face $F^{\ell}\left(x, \pm w \cdot C_{f}^{v}\right)$ for $x \in \mathbb{A}$ and $w \in W^{v}$.
A panel is a spherical face maximal among faces that are not chambers or equivalently a spherical face of dimension $\operatorname{dim} \mathbb{A}-1$.

Chimneys A chimney in $\mathbb{A}$ is associated to a face $F=F\left(x, F_{0}^{v}\right)$ and to a vectorial face $F^{v}$; it is the filter $\mathfrak{r}\left(F, F^{v}\right)=\operatorname{cl}\left(F+F^{v}\right)$. The face $F$ is a basis of the chimney and the vectorial face $F^{v}$ its direction. A chimney is splayed if $F^{v}$ is spherical.

A shortening of a chimney $\mathfrak{r}\left(F, F^{v}\right)$, with $F=F\left(x, F_{0}^{v}\right)$ is a chimney of the shape $\mathfrak{r}\left(F\left(x+\xi, F_{0}^{v}\right), F^{v}\right)$ for some $\xi \in \overline{F^{v}}$. The germ of a chimney $\mathfrak{r}$ is the filter of subsets of $\mathbb{A}$ containing a shortening of $\mathfrak{r}$.

### 2.3.3 Masure

An apartment of type $\mathbb{A}$ is a set $A$ with a nonempty $\operatorname{set} \operatorname{Isom}(\mathbb{A}, A)$ of bijections (called Weyl-isomorphisms) such that if $f_{0} \in \operatorname{Isom}(\mathbb{A}, A)$ then $f \in \operatorname{Isom}(\mathbb{A}, A)$ if and only if, there exists $w \in W$ satisfying $f=f_{0} \circ w$. We will say isomorphism instead of Weyl-isomorphism in the sequel. An isomorphism between two apartments $\phi: A \rightarrow A^{\prime}$ is a bijection such that $\left(f \in \operatorname{Isom}(\mathbb{A}, A)\right.$ if, and only if, $\left.\phi \circ f \in \operatorname{Isom}\left(\mathbb{A}, A^{\prime}\right)\right)$. We extend all the notions that are
preserved by $W$ to each apartment. Thus sectors, enclosures, faces and chimneys are well defined in any apartment of type $\mathbb{A}$.

If $A, A^{\prime}$ are apartments, $\phi: A \rightarrow A^{\prime}$ is an isomorphism of apartments and $E \subset A \cap A^{\prime}$, the notation $\phi: A \xrightarrow{E} A^{\prime}$ means that $\phi$ fixes $E$.

Definition 2.2. A masure of type $\mathbb{A}$ is a set $\mathcal{I}$ endowed with a covering $\mathcal{A}$ of subsets called apartments such that:
(MA i): Any $A \in \mathcal{A}$ admits a structure of an apartment of type $\mathbb{A}$.
( $M A$ ii): if two apartments $A, A^{\prime}$ contain a generic ray, then $A \cap A^{\prime}$ is enclosed and there exists an isomorphism $\phi: A \xrightarrow{A \cap A^{\prime}} A^{\prime}$.
(MA iii): if $\Re$ is the germ of a splayed chimney and if $F$ is a face or a germ of a chimney, then there exists an apartment containing $\mathfrak{R}$ and $F$.

In this definition, one says that an apartment contains a germ of a filter if it contains at least one element of this germ. One says that a map fixes a germ if it fixes at least one element of this germ.

The masure is said to be thick if for each panel of $\Delta$, there exists at least three chambers dominating it.

When $\Delta$ is associated to an affine Kac-Moody group, one can replace (MA ii) by
(MA af ii): if $A, A^{\prime}$ are two apartments then $A \cap A^{\prime}$ is enclosed and there exists an isomorphism $\phi: A \xrightarrow{A \cap A^{\prime}} A^{\prime}$.

