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# On Unilateral Constraints, Friction and Plasticity

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ON UNILATERAL CONSTRAINTS, FRICTION AND PLASTICITY

JEAN JACQUES MOREAU

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ON UNILATERAL CONSTRAINTS, FRICTION  
AND PLASTICITY

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## I INTRODUCTION

### 1. a ORIENTATION

Three intermingled themes run in all the following : variational statements, the duality in paired linear spaces, the convexity of sets or functions. These are precisely three leading themes of Optimization Theory, as it has been developed for several decades ; in fact the study of optimization problems started many progresses of modern convexity theory, in which duality plays an essential part.

In Mechanics these three themes have been present for more than two centuries. There is no need to recall the importance of variational ideas in the development of Analytical Dynamics. Observe, however, that these ideas often served as a mere scaffolding, to be removed before the end of the construction. Lagrange equations arose from the variational properties of a mechanical system subject to frictionless constraints and conservative forces only ; but actually Analytical Dynamics has a much wider scope, so that some modern treatises on the subject may develop it in the framework of Differential Geometry, without reference to any properly variational fact. Variational calculus acted here in suggesting some mathematical structure which eventually supplanted it. In another domain a similar evolution took place quite recently when the

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variational approach of partial differential equations gave rise to the theory of Variational Inequalities which have not much to do with extremum problems.

The classical Calculus of Variations, developed in the context of differentiability, automatically involves the duality of linear spaces, possibly without formalizing it. In Statics, for instance, it is usual to characterize the equilibrium configurations of a "frictionless" system with finite freedom, by equalling to zero the partial derivatives of the potential energy. This induces to consider these partial derivatives as the "components" of mechanical actions or "forces", in a general sense ; in fact this constitutes the correct way to formulate calculation rules about forces, which are preserved under the change of variables ; for example if some evolution of the system takes place, one obtains a simple expression for the work or the power of forces. This benefit in calculation (and also the possible connection with Thermodynamics) promoted the use of energy methods in many domains ; however these methods may have been a hindrance when they happened to prevent scientists from considering phenomena which could not be described by means of potential functions. Here again one improves by forgetting the variational stimulus and considering respectively displacements and forces as the elements of two linear spaces placed in duality by the bilinear form "work". Such

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was already the underlying idea of the traditional method of virtual work.

About convexity, on the other hand, it must be noted that Mechanics was probably the first physical domain to make use of this concept ; this was in formulating the equilibrium condition of a heavy solid body lying on a horizontal plane : the vertical line drawn from the centre of mass must meet the convex hull of the points of support. This is typically a result concerning unilateral constraints. In fact the study of dynamical problems for systems of finite or infinite freedom with unilateral constraints (e.g. the inception of cavitation in a perfect incompressible fluid ; see MOREAU [7], [8], [9]) initially motivated the part taken by the author in the development of convexity theory. It must be stressed that convexity is involved in the theory of unilateral constraints in an essential way ; it is not used as a convenience assumption made to facilitate mathematical treatment, as it often happens, for instance, in Optimization.

These lectures do not deal with dynamics, but only with equilibrium or quasi-static evolution, i.e. evolution problems where inertia is negligible. The motion of a system is studied when resistance phenomena, such as friction or the resistance of a plastic system to yielding, are taken into account. Here again convexity is involved from the stage



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of formulating the resistance law itself. Many mechanists feel that the occurrence of convexity in this connection is essential, probably with some thermodynamical significance.

Classical Coulomb's law of friction enters into our general scheme of resistance laws admitting a (convex) pseudo-potential. It will be objected that this law gives only a rather rough approximation of the friction phenomena ; experimentally, when the sliding velocity increases from zero the friction coefficient begins with decreasing, while the existence of a superpotential would only allow it to increase. The author's position in this matter is the following.

Traditional physics almost always starts from linear laws as first approximations to which improvements have possibly to be added by taking terms of "higher order" into account. The common habit of assuming differentiability in formulations is connected with the same tendency, as the meaning of differentials is precisely to describe some "tangent" linear mappings. On the contrary Coulomb's law of friction is radically nonlinear and nondifferentiable ; nevertheless there is no doubt that this law agrees with the fundamental features of the friction phenomenon and as such it is always used in practice as the first approximation, possibly subject to further improvements. For instance the augmented friction when the sliding velocity is small or vanishes is frequently

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explained as a sort of welding which takes place between the bodies in contact, and has to be broken when sliding occurs.

Let us suggest that, in plasticity as well as in friction, our pseudo-potential formalism describes the primary phenomenon exactly as in other domains of physics the primary phenomena admit linear formulations. This causes no conceptual difficulty ; on the other hand, the considerable amount of work which has been devoted in recent decades to optimization techniques makes now available the computational methods permitting to deal numerically with "subdifferential calculus" and convex analysis.

#### 1. b SUMMARY OF CHAPTER 2

The preparatory Chapter 2 presents the elements of the duality theory of convex functions and the subdifferentials of such functions. The articulation of the concepts is sufficiently detailed but the proofs of the main statements are not given. Except otherwise indicated the reader may find them in MOREAU [10], a multigraph report. Some are also given in the recent book of P.J. LAURENT [1], which devotes a chapter to this subject. Of course, the book of R.T. ROCKAFELLAR [2], yet restricted to finite dimensional spaces, supply much of the fundamental informations.

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The setting is that of a pair of real linear spaces, say  $(X, Y)$ , placed in duality by a bilinear form denoted as  $\langle \cdot, \cdot \rangle$ . This duality is supposed separating, i.e. the two linear forms defined on  $X$  by  $x \mapsto \langle x, y \rangle$  and  $x \mapsto \langle x, y' \rangle$  are identical only if the elements  $y$  and  $y'$  of  $Y$  are equal, and the symmetric assumption is made with exchanging the roles of the two spaces. Therefore, if one of the two spaces has a finite dimension, the dimension of the other is the same ; in this case, every linear form defined on one of the two spaces can be represented in the preceding way and is continuous with regard to the natural topology of finite dimensional linear spaces. The situation is more complicated for infinite dimensional spaces. Recall in that case that each of the two spaces, say  $X$  for instance, may be endowed with various locally convex topologies which are compatible with the duality  $(X, Y)$  in the sense that relatively to any of them, the continuous linear forms are exactly the functions  $x \mapsto \langle x, y \rangle$  with arbitrary  $y$  in  $Y$ . By the separation assumption made above, these topologies are Hausdorff ; it is a classical fact that among them the weak topology  $\sigma(X, Y)$  is the coarsest and the Mackey topology  $\tau(X, Y)$  is the finest. Observe that, by usual separation arguments, the closed convex sets are the same relatively to all these topologies, thus in the following we shall sometimes refer to closed convex sets without specifying the topology. Same remark

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for the lower semi-continuous convex functions.

### 1. c SUMMARY OF CHAPTER 3

Chapter 3 takes up Mechanics by the study of material systems whose set of possible configurations, denoted by  $\mathcal{U}$ , is endowed with a linear space structure. Such is in particular the case, due to the use of linear approximation, in many practical situations where it is supposed that the considered system presents only "infinitely small deviations" from some reference state which constitutes the zero of the linear space  $\mathcal{U}$ . By the bilinear form "work" the linear space  $\mathcal{U}$  is placed in duality with another linear space  $\mathcal{F}$  whose elements represent, in a general sense, forces applied to the system. An example in § 3. a shows why this duality may be supposed separating.

In this framework a statical law is a relation, arising from the study of some of the physical processes in which the system is involved, formulated between the possible configuration, say  $u \in \mathcal{U}$ , of the system and some, say  $f \in \mathcal{F}$ , among the forces it experiences if it happens to come through this configuration. Such a relation may depend on time. The concept of a statical law which admits a potential function is recalled.

At this stage it is stressed that the word constraint

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possesses in Mechanics a stricter sense than it receives, for instance, in Optimization (observe that the French mechanical term is "liaison", while "contrainte" has other meanings). Describing a mechanical constraint requires fundamentally more information than defining some set of permitted configurations ; some precisions must be given about the confining process, in the formulation of which the force of constraint or reaction is involved. Paragraphs 3. c and 3. d emphasize, in the linear framework of this Chapter, that frictionless constraints, bilateral or unilateral, are statical laws. Precisely they come into the general class of the statical laws which possess a superpotential, i.e. the relations between  $u$  and  $f$  which can be written under the form  $-f \in \partial \phi(u)$ , where  $\phi$  denotes a convex numerical function, possibly taking in some part of the space  $\mathcal{U}$  the value  $+\infty$ . The classical laws possessing a potential function also belong to this class, as far as the potential function is convex.

If all the mechanical actions experienced by the system (possibly excepting forces which vanish in any expected equilibrium) are represented by the conjunction of statical laws admitting time-independent superpotentials, the equilibrium configurations trivially possess some extremum properties in the space  $\mathcal{U}$ . Paragraph 3. f supposes that all these mechanical actions have been grouped in order to be summarized as

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the conjunction of two statical laws admitting the respective super-potentials  $\phi_1$  and  $\phi_2$  ; then  $u \in \mathcal{U}$  is an equilibrium configuration if and only if there exists  $f_1 \in \mathcal{F}$  such that  $-f_1 \in \partial \phi_1(u)$  and  $f_1 \in \partial \phi_2(u)$ . The determination of  $f_1$  prior to that of  $u$  is classically called a statical approach to the equilibrium problem ; the duality theory of convex functions immediately yields some extremum formulation for this problem. This involves the respective dual function  $\phi_1^*$  and  $\phi_2^*$  of  $\phi_1$  and  $\phi_2$  , generalizing the so-called complementary energy of linear elastostatics. Similar correspondances between extremum problems formulated in two paired linear spaces are a familiar feature in convex optimization, as well is familiar the connection of such a pair of problems with a saddle-point property concerning some function called a Lagrangian. In fact, Paragraph 3. g gives a simultaneous characterization of  $u$  and  $f_1$  as a saddle-point in the product space  $\mathcal{U} \times \mathcal{F}$  . As all the preceding pattern may usually be applied to each definite mechanical system in several different ways, it is able to generate a great number of extremal or saddle-point characterization of equilibrium. The foregoing concepts were first published as a short Note (MOREAU [ 11]) in which proofs were not given.

Paragraph 3. h illustrates the formalism by some examples of one-dimensional systems. Paragraphs 3. i and 3. j emphasize the

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application to a lattice of bars ; this introduces two pairs of finite dimensional linear spaces  $(X,Y)$  and  $(E,S)$ , a linear mapping  $D$  from  $X$  into  $E$  and the adjoint mapping  $D^*$  from  $S$  into  $Y$  : this is a very common algebraic pattern in elastostatics. Various ways of exploiting it are presented ; in particular the last one is meant to prepare for the evolution problem of elastoplastics, to be treated in Chapter 6. More details about continuous media and the function spaces involved in their study are given by B. Nayroles in his lectures.

#### 1. d SUMMARY OF CHAPTER 4

This Chapter, devoted to resistance laws does not require a linear space structure for the set of the possible configurations. In fact it is a constant feature in Mechanics to associate with each configuration of a system a real linear space  $\mathcal{V}$  ; the elements of  $\mathcal{V}$  constitute, in some sense, the values that may take the velocity of the system if it comes through the considered configuration. A second linear space  $\mathcal{F}$  is also associated with each configuration ; the elements of  $\mathcal{F}$  form, in a generalized sense, the possible values of forces which may be applied to the system at an instant it happens to have the considered configuration. The spaces  $\mathcal{V}$  and  $\mathcal{F}$  corresponding to a given configuration are placed in duality by a bilinear form :  $\langle v, f \rangle$  denotes the

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power of the force  $f \in \mathcal{F}$  if the system possesses the velocity  $v \in \mathcal{V}$ . In the special case of Chapter 3, it turns out that  $\mathcal{V}$  may be identified with  $\mathcal{U}$  and the same  $\mathcal{F}$  is associated with every configuration.

We call in general resistance law a relation formulated between the possible velocity  $v \in \mathcal{V}$  and a force say  $f \in \mathcal{F}$ , arising from some of the physical processes in which the system is involved. This is properly a resistance phenomenon if the relation is dissipative, i.e. if it implies  $\langle v, f \rangle \leq 0$ .

Here again, the case where it exists a function  $\phi$  defined on  $\mathcal{V}$ , called the pseudo-potential of the resistance law, such that the relation takes the form  $-f \in \partial \phi(v)$  deserves special attention. If, in particular  $0 \in \partial \phi(0)$ , the relation is sure to be dissipative; the pseudo-potential is called in this special case a resistance function and one may suppose without loss of generality, that  $\phi(0) = 0$ . An elementary example is that of viscosity laws: then  $\phi$  is a quadratic form, traditionally called the Rayleigh function.

The main application of these ideas concerns dry friction and plasticity; this corresponds to a function  $\phi$  which is sublinear, i.e. convex and positively homogeneous. Equivalently,  $\phi$  is the support function of a closed convex subset of  $\mathcal{F}$ , denoted by  $-C$ , containing the origin. An essential fact in such a case is that the considered



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resistance law, namely  $-f \in \partial \phi(v)$ , neither defines  $f$  as a single-valued function of  $v$  nor  $v$  as a single-valued function of  $f$ ; to  $v = 0$ , in particular, correspond as possible values for  $f$  all the points of  $C$ . This is a familiar feature of the Coulomb law for dry friction or of the Prandtl - Reuss law for perfect plasticity. In their conventional formulation they may, at first sight, look like a piecing together of heterogeneous empirical data; the present formulation on the contrary reveals the strong mathematical consistency of each of these laws. The rest of these lectures is meant to display the efficiency of such an approach. The reader will see, on the other hand, in P. GERMAIN [1] how our pseudo-potential formalism may take place in the more familiar setting of a textbook on Continuum Mechanics.

For what concerns Coulomb's law of dry friction it will be objected that, in most practical problems, the normal component of the contact force, which enters here in the expression of  $\phi$  as a constant, is unknown. Our position is to consider this quantity as one of the state variables of the system.

Paragraph 4. d comes back to perfect constraints as they were introduced by Chapter 3. In the present kinematical context, these constraints are manifested as relations between the velocity of the system and some force acting on it, namely the reaction of the constraint. These relations too can be represented by means of pseudo-potentials and the

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same is true for the nonholonomic perfect constraints of traditional Mechanics (actually an extreme case of friction) : we propose to refer to such relations as velocity constraints.

Friction or plasticity laws, as well as viscosity laws, exhibit a very usual property : the corresponding dissipated power  $-\langle v, f \rangle$  can be expressed as a single-valued function of the velocity, classically called the dissipation function. There is a priori no reason for this function to be related to the pseudo-potential if it exists ; paragraph 4. f characterizes the resistance laws for which such a relation holds.

The chapter ends with remarks about viscoplasticity : adding some viscosity to a resistance law of the plasticity or friction type described above, amounts to replace the indicator function  $\psi_C$  of the set C (the function taking the value 0 on this set and  $+\infty$  outside) by a penalty function of the same set.

#### 1. e SUMMARY OF CHAPTER 5

This is a purely mathematical part. The application of the foregoing mechanical formalism to evolution problems requires, in particular, some investigations about the motion of a set.

By means of Hausdorff distance, the classical concept of the variation of a function defined on a real interval is adapted to moving

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sets in a metric space ; the absolute continuity of such sets is similarly introduced.

As convex subsets of a normed space may be described in terms of their support functions, a special approach of moving sets is developed for this case. In the same setting of normed spaces and convex moving sets, Paragraph 5. c establishes an intersection theorem which formulates sufficient conditions for the intersection of two absolutely continuous convex moving sets to be itself absolutely continuous.

