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# A REMARK ON WHITEHEAD'S LEMMA

MICHAEL HEUSENER AND RICHARD WEIDMANN

ABSTRACT. We observe that Whitehead's lemma is an immediate consequence of Stallings folds.

## INTRODUCTION

Whitehead's lemma [4] states that the Whitehead graph of a primitive element of the free group is either not connected or has a cut vertex. This result was generalized by Stallings to separable subsets of free groups [3]. In this note we observe that this is an essentially trivial consequence of Stallings folds. To do so we relate the connectivity properties of the Whitehead graph of a set of conjugacy classes of elements to the readability of cyclically reduced representatives of these classes in a class of labeled graphs that we call almost-roses.

## 1. FREE GROUPS AND GRAPHS

In the following we denote the free group in  $X = \{x_1, \dots, x_n\}$  by  $F_n$ . Throughout this note  $n \geq 2$  and  $X$  are fixed. As usual we represent elements of  $F_n$  by words in  $\tilde{X} := X \cup X^{-1}$ . For any  $g \in F_n$  we denote the conjugacy class of  $g$  by  $[g]$  and for  $S \subset F_n$  we set  $[S] := \{[s] \mid s \in S\}$ . We say that  $g \in F_n$  is primitive if  $g$  is contained in a basis of  $F_n$ . We further call a set  $S \subset F_n$  separable if there exists a non-trivial decomposition  $F_n = F^1 * F^2$  such that any  $w \in S$  is conjugate into  $F^1$  or  $F^2$ , see [3]. Clearly  $\{g\}$  is separable for any primitive element  $g \in F_n$ .

An  $\tilde{X}$ -labeled graph  $\Gamma$  is a directed graph (with inverse edges) with a labeling map  $\ell : E\Gamma \rightarrow \tilde{X}$  such that  $\ell(e^{-1}) = \ell(e)^{-1}$  for any edge  $e \in E\Gamma$ . Multiple edges and loops are permitted. The label of a path  $\gamma = e_1 \cdots e_k$  in  $\Gamma$  is the word  $\ell(\gamma) = \ell(e_1) \cdots \ell(e_k)$ .

In the following  $R_n$  is the  $\tilde{X}$ -labeled graph whose underlying graph has vertex set  $V = \{v_0\}$  and edge set  $\{e_1^{\pm 1}, \dots, e_n^{\pm 1}\}$  such that  $\ell(e_i^\varepsilon) = x_i^\varepsilon$  for all  $\varepsilon \in \{-1, 1\}$  and  $1 \leq i \leq n$ . As usual we identify  $F_n$  with  $\pi_1(R_n, v_0)$  in the obvious way. For any  $\tilde{X}$ -labeled graph  $\Gamma$  there is a unique label preserving morphism  $f : \Gamma \rightarrow R_n$ .

We say that a word  $w$  is readable in  $\Gamma$  if there exists a closed path in  $\Gamma$  with label  $w$ . We will say that an element  $g \in F_n$  is readable in  $\Gamma$  if the corresponding reduced word is readable in  $\Gamma$ . We will further say that a conjugacy class  $[g]$  is readable in  $\Gamma$  if some (and therefore all) cyclically reduced word representing  $[g]$  is readable in  $\Gamma$ .

We will be mostly interested in connected  $\tilde{X}$ -labeled graphs  $\Gamma$  such that associated map  $f_* : \pi_1(\Gamma) \rightarrow \pi_1(R_n) = F_n$  is surjective. In this case the morphism  $f$  can be written as a product of Stallings folds, see [2].

Recall that a graph  $\Gamma$  is called a core graph if it contains no proper subgraph such that the inclusion is a homotopy equivalence; for a finite graph this just means that  $\Gamma$  has no vertex of valence 1. We also call a pair  $(\Gamma, v_0)$  a core pair if  $\Gamma$  contains no proper