### 2.4 Twin building at infinity and cone topology

### 2.4.1 Buildings at infinity

We now define the twin building of $\Delta$, see also Section 3 of [Rou11] or 2.4.2 of [CMR17].
Let $F$ and $F^{\prime}$ (resp. $\xi$ and $\xi^{\prime}$ ) be two spherical sector-faces (resp. two germs of generic rays) of $\Delta$. By (MA iii), there exists an apartment $A$ containing their germs $\mathfrak{F}$ and $\mathfrak{F}^{\prime}$ (resp. $\xi$ and $\xi^{\prime}$ ). One says that $F$ and $F^{\prime}$ (resp. $\xi$ and $\xi^{\prime}$ ) are parallel if there exists a vectorial face $F_{A}^{v}$ (resp. a generic ray $\Xi_{A}$ ) of $A$ such that $\mathfrak{F}=\operatorname{germ}_{\infty}\left(x+F_{A}^{v}\right)$ and $\mathfrak{F}^{\prime}=\operatorname{germ}_{\infty}\left(x^{\prime}+F_{A}^{v}\right)$ for some $x, x^{\prime} \in A$ (resp. $\xi=\operatorname{germ}_{\infty}\left(x+\Xi_{A}\right)$ and $\xi^{\prime}=\operatorname{germ}_{\infty}\left(x^{\prime}+\Xi_{A}\right)$ ). This does not depend on the choice of $A$ (this is a consequence of (MA ii)). Parallelism is an equivalence relation. The parallelism class of a sector-face germ $\mathfrak{F}$ is denoted $\mathfrak{F}^{\infty}$ and is called the direction of $\mathfrak{F}$. The parallelism class of a generic ray is called its direction.

Let $\partial \Delta_{+}$(resp. $\partial \Delta_{-}$) be the set of directions of positive (resp. negative) spherical sectorfaces of $\Delta$. Then $\partial \Delta_{+}$and $\partial \Delta_{-}$are disjoint unless $\Delta$ is a Bruhat-Tits building (which happens if and only if $W^{v}$ is finite), in which case $\partial \Delta_{+}=\partial \Delta_{-}$. One sets $\partial \Delta=\partial \Delta_{+} \cup \partial \Delta_{-}$.

Let $\epsilon \in\{-,+\}$. One defines an order on $\partial \Delta_{\epsilon}$ as follows. Let $\mathfrak{F}_{1}^{\infty}, \mathfrak{F}_{2}^{\infty} \in \partial \Delta_{\epsilon}$. Let $A$ be an apartment containing $\mathfrak{F}_{1}^{\infty}, \mathfrak{F}_{2}^{\infty}$. Then one says that $\mathfrak{F}_{1}^{\infty}$ dominates $\mathfrak{F}_{2}^{\infty}$ if there exists sector-faces $F_{1}, F_{2}$ of $A$ such that $F_{1}$ dominates $F_{2}$ and whose directions are $\mathfrak{F}_{1}^{\infty}$ and $\mathfrak{F}_{2}^{\infty}$.

If $A$ is an apartment, one denotes by $\partial A_{\epsilon}$ the set of directions of sector-faces of sign $\epsilon$ and one sets $\partial A=\partial A_{+} \cup \partial A_{-}$. Let $\mathcal{A}$ be the set of apartments of $\Delta$ and $\partial \mathcal{A}=\{\partial A \mid A \in \mathcal{A}\}$. The elements of $\partial \mathcal{A}$ are called apartments at infinity. Then by Théorème 3.4 of [Rou11], $\partial \Delta_{\epsilon}$ equipped with its system of apartments and the domination relation is a combinatorial building of type $W^{v}$ (see 2.4.1 of [Rém02] for a definition, it is however not really a simplicial complex, as we consider only directions of spherical sector-faces).

If $\epsilon \in\{-,+, \emptyset\}$, one denotes by $\operatorname{Ch}(\partial \Delta)_{\epsilon} \subset \partial \Delta_{\epsilon}$ the set of directions of sectors. The elements of $\operatorname{Ch}(\partial \Delta)$ are called chambers at infinity.