The rest of the Chapter is restricted to Hilbert spaces. Paragraph 5. b considers among other topics the distance from a moving point  $t \mapsto z(t)$  to a moving convex set  $t \mapsto C(t)$  ; if both are absolutely continuous the distance is an absolutely continuous numerical function and some inequality involving derivatives is established, as a preparation for the following.

Paragraph 5. c introduces the sweeping process associated with a moving convex set in the Hilbert space  $H$ . This gives a fundamental example of an evolution problem under unilateral constraint ; from the mathematical standpoint this process features also as a constituent of several more complicated situations ; in particular it will be met again in the treatment of the elastoplastic problem of Chapter 6. The author has already devoted several studies to this problem, mainly

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published as multigraph seminar reports (cf. MOREAU [17], [18], [20], [21]). The method used in § 5. g to establish an existence theorem consists in a regularization technique, equivalent in the present context to representing the given moving convex set by penalty functions.

The Chapter ends with an algorithm of time discretization for the solution of the sweeping problem ; the convergence of this algorithm is proved by using again regularization, but with a time-dependent "penalty coefficient".

#### 1. f SUMMARY OF CHAPTER 6

This final Chapter shows how all the foregoing operates when applied to the quasi-static evolution problem for elastoplastic systems. This involves a linear space  $\mathcal{U}$  as configuration space and, according to the conventional conception of elastoplasticity, the system is treated as formed by two components : the "visible" or "exposed" component, denoted by  $x \in \mathcal{U}$ , and the "hidden" or "plastic" component denoted by  $p \in \mathcal{U}$ . The elastic restoring force depends only on the difference  $x-p$ . The component  $x$  undergoes perfect constraints and loads, both depending on time in a given way. The component  $p$  undergoes a resistance related to its "velocity"  $\dot{p}$  by a law of the type studied in § 4. This is only perfect plasticity, but a very simple example suggests that

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strain hardening too could be taken into account by a similar pattern, provided a sufficiently large space would be affected to the "hidden" or "internal" variable  $p$  ; this point of view is adopted by several authors.

Great simplification is brought by a notation trick by which the configuration space  $\mathcal{U}$  and the force space  $\mathcal{F}$  are identified with a single Hilbert space  $H$  ; the norm in  $H$  is related to the elastic energy.

An existence theorem is proved by reduction to the sweeping process of Chapter 5 ; thereby a time-discretization algorithm is

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## 2 DUALITY AND SUBDIFFERENTIALS OF CONVEX FUNCTIONS

### 2. a POLAR FUNCTIONS

Let  $X, Y$  be a pair of real linear spaces placed in separating duality by the bilinear form  $\langle \cdot, \cdot \rangle$ . Let  $f$  be a function defined, for instance, on  $X$ , with values in  $\bar{\mathbb{R}} = [-\infty, +\infty]$ . Consider the affine function defined on  $X$  by

$$(2.1) \quad x \mapsto \langle x, y \rangle - \rho$$

with  $y$  fixed in  $Y$ , called the slope of this affine function, and  $\rho$  fixed in  $\mathbb{R}$ ; such is the general form of the affine functions which are continuous for some, then for any, locally convex topology on  $X$  compatible with the duality.

An usual question is that of determining whether this affine function is a minorant of  $f$ ; a trivial necessary and sufficient condition for that is

$$(2.2) \quad \rho \geq \sup_{x \in X} [\langle x, y \rangle - f(x)] .$$

Attention is drawn thereby to the function  $f^*$  defined on  $Y$  by

$$(2.3) \quad f^*(y) = \sup_{x \in X} [\langle x, y \rangle - f(x)]$$

called the polar function of  $f$ .

In particular the equality  $f^*(y) = +\infty$ , for some  $y \in Y$ , means that  $f$  possesses no affine minorant having  $y$  as slope; such is

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the case, for instance, whichever is  $y$ , if  $f$  takes somewhere in  $X$  the value  $-\infty$ .

EXAMPLE. Let  $A$  be a subset of  $X$ ; take as  $f$  the indicator function

$\psi_A$  of  $A$ , i.e.

$$\psi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{if } x \notin A \end{cases} .$$

Its polar function

$$\psi_A^*(y) = \sup_{x \in X} [\langle x, y \rangle - \psi_A(x)] = \sup_{x \in A} \langle x, y \rangle$$

is classically known under the (rather improper) name of the support function of  $A$ . Take  $y$  different from zero in  $Y$  and  $\rho \in \mathbb{R}$ ; the affine function (2.1) is a minorant of  $\psi_A$  iff the closed half space  $\{x \in X : \langle x, y \rangle - \rho \leq 0\}$  contains  $A$ . In view of condition (2.2) this is possible only if  $\psi_A^*(y) < +\infty$ ; in such a case taking exactly  $\rho = \psi_A^*(y)$  yields a half-space which is minimal, with regard to inclusion, among the half-spaces containing  $A$ ; but that does not mean this half-space is necessarily a "supporting half-space"; its boundary hyperplane need not meet  $A$ , even when  $A$  is closed and convex.

## 2. b PAIRS OF DUAL FUNCTIONS

For the construction of the supremum in (2.3) one may equivalently consider only the values of  $x$  such that  $f(x) < +\infty$ . Therefore,

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whichever is  $f$ , the function  $f^*$  belongs to the set, denoted by  
 $\Gamma(Y, X)$ , of the functions on  $Y$  which are the pointwise suprema of col-  
lections of affine functions like  $y \mapsto \langle x, y \rangle - \sigma$ ,  $x \in X$ ,  $\sigma \in \mathbb{R}$ . Using  
Hahn-Banach's theorem, one proves that, besides the constant  $-\infty$  (it  
is the supremum of an empty collection), the set  $\Gamma(Y, X)$  consists  
exactly of the functions on  $Y$ , with values in  $]-\infty, +\infty]$ , which are  
convex and l.s.c. for some locally convex topology on  $Y$  compatible  
with the duality  $(Y, X)$ , then l.s.c. for all such topologies.

The spaces  $X$  and  $Y$  play here symmetric roles ; there is no  
inconvenience in denoting in the same way by the star  $*$  the function  
defined on  $X$  as the polar of a given function on  $Y$ . Then the bipolar  
of  $f$  is defined on  $X$  by

$$f^{**}(x) = \sup_{y \in Y} [\langle x, y \rangle - f^*(y)] .$$

The construction of this supremum may be equivalently be restricted to  
the values of  $y$  such that  $f^*(y)$  is finite ; that means  $f^{**}$  is the  
supremum of the affine functions like (2.1), with  $\rho$  verifying equality  
in (2.2) ; they are the maximal affine minorant of  $f$ , so that  $f^{**}$  may  
also be defined as the pointwise supremum of all the affine function of  
the form (2.1) which minorize  $f$ . This supremum is equivalently charac-  
terized as the greatest element of  $\Gamma(X, Y)$  minorizing  $f$  or  $\Gamma$ -hull



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For instance, if  $A$  is a subset of  $X$ , the  $\Gamma$ -hull of the indicator function  $\psi_A$  is the indicator function of the closed convex hull of  $A$ .

The preceding implies that if it is a priori supposed that

$f \in \Gamma(X,Y)$  and  $g \in \Gamma(Y,X)$  one has the equivalence

$$f = g^* \Leftrightarrow f^* = g .$$

Then  $f$  and  $g$  are said mutually polar or conjugate or dual functions. In this way the star  $*$  induces a one-to-one correspondance between  $\Gamma(X,Y)$  and  $\Gamma(Y,X)$ ; as the constant  $+\infty$  corresponds to the constant  $-\infty$ , the correspondance is also one-to-one between the elements of  $\Gamma(X,Y)$  and  $\Gamma(Y,X)$  other than these singular constants: these elements are called the proper closed convex functions on  $X$  and  $Y$ ; the sets of them will be denoted by  $\Gamma_0(X,Y)$  and  $\Gamma_0(Y,X)$  respectively.

From the definition of polarity it immediately follows

$$\forall x \in X, \forall y \in Y : f(x) + g(y) \geq \langle x, y \rangle$$

called Fenchel's inequality.

REMARK ON TERMINOLOGY. Most of the words introduced by the preceding definitions are the English transcriptions of French terms currently used by French speaking people after the author's multigraph report of 1966 (MOREAU [10]). This involves but slight discrepancies from the book of R.T. ROCKAFELLAR [2]: following the 1949 initiating paper of

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W. FENCHEL [1], Rockafellar prefers the locutions "conjugate function" to "dual functions". It may be inconvenient to call also conjugate of  $f$ , as he does, the function  $f^*$  associated by (2.3) with some  $f$  which does not necessarily belong to  $\Gamma(X, Y)$ . As this so called "conjugacy" is no more a symmetric correspondance, the author chose in the 1966 report, to use in this connotation the term polar function. Unfortunately, in the meantime, Rockafellar applied the word polar to another kind of correspondance (cf. Sec. 15 of his book) concerning nonnegative closed convex functions vanishing at the origin, which generalizes some classical conjugacy of gauge functions (see § 2. h below) ; but there does not seem to be much risk of confusion.

## 2. c IMAGES OF PROPERTIES OR RELATIONS

Many properties or relations concerning functions defined, for instance, on  $X$ , imply some properties or relations concerning the polar of them. Here we restrict ourselves to a few of these "images by polarity" considering exclusively functions  $f, f_1, f_2, \dots$  which belong to  $\Gamma(X, Y)$  and denoting by  $g, g_1, g_2, \dots$  their polar (i.e. dual) functions.

Easy calculation yields :

1° Homothety. If  $\sigma \in \mathbb{R}$  is a non zero constant, the identity

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$$\forall x \in X : f_1(x) = f_2(\sigma x)$$

is equivalent to

$$\forall y \in Y : g_1(y) = g_2\left(\frac{1}{\sigma} y\right) .$$

2° Multiplication by a positive constant. If  $\lambda$  is a strictly positive constant, the identity

$$\forall x \in X : f_1(x) = \lambda f_2(x)$$

is equivalent to

$$\forall y \in Y : g_1(y) = \lambda g_2\left(\frac{1}{\lambda} y\right) ;$$

the right member is sometimes written as a "right product by  $\lambda$ " :

$$\text{notation } g_1 = g_2 \lambda .$$

In particular a function  $g$  belonging to  $\Gamma(Y, X)$  is the support function of a subset of  $X$  (or equivalently the support function of the closed convex hull of this subset) if and only if its dual  $f$  is an indicator, i.e. this dual takes only the values 0 and  $+\infty$ . That means  $f$  remains unchanged under the multiplication by any  $\lambda > 0$ ; in view of the preceding, this is equivalent to  $g$  being positively homogeneous (i.e. sublinear, due to the assumed convexity of  $g$ ). A more special situation is that of a function  $g$  belonging to  $\Gamma(Y, X)$  which at the same time is an indicator function and is sublinear : this happens if and only if  $f$  possesses the same properties ; in such a case  $f$  and  $g$  are respectively the indicator functions of two mutually polar

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(closed, convex) cones,  $P$  and  $Q$ , i.e.

$$Q = \{y \in Y : \forall x \in P, \langle x, y \rangle \leq 0\}$$

and symmetrically

$$P = \{x \in X : \forall y \in Q, \langle x, y \rangle \leq 0\} .$$

3° Translation. If  $a \in X$  and  $\alpha \in \mathbb{R}$ , the identity

$$\forall x \in X : f_1(x) = f_2(x - a) + \alpha$$

is equivalent to

$$\forall y \in Y : g_1(y) = g(y) + \langle a, y \rangle - \alpha .$$

4° Product spaces. Let  $(X_i, Y_i)$ ,  $i = 1, 2, \dots, n$ , be  $n$  pairs of real linear spaces placed in duality by  $n$  bilinear forms respectively denoted by  $\langle \dots \rangle_i$ . If  $x = (x_1, x_2, \dots, x_n)$  denotes the generic element of the linear space

$$X = X_1 \times X_2 \times \dots \times X_n$$

and  $y = (y_1, y_2, \dots, y_n)$  the generic element of the linear space

$$Y = Y_1 \times Y_2 \times \dots \times Y_n$$

the bilinear form

$$\langle x, y \rangle = \langle x_1, y_1 \rangle_1 + \langle x_2, y_2 \rangle_2 + \dots + \langle x_n, y_n \rangle_n$$

places  $X$  and  $Y$  in duality. For each  $i$ , denote by  $f_i, g_i$  a pair of functions defined respectively on  $X_i$  and  $Y_i$  and mutually polar with regard to the bilinear form  $\langle \dots \rangle_i$ . It is easy to see that the functions  $f$  and  $g$  defined on  $X$  and  $Y$  respectively by

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$$f(x) = f_1(x_1) + f_2(x_2) + \dots + f_n(x_n)$$

$$g(y) = g_1(y_1) + g_2(y_2) + \dots + g_n(y_n)$$

are mutually polar with regard to  $\langle \dots \rangle$ .

The following result is less trivial (see proofs in MOREAU [3] or [10]) :

5° Continuity. The setting is again that of single pair of linear spaces  $(X, Y)$ . The function  $f \in \Gamma_0(X, Y)$  <sup>finite and</sup>  $\wedge$  is continuous at the origin for some locally convex topology on  $X$  compatible with the duality (then for the Mackey topology  $\tau(X, Y)$  which is the finest of them) if and only if the dual function  $g \in \Gamma_0(Y, X)$  is inf-compact, i.e. for any  $k \in \mathbb{R}$  the "level set" or "slice"  $\{y \in Y : g(y) \leq k\}$  is compact for some (locally convex) topology on  $Y$  compatible with the duality (then for the weak topology  $\sigma(Y, X)$  which is the coarsest of them). Note that, due to the convexity of  $g$ , a sufficient condition for that is the existence of some  $k > \inf g$  such that this compactness holds.

Using translation (cf. 3° above) one concludes that the continuity of  $f$  at some point  $x_0 \in X$  is equivalent to the compactness of the "oblique slices of  $g$  with slope  $x_0$ ", i.e. the sets  $\{y \in Y : g(y) - \langle x_0, y \rangle \leq k\}$ .

2. d INF - CONVOLUTION AND THE IMAGE OF ADDITION

Let us denote by  $\dot{+}$  the commutative and associative operation extending classical addition to any pair of elements of  $\bar{\mathbb{R}} = [-\infty, +\infty]$  by putting  $(-\infty) \dot{+} (+\infty) = +\infty$  (symmetrically the operations  $\dot{+}$  extends

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classical addition by the convention  $(-\infty) + (+\infty) = -\infty$ .

Let  $f_1, f_2$  be functions defined on the linear space  $X$  with values in  $\bar{\mathbb{R}}$ ; the function  $f$  defined on  $X$  by

$$(2.4) \quad f(x) = \inf_{u \in X} [f_1(u) + f_2(x - u)] = \inf_{v \in X} [f_1(x - v) + f_2(v)]$$

is called the infimal convolute, or shortly inf-convolute, of  $f_1$  and  $f_2$ ; it is denoted by  $f_1 \nabla f_2$  (or also  $f_1 \square f_2$ , as in ROCKAFELLAR [2], when there is no risk of confusion with the supremal convolute  $f_1 \Delta f_2$ , which would be symmetrically defined by using "sup" and  $+$ ). This operation is commutative and associative; if  $f_1$  and  $f_2$  are convex, so is  $f_1 \nabla f_2$ , etc...

Example 1. If  $f_2$  is the indicator function of a singleton  $\{a\}$ , then  $f_1 \nabla f_2$  is a translate of  $f_1$ , namely the function

$$x \mapsto f_1(x - a) .$$

Example 2. If  $A$  is a subset of  $X$  and  $\|\cdot\|$  a norm on this linear space, then  $(\psi_A \nabla \|\cdot\|)(x)$  is the distance from the point  $x$  to the set  $A$ .

