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ON THE STABILITY OF THE SCHWARTZ CLASS UNDER THE MAGNETIC SCHRÖDINGER FLOW

G. BOIL, N. RAYMOND, AND S. VŨ NGỌC

ABSTRACT. We prove that the Schwartz class is stable under the magnetic Schrödinger flow when the magnetic 2-form is non-degenerate and does not oscillate too much at infinity.

1. INTRODUCTION

1.1. Motivation and context. This paper is devoted to describing the solutions to the magnetic Schrödinger equation. Let \mathbf{B} be a smooth and closed 2-form on \mathbb{R}^d . Let $\mathbf{A} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a 1-form (identified with a vector field) such that $d\mathbf{A} = \mathbf{B}$. The magnetic Schrödinger operator is the essentially self-adjoint differential operator

$$\mathcal{L}_h = (-ih\nabla - \mathbf{A})^2 = \sum_{j=1}^d L_j^2,$$

where $h > 0$ and, for all $j \in \{1, \dots, d\}$, $L_j = -ih\partial_j - A_j$. Its domain is given by

$$\begin{aligned} \text{Dom}(\mathcal{L}_h) &= \{\psi \in L^2(\mathbb{R}^d) : (-ih\nabla - \mathbf{A})\psi \in L^2(\mathbb{R}^d), (-ih\nabla - \mathbf{A})^2\psi \in L^2(\mathbb{R}^d)\} \\ &= \{\psi \in L^2(\mathbb{R}^d) : (-ih\nabla - \mathbf{A})^2\psi \in L^2(\mathbb{R}^d)\}. \end{aligned}$$

The time dependent magnetic Schrödinger equation is given by

$$-ih\partial_t\psi = \mathcal{L}_h\psi, \quad \psi(0) = \psi_0 \in \text{Dom}(\mathcal{L}_h). \quad (1.1)$$

By Stone's theorem, this Cauchy problem admits a unique solution, evolving in the domain of \mathcal{L}_h , and it is given by

$$\forall t \in \mathbb{R}, \quad \psi(t) = e^{\frac{it\mathcal{L}_h}{\hbar}} \psi_0.$$

By unitarity of the flow, we have

$$\forall t \in \mathbb{R}, \quad \|\psi(t)\| = \|\psi_0\|, \quad \|\mathcal{L}_h\psi(t)\| = \|\mathcal{L}_h\psi_0\|,$$

where $\|\cdot\|$ denotes the usual norm on $L^2(\mathbb{R}^d)$. This norm controls the rough phase space localization of the quantum state $\psi(t)$; a natural question is to know to which extent a strong phase space localization of ψ_0 is preserved by the flow. More precisely, this paper was inspired by the following rather naive question. Is it true that

$$\psi_0 \in \mathcal{S}(\mathbb{R}^d) \implies \forall t \in \mathbb{R}, \quad \psi(t) \in \mathcal{S}(\mathbb{R}^d) \quad ? \quad (1.2)$$

If so, what kind of explicit control do we have in terms of the Schwartz semi-norms?

These questions are motivated by the recent investigation of the propagation of coherent states by the magnetic Hamiltonian flow in two dimensions (see the Ph. D. thesis of the first author [2]). The present paper gives a positive answer to (1.2). Our explicit estimates of the Schwartz semi-norms (in terms of the semiclassical parameter h),

combined with the use of the Birkhoff normal form from [13], turn out to be the key ingredients in the study by [3] of the propagation of coherent states up to times of order h^{-N} , for all $N \in \mathbb{N}$. This gives a quantum analog to the low energy (say of order ε) classical propagation for times of order $\varepsilon^{-\infty}$ (see [13, Theorem 1.2]). Taking into account the analysis of [6], one can even hope to extend these results to three dimensions where the classical dynamics has a more complex behavior.

Independently of this motivation, the answer to (1.2) has an interest of its own, especially because it lives at the confluence of two closely related domains: hypoellipticity and semiclassical analysis with magnetic fields. On these vast subjects, the literature is enormous, and we only refer to [11, 9, 7, 10, 15, 16, 4, 12]. In this paper, we will use many classical ideas from these two contexts, and provide an elementary and self-contained presentation.

1.2. Main results. Let us now describe our assumptions and results.

Let \mathcal{P} be the class defined by

$$\mathcal{P} = \{\psi \in \mathcal{C}^\infty(\mathbb{R}^d) : \forall \alpha \in \mathbb{N}^d, \exists (C, m) \in \mathbb{R}_+ \times \mathbb{R}_+, \forall x \in \mathbb{R}^d, |\partial^\alpha \psi| \leq C \langle x \rangle^m\}.$$

The following assumption will hold throughout the paper, where we identify \mathbf{B} with its antisymmetric matrix obtained in the usual basis ($dx_j \wedge dx_k; j < k$).