### 2.4.2 $W^{v}$-distance and codistance

Let $\epsilon \in\{-,+\}$. Let $\left(\mathfrak{F}_{1}^{\infty}, \mathfrak{F}_{2}^{\infty}\right)$ in $\left(\operatorname{Ch}(\partial \Delta)_{\epsilon}\right)^{2}$ (resp. in $\left(\operatorname{Ch}(\partial \Delta)_{\epsilon} \times \operatorname{Ch}(\partial \Delta)_{-\epsilon}\right) \cup\left(\operatorname{Ch}(\partial \Delta)_{-\epsilon} \times\right.$ $\left.\operatorname{Ch}(\partial \Delta)_{\epsilon}\right)$ ). Let $A$ be an apartment containing $\mathfrak{F}_{1}^{\infty}, \mathfrak{F}_{2}^{\infty}$, which exists by (MA iii). Let $\phi: A \rightarrow \mathbb{A}$ be an isomorphism of apartments. Let $F_{1}^{v}, F_{2}^{v}$ be vectorial chambers of $\mathbb{A}$ whose directions are $\phi\left(\mathfrak{F}_{1}^{\infty}\right)$ and $\phi\left(\mathfrak{F}_{2}^{\infty}\right)$. Let $w \in W^{v}$ such that $F_{1}^{v}=w \cdot F_{2}^{v}$ (resp. $F_{1}^{v}=-w \cdot F_{2}^{v}$ ). One sets $\delta\left(\mathfrak{F}_{1}^{\infty}, \mathfrak{F}_{2}^{\infty}\right)=w$ (resp. $\left.\delta^{*}\left(\mathfrak{F}_{1}^{\infty}, \mathfrak{F}_{2}^{\infty}\right)=w\right)$. These definitions do not depend on the choices we made. Then $\delta: \operatorname{Ch}(\partial \Delta)_{\epsilon} \times \operatorname{Ch}(\partial \Delta)_{\epsilon} \rightarrow W^{v}$ is a $W^{v}$-distance on $\operatorname{Ch}(\partial \Delta)_{\epsilon}$ for the definition of 2.3.1 of [Rém02].

By Théorème 3.7 of [Rou11], the codistance $\delta^{*}:\left(\operatorname{Ch}(\partial \Delta)_{\epsilon} \times \operatorname{Ch}(\partial \Delta)_{-\epsilon}\right) \cup\left(\operatorname{Ch}(\partial \Delta)_{-\epsilon} \times\right.$ $\left.\operatorname{Ch}(\partial \Delta)_{\epsilon}\right) \rightarrow W^{v}$ defines a twinning of $\mathrm{Ch}(\partial \Delta)_{+}$and $\mathrm{Ch}(\partial \Delta)_{-}$(see 2.5.1 of [Rém02] for the definition of a twinning).

### 2.4.3 Cone topology on $\operatorname{Ch}(\partial \Delta)$

Let $+\infty$ be the germ at infinity of $C_{f}^{v}$. One chooses a ray $\Xi_{+\infty} \subset C_{f}^{v}$ based at 0 and one denotes by $\xi_{+\infty}$ its germ. Let $c \in \operatorname{Ch}(\partial \Delta)$. Then there exists $\epsilon \in\{-,+\}$ and a isomorphism of apartments such that $c=\phi(\epsilon \infty)$. One sets $\xi_{c}=\phi\left(\epsilon \xi_{+\infty}\right)$. This definition does not depend on the choices we made.

Let $x \in \Delta$ and $\mathfrak{F}^{\infty} \in \partial \Delta$ (resp. $\xi$ be the direction of a generic ray). Then there exists a unique sector-face $Q_{x, \mathfrak{F}^{\infty}}$ (resp. a unique generic ray $[x, \xi)$ ) based at $x$ and whose direction is $\mathfrak{F}^{\infty}$ (resp. whose direction is $\xi$ ). Indeed, by (MA iii), there exists an apartment $A$ containing $x$ and $\mathfrak{F}^{\infty}$. Let $F$ be a sector face of $A$ whose direction is $\mathfrak{F}^{\infty}$. Then $Q_{x, \mathfrak{F}^{\infty}}$ is the translate of $F$ in $A$ based at $x$. This does not depend on the choices we made by Proposition 4.7 1) of [Rou11]. One obtains the result for rays similarly.