Example 3. If  $A$  and  $B$  are two subsets of  $X$ , the inf-convolute  $\psi_A \nabla \psi_B$  is the indicator function of the set

$$A + B = \{x \in X : \exists a \in A, \exists b \in B, x = a + b\} .$$

Coming back to the setting of the pair of spaces  $(X, Y)$  in duality, the computation of polar functions yields easily

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$$(f_1 \nabla f_2)^* = f_1^* \dot{+} f_2^* .$$

Suppose now that  $f_1$  and  $f_2$  belong to  $\Gamma(X, Y)$  and that  $g_1$  and  $g_2$  are their polar (i.e. dual) functions ; taking the polars of both members of the preceding equality leads to

$$(2.5) \quad (f_1 \nabla f_2)^{**} = (g_1 \dot{+} g_2)^* .$$

Addition  $\dot{+}$  is a composition law in  $\Gamma(Y, X)$  ; (2.5) describes the composition law in  $\Gamma(X, Y)$  which is the image of it by the one-to-one mapping  $*$  ; this composition law is the  $\Gamma$  - hull of inf-convolution (cf. § 2. b above) ; we denote it by  $\nabla$  ; it may be called  $\Gamma$  - convolution.

Of practical importance are the cases where  $f_1 \nabla f_2$  happens to belong to  $\Gamma(X, Y)$  so that the double star may be omitted in (2.5). Let us just formulate here the two most usual of them.

It is still assumed that  $f_1$  and  $f_2$  belong to  $\Gamma(X, Y)$ .

$l^0$  Suppose that the set, denoted by  $\text{cont } f_1$ , of the points where  $f_1$  is finite and continuous, for some topology compatible with the duality, and the set

$$\text{dom } f_2 = \{x \in X : f_2(x) < +\infty\}$$

are such that

$$\text{cont } f_1 \dot{+} \text{dom } f_2 = X .$$

Then  $f_1 \nabla f_2$  is either the constant  $-\infty$  or is finite and continuous everywhere in  $X$  for the considered topology ; therefore

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$f_1 \nabla f_2 \in \Gamma(X, Y)$ , hence  $f_1 \nabla f_2 = f_1 \underline{\nabla} f_2$ .

2° Suppose that there exists a point  $y_0$  in  $Y$  at which both functions  $g_1$  and  $g_2$  are finite, one of them continuous at this point (for some topology compatible with the duality); then  $f_1 \nabla f_2 \in \Gamma(X, Y)$ ; furthermore this inf-convolution is exact, i.e., whichever is  $x$ , the infimum in (2.4) is a minimum. Note that the hypothesis is equivalent to the following: both functions  $x \mapsto f_1(x) - \langle x, y_0 \rangle$  and  $x \mapsto f_2(x) - \langle x, y_0 \rangle$  are bounded from below and one of them is inf-compact for the weak topology  $\sigma(X, Y)$  (cf. § 2 c).

## 2. e SUBGRADIENTS AND SUBDIFFERENTIALS

Let  $f$  denote a function defined on  $X$ , with values in  $\bar{\mathbb{R}}$ ; an element  $y$  of  $Y$  is called a subgradient of  $f$  at the point  $x \in X$  if  $y$  is the slope of an affine minorant of  $f$  exact at the point  $x$ , i.e. taking at this point the same value as  $f$ . This requires that the value  $f(x)$  is finite and that the expected minorant has the form

$$u \mapsto \langle u - x, y \rangle + f(x).$$

Using condition (2.2) for an affine function to minorize  $f$ , one obtains the following representation for the set, denoted by  $\partial f(x)$ , of the subgradients of  $f$  at the point  $x$

$$\partial f(x) = \{y \in Y : f^*(y) - \langle x, y \rangle \leq -f(x)\}.$$



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This set is called the subdifferential of  $f$  at the point  $x$ . The convexity and the lower semicontinuity of  $f^*$  imply that  $\partial f(x)$  is a convex, possibly empty, subset of  $Y$ , closed for the topologies compatible with the duality  $(Y, X)$ . If  $\partial f(x)$  is not empty the function  $f$  is said to be subdifferentiable at the point  $x$ .

Trivially the function  $f$  possesses a finite minimum attained at the point  $x$  if and only if  $\partial f(x)$  contains the zero of  $Y$ .

Recall that the function  $f$  is said weakly differentiable, or Gâteaux-differentiable, at the point  $x$ , relatively to the duality  $(X, Y)$ , if there exists  $y \in Y$  (necessarily unique) such that for any  $u \in X$ , the function  $t \mapsto f(x + t u)$  of the real variable  $t$  possesses for  $t = 0$  a derivative equal to  $\langle u, y \rangle$ ; the element  $y$  is called the weak gradient, or Gâteaux-gradient, of the function  $f$  at the point  $x$ , relatively to the duality  $(X, Y)$ . If in addition the function  $f$  is convex, one easily finds that the subgradient  $\partial f(x)$  consists of the single element  $y$ . When  $X$  is a normed space,  $Y$  its topological dual, all this a fortiori holds if  $f$  is Fréchet-differentiable at the point  $x$ .

Subdifferentiability finds its clearest setting when a pair of dual, i.e. mutually polar functions  $f \in \Gamma_0(X, Y)$  and  $g \in \Gamma_0(Y, X)$  is considered. Then, for  $x$  in  $X$  and  $y$  in  $Y$  the three following properties are equivalent :

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$$(2.6) \quad y \in \partial f(x)$$

$$(2.7) \quad x \in \partial g(y)$$

$$(2.8) \quad f(x) + g(y) - \langle x, y \rangle = 0 ;$$

observe that, by Fenchel's inequality, the  $=$  sign above may equivalently be replaced by  $\leq$ . If these properties hold, the points  $x$  and  $y$  are said conjugate relative to the pair of mutually polar functions  $(f, g)$ .

EXAMPLE. Take as  $f$  the indicator function  $\psi_C$  of a nonempty closed convex subset of  $X$ . Then the relation  $y \in \partial \psi_C(x)$  is trivially equivalent to the following : the point  $x$  belongs to  $C$  and the set

$\{u \in X : \langle u - x, y \rangle \leq 0\}$  contains  $C$ . If  $y$  differs from the zero of  $Y$  this set is a closed half-space whose boundary is a supporting hyper-plane of the set  $C$  at the point  $x$  ; then one classically says that

$y \in Y$  is an outward normal vector at the point  $x$  of the convex set  $C \subset X$ . Let us agree to take this locution in a weak sense, by considering also the zero of  $Y$  as a normal vector at the point  $x$  if it belongs to  $\partial \psi_C(x)$  ; thus the set  $\partial \psi_C(x)$  will be called the outward normal cone at the point  $x$ . This cone is empty if  $x \notin C$  ; if  $x \in C$  it contains at least the zero of  $Y$  and reduces to this single element, in particular, when  $x$  is an internal point of  $C$  (i.e. every straight line drawn to  $x$  intersects  $C$  along a segment to which  $x$  is interior). In terms of the support function  $\psi_C^*$  of  $C$ , condition (2.8) yields that if  $x$

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belongs to C one has

$$\begin{aligned}\partial \psi_C(x) &= \{y \in Y : \psi_C^*(y) = \langle x, y \rangle\} \\ &= \{y \in Y : \psi_C^*(y) \leq \langle x, y \rangle\}.\end{aligned}$$

REMARK. For a pair of spaces  $(X, Y)$  with finite dimension and convex functions  $f, g$  which are differentiable, relations (2.6), (2.7), (2.8) show that the correspondance between  $f$  and  $g$  reduces to the classical Legendre transform.

Let us come back to the case of an arbitrary  $f$  and possibly infinite dimensional spaces. By associating with every  $x \in X$  the subset  $\partial f(x)$  of  $Y$  one defines a multimapping (also called a multifunction, or a multivalued mapping, or a set-valued mapping) from  $X$  into  $Y$ . Independently of the formalization of subgradients and the "subdifferential calculus" (MOREAU [2] ; similar ideas were also present in Rockafellar's Thesis, Harvard, 1963) this multimapping was considered in G.J. MINTY [1] as the leading example of monotone, possibly multivalued, operator. In fact whichever are  $x$  and  $x'$  in  $X$ , whichever are  $y$  in  $\partial f(x)$  and  $y'$  in  $\partial f(x')$ , if any, one finds easily

$$\langle x - x', y - y' \rangle \geq 0$$

which is, by definition, the monotony property of the multimapping  $\partial f$ .

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## 2. f ADDITION RULE

The main calculation rule for subdifferentials concerns addition. If  $f_1$  and  $f_2$  are two numerical functions, defined for instance on  $X$ , the inclusion

$$(2.9) \quad \partial f_1(x) + \partial f_2(x) \subset \partial(f_1 + f_2)(x)$$

is trivial. If this inclusion holds as an equality of sets the functions  $f_1$  and  $f_2$  are said to possess the additivity of the subdifferentials at the point  $x$ .

Let us indicate two usual sufficient conditions for that :

1° If both functions  $f_1$  and  $f_2$  are convex, one of them weakly differentiable at the point  $x$ , inclusion (2.9) holds as an equality of sets.

2° If both functions  $f_1$  and  $f_2$  are convex and if there exists a point  $x_0$  in  $X$  at which one of them is continuous, with both values  $f_1(x_0)$  and  $f_2(x_0)$  finite, inclusion (2.9) holds as an equality of sets for every  $x$  in  $X$ . Continuity must be understood here in the sense of some (locally convex) topology compatible with the duality  $(X,Y)$  : thus the less stringent hypothesis is obtained by taking the finest of them, i.e. the Mackey topology  $\tau(X,Y)$ .

EXAMPLE. Make  $f_1 = f$ , a function defined on  $X$ , with values in  $]-\infty, +\infty]$  and  $f_2 = \psi_C$ , the indicator function of a non empty subset  $C$  of  $X$ . The problem of minimizing the restriction of  $f$  to  $C$  is

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clearly equivalent to that of minimizing, over the whole of  $X$ , the function  $f + \psi_C$ ; a minimizing point  $x$  is characterized by

$$(2.10) \quad 0 \in \partial(f + \psi_C)(x)$$

a condition which is implied by

$$(2.11) \quad 0 \in \partial f(x) + \partial \psi_C(x)$$

When the additivity of the subdifferentials holds, conditions (2.10) and (2.11) are equivalent.

Such is the case for instance, by 1° above, if the set  $C$  is convex, and the function  $f$  convex, everywhere weakly differentiable : then (2.11), written as

$$(2.12) \quad - \text{grad } f(x) \in \partial \psi_C(x)$$

is a necessary and sufficient condition for  $x$  to be a solution of our "constrained minimization problem". Make in particular  $X = Y = H$ , a separated pre-Hilbert space with the inner product  $(\cdot | \cdot)$  playing the role of the bilinear form  $\langle \cdot, \cdot \rangle$ . Let  $a$  be an arbitrary element of  $H$ ; define the function  $f$  by

$$f(x) = \frac{1}{2} (x-a | x-a) = \frac{1}{2} \|x-a\|^2 .$$

Elementary calculation proves that this function is convex and weakly differentiable relatively to the duality  $(H,H)$ , with

$$\text{grad } f(x) = x - a .$$

Then (2.12) yields a necessary and sufficient condition for  $x$  to be

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the nearest point to a in C

$$(2.13) \quad a - x \in \theta \psi_C(x) ;$$

such an  $x$  is denoted by  $\text{proj}_C(a)$  or  $\text{proj}(a, C)$ , if it exists. Uniqueness of this points results from  $f$  being strictly convex ; recall on the other hand that if  $H$  is complete, i.e. if it is a Hilbert space, the existence of  $\text{proj}_C a$  is secured for any  $a \in H$ .

## 2. g IMAGES BY LINEAR MAPPINGS

Let  $(F,G)$  be another pair of linear spaces, placed in separating duality by a bilinear form denoted by  $\langle\langle \cdot, \cdot \rangle\rangle$ . Let  $A$  be a linear mapping from  $F$  into  $X$ , weakly continuous (i.e. continuous from  $F$  endowed with any topology compatible with the duality  $(F,G)$ , to  $X$  endowed with the weak topology  $\sigma(X,Y)$ ). Weak continuity implies the existence of the adjoint (or transpose) of  $A$ , i.e. the linear mapping  $A^*$  from  $Y$  into  $G$  such that

$$\forall u \in F, \forall y \in Y : \langle Au, y \rangle = \langle\langle u, A^* y \rangle\rangle .$$

Let  $f \in \Gamma(X,Y)$  ; clearly the function

$$f \circ A : u \mapsto f(Au)$$

belongs to  $\Gamma(F,G)$  ; one proves (see ROCKAFELLAR [3] ) that its dual function  $(f \circ A)^*$  is the  $\Gamma$ - hull of the function defined on  $G$  by

$$(2.14) \quad v \mapsto \inf \{ f^*(y) : A^* y = v \} .$$

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If in addition there exists a point in the range of A at which f is finite and continuous (for some topology compatible with the duality (X,Y)) then  $(f \circ A)^*$  equals the function (2.14) itself. Under the same assumption, for every  $u \in F$ , the subdifferential  $\partial(f \circ A)(u)$  is the image of  $\partial f(A u) \subset Y$  under the mapping  $A^*$ ; this may be expressed by writing

$$(2.15) \quad \partial(f \circ A) = A^* \circ \partial f \circ A .$$

2. h CONJUGATE GAUGE FUNCTIONS AND QUASI - HOMOGENEOUS CONVEX FUNCTIONS

The setting is again that of a single pair of spaces  $(X,Y)$ .

Let A be a closed convex subset of X containing the origin ; denote by B the polar set of A, i.e.

$$B = \{y \in Y : \forall x \in A , \langle x,y \rangle \leq 1\} .$$

Then A is, symmetrically, the polar set of B. It is easily seen that

the gauge function of A, namely the function a defined on X by

$$a(x) = \inf \{ \lambda \in ]0, +\infty[ : \frac{1}{\lambda} x \in A \} ,$$

is the support function of B ; symmetrically the gauge function b of

B is the support function of A. We shall refer to this situation by

saying that  $(a, b)$  is a pair of conjugate gauge functions.

For sake of simplicity let us restrict ourselves here to the

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case where both functions take only finite values ; this means that  $A$  is absorbent in  $X$  (i.e. the origin is an internal point) and that it is bounded relatively to the topologies compatible with the duality ; equivalently  $B$  possesses the same properties in  $Y$ . Such is the case, for instance if  $X$  is a given normed space,  $Y$  its dual endowed with the usual norm : the respective norms form a pair of conjugate gauge functions and the corresponding mutually polar sets are the closed unit balls of the two spaces.

One finds

$$b(y) = \sup_{a \in X} \frac{\langle x, y \rangle}{a(x)}$$

and the symmetrical relation (this can be extended to possibly infinite valued conjugate gauge functions, under some notational precautions).

Consider on the other hand a mapping  $\phi$  from  $[0, +\infty[$  into  $[0, +\infty]$  possessing the following properties :  $\phi$  is convex, non decreasing, lower semi continuous and  $\phi(0) = 0$  (actually  $\phi$  is continuous on the interior of  $\text{dom } \phi = \{\xi \in [0, +\infty[ : \phi(\xi) < +\infty\}$ ). Classically, with such a function is associated its Young conjugate  $\gamma$  defined on  $[0, +\infty[$  by

$$\gamma(\eta) = \sup (\xi \eta - \phi(\xi))$$

which possesses the same properties ;  $\phi$  is, in turn, the Young conjugate of  $\gamma$ .