Assumption 1.1. We assume that

- i. \mathbf{A} belongs to \mathcal{P} (in particular $\mathbf{B} \in \mathcal{P}$),
- ii. there exists $b_0 > 0$ such that, for all $x \in \mathbb{R}^d$,

$$\mathrm{Tr}^+ \mathbf{B}(x) \geq b_0,$$

where $\mathrm{Tr}^+ \mathbf{B}(x)$ denotes the sum of the moduli of the eigenvalues with positive imaginary part of the matrix $\mathbf{B}(x)$,

- iii. for all $\alpha \in \mathbb{N}^d$, there exists $C > 0$ such that, for all $x \in \mathbb{R}^d$, $\|\partial^\alpha \mathbf{B}(x)\| \leq C \|\mathbf{B}(x)\|$, where $\|\cdot\|$ denotes a norm on the space of matrices.

Assumption 1.1 is stronger than really necessary as we can see in our proofs. In this context, we will use the following lemma (see [8, Theorem 2.2]).

Lemma 1.2. *We have*

$$\inf \mathrm{sp}(\mathcal{L}_h) = h \inf_{x \in \mathbb{R}^d} \mathrm{Tr}^+ \mathbf{B}(x) + o(h).$$

In particular, there exist $C > 0$ and $h_0 > 0$, such that, for all $h \in (0, h_0)$, \mathcal{L}_h is invertible and

$$\|\mathcal{L}_h^{-1}\| \leq Ch^{-1}.$$

In the following, we will always assume that h is small enough and such that \mathcal{L}_h is invertible.

Definition 1.3. For all $n \in \mathbb{N}$, we let

$$\mathrm{Dom}(\mathcal{L}_h^n) = \{\psi \in L^2(\mathbb{R}^d) : \forall \ell \in \{1, \dots, n\} : \mathcal{L}_h^{\ell-1} \psi \in \mathrm{Dom}(\mathcal{L}_h)\}.$$

The operator \mathcal{L}_h^n is defined by induction by

$$\forall \psi \in \mathrm{Dom}(\mathcal{L}_h^n), \quad \mathcal{L}_h^n \psi = \mathcal{L}_h(\mathcal{L}_h^{n-1} \psi).$$

The operator $(\text{Dom}(\mathcal{L}_h^n), \mathcal{L}_h^n)$ is self-adjoint and invertible. The following theorem proves some magnetic elliptic estimates, showing that iterations of the magnetic laplacian \mathcal{L}_h control iterations of the magnetic derivatives $(L_j)_{1 \leq j \leq d}$. This will be an important tool on the proof of the main result of the paper.

Theorem 1.4. *Let Assumption 1.1 hold. Let $n \in \mathbb{N}$. There exist $h_0 > 0$ and $C > 0$ such that, for all $h \in (0, h_0)$, and all $\psi \in \text{Dom}(\mathcal{L}_h^n)$,*

$$\sum_{\sigma \in \mathfrak{A}(2n)} \|L_\sigma \psi\| \leq C h^{-3n/2} \|\mathcal{L}_h^n \psi\|, \quad (1.3)$$

where, for $k \in \mathbb{N}$, $\mathfrak{A}(k) = \cup_{p=0}^k \{1, \dots, d\}^{\{1, \dots, p\}}$, and for $p \in \mathbb{N}$, for $\sigma \in \{1, \dots, d\}^{\{1, \dots, p\}}$, $L_\sigma := L_{\sigma(1)} \dots L_{\sigma(p)}$, with the convention $L_\emptyset = \text{Id}$.

In the case where \mathbf{A} is bounded, Theorem 1.3 is closely related to [15, Theorem 3], which deals with the context of general Gårding inequalities.

Definition 1.5. For all $k \in \mathbb{N}$ and all $\psi \in \mathcal{S}(\mathbb{R}^d)$, we let

$$p_k(\psi) = \max_{\substack{(\alpha, \beta) \in \mathbb{N}^{2d} \\ |\alpha| + |\beta| \leq k}} \|x^\alpha \partial^\beta \psi\|_\infty.$$

We can now state the main result of this paper.

Theorem 1.6. *Let Assumption 1.1 hold. For all $t \in \mathbb{R}$, we have*

$$e^{\frac{it\mathcal{L}_h}{h}} \mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d).$$

More precisely, for all $M \in \mathbb{N}^*$, for all $k \in \mathbb{N}$, there exist $h_0 > 0$, $C > 0$, $N \in \mathbb{N}^*$ and $K \in \mathbb{N}$, such that, for all $h \in (0, h_0)$, and for all $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$, and all $t \in [0, h^{-M}]$,

$$p_k(e^{\frac{it\mathcal{L}_h}{h}} \psi_0) \leq C h^{-N} p_K(\psi_0).$$

Theorem 1.6 is related to the (pseudo-differential) analysis in [16, Section 7]. In this work, under the assumption that the derivatives of order two or higher of the symbol of the propagator should be bounded, a parametrix of the evolution operator was constructed. Closely related is also the paper [14, Corollary 2.11], based on the analysis of coherent states, where the derivatives of order three or higher of the symbol have to be bounded. Our approach here is more directly related to the structure of the magnetic Laplacian, and is reminiscent of the analysis in [11] of the ellipticity of certain algebras of non-commuting vector fields.

1.3. Organisation of the proofs. In Section 2, we prove Theorem 1.4 by using a regularization argument involving exponentially weighted estimates and commutator estimates. In Section 3, we apply our magnetic elliptic estimates to prove Theorem 1.6.

2. MAGNETIC ELLIPTIC ESTIMATES

This section is devoted to the proof of Theorem 1.4.