Let $x \in \Delta, c \in \operatorname{Ch}(\partial \Delta)$ and $r \in\left[x, \xi_{c}\right)$. One sets

$$
U_{x, r, c}=\left\{c^{\prime} \in \operatorname{Ch}(\partial \Delta) \mid[x, r] \subset\left[x, \xi_{c^{\prime}}\right)\right\} .
$$

The cone topology on $\operatorname{Ch}(\partial \Delta)$ is the topology generated by the $U_{x, r, c}$ such that $c \in \operatorname{Ch}(\partial \Delta)$ and $r \in\left[x, \xi_{c}\right)$. By Lemma 3.3 and Proposition 3.5 of [CMR17], this topology depend neither on the choice of $\xi_{+\infty}$ nor on the choice of $x \in \Delta$.

The cone topology is Hausdorff (Lemma 3.13 of [CMR17]). Indeed, if $c \neq c^{\prime} \in \operatorname{Ch}(\partial \Delta)$, one chooses an apartment $A$ containing $c, c^{\prime}$. One can suppose that $x \in A$. Then for all $r \in\left(x, \xi_{c}\right)$ and $r^{\prime} \in\left(x, \xi_{c^{\prime}}\right)$, one has $c \in U_{x, r, c}, c^{\prime} \in U_{x, r, c^{\prime}}$ and $U_{x, r, c} \cap U_{x, r^{\prime}, c^{\prime}}=\emptyset$.

From now on, we assume that $x=0$.

## 3 Non-satisfaction of the axiom (TTB3)

If $c \in \operatorname{Ch}(\partial \Delta)$ and $w \in W^{v}$, one sets $E_{\leq w}(c)=\left\{d \in \operatorname{Ch}(\partial \Delta)_{\epsilon} \mid \delta(c, d) \leq w\right\}$, where $\epsilon$ is the sign of $c$ and $\leq$ is the Bruhat order.

On says that $\operatorname{Ch}(\partial \Delta)_{\epsilon}=\lim _{\rightarrow} E_{\leq w}(c)$ topologically if it satisfies:
" subset $U \subset \operatorname{Ch}(\partial \Delta)$ is open if and only if $U \cap E_{\leq w}(c)$ is open in $E_{\leq w}(c)$ (for the topology induced by the cone topology) for all $w \in W^{v}$ ".

The aim of this note is to prove the following proposition:
Proposition 3.1. Let $\Delta$ be a thick masure such that $W^{v}$ is infinite. We equip $\operatorname{Ch}(\partial \Delta)$ with the cone topology. Then the axiom (TTB3) of [HKM13] is not satisfied, which means that for all $c \in \operatorname{Ch}(\partial \Delta)$, one has topologically $\operatorname{Ch}(\partial \Delta)_{\epsilon} \neq \lim _{\rightarrow} E_{\leq w}(c)$, where $\epsilon$ is the sign of $c$. In particular, $\operatorname{Ch}(\partial \Delta)$ is not a topological twin building in the definition of [HKM13].

In order to prove this, we construct for each chamber $c$ in $\operatorname{Ch}(\partial \Delta)$ a set $U \subset \operatorname{Ch}(\partial \Delta)$ such that:

- $U \cap E_{\leq w}(c)$ is open for all $w \in W^{v}$
- $U$ is not open.

Let us fix some notation. Let $\Delta$ be a thick masure with $W^{v}$ infinite. Let $c \in \operatorname{Ch}(\partial \Delta)$ and $\epsilon$ be its sign. We will consider only chambers of $\operatorname{sign} \epsilon$. Let $A$ be an apartment containing $c$. One identifies $A$ and $\mathbb{A}$. If $d \in \operatorname{Ch}(\partial \Delta)$ and $r \in\left[0, \xi_{d}\right)$, one writes $U_{r, d}$ instead of $U_{0, r, d}$.