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Examples :

$$1^{\circ} \quad \phi(\xi) = \frac{1}{p} \xi^p, \quad \gamma(\eta) = \frac{1}{q} \eta^q$$

where  $p$  and  $q$  denote two constants in  $]1, +\infty[$ , such that

$$1/p + 1/q = 1.$$

$$2^{\circ} \quad \phi(\xi) = \lambda \xi, \quad \gamma(\eta) = \begin{cases} 0 & \text{if } 0 \leq \eta \leq \lambda \\ +\infty & \text{if } \lambda < \eta < +\infty \end{cases}$$

where  $\lambda \in [0, +\infty[$  is a constant.

Exclude the singular case where one of the two functions  $\phi$  and  $\gamma$  is the constant zero. Then one proves that the functions

$f = \phi \circ a$ ,  $g = \gamma \circ b$  respectively defined on X and Y, i.e.

$$f(x) = \phi(a(x)), \quad g(y) = \gamma(b(y))$$

are a pair of dual functions in the sense of the preceding paragraphs.

Each of these functions is said quasi-homogeneous (or gauge-like in ROCKAFELLAR [2]) ; in fact in the special case where

$$\phi(\xi) = \frac{1}{p} \xi^p, \text{ the function } f \text{ is } \underline{\text{positively homogenous with degree } p}.$$

The functions defined in this way, for instance on  $X$ , may be characterized as follows :

they are the elements of  $\Gamma_0(X, \mathbb{R})$  such that the various sets

$\{x \in X : f(x) \leq k\}$  (the "slices" of  $f$ ), for  $k \in \mathbb{R}$

are homothetic to  $A$ . (they are empty for  $k < 0$ ).

Concerning the determination of the subdifferentials of these functions, let us only indicate : Two points  $x \in X$  and  $y \in Y$  are

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conjugate relatively to (f,g) if and only if

$$\phi(a(x)) + \gamma(b(y)) = a(x)b(y) = \langle x, y \rangle .$$

The first equality may be interpreted by saying that the real numbers  $a(x)$  and  $b(y)$  are conjugate points with regard to the pair of Young conjugate functions  $(\phi, \gamma)$ ; if  $x$  and  $y$  are different from the respective origins of  $X$  and  $Y$ , the second one expresses a property of the "rays" (i.e. one-dimensional cones) they generate in  $X$  and  $Y$ ; such rays may be said conjugate relative to the pair of conjugate gauge functions  $a$  and  $b$ .

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### 3 SUPERPOTENTIALS AND PERFECT CONSTRAINTS

#### 3. a CONFIGURATIONS AND FORCES

In this Chapter is considered a mechanical system  $\mathcal{O}$  whose set of possible configurations, denoted by  $\mathcal{U}$ , is endowed with a linear space structure. Such is traditionally the case, due to the use of linear approximation, if the system presents only "small deviations" from a certain reference configuration which constitutes the zero of  $\mathcal{U}$ .

The bilinear form work places the linear space  $\mathcal{U}$  in duality with a linear space  $\mathcal{F}$  whose elements constitutes, in a general sense, the possible values of forces experienced by the system. Precisely  $\langle u, f \rangle$  denotes the work of the force  $f \in \mathcal{F}$  for the displacement  $u \in \mathcal{U}$  of the system. For sake of clarity, we shall in some cases comply with the habit of denoting a displacement by such a symbol as  $\delta u$ ; this symbol is meant to recall that the considered displacement equals the difference between two elements of  $\mathcal{U}$  representing some configurations; actually, in the present framework, due to the existence of the privileged configuration "zero", configurations as well as displacements are elements of  $\mathcal{U}$ , thus have the same algebraic nature.

After replacing, if necessary, the considered spaces by some quotients, it may be supposed that this duality is separating.

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EXAMPLE. Take as  $\mathcal{S}$  a perfectly rigid body performing only "infinitely small" motions in the neighborhood of the reference configuration. From this reference state, each possible configuration of the body may be described by the corresponding field of displacement vectors, say  $u : x \mapsto \vec{u}(x)$ . Due to the rigidity of the body and to the fact that displacements are, by approximation, treated as infinitely small this field possesses the property of equiprojectivity ; the totality of equiprojective vector fields is well known to form a linear space of dimension 6 : such is  $\mathcal{U}$  in the present case. For sake of brevity let us accept only as acting on  $\mathcal{O}$  finite families of forces in the sense of elementary Mechanics. Such a family may be described as a vector field  $\phi : x \mapsto \vec{\phi}(x)$  taking the value zero everywhere except on a finite set of points and its work for a displacement field  $u \in \mathcal{U}$  is classically defined as the finite sum  $w = \sum \vec{u}(x) \cdot \vec{\phi}(x)$ . For a fixed  $\phi$  the mapping  $u \mapsto w$  is clearly a linear form on the space  $\mathcal{U}$  ; on the other hand, the set  $\Phi$  of the possible  $\phi$ 's is naturally endowed with a linear space structure which makes that, for a fixed  $u$ , the work  $w$  is a linear form of  $\phi$ . But the space  $\Phi$  clearly has an infinite dimension, so that this bilinear form cannot place  $\mathcal{U}$  and  $\Phi$  in separating duality. The classical procedure consists in treating as equivalent two families of forces, say  $\phi$  and  $\phi'$ , such that

$$\forall u \in \mathcal{U} \quad : \quad \sum \vec{u}(x) \cdot \vec{\phi}(x) = \sum \vec{u}(x) \cdot \vec{\phi}'(x).$$

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The corresponding equivalence classes are called torsors. In other words, if  $\Phi_0$  denotes the linear subspace of  $\Phi$  formed by the families of forces which yield a zero work for any  $u \in \mathcal{U}$ , torsors are the elements of the quotient space  $\Phi / \Phi_0$ , with dimension 6. Such is  $\mathcal{F}$  in the present case ; the duality between  $\mathcal{U}$  and  $\mathcal{F}$  is then separating.

PRODUCT SPACES. Suppose the mechanical system  $\mathcal{O}$  consists in the conjunction of  $n$  possibly interacting systems  $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_n$  whose respective configuration spaces are the linear spaces  $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_n$ .

Then the configuration space of  $\mathcal{O}$  is the product space  $\mathcal{U}_1 \times \mathcal{U}_2 \times \dots$

$\dots \times \mathcal{U}_n$ , naturally endowed with a linear space structure. Denote by  $\mathcal{F}_i$

the force space corresponding to the system  $\mathcal{O}_i$ , a linear space placed

in separating duality with  $\mathcal{U}_i$  by the bilinear form  $\langle \dots \rangle_i$ . A force  $f$

exerted on the total system  $\mathcal{O}$  is a  $n$ -tuple  $(f_1, f_2, \dots, f_n)$ ,

$f_i \in \mathcal{F}_i$ ; this is the generic element of the product space

$\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 \times \dots \times \mathcal{F}_n$ . The work of  $f$  for a displacement

$u = (u_1, u_2, \dots, u_n)$  of  $\mathcal{U}$  is by definition the sum

$$\langle u, f \rangle = \sum_i \langle u_i, f_i \rangle_i$$

in which we recognize the natural bilinear form placing the product spaces

$\mathcal{U}$  and  $\mathcal{F}$  in separating duality (cf. § 2. c).

This construction of  $\mathcal{U}$  and  $\mathcal{F}$  as the products of the respective spaces corresponding to subsystems of  $\mathcal{O}$  is a customary procedure in computation. It prepares also for the application of our general

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pattern to continuous media, as developed in B. Nayroles's lectures : then  $\mathcal{U}$  and  $\mathcal{F}$  are some linear spaces of measurable functions, with regard to a certain non-negative measure. The sum which above defines the work is replaced by an integral.

### 3. b STATICAL LAWS

A statical law is a relation, denote it by  $\mathcal{R}$ , between the configuration  $u \in \mathcal{U}$  that the system  $\mathcal{C}$  may occupy and some, say  $f \in \mathcal{F}$ , among the forces it may experience when it comes through this configuration. Such a relation arises from the study of some of the physical processes in which the system is involved.

Instead of relations as  $\mathcal{R}$ , one may ~~equivalently~~ speak of multi-mappings from one of the two spaces into the other ; for instance, to every  $u$  in  $\mathcal{U}$  corresponds the (possibly empty) set, denote it by  $R(u)$ , of the elements  $f$  of  $\mathcal{F}$  which are related to  $u$  by  $\mathcal{R}$ .

In particular it may happen that the set  $R(u)$  consists, for each  $u$ , of a single element ; then the statical law is described as a single-valued mapping  $u \mapsto f$  from  $\mathcal{U}$  into  $\mathcal{F}$ . If, in addition, there exists a numerical function  $W : \mathcal{U} \rightarrow \mathbb{R}$  such that this mapping is expressed by

$$f = - \text{grad } W(u)$$

(weak gradient or "Gâteaux differential" relative to the duality defined

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above) it is classically said that the considered statical law admits  $W$  as potential.

The simplest statical law imposes the value  $f_0 \in \mathcal{F}$  of a certain force acting on the system, independently of the configuration  $u$ . Such a constant mapping from  $\mathcal{U}$  into  $\mathcal{F}$  evidently admits the potential  $W$  expressed by

$$W(u) = -\langle u, f_0 \rangle .$$

**EQUILIBRIUM.** Suppose that all the physical processes in which the system  $\mathcal{O}$  takes part imply forces, acting on it, which either vanish in any expected equilibrium or are  $n$  forces  $f_1, f_2, \dots, f_n$  respectively related to the configuration  $u$  by  $n$  statical laws independent of time, denoted by  $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_n$ . Then the equilibrium problem consists in determining the values of  $u$  in  $\mathcal{U}$  possessing the following property : there exist  $f_1, f_2, \dots, f_n$  in  $\mathcal{F}$  respectively related to  $u$  by the relations  $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_n$  and such that  $f_1 + f_2 + \dots + f_n = 0$ . According to the "principle of virtual work" and due to the way in which  $\mathcal{F}$  has been constructed as a quotient space placed in separating duality with  $\mathcal{U}$ , these values of  $u$  correspond in fact to the equilibrium configurations of  $\mathcal{O}$ , i.e. the configurations in which immobility is a motion compatible with our physical information about this system.

Equivalently, if  $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_n$  denote the multimappings corresponding as above to the  $n$  statical laws, the equilibrium

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configurations are characterized by

$$0 \in R_1(u) + R_2(u) + \dots + R_n(u) .$$

Let us stress at last that the concept of statical law, as we just defined it, is not restricted to the study of equilibrium problems. In evolution problems also, statical laws will be considered, possibly depending on time.

### 3. c FRICTIONLESS BILATERAL CONSTRAINTS

The description of a constraint in Mechanics requires fundamentally more information than merely defining a set of permitted configurations. This description always includes some indication concerning the forces of constraint or reactions experienced by the system and implied by the material process which restricts its freedom. Let us emphasize that perfect, i.e. frictionless, constraints are a special type of statical law.

Consider for instance the situation described in the language of elementary Mechanics as follows : a certain particle  $s$  of the system  $\mathcal{O}$  is maintained bilaterally, without friction, on a given regular material surface  $S$ . Let

$$(3.1) \quad h(\vec{x}) = 0$$

be the equation of  $S$ , where  $\vec{x}$  denotes the generic element of a three-dimensional frame of reference  $E_3$ , treated as a three-dimensional



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linear space, and  $h$  a smooth numerical function defined on  $E_3$ , with nonzero gradient. Let  $\vec{p}_0$  denote the position of the particle  $s$  in  $E_3$  when the system  $\mathcal{C}$  presents the configuration corresponding to the zero of  $\mathcal{U}$ . For the configuration corresponding to some element  $u$  of  $\mathcal{U}$ , this position is  $\vec{p}$  and, due to our framework of small deviations and linearization, the mapping  $\vec{\ell} : u \mapsto \vec{p} - \vec{p}_0$  is treated as linear from  $\mathcal{U}$  into  $E_3$ ; in all the following, this linear mapping is supposed continuous with regard to some locally convex topology compatible with the duality  $(\mathcal{U}, \mathcal{F})$ , thus continuous for all such topologies. Similarly, the linearization procedure replaces the function  $h$  by its first order expansion in the neighborhood of  $\vec{p}_0$  so that the condition  $\vec{p} \in S$  takes the form

$$h(\vec{p}_0) + (\vec{p} - \vec{p}_0) \cdot \overrightarrow{\text{grad}} h(\vec{p}_0) = 0$$

(scalar product and gradient are understood here in the sense of the three-dimensional Eucliden space  $E_3$ ) i.e.

$$(3.2) \quad h(\vec{p}_0) + \vec{\ell}(u) \cdot \overrightarrow{\text{grad}} h(\vec{p}_0) = 0$$

Here arises the need of an additional hypothesis concerning  $\vec{\ell}$  for the continuous linear form  $u \mapsto \vec{\ell}(u) \cdot \overrightarrow{\text{grad}} h(\vec{p}_0)$  not to be identically zero; as the vector  $\overrightarrow{\text{grad}} h(\vec{p}_0)$  has been supposed different from zero, the sufficient assumption we shall make in all the following is: the linear mapping  $\vec{\ell}$  from  $\mathcal{U}$  into the three-dimensional space of the "physical" vectors is surjective. One may express this by saying that the

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particle  $s$  of the system is regular regarding the use of  $\mathcal{U}$  as the configuration space of the system. Then the values of  $u$  satisfying (3.2) constitute a closed hyperplane

$$(3.3) \quad \mathcal{L} = U + a \quad ,$$

where  $a$  represents some known element of  $\mathcal{U}$  and  $U$  denotes the linear subspace with codimension 1

$$U = \{u \in \mathcal{U} : \vec{\ell}(u) \cdot \overrightarrow{\text{grad}} h(\vec{p}_0) = 0\} \quad .$$

For the particle  $s$  to be maintained in  $S$  it must experience in addition to other possible actions, the force of constraint  $\vec{R}$ , or reaction, arising from this material surface. In the language of the pair of spaces  $(\mathcal{U}, \mathcal{F})$  the representation of this force consists, by definition, in the element  $r \in \mathcal{F}$  possessing the following property : for any  $\delta u \in \mathcal{U}$ , to which corresponds in the "physical" space  $E_3$  the displacement  $\delta \vec{p} = \vec{\ell}(\delta u)$  of the particle  $s$ , the work of  $\vec{R}$  equals  $\langle \delta u, r \rangle$ , i.e.

$$(3.4) \quad \langle \delta u, r \rangle = \vec{\ell}(\delta u) \cdot \vec{R} \quad .$$

Let us make use now of the hypothesis that the constraint is frictionless. By definition this means  $\vec{R}$  is normal to the surface  $S$  at the point  $p$ ; equivalently  $\vec{R}$  yields a zero work for any displacement vector  $\delta \vec{p}$  which is tangent to  $S$  at this point. Due to the linearization procedure which replaces the equation of  $S$  by (3.2), this amounts to

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$$(3.5) \quad \forall \delta u \in U : \langle \delta u, r \rangle = 0 .$$

In other words  $r$  belongs to  $V$ , the subspace of  $\mathcal{F}$  orthogonal to  $U$ .

It will be supposed that conversely any value of  $r$ , i.e. of  $\vec{R}$ , satisfying this condition can be produced by the device enforcing the constraint. Physically, this means first the constraint is bilateral : the particle  $s$  should more exactly be visualized as guided without friction between two parallel surfaces infinitely close to each other ; secondly these surfaces are strong enough to exert arbitrarily large normal reactions. We propose to summarize these facts by saying that the considered perfect constraint is firm (cf. MOREAU [14], vol. 2, § 9. 2) Except otherwise stated, firmness will always be implicitly assumed in the following.