2.1. Density argument. Let us explain here why it is sufficient to prove Theorem 1.4 for $\psi \in \mathcal{S}(\mathbb{R}^d)$.

Let us consider $f \in L^2(\mathbb{R}^d)$. There is a unique $\psi \in \text{Dom}(\mathcal{L}_h^n)$ such that $\mathcal{L}_h^n \psi = f$. Consider $f_k \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ converging to f in $L^2(\mathbb{R}^d)$ and consider the unique $\psi_k \in \text{Dom}(\mathcal{L}_h^n)$ such that $\mathcal{L}_h^n \psi_k = f_k$. Note that, by continuity of $(\mathcal{L}_h^n)^{-1}$, ψ_k converges to ψ in $L^2(\mathbb{R}^d)$.

Lemma 2.1. *We let*

$$H_{\text{exp}}^\infty(\mathbb{R}^d) = \{\psi \in L^2(\mathbb{R}^d) : \forall \alpha \in \mathbb{N}^d, \exists \beta > 0 : e^{\beta \langle x \rangle} \partial^\alpha \psi \in L^2(\mathbb{R}^d)\}.$$

Consider $f \in H_{\text{exp}}^\infty(\mathbb{R}^d)$. Then, the unique solution $u \in \text{Dom}(\mathcal{L}_h)$ to $\mathcal{L}_h u = f$ satisfies $u \in H_{\text{exp}}^\infty(\mathbb{R}^d)$.

Proof. The proof follows from the classical Agmon estimates (see [1, 5]). Consider $\beta > 0$ such that $e^{\beta \langle x \rangle} f \in L^2(\mathbb{R}^d)$. Let $\varepsilon > 0$ and $\Phi_\varepsilon = \beta \min(\langle x \rangle, \varepsilon^{-1})$. We have the Agmon formula (see [12, Section 4.2]):

$$\langle \mathcal{L}_h u, e^{2\Phi_\varepsilon} u \rangle = \int_{\mathbb{R}^d} |(-ih\nabla - \mathbf{A})e^{\Phi_\varepsilon} u|^2 dx - h^2 \|\nabla \Phi_\varepsilon e^{\Phi_\varepsilon} u\|^2.$$

In particular,

$$\int_{\mathbb{R}^d} |(-ih\nabla - \mathbf{A})e^{\Phi_\varepsilon} u|^2 dx - \beta^2 h^2 \|e^{\Phi_\varepsilon} u\|^2 \leq \|e^{\Phi_\varepsilon} f\| \|e^{\Phi_\varepsilon} u\|. \quad (2.1)$$

On the other hand, by Lemma 1.2, we get, for some $c > 0$,

$$(hc - \beta^2 h^2) \|e^{\Phi_\varepsilon} u\|^2 \leq \int_{\mathbb{R}^d} |(-ih\nabla - \mathbf{A})e^{\Phi_\varepsilon} u|^2 dx - \beta^2 h^2 \|e^{\Phi_\varepsilon} u\|^2.$$

Choosing β small enough, we get, for some $C(h) > 0$ independent of ε ,

$$\|e^{\Phi_\varepsilon} u\| \leq C(h) \|e^{\Phi_\varepsilon} f\|.$$

Then, we take the limit $\varepsilon \rightarrow 0$ and apply Fatou's Lemma to find

$$\|e^{\beta \langle x \rangle} u\| \leq C(h) \|e^{\beta \langle x \rangle} f\|. \quad (2.2)$$

Coming back to (2.1), and replacing β by $\tilde{\beta} < \beta$, we get

$$\int_{\mathbb{R}^d} |e^{\Phi_\varepsilon} (-ih\nabla - \mathbf{A} - ih\nabla \Phi_\varepsilon) u|^2 dx - \tilde{\beta}^2 h^2 \|e^{\Phi_\varepsilon} u\|^2 \leq \|e^{\Phi_\varepsilon} f\| \|e^{\Phi_\varepsilon} u\|.$$

Thus,

$$\begin{aligned} \frac{h^2}{2} \int_{\mathbb{R}^d} |e^{\Phi_\varepsilon} \nabla u|^2 dx &\leq \|e^{\Phi_\varepsilon} f\| \|e^{\Phi_\varepsilon} u\| + \tilde{\beta}^2 h^2 \|e^{\Phi_\varepsilon} u\|^2 + 2\|(ih\nabla \Phi_\varepsilon + \mathbf{A})e^{\Phi_\varepsilon} u\|^2 \\ &\leq \|e^{\Phi_\varepsilon} f\| \|e^{\Phi_\varepsilon} u\| + \tilde{\beta}^2 h^2 \|e^{\Phi_\varepsilon} u\|^2 + 4\|\nabla \Phi_\varepsilon e^{\Phi_\varepsilon} u\|^2 + 4\|\mathbf{A}e^{\Phi_\varepsilon} u\|^2. \end{aligned}$$

Using that $\mathbf{A} \in \mathcal{P}$, (2.2), and Fatou's Lemma, we get $e^{\tilde{\beta} \langle x \rangle} \nabla u \in L^2(\mathbb{R}^d)$.