If $w \in W^{v}$, one sets $E_{\leq w}=E_{\leq w}(c)$ and $F_{w}=\bigcup_{v \in W^{v} \mid \ell(v) \leq \ell(w)} E_{\leq v}$.
Let $\alpha \in \Phi$ such that $D(\alpha, k) \nsupseteq c$ for all $k \in \mathbb{R}$.
If $w \in W^{v}$, one chooses an apartment $A_{w}$ such that $\mathbb{A} \cap A_{w}=D(\alpha, \ell(w))$, which is possible by Proposition 2.9 of [Rou11]. By (MA ii), there exists a unique isomorphism $\phi_{w}: \mathbb{A} \rightarrow A_{w}$ fixing $\mathbb{A} \cap A_{w}$ pointwise. One identifies $\mathbb{R}_{+}$and $\left[0, \xi_{c}\right)$. Let $r_{w} \in \mathbb{R}_{+}$such that $D(\alpha, \ell(w)) \cap\left[0, \xi_{c}\right)=\left[0, r_{w}\right]$.

If $A$ is an apartment such that $A \cap \mathbb{A}$ contains 0 in its interior, there exists a unique chamber $d$ of $\partial A$ such that $\left[0, \xi_{d}\right) \cap\left[0, \xi_{c}\right) \neq\{0\}$ and we denote it by $c_{A}$. Indeed, the enclosure $d$ of the ray of $A$ based at 0 and containing the germ of $\left[0, \xi_{c}\right)$ in 0 satisfies $\left[0, \xi_{d}\right) \cap\left[0, \xi_{c}\right) \neq 0$. The uniqueness is a consequence of the fact that two vectorial faces of $\mathbb{A}$ are equal or disjoint. In particular if $\phi: \mathbb{A} \rightarrow A$ is an isomorphism fixing a neighborhood of 0 , then $c_{A}=\phi(c)$.

Let $d \in \operatorname{Ch}(\partial \Delta), A$ be an apartment containing $Q_{0, d}$ and $\phi: \mathbb{A} \rightarrow A$ be an isomorphism sending $Q_{0, c}$ on $Q_{0, d}$. If $r \in \mathbb{R}_{+}$, one sets $[0, r]_{d}=\phi([0, r])$ and $U_{r, d}=U_{\phi(r), d}$.

### 3.1 Construction of a sequence $\left(d_{w}\right)_{w \in W^{v}}$

The aim of this subsection is to construct a sequence $\left(d_{w}\right)_{w \in W^{v}}$ of chambers such that for all $w \in W^{v}, d_{w} \notin F_{w}$ and $\left[0, \xi_{d_{w}}\right) \cap\left[0, \xi_{c}\right)=\left[0, r_{w}\right]$.

An element $s \in W^{v}$ is called a reflection of $\mathbb{A}$ if it is of the shape $w \cdot r_{i} \cdot w^{-1}$ for some $w \in W^{v}$ and $i \in I$. Using isomorphisms of apartments, we extend this notion to each apartment of $\Delta$. Let $M$ be a wall of $\mathbb{A}$. One writes $M=\left(w \cdot \alpha_{i}\right)^{-1}(\{0\})$ for some $i \in I$ and $w \in W^{v}$. Then $w \cdot r_{i} \cdot w^{-1}$ is a reflection fixing $M$ and thus the number of reflections of $\mathbb{A}$ fixing 0 is infinite.