In short, all our information about the constraint is contained in the two conditions  $u \in \mathcal{L}$  ,  $r \in V$  ; equivalently it may be said that the pair  $(u, r)$  belongs to the subset  $\mathcal{L} \times V$  of  $\mathcal{U} \times \mathcal{F}$  and this indeed constitutes a statical law in the sense defined by § 3. b, i.e. a relation between the possible configuration  $u$  of the system and some of the forces it undergoes.

This relation is subdifferential.

In fact consider the indicator function  $\psi_{\mathcal{L}}$  of the affine manifold described by (3.3) ; the subdifferential of this closed convex function is easily found to be

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$$\partial \psi_{\mathcal{L}}(u) = \begin{cases} V & \text{if } u \in \mathcal{L} \\ \emptyset & \text{if } u \notin \mathcal{L} \end{cases} .$$

Therefore the relation  $(u,r) \in \mathcal{L} \times V$  is equivalent to

$$(3.6) \quad -r \in \partial \psi_{\mathcal{L}}(u) ,$$

which is another way of conveying the whole of our information about the considered constraint. The minus sign in the left member is immaterial as the right member is a linear space : this is only for sake of consistency with further developments.

More generally, the system  $\mathcal{C}$  may be submitted at the same time to several constraints of the preceding sort, respectively defined by  $n$  closed hyperplanes  $\mathcal{L}_i = U_i + a_i$ ,  $i = 1, 2, \dots, n$ . The set of the permitted configurations is then  $\bigcap_i \mathcal{L}_i$ ; if this intersection is not empty let us use again the notation  $\mathcal{L} = U + a$  to represent it, where  $U$  is now the intersection of the closed linear subspaces  $U_i$ , each with codimension 1. As the reaction  $r_i$  implied by the  $i$ -th constraint belongs to  $V_i$ , the one-dimensional subspace orthogonal to  $U_i$  in  $\mathcal{F}$ , the sum  $r$  of the  $n$  reactions belongs to  $V$ , the subspace orthogonal to  $U$ . Conversely, any element of  $V$  possesses at least one decomposition into a sum  $\sum_i r_i$ ,  $r_i \in V_i$  (this is merely the classical theorem of Lagrange multipliers : the duality between  $\mathcal{U}$  and  $\mathcal{F}$  being separating, the bi-orthogonal of a finitely generated subspace equals this subspace itself). Therefore, each of the  $n$  perfect constraints being assumed firm, the

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joint effect of them is fully represented by the same writing as (3.6) and this is also trivially true in the case  $\mathcal{L}$  is empty.

Thereby we are induced to consider, in general, statical laws expressed under the form (3.6), where  $\mathcal{L}$  represents a closed affine manifold whose codimension is not necessarily finite : we shall refer to such statical laws as (firm) perfect affine constraints.

Note at last that, when studying evolution problems, a perfect constraint described as above may be moving : i.e. the affine manifold  $\mathcal{L}$  may depend on time in a given way. Just keep in mind at such event that the so-called displacements, labelled in the preceding by the symbol  $\delta$ , merely express the comparison between possible configurations at a definite instant ; traditionnally they are qualified as virtual in contrast with the real displacements which occur as a consequence of the actual motion. In most practical cases the subspace  $U$  which defines the dimension and the direction of  $\mathcal{L}$  is independent of time ; only the element  $a$  of  $\mathcal{U}$  is moving ; we shall meet such a situation in Chapter 6.

### 3. d PERFECT UNILATERAL CONSTRAINTS

With the same notations as in the preceding, suppose now that the particle  $s$  of the system  $\mathcal{C}^s$ , instead of being bilaterally maintained in the surface  $S$ , is only confined by some impenetrable block

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whose  $S$  constitutes the boundary. Suppose the function  $h$  chosen in such a way that the region of  $E_3$  permitted thereby to the position  $\vec{p}$  of  $s$  is defined by the inequality

$$h(\vec{p}) \geq 0 .$$

Then, using the same linearization procedure as before, the set of the permitted values of  $u$  is characterized by the inequality

$$(3.7) \quad h(\vec{p}_0) + \vec{\ell}(u) \cdot \overrightarrow{\text{grad}} h(\vec{p}_0) \geq 0$$

which defines in  $\mathcal{U}$  a closed half-space  $\mathcal{D}$  with the affine manifold  $\mathcal{S}$  as boundary.

Here again, the description of the mechanical situation requires some information about the force of constraint  $\vec{R}$  that the block must exert on  $s$  to prevent penetration ; this information will rather be formulated by means of the element  $r \in \mathcal{F}$  which represents the force according to (3.4).

First, this reaction vanishes when  $s$  does not touch the block, i.e. when (3.7) holds as a strict inequality ; in other words one has the implication

$$(3.8) \quad u \in \text{int } \mathcal{D} \Rightarrow r = 0 .$$

When, on the contrary,  $s$  lies in contact with the boundary  $S$ , we still make the no-friction hypothesis, i.e.  $\vec{R}$  is normal to  $S$ . In addition the unilaterality of the contact imposes that the vector  $\vec{R}$  is directed toward the permitted region i.e. directed in concordance with

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the vector  $\overrightarrow{\text{grad}} h(\vec{p})$ . In terms of work this is expressed as follows : any  $\delta \vec{p}$  such that  $\delta \vec{p} \cdot \overrightarrow{\text{grad}} h(\vec{p}) \geq 0$  yields  $\delta \vec{p} \cdot \vec{R} \geq 0$ . Now, recalling the regularity assumption made about the mapping  $\vec{\ell}$ , take as  $\delta \vec{p}$  the displacement of  $s$  in  $E_3$  associated as before with the element  $\delta u$  of  $\mathcal{U}$  by  $\delta \vec{p} = \vec{\ell}(\delta u)$ . The contact between  $s$  and the block means that (3.7) holds as an equality. Besides, due to the linearization procedure,  $\overrightarrow{\text{grad}} h(p)$  is treated as independent of  $p$ . Then  $\delta \vec{p} \cdot \overrightarrow{\text{grad}} h(\vec{p}) \geq 0$  holds if and only if  $\vec{\ell}(\delta u) \cdot \overrightarrow{\text{grad}} h(\vec{p}_0) \geq 0$ ; by putting  $u' = u + \delta u$  the latter is equivalent to  $u' \in \mathcal{D}$  so that finally, in view of (3.4), all our information about  $\vec{R}$  comes to be equivalent to the following

$$(3.9) \quad \forall u' \in \mathcal{D} : \langle u' - u, r \rangle \geq 0 .$$

This actually implies also (3.8) ; in fact, if  $u \in \text{int } \mathcal{D}$  the difference  $u' - u$ , for  $u' \in \mathcal{D}$  can be a non zero element of  $\mathcal{U}$  with arbitrary direction; hence  $r = 0$  for the duality is separating.

In conclusion the geometric condition  $u \in \mathcal{D}$  of the constraint is expressed jointly with (3.9) by writing

$$(3.10) \quad -r \in \partial \psi_{\mathcal{D}}(u) .$$

Here as in § 3. c let us make conversely the firmness assumption : the block is supposed strong enough to exert any value of  $\vec{R}$  agreeing with the preceding requirements ; in other words any value of  $r$  satisfying (3.9) is possible. Then relation (3.10) conveys all our

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information about the considered constraint.

More generally suppose the system  $\mathcal{S}$  subjected to  $n$  constraints of the preceding sort, corresponding to half-spaces  $\mathcal{D}_i$ ,  $i = 1, 2, \dots, n$ . Then the set of the permitted configurations is the closed convex set  $C = \bigcap_i \mathcal{D}_i$ . As each of the reactions  $r_i$  satisfies a relation of the form (3.10), their sum  $r$  satisfies

$$-r \in \partial \psi_{\mathcal{D}_1}(u) + \partial \psi_{\mathcal{D}_2}(u) + \dots + \partial \psi_{\mathcal{D}_n}(u) .$$

The right member is trivially contained in  $\partial \psi_C(u)$ ; actually this sum of sets equals exactly  $\partial \psi_C(u)$  because of the "unilateral" counterpart of the multipliers theorem (as the duality  $(\mathcal{U}, \mathcal{F})$  is separating, a finitely generated convex cone in  $\mathcal{F}$  is closed, thus equal to its bipolar). In conclusion the conjunction of our  $n$  unilateral constraints is equivalent to the following statical law

$$(3.11) \quad -r \in \partial \psi_C(u) .$$

Hence we are induced to consider more generally the statical laws defined in the same way by taking as  $C$  arbitrary closed convex subsets of  $\mathcal{U}$ : we call these laws (firm) perfect convex constraints.

Evidently the bilateral constraint studied in § 3. e are a special case of this: take as  $C$  a closed affine manifold.

### 3. e SUPERPOTENTIALS

We shall say that a statical law admits a function



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$\phi \in \Gamma_0(\mathcal{U}, \mathcal{F})$  as superpotential if this law consists in the following relation between the configuration  $u$  and some force  $f$

$$-f \in \partial \phi(u) .$$

In particular, if a statical law admits some numerical function  $W$  as potential,  $W$  is also a superpotential if and only if this function is convex. For instance the constant law  $f = f_0$  (independent of  $u$ ) admits as superpotential the linear form  $u \mapsto -\langle u, f_0 \rangle$ .

Another fundamental example is that of a perfect convex constraint, as presented in the preceding paragraph : (3.11) means that the function  $\psi_C$  is a superpotential for such a statical law ; by taking as  $C$  a closed affine manifold, this includes, according to § 3. c, the traditional bilateral constraints.

Suppose the system subjected at the same time to a finite family of statical laws admitting the respective superpotentials  $\phi_1, \phi_2, \dots, \phi_n$ . Then the sum of  $f = f_1 + f_2 + \dots + f_n$  of the corresponding forces is related to  $u$  by

$$-f \in \partial \phi_1(u) + \partial \phi_2(u) + \dots + \partial \phi_n(u) .$$

This relation implies

$$(3.12) \quad -f \in \partial (\phi_1 + \phi_2 + \dots + \phi_n)(u)$$

but is equivalent to it only if some conditions ensuring the additivity of subdifferentials are fulfilled ; according to § 2.f, the usual case where such additivity holds is described as follows : 1° some of the

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functions  $\phi_i$  are weakly differentiable everywhere in  $\mathcal{U}$  ; 2° there exists a point  $u_0 \in \mathcal{U}$  at which the others, but possibly one, are finite and continuous for some topology compatible with the duality  $(\mathcal{U}, \mathcal{F})$  ; 3° the last one is finite at  $u_0$  .

EQUILIBRIUM. Suppose first that all the mechanical actions to which the system is subjected (except possibly those which vanish in any expected equilibrium) are summarized under the form of a single statical law admitting a superpotential  $\phi$  independent of time. Then, as explained in § 3. b, the equilibrium configurations are characterized by

$$0 \in \partial \phi (u) \quad ;$$

this is a necessary and sufficient condition for  $u$  to be one of the points of  $\mathcal{U}$  where the numerical function  $\phi$  attains its infimum (cf. § 2. e). Such values of  $u$  form a closed convex subset of  $\mathcal{U}$ , possibly empty.

Suppose more generally that the considered mechanical actions are described by the conjunction of  $n$  statical laws admitting as above the respective superpotentials  $\phi_i$ , independent of time. A necessary and sufficient condition for  $u$  to be an equilibrium configuration is now

$$0 \in \partial \phi_1 (u) + \partial \phi_2 (u) + \dots + \partial \phi_n (u) \quad .$$

This implies  $0 \in \partial \phi (u)$ , with  $\phi$  equal to the sum of the functions  $\phi_i$  ; therefore this sum attains its infimum at the point  $u$ . But the converse may not be true, unless the additivity of subdifferentials

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holds. Actually such a reserve does not seem to be of great practical importance and B. Nayroles suggests in his lectures a logical attitude which would overcome the difficulty.

EXAMPLE. Make  $n = 2$  and suppose that  $\phi_1 = \psi_C$ , the superpotential of a perfect convex constraint. Then equilibrium is characterized by

$$0 \in \partial \psi_C (u) + \partial \phi_2 (u) .$$

This implies that  $u$  is a point in  $C$  where the restriction of the function  $\phi_1$  to this set attains its infimum ; in the vocabulary of mathematical programming,  $u$  is one of the solutions of a "constrained" minimization problem. But the converse may not be true, unless the additivity of subdifferentials holds ; particularizing the situation described above, one finds that any of the three following conditions ensures this additivity :

1° The function  $\phi_2$  is weakly differentiable everywhere in  $\mathcal{U}$ , i.e. it is a potential in the classical sense.

2° There exists a point in the interior of  $C$  where the function  $\phi_2$  takes a finite value.

3° There exists a point in  $\mathcal{U}$  at which the function  $\phi_2$  is finite and continuous and which belongs to  $C$ .

Recall that "interior" or "continuous" may here be understood in the sense of any locally convex topology compatible with the duality  $(\mathcal{U}, \mathcal{F})$ : the weakest assumption is thus obtained by choosing the finest

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of these topologies, i.e. the Mackey topology  $\tau(\mathcal{U}, \mathcal{F})$ ; this remark is of course without object in finite dimensional cases.

### 3. f DUAL MINIMUM PROPERTIES

This paragraph is devoted to the equilibrium problem, in the case where all the mechanical actions exerted on the system  $\mathcal{O}$  (except possibly those which vanish at any expected equilibrium) are expressed as the conjunction of two statical laws respectively admitting the superpotentials  $\phi_1$  and  $\phi_2$ , independent of time. Of course, each of these two superpotentials may in its turn describe the conjunction of several laws; in practical situations there are usually various possibilities of classifying the mechanical actions into such two groups, so that the statements presented below can generate a great number of different variational properties. It may be imagined that  $\phi_1$  and  $\phi_2$  correspond to two different sorts of mechanical action: for instance  $\phi_1$  is the superpotential of a perfect constraint, while  $\phi_2$  represents "active forces".

An element  $u$  of  $\mathcal{U}$  is an equilibrium configuration if and only if there exist  $f_1 \in -\partial \phi_1(u)$  and  $f_2 \in -\partial \phi_2(u)$  such that  $f_1 + f_2 = 0$ . The determination of such  $f_1$  (or equivalently  $f_2$ ) prior to that of  $u$ , is sometimes called a statical approach of the equilibrium problem (we should prefer to call it sthenic, an adjective meaning

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"relative to forces"). Privileging  $\phi_1$ , let us agree to call an equilibrium force any value of  $f_1$  associated in this way with some equilibrium configuration.

PROPOSITION 1. Let  $\gamma_1$  and  $\gamma_2$  be the respective polar (i.e. dual) functions of  $\phi_1$  and  $\phi_2$ , relative to the duality  $(\mathcal{U}, \mathcal{F})$ ; denote by  $\hat{\gamma}_1$  the function  $f \mapsto \gamma_1(-f)$  (it is the polar function of  $\hat{\phi}_1 : u \mapsto \phi_1(-u)$ ). Then any equilibrium force minimizes the function  $\hat{\gamma}_1 + \gamma_2$  over  $\mathcal{F}$ ; conversely, if  $f_1$  is a minimizing point of this sum and if  $\hat{\gamma}_1$  and  $\gamma_2$  possess the additivity of subdifferentials at this point,  $f_1$  is an equilibrium force.