Considering the equation $\mathcal{L}_h u = f$, we get, in the sense of tempered distributions,

$$-h^2 \Delta u = f - |\mathbf{A}|^2 u - ih(\nabla \cdot \mathbf{A})u - 2ih(\mathbf{A} \cdot \nabla)u. \quad (2.3)$$

Noticing that we have just controlled the terms of order at most one, we deduce that, for some $\beta > 0$,

$$e^{\beta\langle x \rangle} \Delta u \in L^2(\mathbb{R}^d),$$

and also

$$\Delta \left(e^{\beta\langle x \rangle} u \right) \in L^2(\mathbb{R}^d).$$

By Fourier transform, we get

$$e^{\beta\langle x \rangle} u \in H^2(\mathbb{R}^d),$$

which implies that, for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq 2$,

$$e^{\beta\langle x \rangle} \partial^\alpha u \in L^2(\mathbb{R}^d).$$

The higher order derivatives can be controlled by induction (taking successive derivatives of (2.3)). \square

Remark 2.2. By the Sobolev embeddings, we have $H_{\text{exp}}^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$.

By Lemma 2.1, we see that $\psi_k \in \mathcal{S}(\mathbb{R}^d)$. Assume that (1.3) holds for any $\psi \in \mathcal{S}(\mathbb{R}^d)$, with $\psi = \psi_k - \psi_\ell$, for all $(k, \ell) \in \mathbb{N}^2$, this shows that, for all $\sigma \in \mathfrak{A}(2n)$, $(L_\sigma \psi_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^d)$. Thus, in the sense of distributions, $L_\sigma \psi \in L^2(\mathbb{R}^d)$. It remains to use again (1.3) with $\psi = \psi_k$ and to take the limit $k \rightarrow +\infty$.

2.2. Preliminary lemmas.

Lemma 2.3. *For all $\psi \in \text{Dom}(\mathcal{L}_h)$, we have*

$$\|(-ih\nabla - \mathbf{A})\psi\|^2 \leq \frac{1}{2} \|\psi\|_{\mathcal{L}_h}^2$$

where $\|\psi\|_{\mathcal{L}_h}^2 = \|\psi\|^2 + \|\mathcal{L}_h \psi\|^2$.

Proof. We recall that, by definition of the domain, for all $\psi \in \text{Dom}(\mathcal{L}_h)$,

$$\langle \mathcal{L}_h \psi, \psi \rangle = \|(-ih\nabla - \mathbf{A})\psi\|^2 = \sum_{j=1}^d \|L_j \psi\|^2,$$

so that

$$\sum_{j=1}^d \|L_j \psi\|^2 \leq \|\psi\| \|\mathcal{L}_h \psi\|.$$

\square

Lemma 2.4. *There exist $C > 0$ and $h_0 > 0$ such that, for all $\psi \in \mathcal{S}(\mathbb{R}^d)$ and all $h \in (0, h_0)$,*

$$\|\langle \mathbf{B} \rangle \psi\| + \|\langle \mathbf{B} \rangle^{\frac{1}{2}} (-ih\nabla - \mathbf{A})\psi\| \leq Ch^{-1} \|\psi\|_{\mathcal{L}_h}. \quad (2.4)$$

Moreover, for all $\psi \in \text{Dom}(\mathcal{L}_h)$, we have

$$\mathbf{B}\psi \in L^2(\mathbb{R}^d) \quad \text{and} \quad |\mathbf{B}|^{\frac{1}{2}} (-ih\nabla - \mathbf{A})\psi \in L^2(\mathbb{R}^d),$$

and (2.4) holds for $\psi \in \text{Dom}(\mathcal{L}_h)$.

Proof. By integration by parts and using Assumption 1.1,

$$\begin{aligned} \int_{\mathbb{R}^d} \langle \mathbf{B} \rangle |(-ih\nabla - \mathbf{A})\psi|^2 dx &= \langle \langle \mathbf{B} \rangle (-ih\nabla - \mathbf{A})\psi, (-ih\nabla - \mathbf{A})\psi \rangle \\ &\leq \langle \mathcal{L}_h \psi, \langle \mathbf{B} \rangle \psi \rangle + C \| \langle \mathbf{B} \rangle \psi \| \| (-ih\nabla - \mathbf{A})\psi \| \\ &\leq C \| \langle \mathbf{B} \rangle \psi \| \| \psi \|_{\mathcal{L}_h}. \end{aligned} \quad (2.5)$$

Then, we have

$$\int_{\mathbb{R}^d} |h\mathbf{B}|^2 |\psi|^2 dx = \sum_{(k,\ell) \in \{1,\dots,d\}^2} \int_{\mathbb{R}^d} |hB_{k,\ell}|^2 |\psi|^2 dx$$

and we write

$$\begin{aligned} \sum_{(k,\ell) \in \{1,\dots,d\}^2} \int_{\mathbb{R}^d} |hB_{k,\ell}|^2 |\psi|^2 dx &= \sum_{(k,\ell) \in \{1,\dots,d\}^2} | \langle [L_k, L_\ell] \psi, hB_{k,\ell} \psi \rangle | \\ &\leq Ch \| \langle \mathbf{B} \rangle \psi \| \| (-ih\nabla - \mathbf{A})\psi \| + Ch \int_{\mathbb{R}^d} \langle \mathbf{B} \rangle |(-ih\nabla - \mathbf{A})\psi|^2 dx, \end{aligned}$$

where we used an integration by parts and Assumption 1.1.