Lemma 3.2. Let $A$ be an apartment such that $A \cap \mathbb{A}$ contains 0 in its interior. Let $A_{1}$ be an apartment such that $A_{1} \cap \mathbb{A}$ contains 0 in its interior and such that $A \cap A_{1}$ is a half-apartment which does not contain $c_{A}$. Let $M$ be the wall of $A \cap A_{1}$ and $A_{2}=M \cup A_{1} \backslash A \cup A \backslash A_{1}$. Let $s$ be the reflection of $A_{2}$ fixing $M$. Then:

1. One has $c_{A_{1}}=s\left(c_{A}\right)$.
2. Let $s^{\prime}: A \rightarrow A$ be a reflection fixing $M$ and $f: A \rightarrow \mathbb{A}$ be a isomorphism. Let $s^{\prime \prime} \in W^{v}$ be the vectorial part of $f \circ s^{\prime} \circ f^{-1} \in W^{v}$. Then $\delta\left(c_{A}, c_{A_{1}}\right)=s^{\prime \prime}$.

Proof. Let $\phi_{0}: A_{1} \xrightarrow{A_{1} \cap A_{2}} A_{2}, \phi_{1}: A \xrightarrow{A \cap A_{2}} A_{2}$ and $\phi_{2}: A \xrightarrow{A \cap A_{1}} A_{1}$. Then by Lemma 3.4 of [Héb16], the following diagram is commutative:


One has $c_{A_{1}}=\phi_{2}\left(c_{A}\right)=\phi_{0} \circ \phi_{2}\left(c_{A}\right)$ and $c_{A}=\phi_{1}\left(c_{A}\right)$, which proves 1 , from which we deduce 2.

Lemma 3.3. The set $\left\{\delta\left(c_{A_{w}}, c\right) \mid w \in W^{v}\right\}$ is finite.
Proof. Let $c^{\prime}$ be a chamber of $\mathbb{A}$ such that $c^{\prime} \subset D(\alpha, k)$ for all $k \in \mathbb{Z}$. Let $w \in W^{v}$. One has $c_{A_{w}}=\phi_{w}(c)$. One has $\delta\left(c^{\prime}, c\right)=\delta\left(\phi_{w}\left(c^{\prime}\right), \phi_{w}(c)\right)=\delta\left(c^{\prime}, c_{A_{w}}\right)$ and the lemma follows.

For the next lemma, one uses the Tits preorder on $\Delta$. Let us define it. As the Tits preorder $\leq$ on $\mathbb{A}$ is $W$-invariant, one can define a Tits preorder $\leq_{A}$ on each apartment. Let $x, y \in \Delta$. By Proposition 5.4 of [Rou11], if there exists an apartment $A$ containing $\{x, y\}$ for which $x \leq_{A} y$, then $x \leq_{A^{\prime}} y$ for each apartment $A^{\prime}$ containing $\{x, y\}$. Thus this defines a relation $\leq$ on $\Delta$. By Théorème 5.9 of [Rou11], this relation is a preorder and we call it the Tits preorder.

Lemma 3.4. Let $w \in W^{v}$. Then there exists $d_{w} \in \operatorname{Ch}(\partial \Delta)$ such that $d_{w} \notin F_{w}$ and $\left[0, \xi_{d_{w}}\right) \cap$ $\left[0, \xi_{c}\right)=\left[0, r_{w}\right]$.

Proof. Let $s^{\prime \prime} \in W^{v}$ be a reflection such that $\ell\left(s^{\prime \prime}\right)>\ell(w)+1+\max \left\{\ell\left(\delta\left(c_{A_{w}}, c\right)\right) \mid w \in W^{v}\right\}$. Let $M^{\prime \prime}$ be the fixed wall of $s^{\prime \prime}$ and $M^{\prime}=\phi_{w}\left(M^{\prime \prime}\right)$. Let $A_{1}$ be an apartment satisfying the following conditions:

- $A_{w} \cap A_{1}$ is a half-apartment containing $\left[0, r_{w}\right]$ in its interior,
- the wall of $A_{w} \cap A_{1}$ is parallel to $M^{\prime}$,
- $A_{w} \cap A_{1}$ does not contain $c_{A_{w}}$.

