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► **To cite this version:**

Michael Heusener, Richard Weidmann. A remark on Whitehead's cut-vertex lemma. Journal of Group Theory, 2019. hal-01794982v1

HAL Id: hal-01794982

<https://hal.science/hal-01794982v1>

Submitted on 25 May 2018 (v1), last revised 23 Feb 2019 (v2)

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A REMARK ON WHITEHEAD'S LEMMA

MICHAEL HEUSENER AND RICHARD WEIDMANN

ABSTRACT. We observe that Whitehead's lemma is an immediate consequence of Stallings folds.

INTRODUCTION

Whitehead's lemma [4] states that the Whitehead graph of a primitive element of the free group is either not connected or has a cut vertex. This result was generalized by Stallings to separable subsets of free groups [3]. In this note we observe that this is an essentially trivial consequence of Stallings folds. To do so we relate the connectivity properties of the Whitehead graph of a set of conjugacy classes of elements to the readability of cyclically reduced representatives of these classes in a class of labeled graphs that we call almost-roses.

1. FREE GROUPS AND GRAPHS

In the following we denote the free group in $X = \{x_1, \dots, x_n\}$ by F_n . Throughout this note $n \geq 2$ and X are fixed. As usual we represent elements of F_n by words in $\tilde{X} := X \cup X^{-1}$. For any $g \in F_n$ we denote the conjugacy class of g by $[g]$ and for $S \subset F_n$ we set $[S] := \{[s] \mid s \in S\}$. We say that $g \in F_n$ is primitive if g is contained in a basis of F_n . We further call a set $S \subset F_n$ separable if there exists a non-trivial decomposition $F_n = F^1 * F^2$ such that any $w \in S$ is conjugate into F^1 or F^2 , see [3]. Clearly $\{g\}$ is separable for any primitive element $g \in F_n$.

An \tilde{X} -labeled graph Γ is a directed graph (with inverse edges) with a labeling map $\ell : E\Gamma \rightarrow \tilde{X}$ such that $\ell(e^{-1}) = \ell(e)^{-1}$ for any edge $e \in E\Gamma$. Multiple edges and loops are permitted. The label of a path $\gamma = e_1 \cdots e_k$ in Γ is the word $\ell(\gamma) = \ell(e_1) \cdots \ell(e_k)$.

In the following R_n is the \tilde{X} -labeled graph whose underlying graph has vertex set $V = \{v_0\}$ and edge set $\{e_1^{\pm 1}, \dots, e_n^{\pm 1}\}$ such that $\ell(e_i^\varepsilon) = x_i^\varepsilon$ for all $\varepsilon \in \{-1, 1\}$ and $1 \leq i \leq n$. As usual we identify F_n with $\pi_1(R_n, v_0)$ in the obvious way. For any \tilde{X} -labeled graph Γ there is a unique label preserving morphism $f : \Gamma \rightarrow R_n$.

We say that a word w is readable in Γ if there exists a closed path in Γ with label w . We will say that an element $g \in F_n$ is readable in Γ if the corresponding reduced word is readable in Γ . We will further say that a conjugacy class $[g]$ is readable in Γ if some (and therefore all) cyclically reduced word representing $[g]$ is readable in Γ .

We will be mostly interested in connected \tilde{X} -labeled graphs Γ such that associated map $f_* : \pi_1(\Gamma) \rightarrow \pi_1(R_n) = F_n$ is surjective. In this case the morphism f can be written as a product of Stallings folds, see [2].

Recall that a graph Γ is called a core graph if it contains no proper subgraph such that the inclusion is a homotopy equivalence; for a finite graph this just means that Γ has no vertex of valence 1. We also call a pair (Γ, v_0) a core pair if Γ contains no proper