By (2.5), it follows

$$\int_{\mathbb{R}^d} |\mathbf{B}|^2 |\psi|^2 dx \leq Ch^{-1} \| \langle \mathbf{B} \rangle \psi \| \| \psi \|_{\mathcal{L}_h}$$

and then

$$\| \langle \mathbf{B} \rangle \psi \| \leq Ch^{-1} \| \psi \|_{\mathcal{L}_h}.$$

Using again (2.5), the conclusion follows. \square

2.3. Case $n = 1$. The estimate of Theorem 1.4 is obvious when $n = 0$. Let us consider the case when $n = 1$ to explain the principle producing these estimates.

Lemma 2.5. *There exist $C > 0$, $h_0 > 0$ such that, for all $\psi \in \mathcal{S}(\mathbb{R}^d)$ and all $h \in (0, h_0)$,*

$$\|L_1^2 \psi\| + \|L_2^2 \psi\| + \|L_1 L_2 \psi\| + \|L_2 L_1 \psi\| \leq Ch^{-1} \|\mathcal{L}_h \psi\|.$$

Proof. Let us consider $\psi \in \mathcal{S}(\mathbb{R}^d)$ and let

$$\mathcal{L}_h \psi = f.$$

Consider $j \in \{1, \dots, d\}$. We have

$$\mathcal{L}_h(L_j \psi) = L_j f + [\mathcal{L}_h, L_j] \psi,$$

and

$$\|(-ih\nabla - \mathbf{A})(L_j \psi)\|^2 = \langle L_j f, L_j \psi \rangle + \langle [\mathcal{L}_h, L_j] \psi, L_j \psi \rangle,$$

We have

$$[\mathcal{L}_h, L_j] = \sum_{k=1}^d [L_k^2, L_j] = \sum_{k=1}^d ([L_k, L_j] L_k + L_k [L_k, L_j]).$$

Thus,

$$| \langle [\mathcal{L}_h, L_j] \psi, L_j \psi \rangle | \leq C \| \mathbf{B} \psi \| \| (-ih\nabla - \mathbf{A})\psi \| + C \int_{\mathbb{R}^d} |\mathbf{B}| |(-ih\nabla - \mathbf{A})\psi|^2 dx.$$

We have, for all $\varepsilon \in (0, 1)$,

$$|\langle L_j f, L_j \psi \rangle| \leq \varepsilon \|L_j^2 \psi\|^2 + C \|f\|^2,$$

and then

$$\begin{aligned} & \|(-ih\nabla - \mathbf{A})(L_j \psi)\|^2 \\ & \leq C \|f\|^2 + C \|\mathbf{B}\psi\| \|(-ih\nabla - \mathbf{A})\psi\| + C \int_{\mathbb{R}^d} |\mathbf{B}| |(-ih\nabla - \mathbf{A})\psi|^2 dx. \end{aligned}$$

With Lemma 2.3, noting that $\|\psi\|_{\mathcal{L}_h}^2 \leq C(1 + h^{-2})\|f\|^2$, we find

$$\|(-ih\nabla - \mathbf{A})(L_j \psi)\|^2 \leq Ch^{-3}\|f\|^2.$$

□

2.4. Induction. Let $n \in \mathbb{N}^*$. Let us assume that, for all $k \in \{1, \dots, n\}$, the ellipticity property (1.3) is true. Let us consider $f, \psi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\mathcal{L}_h \psi = f.$$

Consider $\sigma \in \mathfrak{A}(2n)$. Since the functions are in the Schwartz class, all the following computations are justified.

We have

$$\mathcal{L}_h L_\sigma \psi = L_\sigma f + [\mathcal{L}_h, L_\sigma] \psi,$$

and then

$$\langle \mathcal{L}_h L_\sigma \psi, L_\sigma \psi \rangle = \langle L_\sigma f, L_\sigma \psi \rangle + \langle [\mathcal{L}_h, L_\sigma] \psi, L_\sigma \psi \rangle. \quad (2.6)$$

By using the Cauchy-Schwarz inequality and the induction assumption, we have for all $\varepsilon \in (0, 1)$,

$$|\langle L_\sigma f, L_\sigma \psi \rangle| \leq Ch^{-3n} \|\mathcal{L}_h^n f\| \|\mathcal{L}_h^n \psi\| \leq Ch^{-3n} \|\mathcal{L}_h^{n+1} \psi\|^2 + Ch^{-3n} \|\mathcal{L}_h^n \psi\|^2,$$

so that, by Lemma 1.2,

$$|\langle L_\sigma f, L_\sigma \psi \rangle| \leq Ch^{-(3n+2)} \|\mathcal{L}_h^{n+1} \psi\|^2.$$

Let us now deal with $\langle [\mathcal{L}_h, L_\sigma] \psi, L_\sigma \psi \rangle$. The commutator $[\mathcal{L}_h, L_\sigma]$ is the sum of various terms. Each of them is the composition of at most $2n - 2$ of the L_j and with exactly one of the $B_{k,\ell}$. By commuting the $B_{k,\ell}$ to put it on the left, and using Assumption 1.1, we get

$$\|[\mathcal{L}_h, L_\sigma] \psi\| \leq C \sum_{\tau \in \mathfrak{A}(2n-2)} \|(\mathbf{B}) L_\tau \psi\|.$$