Let $d_{w}=c_{A_{1}}$. Then $\ell\left(\delta\left(d_{w}, c\right)\right) \geq \ell\left(\delta\left(d_{w}, c_{A_{w}}\right)\right)-\ell\left(\delta\left(c_{A_{w}}, c\right)\right)=\ell\left(s^{\prime \prime}\right)-\ell\left(\delta\left(c_{A_{w}}, c\right)\right) \geq$ $\ell(w)+1$.

By construction, $\left[0, \xi_{d_{w}}\right) \cap\left[0, \xi_{c}\right)$ contains $\left[0, r_{w}\right]$. Suppose that there exists $z \in\left[0, \xi_{d_{w}}\right) \cap$ $\left[0, \xi_{c}\right) \backslash\left[0, r_{w}\right]$. Then $z \geq r_{w}$. By Proposition 5.4 of [Rou11], $\left[r_{w}, z\right]_{A_{1}}=\left[r_{w}, z\right]_{\mathbb{A}}$. Moreover for $z^{\prime} \in\left(r_{w}, z\right)_{A_{1}}$ near enough from $r_{w}, z^{\prime} \in A_{w}$ and consequently $z^{\prime} \in A_{w} \cap \mathbb{A}$. This is absurd because $A_{w} \cap \mathbb{A} \cap\left[0, \xi_{c}\right)=\left[0, r_{w}\right]$. Therefore $\left[0, \xi_{d_{w}}\right) \cap\left[0, \xi_{c}\right)=\left[0, r_{w}\right]$ and the lemma is proved.

### 3.2 Construction of $U$

Let $\left(d_{w}\right)_{w \in W^{v}}$ such that for all $w \in W^{v}, d_{w} \notin F_{w}$ and $\left[0, \xi_{d_{w}}\right) \cap\left[0, \xi_{c}\right)=\left[0, r_{w}\right]$, where $\left[0, r_{w}\right]=\left[0, \xi_{c}\right) \cap D(\alpha$, ell $(w))$. Let $\mathcal{D}=\left\{d_{w} \mid w \in W^{v}\right\}$ and $\overline{\mathcal{D}}=\mathcal{D} \cup\{c\}$. We now construct a set $U$ containing $c$, such that $U \cap \mathcal{D}=\emptyset$ and such that $U \cap E_{\leq w}$ is open for all $w \in W^{v}$.

Lemma 3.5. Let $V$ be an nonempty open set and $d \in V$. Then there exists $r \in \mathbb{R}_{+}^{*}$ such that $V \supset U_{r, d}$.

Proof. By definition, there exist $J \subset \operatorname{Ch}(\partial \Delta)$ and $\left(r_{d^{\prime}}\right) \in\left(\mathbb{R}_{+}^{*}\right)^{J}$ such that $V=\bigcup_{d^{\prime} \in J} U_{r_{d^{\prime}}, d^{\prime}}$. Let $d^{\prime} \in J$ such that $d \in U_{r_{d^{\prime}}, d^{\prime}}$. Then $U_{r_{d^{\prime}}, d}=U_{r_{d^{\prime}}, d^{\prime}}$, thus $U_{r_{d^{\prime}}, d} \subset V$, which proves the lemma.

Lemma 3.6. Let $d \in \operatorname{Ch}(\partial \Delta)$. Then $\bigcap_{r \in \mathbb{R}_{+}^{*}} U_{r, d}=\{d\}$.
Lemma 3.7. Let $d \in \operatorname{Ch}(\partial \Delta) \backslash \overline{\mathcal{D}}$. Then there exists $a_{d} \in \mathbb{R}_{+}$such that $U_{a_{d}, d} \cap \overline{\mathcal{D}}=\emptyset$.