In fact if  $f_1 \in \mathcal{F}$  corresponds to some  $u \in \mathcal{U}$  such that  $-f_1 \in \partial \phi_1(u)$  and  $f_1 \in \partial \phi_2(u)$  one has equivalently  $u \in \partial \gamma_2(f_1)$  and  $u \in \partial \hat{\gamma}_1(-f_1)$ ; the latter is the same as  $-u \in \partial \hat{\gamma}_1(f_1)$ ; therefore

$$0 \in \partial \hat{\gamma}_1(f_1) + \partial \gamma_2(f_1) \subset \partial (\hat{\gamma}_1 + \gamma_2)(f_1).$$

Conversely, the assumption that  $f_1$  is a minimizing point of  $\hat{\gamma}_1 + \gamma_2$  means that the zero of  $\mathcal{U}$  belongs to  $\partial (\hat{\gamma}_1 + \gamma_2)(f_1)$ ; if this set equals  $\partial \hat{\gamma}_1(f_1) + \partial \gamma_2(f_1)$ , one has

$$0 \in \partial \gamma_2(f_1) - \partial \gamma_1(-f_1)$$

which precisely expresses the existence of some  $u$  associated with  $f_1$  in the preceding way.

As far as we can see this Proposition contains as special cases,

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all the extremal properties of "statical" type in elastostatics. Observe in this connection that if  $\phi_2$ , for instance, is the superpotential of the perfect bilateral constraint defined by the affine manifold

$\mathcal{L} = U + a$  (cf. § 3. e), its dual function is defined by

$$\gamma_2(f) = \psi_V(f) + \langle a, f \rangle .$$

Thus minimizing  $\hat{\gamma}_1 + \gamma_2$  over  $\mathcal{F}$  is the same as minimizing  $\hat{\gamma}_1 + \langle a, \cdot \rangle$  over  $V$ , the linear subspace of  $\mathcal{F}$  orthogonal to  $U$ .

On the other hand, in the usual situations of linear elastostatics, one may take as  $\phi_1$  the potential of elastic forces, which is a nonnegative quadratic form on  $\mathcal{U}$ . Calculating its dual  $\gamma_1$  (equal to  $\hat{\gamma}_1$ , since quadratic forms are even functions) yields a nonnegative quadratic form defined on some linear subspace of  $\mathcal{F}$  and  $+\infty$  outside of this subspace ; a special property of the quadratic case is that, if  $u$  and  $-f$  are conjugate points with regard to  $\phi_1$ ;  $\gamma_1$ , one has

$$\phi_1(u) = \gamma_1(f) = -\frac{1}{2} \langle u, f \rangle .$$

Thus,  $\gamma_1$  may be interpreted as "the expression of the elastic energy in terms of the elastic force" and sometimes called the complementary energy. This does not hold anymore in non linear elasticity ; however in the very usual case where the elastic potential  $\phi_1$  is a quasi-homogeneous convex function, there is still a relation between  $\phi_1(u)$  and  $\hat{\gamma}_1(f)$ , if  $f$  is the elastic force corresponding to  $u$ .

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3. g SADDLE - POINT PROPERTY

The notations are the same as in the preceding paragraph. Determining the equilibrium configurations of  $\mathcal{C}$  as minimizing points of  $\phi_1 + \phi_2$  (cf. § 3. e) and determining the equilibrium forces as minimizing points of  $\hat{\gamma}_1 + \gamma_2$  may be considered as dual extremum problems. This is a familiar feature of convex programming and it is habitual to relate such a pair of problems to a saddle-point property for a function called Lagrangian.

PROPOSITION. Define the concave-convex function L on the product space  $\mathcal{U} \times \mathcal{F}$  by

$$L(u, f) = \langle u, f \rangle + \hat{\gamma}_1(f) - \phi_2(u)$$

with the convention  $+\infty - \infty = +\infty$  (or equivalently the convention  $+\infty - \infty = -\infty$ ). A point  $u_0 \in \mathcal{U}$  is an equilibrium configuration of  $\mathcal{C}$ , with  $f_1 \in \mathcal{F}$  as corresponding equilibrium force, if and only if the element  $(u_0, f_1)$  of  $\mathcal{U} \times \mathcal{F}$  is a saddle point of L with finite value, i.e.  $L(u_0, f_1)$  is finite and for any  $u \in \mathcal{U}$  and any  $f \in \mathcal{F}$ ,

$$(3.13) \quad L(u, f_1) \leq L(u_0, f_1) \leq L(u_0, f) .$$

In fact, suppose first that  $u_0$  is an equilibrium configuration with  $f_1$  as equilibrium force, i.e.  $-u_0 \in \partial \hat{\gamma}_1(f_1)$  and  $f_1 \in \partial \phi_2(u_0)$ ; the former of these conditions means

$$(3.14) \quad \forall f \in \mathcal{F} : -\langle u_0, f-f_1 \rangle + \hat{\gamma}_1(f_1) \leq \hat{\gamma}_1(f)$$

and the latter

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$$(3.15) \quad \forall u \in \mathcal{U} : \langle u - u_0, f_1 \rangle + \phi_2(u_0) \leq \phi_2(u) \quad .$$

Adding the finite number  $-\phi_2(u_0)$  to both members of (3.14) yields the second of inequalities (3.13) ; adding the finite number  $\hat{\gamma}_1(f_1)$  to both members of (3.15) yields the first one. The value  $L(u_0, f_1)$  is clearly finite.

Conversely, supposing  $L(u_0, f_1)$  finite implies that  $\hat{\gamma}_1(f_1)$  and  $\phi_2(u_0)$  are finite ; then the preceding calculation may be effected backward to deduce (3.14) and (3.15) from (3.13).

REMARK. Exchanging the roles of  $\phi_1$  and  $\phi_2$  would yield a quite different function  $L$ . Since, in practical situations, there are usually several ways of classifying mechanical action into two groups corresponding to  $\phi_1$  and  $\phi_2$ , since, on the other hand the  $(\mathcal{U}, \mathcal{F})$  pattern may usually be applied in several ways (see § 3. j below), the preceding Proposition generates a pretty great number of saddle point characterisations of the equilibrium in elastostatics.

### 3. h ONE - DIMENSIONAL EXAMPLES

We consider in this paragraph a system  $\mathcal{O}$  whose configuration can be specified by a single numerical variable : it is for instance a rectilinear bar or a string, as far as we are only interested in the distance between its extremities. Denote by  $\ell_0 + e$  this distance ; in



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other words,  $e$  denotes the elongation of the bar by comparison with some reference state in which the length was  $l_0$ . As we are only concerned with static or quasi-static situations, the state of stress of the bar is sufficiently described by the tension  $s$ . Classically, for the application of the principle of virtual work to systems comprising the considered bar, the expression of the work of the internal actions must be  $-s \delta e$ . Thus the pattern of the preceding paragraph applies by taking for the linear space  $\mathcal{U}$  a copy of the real line  $\mathbb{R}$ , with  $e$  as generic element, and for the linear space  $\mathcal{F}$  another copy of  $\mathbb{R}$ , with  $s$  as generic element ; these two one-dimensional linear spaces are placed in separating duality by the bilinear form  $\langle ., . \rangle$

$$(3.16) \quad \langle e, s \rangle = - e s .$$

This unpleasant minus sign merely comes from our complying with the common habit in solid mechanics of measuring the state of stress by a positive number when it is properly a tension, by a negative number when it is a proper pressure. It has nothing to do with the fact that the considered "actions" are internal : in our formalism, stress is a "force" like any other mechanical action.

This framework permits the formulation of usual behavioral laws of the rectilinear system.

1° Regular elasticity. Suppose that the behavioral law of the bar

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defines the tension  $s$  as a continuous strictly increasing function of the elongation  $e$ , namely  $s = j(e)$  or equivalently  $s = \theta'(e)$ , where  $\theta$  denotes a primitive of  $j$ ; observe that  $\theta$  is then a convex function. Let  $e_0$  be some definite value of  $e$  and  $s_0 = \theta'(e_0)$ . The affine function

$$e \mapsto (e - e_0) s_0 + \theta(e_0)$$

is tangent to  $\theta$  at the point  $e_0$ ; now, with regard to the duality defined by (3.16), the slope of this affine function is  $-s_0$ . In other words the relation  $s = \theta'(e)$  may be written as

$$-s = \text{grad } \theta(e) .$$

This means that  $\theta$  is a potential for the considered statical law (and also, as usual, the expression of the potential energy); due to the convexity of  $\theta$  it is also a superpotential. As we have supposed the function  $\theta' = j$  continuous and strictly increasing, it possesses an inverse function  $j^{-1}$ , defined on the range of  $j$ ; this range is an interval  $I$ , possibly unbounded or not closed. The characterization of  $e$  and  $-s$  as conjugate points

$$\theta(e) + \theta^*(-s) = \langle -s, e \rangle$$

permits the calculation of  $\theta^*$  by the formula

$$\theta^*(-s) = s j^{-1}(s) - \theta [j^{-1}(s)]$$

valid for any  $s$  in  $I$ . The function  $\theta$  takes the value  $+\infty$  outside

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of the closure of  $-I$ .

2° Elastic string. We agreed that  $l_0 + e$  represented the distance between the extremities of the considered one-dimensional system. If  $l_0$  denotes exactly the length at rest of an elastic string, the corresponding static law has the form  $s = j(e)$  where the function  $j$  takes now the value zero for  $e \leq 0$ . A primitive of  $j$  is a superpotential; its dual function  $\theta^*$  with regard to the bilinear form (3.16) takes the value  $+\infty$  on  $]0, +\infty[$ ; the values of  $\theta^*(-s)$  for  $s$  belonging to the range of  $j$  are constructed as above if  $j$  is continuous and strictly increasing on  $[0, +\infty[$ .

3° Inelastic string. This may be considered as a boundary case of the preceding. Supposing that  $l_0$  is the proper length of the string and that the breaking load is infinite, one finds the following superpotential for the relation between  $e$  and  $s$

$$\theta(e) = \begin{cases} +\infty & \text{if } e > 0 \\ 0 & \text{if } e \leq 0 \end{cases} .$$

This is the indicator function of the closed convex subset  $C = ]-\infty, 0]$  of  $\mathcal{U}$ , so that the present law comes to be a perfect convex constraint. As  $C$  is actually a convex cone (see § 2. c) the dual function  $\theta^*$  is the indicator function of the polar cone, i.e. the subset  $]-\infty, 0]$  of  $\mathcal{F}$  (it is the set of the possible values of  $-s$ ).

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The reader will study other examples such as a cylindrical helix spring, enclosed in a guide tube to prevent buckling ; the length of this spring cannot be less than the length it has when all the spires come into contact. The corresponding behavioral law is equivalent to the conjunction of a law of elasticity and of a perfect convex constraint. This gives a very elementary model of an elastic solid with limited compressibility, a type of material which was studied in generality by W. PRAGER [1] ; the behavior of such a material can be formulated as a statical law admitting a superpotential.

### 3. 1 AN EXAMPLE OF COMPOUND SYSTEM

Take as  $\mathcal{O}^p$  a lattice of bars (a truss) whose extremities are articulated with one another through spherical joints. The joints are represented by  $n$  points  $A_1, A_2, \dots, A_n$  the nodes of the lattice. To make the description simpler suppose that between each pair of nodes, say  $A_i$  and  $A_j$  with  $i < j$  to avoid repetition, there exists one of the bars denoted by  $B_{ij}$ , thus  $\frac{1}{2} n (n-1)$  bars in all. The behavior of each bar is treated as one-dimensional ; denote by  $s_{ij}$  the tension of the bar  $B_{ij}$  and by  $e_{ij}$  its elongation with respect to the "zero" state.

Any configuration of the system  $\mathcal{O}^p$  is fully determined by the

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corresponding positions of the  $n$  nodes  $A_i$  relative to some three-dimensional Cartesian frame ; these respective positions may be described by the  $n$  three-dimensional displacement vectors  $\vec{x}_i$  by which they differ from the positions corresponding to the "zero" configuration of the system. Thereby we are induced to consider as the configuration space of  $\mathcal{C}$  the  $3n$ -dimensional linear space  $X$  whose generic element  $x$  consists in the  $n$ -tuple  $(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n)$ .

Here again we restrict ourselves to linearized geometry, by treating the displacements as infinitely small with regard to the lengths of all the bars. Denote by  $\vec{\alpha}_{ij}$  (with  $i < j$ ) the unit vector of the oriented line  $A_i A_j$  (taken, to fix the ideas, in the zero configuration ; but this precision is immaterial since the bars present only infinitesimal rotations). The elongation of the bar  $B_{ij}$  is related to  $u$  by

$$(3.17) \quad e_{ij} = \vec{\alpha}_{ij} \cdot (\vec{x}_j - \vec{x}_i)$$

(three-dimensional scalar product).

An external action is a  $n$ -tuple of forces  $(\vec{y}_1, \vec{y}_2, \dots, \vec{y}_n)$  respectively exerted on the  $n$  nodes ; this  $n$ -tuple of three-dimensional vectors, denoted by  $y$ , constitutes the generic element of a  $3n$ -dimensional linear space  $Y$ . The bilinear form "work", placing the spaces  $X$  and  $Y$  in separated duality will be noted  $\langle\langle \cdot, \cdot \rangle\rangle$  to

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prevent confusion in the following, and has the familiar expression

$$(3.18) \quad \langle\langle x, y \rangle\rangle = \sum_{i=1}^n \vec{x}_i \cdot \vec{y}_i .$$

In order to formulate the equilibrium problem for the considered system one has to specify the statical laws to which it is subjected. These statical laws are of two sorts:

Some of them concern external actions ; for instance given loads may be applied to some nodes ; or some nodes may be submitted to bilateral or unilateral constraints ; or also some nodes may be subjected to statical laws relating to their positions some of the forces they experience. All this has to be described in the framework of the pair of linear spaces  $(X, Y)$ .

The other laws, said internal, concern the behavior of the bars and are formulated in terms of the elongations  $e_{ij}$  and the tensions  $s_{ij}$  : this induces to consider the  $\frac{1}{2} n(n-1)$ -dimensional linear space  $E$  whose generic element, denoted by  $e$ , is the  $\frac{1}{2} n(n-1)$ -tuple of real numbers  $e_{ij}$ ,  $i < j$ , and the similar space  $S$  whose generic element is  $s$ , consisting of the  $s_{ij}$ ,  $i < j$ . As explained in § 3. h, the expression of the internal work in the bar  $B_{ij}$ , corresponding to a tension measured by the real number  $s_{ij}$  and an (increase of) elongation measured by the real number  $e_{ij}$  is  $- e_{ij} s_{ij}$ . Therefore the total internal work in the bars corresponding to given  $e = (e_{ij})$  and  $s = (s_{ij})$  is

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$$(3.19) \quad \langle e, s \rangle = - \sum_{i < j} e_{ij} s_{ij}$$

a bilinear form which places the two linear spaces E and S in separating duality : keep in mind that it differs by the presence of the minus sign from the natural "scalar product" between two spaces whose elements are such  $\frac{1}{2} n(n-1)$ -tuples of real numbers.