By applying Lemma 2.4, and then the induction assumption, we get

$$\|[\mathcal{L}_h, L_\sigma] \psi\| \leq Ch^{-1} \sum_{\tau \in \mathfrak{A}(2n-2)} \|L_\tau \psi\|_{\mathcal{L}_h} \leq Ch^{-(3n-1)/2} \|\mathcal{L}_h^n \psi\|.$$

Thus, we deduce

$$|\langle \mathcal{L}_h L_\sigma \psi, L_\sigma \psi \rangle| \leq Ch^{-3n-1/2} \|\mathcal{L}_h^{n+1} \psi\|^2.$$

This shows that, for all $\gamma \in \mathfrak{A}(2n+1)$,

$$\|L_\gamma \psi\| \leq Ch^{-(3n/2+1)} \|\mathcal{L}_h^{n+1} \psi\|. \quad (2.7)$$

Now, we want to get the control for $\gamma \in \mathfrak{A}(2n+2)$. Let $\sigma \in \mathfrak{A}(2n+1)$. We consider again (2.6). By integration by parts, we can write

$$\langle L_\sigma f, L_\sigma \psi \rangle = \langle L_{\check{\sigma}} f, L_{\check{\sigma}} \psi \rangle,$$

with $\check{\sigma} \in \mathfrak{A}(2n)$ and $\hat{\sigma} \in \mathfrak{A}(2n+2)$. Thus, by Cauchy-Schwarz, and the induction assumption, for all $\varepsilon > 0$, there exists $C > 0$ such that

$$|\langle L_\sigma f, L_\sigma \psi \rangle| \leq \varepsilon \|L_{\hat{\sigma}} \psi\|^2 + Ch^{-3n/2} \|\mathcal{L}_h^{n+1} \psi\|^2. \quad (2.8)$$

As previously, we have

$$\|[\mathcal{L}_h, L_\sigma] \psi\| \leq C \sum_{\tau \in \mathfrak{A}(2n-1)} \|\langle \mathbf{B} \rangle L_\tau \psi\|.$$

We use Lemma 2.4 and (2.7) to find

$$\|[\mathcal{L}_h, L_\sigma] \psi\| \leq Ch^{-1} \sum_{\tau \in \mathfrak{A}(2n-1)} \|L_\tau \psi\|_{\mathcal{L}_h} \leq Ch^{-(3n/2+2)} \|\mathcal{L}_h^{n+1} \psi\|.$$

Then, with Cauchy-Schwarz and (2.7), we get

$$|\langle [\mathcal{L}_h, L_\sigma] \psi, L_\sigma \psi \rangle| \leq Ch^{-(3n+3)} \|\mathcal{L}_h^{n+1} \psi\|^2. \quad (2.9)$$

From (2.6), (2.8), and (2.9), summing over $\sigma \in \mathfrak{A}(2n+1)$, and choosing ε small enough, we get (1.3) with n replaced by $n+1$.

This achieves the proof of Theorem 1.4 when $\psi \in \mathcal{S}(\mathbb{R}^d)$ and it remains to use the discussion of Section 2.1.

3. APPLICATION TO THE EVOLUTION PROBLEM

We can now prove Theorem 1.6. Let $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$. We denote by $\psi(\cdot)$ the solution to the Schrödinger equation (1.1).

Notation. For $\kappa, \lambda \in \mathbb{N}$, we define $\Pi_{\kappa, \lambda}$ the set of the operators P that are composition of operators taken among $(L_j)_{1 \leq j \leq d}$ and $(x_k)_{1 \leq k \leq d}$ with κ occurrences of x and λ occurrences of L . We also set $\Pi = \bigcup_{(\kappa, \lambda) \in \mathbb{N}^2} \Pi_{\kappa, \lambda}$.

The aim of this section is to prove the following proposition.

Proposition 3.1. *Let $(\kappa, \lambda) \in \mathbb{N}^2$ and consider $P \in \Pi_{\kappa, \lambda}$. There exist $h_0 > 0$, $C \geq 0$ and $N \in \mathbb{N}$ such that, for all $t \geq 0$, and all $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$, for $\lambda \leq 2n \leq \lambda + 1$,*

$$\|P\psi(t)\| \leq Ch^{-N} (1 + t^\kappa) \sum_{|\alpha| \leq \kappa: |\alpha| + \nu \leq \kappa + n} \|\mathcal{L}_h^\nu x^\alpha \psi_0\|.$$