Proof. As $\operatorname{Ch}(\partial \Delta)$ is Hausdorff, there exist open sets $V_{d} \ni d$ and $V_{c} \ni c$ such that $V_{c} \cap V_{d}=\emptyset$. One has $\lim _{\ell(w) \rightarrow+\infty} d_{w}=c$ and thus for $\ell(w)$ large enough, $d_{w} \in V_{c}$. Therefore $V_{d} \cap \overline{\mathcal{D}}$ is finite. By Lemma 3.5, one can suppose, reducing $V_{d}$ if necessary that $V_{d}=U_{r, d}$ for some $r \in \mathbb{R}_{+}$. We conclude with Lemma 3.6.

Lemma 3.8. Let $w \in W^{v}$. Then $\left(U_{r_{w}, c} \cap E_{\leq w}\right) \cap \mathcal{D}$ is empty.
Proof. Let $v \in W^{v}$. If $\ell(v)<\ell(w)$, then $\left[0, \xi_{d_{v}}\right) \cap\left[0, \xi_{c}\right)=\left[0, r_{v}\right] \subsetneq\left[0, r_{w}\right]$ and thus $d_{v} \notin U_{r_{w}, c}$. If $\ell(w) \leq \ell(v)$, then $d_{v} \notin F_{v}$ by construction. As $F_{v} \supset E_{\leq w}$, the lemma follows.

If $w \in W^{v}$, one sets

$$
U_{w}=\bigcup_{d \in\left(U_{r_{w}, c} \cap E_{\leq w}\right) \backslash\{c\}} U_{r_{w}+a_{d}, d},
$$

which is well-defined by the lemma above. One sets $U=\bigcup_{w \in W^{v}} U_{w} \cup\{c\}$. By construction, $U \cap \mathcal{D}=\emptyset$.

Lemma 3.9. Let $w \in W^{v}$. Then $U_{w} \subset U_{r_{w}, c}$.
Proof. Let $d \in U_{r_{w}, c}$ and $d^{\prime} \in U_{r_{w}+a_{d}, d}$. Then

$$
\left[0, \xi_{d^{\prime}}\right) \cap\left[0, \xi_{d}\right) \supset\left[0, r_{w}+a_{d}\right]_{d} \supset\left[0, r_{w}\right]_{d}=\left[0, r_{w}\right]_{c},
$$

thus $d^{\prime} \in U_{r_{w}, c}$ and the lemma follows.
Lemma 3.10. Let $w \in W^{v}$. Then $\left(U_{w} \cup\{c\}\right) \cap E_{\leq w}=U_{r_{w}, c} \cap E_{\leq w}$.
Proof. Let $d \in\left(U_{r_{w}, c} \cap E_{\leq w}\right) \backslash\{c\}$. Then $d \in U_{r_{w}+a_{d}, d}$ and thus

$$
d \in \bigcup_{d^{\prime} \in\left(U_{r_{w}}, c \cap E_{\leq w}\right) \backslash\{c\}} U_{r_{w}+a_{d^{\prime}}, d^{\prime}}=U_{w} .
$$

Therefore $U_{r_{w}, c} \cap E_{\leq w} \subset\left(U_{w} \cup\{c\}\right) \cap E_{\leq w}$.
By Lemma 3.9, $\left(U_{w} \cup\{c\}\right) \cap E_{\leq w} \subset U_{r_{w}, c} \cap E_{\leq w}$ and the lemma follows.
The following lemma implies Proposition 3.1.
Lemma 3.11. The set $U$ is not open but for all $w \in W^{v}, U \cap E_{\leq w}$ is open in $E_{\leq w}$.
Proof. One has $d_{w} \rightarrow c, U \ni c$ but $U \cap \mathcal{D}=\emptyset$ and thus $U$ is not open.
Let $w \in W^{v}$. Then

$$
U \cap E_{\leq w}=\left(\left(U_{w} \cup\{c\}\right) \cap E_{\leq w}\right) \cup\left(\bigcup_{v \in W^{v} \backslash\{w\}}\left(U_{v} \cap E_{\leq w}\right)\right)=\left(U_{r_{w}, c} \cup \bigcup_{v \in W^{v} \backslash\{w\}} U_{v}\right) \cap E_{\leq w}
$$

is open in $E_{\leq w}$.

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