At the present stage, where plasticity is not taken into account, the behavioral laws of the bars are relations between  $e_{ij}$  and  $s_{ij}$  formulated in the same ways as in § 3. h. This introduces, for each  $(i, j)$ ,  $i < j$ , a superpotential  $\theta_{ij}$  which is a closed convex functions on R and the corresponding statical laws takes the form

$$(3.20) \quad - s_{ij} \in \partial \theta_{ij} (e_{ij}) .$$

By the remarks made in § 2. c about the product of linear spaces, the function  $\theta$  defined on E by

$$\theta(e) = \sum_{i < j} \theta_{ij} (e_{ij})$$

permits to summarize the  $\frac{1}{2} n(n-1)$  relations (3.20) by writing

$$(3.21) \quad - s \in \partial \theta (e) .$$

### 3. j VARIOUS TREATMENTS OF THE EQUILIBRIUM PROBLEM

Let us pursue the study of the system described above. Continuously distributed external actions, such as gravity, are not taken into account, so that the equilibrium condition of the system consists in the

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vanishing of the total force experienced by each of the  $n$  nodes, i.e. for each value of  $i = 1, 2, \dots, n$  the following three-dimensional vector equation

$$(3.22) \quad \vec{y}_i + \sum_{j \in \langle i \rangle} s_{ij} \vec{\alpha}_{ij} - \sum_{j \in \langle i \rangle} s_{ji} \vec{\alpha}_{ji} = 0 \quad .$$

On the other hand, equalities (3.17) define a linear mapping from  $X$  into  $E$  which will be denoted by  $D$ . By definition the adjoint  $D^*$  of  $D$  is the linear mapping from  $S$  into  $Y$  defined by

$$\forall x \in X, \quad \forall s \in S : \quad \langle D x, s \rangle = \langle \langle x, D^* s \rangle \rangle \quad .$$

Referring to the definitions of  $\langle \dots \rangle$  and  $\langle \langle \dots \rangle \rangle$ , then identifying the terms of each member yields that the element  $D^* s$  of  $Y$  consists of the  $n$ -tuple of three-dimensional vectors  $\overrightarrow{(D^* s)}_i$

$$\overrightarrow{(D^* s)}_i = \sum_{j \in \langle i \rangle} s_{ij} \vec{\alpha}_{ij} - \sum_{j \in \langle i \rangle} s_{ji} \vec{\alpha}_{ji} \quad .$$

Therefore the equilibrium condition (3.22) takes the form

$$(3.23) \quad y + D^* s = 0$$

which of course is equivalent to the principle of virtual work, namely

$$(3.24) \quad \forall x \in X : \quad \langle \langle x, y \rangle \rangle + \langle D x, s \rangle = 0 \quad .$$

1° The method of big spaces.

We give this name to the method which consists in using the pair  $(x, e)$ , denoted by  $u$  as the element which specifies the configuration of our system. Then, with the notations of § 3, a the configuration space is  $\mathcal{U} = X \times E$  ; the corresponding  $\mathcal{F}$  is the space  $Y \times S$ ,



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whose generic element is the pair  $(y,s)$  denoted by  $f$ . These spaces are placed in separating duality by the expression of the total work  $\langle\langle x,y \rangle\rangle + \langle e,s \rangle$ , to be denoted by  $\langle u,f \rangle$ .

Clearly the whole of the space  $\mathcal{U}$  is not permitted to  $u$ , since the pair  $(x,e)$  must belong to the following linear subspace of  $\mathcal{U}$

$$U = \{ (x,e) \in X \times E : e = D x \}$$

i.e. the graph of  $D$ . Let us show that this restriction of freedom may be treated as a perfect constraint.

In fact the equilibrium condition of the system is not the vanishing of the element  $f = (y,s)$  but merely equality (3.23). Putting

$$V = \{ (y,s) \in Y \times S : y + D^* s = 0 \}$$

we observe that  $V$  is precisely the subspace of  $\mathcal{F}$  orthogonal to  $U$  : this is the same as the equivalence between (3.23) and (3.24). Condition (3.23) is equivalent to asserting the existence of some  $r$  in  $V$  such that  $f + r$  vanishes. Interpreting  $r$  as the reaction associated with the considered constraint agrees with our general definition of a perfect affine constraint.

Actually this conception may be related to a physical realization of the constraint : considering  $X \times E$  as the configuration space amounts to regarding our system as the conjunction of the following subsystems : the nodes  $A_i$ , whose respective configurations are described

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by the three-dimensional vectors  $\vec{x}_i$  and the bars  $B_{ij}$ , whose respective states are described by the elongations  $e_{ij}$ . The constraint whose geometric effect is expressed by (3.17) merely consists in connecting the bars with the nodes. However, our main motivation in developing the present example is to prepare for the case of continuous media, (cf. B. Nayroles's lectures) ; in this case  $x$  is replaced by a field of displacement vectors defined on a region of  $R^3$  and  $e$  is replaced by a field of strain tensors ; then  $e = D x$  is the condition of geometric compatibility between displacements and strains ; this restriction of freedom may be formally considered as a perfect constraint in the same way as above but it does not seem wise to try and visualize a mechanical realization for it.

Suppose that the statical laws concerning the external actions experienced by the system (possibly including constraints acting on the nodes) can be globally described in the framework of the spaces  $(X, Y)$  by a superpotential  $\zeta \in \Gamma_0(X, Y)$  ; in other words the external force  $y \in Y$  is related to the "external" configuration  $x \in X$  by

$$(3.25) \quad - y \in \partial \zeta(\tau) \quad ,$$

where the subdifferential is understood in the sense of the duality  $(X, Y)$ . Suppose on the other hand that the internal statical laws are expressed by (3.21). By the rules formulated in § 2. c about product

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spaces, (3.21) and (3.25) are equivalently summarized as

$$- f \in \partial \phi (u)$$

in the sense of the duality between the big spaces with  $u = (x, e)$ ,

$f = (y, s)$ , and the superpotential  $\phi$  defined by

$$\phi (u) = \zeta (x) + \theta (e) .$$

The equilibrium of the system may then be studied by the methods of §§ 3. e, f, g.

## 2° The elimination of (E,S)

As the configuration of the system is fully specified when  $x \in X$  is given, one may prefer to consider only  $X$  as the configuration space, and  $Y$  as the force space. Then every mechanical action experienced by the system must be described in terms of elements of  $Y$  : precisely it is represented by the element  $y$  of  $Y$  such that for every displacement  $\delta x$  of the system, the work of the considered action is  $\langle\langle \delta x, y \rangle\rangle$ . In this way an internal stress  $s \in S$  is represented by the element  $y_s$  of  $Y$  such that

$$\forall \delta x \in X : \langle\langle \delta x, y_s \rangle\rangle = \langle D \delta x, s \rangle ,$$

i.e.

$$(3.26) \quad y_s = D^* s .$$

Thus the statical law (3.21) is transcribed in terms of the pair of spaces  $(X, Y)$  as follows

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$$(3.27) \quad -y_s \in D^* (\theta \circ D(x)) .$$

If, in particular, there exists a point in the range of  $D$  at which  $\theta$  is finite and continuous (for some topology compatible with the duality  $(E,S)$ ), the calculation rule (2.15) holds, so that (3.27) amounts to

$$(3.28) \quad -y_s \in D(\theta \circ D)(x)$$

in the sense of the duality  $(X,Y)$ ; this constitutes a statical law admitting the function  $\theta \circ D$  as superpotential. In this way the techniques of the foregoing paragraphs may be applied with regard to the pair of spaces  $(X,Y)$ .

### 3° The elimination of $(X,Y)$

The mapping  $D : X \rightarrow E$  is not injective; this means that the element  $e = D x$  does not convey enough information to specify completely the configuration of the system. However one may wish to determine the equilibrium values of  $e$  or  $s$  prior to that of  $x$  or  $y$  and in some instances one may be interested in these elements only (in order to discuss strength, for example).

In the principle, the elimination is similar to that of the preceding case. Suppose that all the external laws to which the system is jointly submitted are summarized under the form

$$(3.29) \quad x \in P(y)$$

where  $P$  denotes a given multimapping from  $Y$  into  $X$ . Similarly suppose

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that all the internal laws are summarized as

$$(3.30) \quad s = R(e)$$

where  $R$  denotes a given multimapping from  $E$  into  $S$ . A system of values of  $x, y, e, s$  defines an equilibrium state if and only if it satisfies  $e = D x$  and (3.23), (3.29), (3.30). Thus, as far as  $e$  and  $s$  only are concerned, the equilibrium condition (i.e. a necessary and sufficient condition for the existence of at least one pair  $(x, y)$  associated with  $(e, s)$  in such a way that the preceding equilibrium conditions hold) consists in the conjunction of (3.30) with

$$(3.31) \quad e \in D(P(-D^*s)) .$$

In the principle, (3.31) may as well be written under the form

$$(3.32) \quad -s \in Q(e) .$$

Now as far as the interesting unknown is  $e$ , the conjunction of (3.30) with (3.32) is equivalently formulated as follows : there exist  $s_1$  and  $s_2$  in  $S$  such that

$$s_1 \in R(e)$$

$$s_2 \in Q(e)$$

$$s_1 + s_2 = 0 .$$

Formally we are reduced to the usual pattern of the equilibrium of a system submitted to two statical laws. From this standpoint the relation  $s \in Q(e)$  should be considered as the "internal image" of the external

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statical law (3.29).

The reader is invited to apply this procedure to an external law of the form  $-y \in \partial \zeta(x)$ , equivalently written as  $x \in \partial \zeta^*(-y)$ . Here again the calculation rule (2.15), under some continuity assumption, will yield an image in  $(E,S)$  which admits a superpotential. As a first example, take as external statical law a given load  $y_0 \in Y$ , applied to the system ; this may be written under the form (3.29) with

$$P(y) = \begin{cases} X & \text{if } y = y_0 \\ \emptyset & \text{if } y \neq y_0 \end{cases} .$$

Another primary example is that of a perfect affine constraint formulated relatively to the pair  $(X,Y)$ .

But it will be more in the spirit of this Chapter to operate with the pair  $(E,S)$  in the following way :

Since we choose to deal only with informations formulated in the framework of the paired spaces  $(E,S)$ , we accept only to speak of the state of the system in terms of  $e$  ; on the other hand, a mechanical action experienced by the system will be taken into account only if it can be represented by an element  $\sigma \in S$ , in such a way that the work of this action for every displacement of the system has the expression  $\langle \delta e, \sigma \rangle$ . Therefore, if in particular the considered action is an external force  $y \in Y$  treated as given, the corresponding  $\sigma$  must be such

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that

$$(3.33) \quad \forall \delta x \in X : \langle\langle \delta x, y \rangle\rangle = \langle D \delta x, \sigma \rangle$$

Such a  $\sigma$  does not necessarily exist ; an evident condition for its existence is that  $y$  belongs to  $D^* S$ , the image of  $S$  under the linear mapping  $D^*$ . The linear subspace  $D^* S$  of  $Y$  is the orthogonal, in the sense of the duality  $(X,Y)$ , of the subspace  $\text{Ker } D$  of  $X$ . Actually the impossibility of representing in the  $(E,S)$  framework a load  $y$  which would not belong to  $D^* S$  does not make any hindrance. In fact suppose, for sake of simplicity, that this load is the only external action exerted on the system ; clearly by (3.23) or by (3.24),  $y \in D^* S$  is a necessary condition for the existence of an equilibrium ; this is a familiar fact ; only a family of external forces with zero resultant and zero moment is compatible with equilibrium.

Another fundamental remark about the use of the  $(E,S)$  pattern is that all the values of  $e$  are not permitted, since necessarily  $e$  belongs to the subspace  $D X$  (the subspace of  $E$  consisting of the "states of strain" which are "geometrically compatible"). On the other hand, if  $s \in S$  denotes the sum of all the elements of  $S$  representing the mechanical actions exerted on the system, the equilibrium condition is not  $s = 0$ , but the principle of virtual work, namely

$$\forall \delta x \in X : \langle D \delta x, s \rangle = 0$$

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which means that  $s$  belongs to the subspace of  $S$  orthogonal to  $D X$  (actually the kernel of  $D^*$ ).

In conclusion the equilibrium problem in  $(E,S)$  must be treated by considering the condition  $e \in D X$  as a perfect constraint.

The reader will check that given external loads and external perfect affine constraints are transcribed in the  $(E,S)$  language by given forces and perfect affine constraints.

It is from this standpoint that the elastoplastic evolution problem will be studied in Chapter 6.



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## 4 LAWS OF RESISTANCE

### 4. a VELOCITIES AND FORCES

A habitual procedure, when studying a mechanical system, is to associate with each possible configuration of this system a linear space -let us denote it by  $\mathcal{V}^p$ - whose elements constitute, in a general sense, the possible values of the velocity of the system if it happens to pass through the considered configuration. Roughly speaking,  $\mathcal{V}^p$  may be interpreted as the tangent space at the corresponding point of the configuration manifold but this need not be made more precise here. This space is of infinite dimension if the system has an infinite degree of freedom.

In the special framework of Chapter 3, where the configuration manifold is treated as a linear space  $\mathcal{U}$ , a motion of the system is described by a mapping  $t \mapsto u(t)$  from some interval of time into  $\mathcal{U}$ . The velocity is naturally defined in this case as the derivative  $\dot{u}(t)$  (taken in the sense of some topology on  $\mathcal{U}$ ) if it exists ; then  $\mathcal{V}^p = \mathcal{U}$ , the same for all the configurations.

Let us come back to the general setting. With each configuration is also associated a linear space -denote it by  $\mathcal{F}$ - whose elements represent in a more or less abstract way, the mechanical actions which may be exerted on the system when it happens to come through the consi-

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dered configuration : see the construction of the space of torsors in § 3. a. By extension, the elements of  $\mathcal{F}$  are called forces. An essential feature in the practice of Mechanics is that several forces are usually applied to the system at the same time. This produces a fundamental dissymmetry between the roles played by  $\mathcal{U}$  and  $\mathcal{F}$ .

To any pair  $v \in \mathcal{V}$ ,  $f \in \mathcal{F}$  corresponds the power of the force  $f$  if the system possesses the velocity  $v$ , a real number denoted by  $\langle v, f \rangle$ ; this defines a bilinear form which places  $\mathcal{V}$  and  $\mathcal{F}$  in duality.

In the linear framework of Chapter 3 where  $\mathcal{V} = \mathcal{U}$ , there is no inconsistency in considering the single space  $\mathcal{F}$  as the force space associated with any configuration and in using the same bracket as above to denote by  $\langle \delta u, f \rangle$  the work of  $f \in \mathcal{F}$  corresponding to the displacement  $\delta u \in \mathcal{U}$ . In fact, suppose this displacement results from a motion  $t \mapsto u(t)$  with velocity  $\dot{u}$  (derivative understood in the sense of some topology compatible with the duality  $(\mathcal{U}, \mathcal{F})$ ) taking place during a time interval  $[t_1, t_2]$ , while  $f$  is constant in  $\mathcal{F}$ . The general definition of work as the integral of power yields

$$\int_{t_1}^{t_2} \langle \dot{u}(t), f \rangle dt = \int_{t_1}^{t_2} \frac{d}{dt} \langle u(t), f \rangle dt = \langle u(t_2), f \rangle - \langle u(t_1), f \rangle = \langle \delta u, f \rangle.$$

#### 4. b PSEUDO - POTENTIALS

Let us agree to call a resistance law a relation, denote it by

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$\mathcal{R}$ , formulated between the possible velocity  $v \in \mathcal{V}^0$  of the considered system in the considered configuration and one, say  $f \in \mathcal{F}$ , of the forces it experiences at the same instant. Such a law arises from the study of some of the physical processes in which the system takes part.

It will be said that the law  $\mathcal{R}$  is dissipative if the following implication holds

$$(4.1) \quad v \mathcal{R} f \Rightarrow \langle v, f \rangle \leq 0 ,$$

which makes it a resistance law in the usual sense.