This proposition implies the control of the Schwartz semi-norms and achieves the proof of Theorem 1.6. Indeed, from Sobolev embeddings, for all $k \in \mathbb{N}$, there exists $K \in \mathbb{N}$ such that for all $f \in \mathcal{S}(\mathbb{R}^d)$,

$$p_k(f) \leq \max_{|\alpha|, |\beta| \leq K} \|x^\alpha \partial^\beta f\|.$$

Using now that $\partial_j = (-ih)^{-1}(L_j + A_j)$, and $A \in \mathcal{P}$, there exists $C > 0$ and $N \in \mathbb{N}$ such that, for all $f \in L^2(\mathbb{R}^d)$, if $\|Pf\| < +\infty$ for all $P \in \Pi$, then $x^\alpha \partial^\beta f \in L^2(\mathbb{R}^d)$ for all $\alpha, \beta \in \mathbb{N}^d$, and

$$\|x^\alpha \partial^\beta f\| \leq Ch^{-N} \sum_{P \in R} \|Pf\|,$$

where R is a finite part of Π . Then, Proposition 3.1 implies, for N , κ and n large enough,

$$\|x^\alpha \partial^\beta \psi(t)\| \leq Ch^{-N} \sum_{P \in R} \|P\psi(t)\| \leq Ch^{-N}(1+t^\kappa) \sum_{|\gamma| \leq \kappa, |\nu| \leq n} \|\mathcal{L}_h^\nu x^\gamma \psi_0\|.$$

Finally, we conclude by

$$\|\mathcal{L}_h^\nu x^\alpha \psi_0\| \leq C\|(1+x^2)^m \mathcal{L}_h^\nu x^\alpha \psi_0\|_\infty \leq Cp_K(\psi_0),$$

for m , K large enough.

3.1. Case when $\kappa = 0$. For all $t \geq 0$, we have, by definition,

$$\psi(t) = e^{\frac{it}{h} \mathcal{L}_h} \psi_0.$$

For all $\ell \in \mathbb{N}$, we get

$$\mathcal{L}_h^\ell \psi(t) = e^{\frac{it}{h} \mathcal{L}_h} \mathcal{L}_h^\ell \psi_0.$$

Applying Theorem 1.4, this establishes the estimate of Proposition 3.1 when $\kappa = 0$.

3.2. Case when $\kappa = 1$. Before starting the induction procedure, let us understand first the mechanism with only one occurrence of x . Let $j \in \{1, \dots, d\}$. We have

$$-ih\partial_t(x_j\psi) = \mathcal{L}_h(x_j\psi) + [x_j, \mathcal{L}_h]\psi.$$

Note that $[x_j, \mathcal{L}_h] = 2hL_j$. With the Duhamel formula, we have, for all $t \geq 0$,

$$x_j\psi(t) = e^{\frac{it}{h} \mathcal{L}_h}(x_j\psi_0) + \int_0^t e^{\frac{i(t-s)}{h} \mathcal{L}_h} 2L_j\psi(s) ds. \quad (3.1)$$

With Lemma 2.3, we get

$$\|x_j\psi(t)\| \leq \|x_j\psi_0\| + 2 \int_0^t \|L_j\psi(s)\| ds \leq \|x_j\psi_0\| + \sqrt{2} \int_0^t \|\psi(s)\|_{\mathcal{L}_h} ds.$$

Since the evolution is unitary, we get, for all $t \geq 0$,

$$\|x_j\psi(t)\| \leq \|x_j\psi_0\| + 2 \int_0^t \|L_j\psi(s)\| ds \leq \|x_j\psi_0\| + t\sqrt{2}\|\psi_0\|_{\mathcal{L}_h}.$$

More generally, with (3.1), we have, for all $\ell \in \mathbb{N}$,

$$\mathcal{L}_h^\ell x_j\psi(t) = e^{\frac{it}{h} \mathcal{L}_h} \mathcal{L}_h^\ell(x_j\psi_0) + \int_0^t e^{\frac{i(t-s)}{h} \mathcal{L}_h} 2\mathcal{L}_h^\ell L_j\psi(s) ds.$$

so that

$$\begin{aligned} \|\mathcal{L}_h^\ell x_j \psi(t)\| &\leq \|\mathcal{L}_h^\ell(x_j \psi_0)\| + 2 \int_0^t \|\mathcal{L}_h^\ell L_j \psi(s)\| ds \\ &\leq \|\mathcal{L}_h^\ell(x_j \psi_0)\| + Ch^{-N} \int_0^t \|\mathcal{L}_h^{\ell+1} \psi(s)\| ds \\ &\leq \|\mathcal{L}_h^\ell(x_j \psi_0)\| + Cth^{-N} \|\mathcal{L}_h^{\ell+1} \psi_0\|. \end{aligned}$$

It remains to apply Theorem 1.4 and to commute the x_j with the L_k .

3.3. Induction. Let us now end the proof of Proposition 3.1 by induction. We set the following two induction assumptions. For $\kappa \in \mathbb{N}$, let

$$\mathcal{Q}_\kappa : \forall n \in \mathbb{N}, \forall \alpha \in \mathbb{N}^d, |\alpha| = \kappa, \exists N \in \mathbb{N} \text{ s.t.}$$

$$\|\mathcal{L}_h^n x^\alpha \psi(t)\| \leq \|\mathcal{L}_h^n x^\alpha \psi_0\| + C \sum_{|\beta| \leq \kappa - 1: |\beta| + \nu \leq \kappa + n} \|\mathcal{L}_h^\nu x^\beta \psi_0\| h^{-N} (1 + t^\kappa);$$

and

$$\mathcal{P}_\kappa : \forall P \in \Pi_{\kappa, \lambda}, \|P\psi(t)\| \leq C \sum_{|\alpha| \leq \kappa: |\alpha| + \nu \leq \kappa + n} \|\mathcal{L}_h^\nu x^\alpha \psi_0\| h^{-N} (1 + t^\kappa), \text{ for } \lambda \leq 2n \leq \lambda + 1.$$

