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A REMARK ON WHITEHEAD'S CUT-VERTEX LEMMA

MICHAEL HEUSENER AND RICHARD WEIDMANN

ABSTRACT. We observe that Whitehead's cut-vertex lemma is an immediate consequence of Stallings folds.

INTRODUCTION

Whitehead's cut-vertex lemma [5] implies that the Whitehead graph of a primitive element of the free group is either not connected or has a cut vertex. This result was generalized by Stong [4] and Stallings [3] to separable subsets of free groups. In this note we observe that this is an essentially trivial consequence of Stallings folds. To do so we relate the connectivity properties of the Whitehead graph of a set of conjugacy classes of elements to the readability of cyclically reduced representatives of these classes in a class of labeled graphs that we call almost-roses.

1. FREE GROUPS AND GRAPHS

In the following we denote the free group in $X = \{x_1, \dots, x_n\}$ by F_n . Throughout this note $n \geq 2$ and X are fixed. As usual we represent elements of F_n by words in $\tilde{X} := X \cup X^{-1}$. For any $g \in F_n$ we denote the conjugacy class of g by $[g]$ and for $S \subset F_n$ we set $[S] := \{[s] \mid s \in S\}$. We say that $g \in F_n$ is primitive if g is contained in a basis of F_n . We further call a set $S \subset F_n$ separable if there exists a non-trivial decomposition $F_n = F^1 * F^2$ such that any $w \in S$ is conjugate into F^1 or F^2 , see [3]. Clearly $\{g\}$ is separable for any primitive element $g \in F_n$.

An \tilde{X} -labeled graph Γ is a directed graph (with inverse edges) with a labeling map $\ell : E\Gamma \rightarrow \tilde{X}$ such that $\ell(e^{-1}) = \ell(e)^{-1}$ for any edge $e \in E\Gamma$. Multiple edges and loops are permitted. The label of a path $\gamma = e_1 \cdots e_k$ in Γ is the word $\ell(\gamma) = \ell(e_1) \cdots \ell(e_k)$.

In the following R_n is the \tilde{X} -labeled graph whose underlying graph has vertex set $V = \{v_0\}$ and edge set $\{e_1^{\pm 1}, \dots, e_n^{\pm 1}\}$ such that $\ell(e_i^\varepsilon) = x_i^\varepsilon$ for all $\varepsilon \in \{-1, 1\}$ and $1 \leq i \leq n$. As usual we identify F_n with $\pi_1(R_n, v_0)$ in the obvious way. For any \tilde{X} -labeled graph Γ there is a unique label preserving morphism $f : \Gamma \rightarrow R_n$.

We say that a word w is readable in Γ if there exists a closed path in Γ with label w . We will say that an element $g \in F_n$ is readable in Γ if the corresponding reduced word is readable in Γ . We will further say that a conjugacy class $[g]$ is readable in Γ if some (and therefore all) cyclically reduced word representing $[g]$ is readable in Γ .

We will be mostly interested in connected \tilde{X} -labeled graphs Γ such that associated map $f_* : \pi_1(\Gamma) \rightarrow \pi_1(R_n) = F_n$ is surjective. In this case the morphism f can be written as a product of Stallings folds, see [2].

Recall that a graph Γ is called a core graph if it contains no proper subgraph such that the inclusion is a homotopy equivalence; for a finite graph this just means that Γ has no vertex of valence 1. We also call a pair (Γ, v_0) a core pair if Γ contains no proper