We have proved propositions \mathcal{P}_0 , \mathcal{Q}_0 and \mathcal{Q}_1 . We assume now that for a given $\kappa \in \mathbb{N}^*$, for any $k \leq \kappa$, \mathcal{Q}_k and \mathcal{P}_k hold, and we prove $\mathcal{P}_{\kappa+1}$ and $\mathcal{Q}_{\kappa+1}$. We begin with $\mathcal{Q}_{\kappa+1}$. Let $\alpha \in \mathbb{N}^d$, with $|\alpha| = \kappa + 1$ and $n \in \mathbb{N}$. We have

$$-ih\partial_t(\mathcal{L}_h^n x^\alpha \psi(t)) = \mathcal{L}_h^{n+1} x^\alpha \psi(t) + \mathcal{L}_h^n [x^\alpha, \mathcal{L}_h] \psi(t).$$

Then, noting that

$$[x^\alpha, \mathcal{L}_h] = hP_1$$

with P_1 a sum of elements in $\Pi_{\kappa, 1}$, we get from the Duhamel formula

$$\mathcal{L}_h^n x^\alpha \psi(t) = e^{\frac{it}{h} \mathcal{L}_h} \mathcal{L}_h^n x^\alpha \psi_0 + i \int_0^t e^{i\frac{t-s}{h} \mathcal{L}_h} \mathcal{L}_h^n P_1 \psi(s) ds.$$

As $\mathcal{L}_h^n P_1$ is a sum of elements in $\Pi_{\kappa, (2n+1)}$, we can apply \mathcal{P}_κ , and integrating in time and using the unitariness of $e^{i\frac{t}{h} \mathcal{L}_h}$, we find, for some integer $N \in \mathbb{N}$

$$\|\mathcal{L}_h^n x^\alpha \psi(t)\| \leq \|\mathcal{L}_h^n x^\alpha \psi_0\| + Ch^{-N} \sum_{|\beta| \leq \kappa: |\beta| + \nu \leq \kappa + n + 1} \|\mathcal{L}_h^\nu x^\beta \psi_0\| (1 + t^{\kappa+1})$$

that proves $\mathcal{Q}_{\kappa+1}$. It remains to prove $\mathcal{P}_{\kappa+1}$. We consider so some $P \in \Pi_{\kappa+1, \lambda}$, for a given $\lambda \in \mathbb{N}$. Then, because of the commutation relation of the $(x_k)_{1 \leq k \leq d}$ with the $(L_j)_{1 \leq j \leq d}$, that is $[x_k, L_j] = -ih\delta_{k,j}$, there is $\alpha \in \mathbb{N}^d$, $|\alpha| = \kappa + 1$ such that

$$P = P_2 + P_3 x^\alpha$$

with P_2 a sum of elements in $\Pi_{\kappa, \lambda-1}$ and $P_3 \in \Pi_{0, \lambda}$. So

$$\|P\psi(t)\| \leq \|P_2\psi(t)\| + \|P_3\psi(t)\|.$$

Then, applying \mathcal{P}_κ , we get, for some integer $N \in \mathbb{N}$

$$\|P_2\psi(t)\| \leq Ch^{-N} (1 + t^\kappa) \sum_{|\beta| \leq \kappa: |\beta| + \nu \leq \kappa + n} \|\mathcal{L}_h^\nu x^\beta \psi_0\| \quad (3.2)$$

and applying Theorem 1.4 along with $\mathcal{Q}_{\kappa+1}$, for some integer $N \in \mathbb{N}$, and for $\lambda \leq 2n \leq \lambda + 1$, we have

$$\begin{aligned} \|P_3 x^\alpha \psi(t)\| &\leq Ch^{-n} \|\mathcal{L}^n x^\alpha \psi(t)\| \\ &\leq Ch^{-N} (1 + t^{\kappa+1}) \left(\|\mathcal{L}^n x^\alpha \psi_0\| + \sum_{|\beta| \leq \kappa; |\beta| + \nu \leq \kappa + n + 1} \|\mathcal{L}_h^\nu x^\beta \psi_0\| \right). \end{aligned} \quad (3.3)$$

Now, gathering (3.2) and (3.3), we find, for some integer N ,

$$\|P\psi(t)\| \leq Ch^{-N} (1 + t^{\kappa+1}) \sum_{|\beta| \leq \kappa+1; |\beta| + \nu \leq \kappa + n + 1} \|\mathcal{L}_h^\nu x^\beta \psi_0\|$$

that proves $\mathcal{P}_{\kappa+1}$ and achieves the proof of Proposition 3.1.

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E-mail address: `gregory.boil@univ-rennes1.fr`

E-mail address: `nicolas.raymond@univ-rennes1.fr`

E-mail address: `san.vu-ngoc@univ-rennes1.fr`

IRMAR, UNIVERSITÉ DE RENNES 1, CAMPUS DE BEAULIEU, F-35042 RENNES CEDEX, FRANCE