It will be said that  $\mathcal{R}$  admits a function  $\phi \in \Gamma_0(\mathcal{V}^0, \mathcal{F})$  as pseudo-potential if the relation  $\mathcal{R}$  is equivalent to

$$(4.2) \quad -f \in \partial \phi (v) .$$

Recall that any subdifferential relation is monotone ; then a law  $\mathcal{R}$  of the form (4.2) ensures the implication :

$$(4.3) \quad v \mathcal{R} f , v' \mathcal{R} f' \Rightarrow \langle v-v' , f-f' \rangle \leq 0 .$$

Make in addition the frequently verified hypothesis that zero is among the values that the relation  $\mathcal{R}$  permits to  $f$  when  $v$  is zero, i.e.

$$(4.4) \quad 0 \in \partial \phi (0) .$$

then (4.1) ensues from (4.3) : the corresponding resistance law is dissipative. Observe that (4.4) implies that  $\phi (0)$  is finite and constitutes the minimal value of  $\phi$  ; since adding a finite constant to  $\phi$  does not

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affect the subdifferential, there is no loss of generality in supposing here

$$(4.5) \quad \phi(0) = 0 \quad ;$$

then the function  $\phi$  takes only nonnegative values.

In the following, we shall refer to the situation characterized by (4.2), (4.4), (4.5) by saying that the pseudo-potential  $\phi$  is the resistance function of the considered law.

Recall that, a priori, the pair of linear spaces  $\mathcal{V}, \mathcal{F}$  is relative to a definite configuration of the system, so that the foregoing concerns only this configuration. However in the usual linear case of Chapter 3, by making  $\mathcal{V} = \mathcal{U}$  and considering the single force space  $\mathcal{F}$ , it will be possible to formulate resistance laws independently of configurations.

REMARK. The example developed in § 3. i, 3. j makes understand also that the pattern of the present Chapter may usually be applied to a definite mechanical situation in several different ways.

A similar example is that of a continuous medium, occupying in the considered configuration a region  $\Omega$  of the physical space. A first possibility is to interpret as  $\mathbf{v}$  the vector field defined on  $\Omega$  by the velocities of the various particles forming the medium : then the linear space  $\mathcal{V}$  will consist of vector fields satisfying some assumptions

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of integrability, derivability, etc... But in some theories it will be more convenient to consider  $v$  as the strain rate tensor field of the medium. Or else, as in § 3. j, one may take for  $\mathcal{V}^0$  a "big space" whose generic element is the couple of a velocity vector field and of a tensor field presumed to be the strain rate field ; then the geometric compatibility between velocity field and strain rate field will be seen as a constraint. To these various standpoints correspond natural choices for the elements  $f$  forming the space  $\mathcal{F}$  : rates of distributed forces, stress tensor fields, etc...

The same pattern will also be applied to formulate local laws : a point of the continuous medium being specified, one considers as  $\mathcal{V}$  the linear space of dimension 6 whose elements are the possible values of the local strain rate tensor  $\dot{\epsilon}$  of the medium ; the associated  $\mathcal{F}$  is the linear space formed by the possible values of the local stress tensor  $\sigma$  ; the bilinear form which places these two spaces in duality is the classical expression of the density of internal power. A local law, i.e. a relation between the strain rate tensor and the stress tensor at the considered point of the medium, will be formulated by means of a local pseudo-potential, which is a numerical function defined on  $\mathcal{V}^0$ . This being done for each point of the medium, it generates a behavioral law of the medium as a whole, i.e. a relation between elements of two

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function spaces whose generic elements are the strain rate tensor field and the stress tensor field. Under suitable integrability assumptions, these two function spaces are placed in separating duality by the bilinear form defined as the integral of the density of internal power. This permits the description of the considered behavioral law by means of a superpotential which is an integral convex functional. The reader will refer to B. Nayroles's lecture for more details about this mechanical situation and to C. Castaing's lecture for more details about the functional analytic aspect. The basic mathematical material may be found in R. T. ROCKAFELLAR [1], [3], [4].

#### 4. c VISCOUS RESISTANCE

As a first example consider a relation  $\mathcal{R}$  of the form

$$(4.6) \quad -f = L v$$

where  $L$  denotes a linear mapping from  $\mathcal{V}^0$  into  $\mathcal{F}$ . In all the phenomena classified as viscosity effects it is always admitted that  $L$  is self-adjoint (or "symmetric") with regard to the duality  $\langle \cdot, \cdot \rangle$ , i.e., for any  $v$  and  $v'$  in  $\mathcal{V}^0$  :

$$\langle v, L v' \rangle = \langle v', L v \rangle .$$

From this, one easily deduces that  $L v$  is the weak gradient at the point  $v$  of the quadratic form  $\phi$  defined on  $\mathcal{V}^0$  by

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$$\phi (v) = \frac{1}{2} \langle v, L v \rangle$$

This quadratic form is usually called the Rayleigh function of the considered viscosity law.

Making the additional assumption that the viscosity law is dissipative yields that this quadratic form is nonnegative, thus convex. And at any point  $v$  the weak gradient  $L v$  constitutes the whole of the subdifferential  $\partial \phi (v)$ . This means that in the present case, the relation (4.6) may equivalently be written as

$$- f \in \partial \phi (v) .$$

Thus  $\phi$  is pseudo-potential and, more precisely, resistance function of the considered law.

The power of the force  $f$  associated with  $v$  in this way is

$$\langle v, f \rangle = - \langle v, L v \rangle = - 2 \phi (v) ;$$

the negative of it is frequently called the dissipated power corresponding to  $v$  ; hence the name of dissipation function which is given in the present case to the quadratic form  $v \mapsto 2 \phi (v)$ .

REMARK. Gyroscopic forces give an example of a law of the form (4.6) with a linear mapping  $L$  which is not self-adjoint ; on the contrary

$$\langle v, L v' \rangle = - \langle v', L v \rangle .$$

Such a law admits no pseudo-potential unless  $L$  is the zero mapping ; the dissipated power is essentially zero, so that (4.1) is satisfied :

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this law may be said dissipative.

#### 4. d VELOCITY CONSTRAINT

Take back the framework of § 3. e, i.e. the example of the firm perfect constraint whose geometric condition is  $u \in \mathcal{L}$ , with  $\mathcal{L} = U + a$ ,  $a$  a possibly moving affine manifold. The linear subspace  $U$  is supposed independent of time thus also  $V$  which is the subspace of  $\mathcal{F}$  orthogonal to  $U$ . This geometric condition may equivalently be written, for every  $t$ ,

$$\forall w \in V : \langle u - a, w \rangle = 0 .$$

Supposing that the known function  $t \mapsto a$  possesses a weak derivative  $\dot{a}$ , this yields, by choosing  $w$  independent of  $t$ , that the velocity  $v = \dot{u}$  satisfies

$$\forall w \in V : \langle \dot{u} - \dot{a}, w \rangle = 0$$

i.e.

$$(4.7) \quad v \in U + \dot{a} .$$

Recall on the other hand that, by the definition of a firm perfect constraint, the reaction  $r \in \mathcal{F}$  exerted on the system by the enforcing device may be an arbitrary element of  $V$ . Exactly like in § 3. b, this fact may be expressed jointly with (4.7) by writing :

$$(4.8) \quad - r \in \partial \psi_{\mathcal{L}}(v)$$



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where  $\dot{\mathcal{L}}$  denotes the affine manifold  $U + \dot{a}$ .

This constitutes a resistance law admitting the function  $\psi_{\dot{\mathcal{L}}}$  as pseudo-potential. Let us call it a velocity constraint.

It is no place to explain how, in the general setting of a configuration-depending pair of spaces  $(\mathcal{V}, \mathcal{F})$ , the usual differentiability assumptions let any firm perfect bilateral smooth constraint be expressed under the form (4.8). This form includes more generally the relations between reaction and velocity classically known as non-holonomic perfect constraints; the standard example of it consists in the perfect rolling without sliding of solid bodies, actually an extreme case of friction.

4. e FRICTION AND PLASTICITY

Suppose given a weakly closed non empty convex subset  $C$  of  $\mathcal{F}$ . Let us formulate a relation  $\mathcal{R}$  between  $v$  and  $f$  by the principle of maximal dissipation namely : the values of  $f \in \mathcal{F}$  which this relation associates with a given  $v \in \mathcal{V}$  are the elements of  $C$  which minimize the power, i.e. minimize the function  $\langle v, \cdot \rangle$ . In other words  $v \mathcal{R} f$  means

$$\left\{ \begin{array}{l} f \in C \\ \forall f' \in C : \langle v, f' \rangle \geq \langle v, f \rangle \end{array} \right.$$

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which is immediately found equivalent to

$$\forall f' \in \mathcal{F}: -\langle v, f'-f \rangle + \psi_C(f) \leq \psi_C(f')$$

i.e.

$$(4.9) \quad -v \in \partial \psi_C(f)$$

which in turn is equivalent to

$$(4.10) \quad f \in \partial \psi_C^*(-v)$$

(cf. § 2. e) and also to

$$(4.11) \quad \psi_C^*(-v) + \psi_C(f) + \langle v, f \rangle = 0.$$

Denote by  $\phi$  the function  $v \mapsto \psi_C^*(-v)$ , i.e.

$$\phi(v) = \sup_{f \in C} \langle -v, f \rangle = \sup_{g \in -C} \langle v, g \rangle ;$$

it is the support function of the set  $-C$ .

Then (4.10) is transcribed as

$$-f \in \partial \phi(v) ;$$

this means that the considered resistance law admits  $\phi$  as pseudo-potential (or resistance function in the usual case where  $C$  contains the origin of  $\mathcal{F}$  ; such is the condition for the present law to be dissipative).

Relation (4.11) may equivalently be written as

$$(4.12) \quad \left\{ \begin{array}{l} f \in C \\ -\langle v, f \rangle = \phi(v) \end{array} \right. ;$$

in other words the values of  $f$  that the considered relation associates

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with a given  $v$  are those elements of  $C$  for which the dissipated power  $-\langle v, f \rangle$  equals exactly  $\phi(v)$ .

The reader will check that all the preceding pattern applies to Coulomb's law of friction between two solid bodies  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , when the pressure  $N$ , i.e. the normal component of the reaction, is treated as known. Take as  $v$  the sliding velocity of  $\mathcal{C}_2$  with respect to  $\mathcal{C}_1$ ; then  $\mathcal{V}^0$  is the linear space of dimension 2 consisting of the vectors whose direction is contained in the common tangent plane to the two bodies at the point of contact (this space is not exactly the velocity space for the considered system as a whole, but it is visibly isomorphic to a subspace of it). Take as  $f$  the tangential component of the reaction that  $\mathcal{C}_2$  undergoes from  $\mathcal{C}_1$  so that  $\mathcal{F}$  may be considered as the same space as  $\mathcal{V}^0$ , the bilinear form  $\langle ., . \rangle$  reducing then to the conventional Euclidian scalar product. The customary Coulomb law of isotropic friction consists in taking as  $C$  the closed disk centered at the origin, with radius equal to the product of  $N$  by the friction coefficient. But anisotropic friction may be described as well, by using convex sets of different shape. See MOREAU [12] about the application of this to discuss the sliding of a vehicle wheel when brake is applied : if the inertia of the wheel is neglected, the resulting effect comes to be equivalent to some anisotropic friction which would take place directly

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between the vehicle and the ground.

However, the main domain of application of the preceding is plasticity. In its local form the classical law of perfect plasticity (i.e. without strain hardening) is formulated as a relation between the local values of the stress tensor  $\sigma$  and of the plastic strain rate  $\dot{\epsilon}_p$ . Giving the yield locus defines a closed convex set  $C$  in the six-dimensional space of the variable  $\sigma$ ; among various equivalent formulations, the considered law may be stated as a principle of maximal dissipation which was precisely the starting point of this paragraph. From the local law one obtains the global one by the functional analytic procedure described at the end of § 4. b.

In the study of plasticity as well as in that of friction, an essential feature is the occurrence of a relation between the velocity  $v$  and the force  $f$  which cannot be "solved" to define one of these two elements as a function of the other: to the value zero of  $v$  correspond for  $f$  all the points of  $C$  and to a value of  $f$  corresponds as values of  $v$  all the elements of the cone  $-\partial \psi_C(f)$ . This causes much trouble in traditional treatments; our purpose in Chapter 6, will be to show that such formulations as (4.9), (4.10) or (4.11) permit a very efficient handling in this situation.

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#### 4. f DISSIPATION FUNCTION

The relation  $\mathcal{R}$  between  $v$  and  $f$  may equivalently be written under the form

$$f \in R(v)$$

where  $R$  denotes a multimapping from  $\mathcal{V}^o$  into  $\mathcal{F}$ . Given  $v$  in  $\mathcal{F}$ , there is a priori no reason for all the values of  $f$  in the set  $R(v)$  to yield the same value for the dissipated power  $-\langle v, f \rangle$ . However this precisely happens in many practical instances : in such cases, the dissipated power appears as a single-valued numerical function of the variable  $v$ , defined on  $\text{dom } R = \{v \in \mathcal{V}^o : R(v) \neq \emptyset\}$ . Let us denote by  $D$  this function, usually called the dissipation function of the considered law.

In the case of viscous resistance presented in § 4. c, the set  $R(v)$  reduces to a single element for each  $v$  in  $\mathcal{V}^o$ ; hence the existence of a dissipation function is trivial. In fact we found

$$D(v) = 2 \phi(v) .$$

In the case of friction or plasticity presented in § 4. e, (4.12) proves the existence of a dissipation function expressed now, for every  $v$  in  $\text{dom } \partial \phi$ , as

$$D(v) = \phi(v) .$$

Both preceding examples exhibit a close connection between the superpotential, or resistance function,  $\phi$  and the dissipation

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function  $D$ . Actually in both cases, the resistance function  $\phi$  happens to be positively homogeneous, with degree  $m$  ; this implies  $-\langle v, f \rangle = m \phi(v)$ , which may be considered as a generalization of Euler's identity to "subdifferential calculus". Many practical resistance functions possess such a homogeneity (e.g. usual laws of creep). More generally :

PROPOSITION. Let  $\phi$  be a resistance function (i.e.  $\phi$  is the pseudo-potential of a resistance law, with  $0 \in \partial \phi(0)$  and  $\phi(0) = 0$ ) ; suppose  $\partial \phi(v) \neq \emptyset$  whichever is  $v$  in  $\mathcal{V}^0$ . For the existence of a function  $h : \mathbb{R} \rightarrow \mathbb{R}$  ensuring the implication

$$-f \in \partial \phi(v) \Rightarrow -\langle v, f \rangle = h(\phi(v))$$

(in other words, for the function  $h \circ \phi$  to be dissipation function) it is necessary and sufficient that  $\phi$  has the quasi-homogeneous form  $\phi = \alpha \circ j$ , where  $j$  is an everywhere subdifferentiable gauge function on  $\mathcal{V}^0$  and  $\alpha$  a convex differentiable mapping from  $[0, +\infty[$  into itself, with  $\alpha(0) = 0$ .

A sketched proof is given in MOREAU [13], and for more details [16]. It may be remarked that the function  $h$  is then strictly increasing. The dissipation function  $D = h \circ \phi$  is not convex in general, but only quasi-convex i.e. its "slices"  $\{v \in \mathcal{V}^0 : D(v) \leq \rho\}$  for  $\rho \in \mathbb{R}$ , are (closed) convex sets ; all these sets are homothetic of

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$J = \{v \in \mathcal{V}^0 : j(v) \leq 1\}$ , the set whose  $j$  is the gauge.

By the facts indicated in § 2. h , the dual function of  $\phi = \alpha \circ j$  is also a quasi-homogeneous function, namely  $\phi^* = \beta \circ k$  , where  $\beta$  is the Young conjugate of  $\alpha$  and  $k$  the gauge function of the polar set  $K$  of  $J$ .

In the case of plasticity or friction the function  $\alpha$  is identity , so that  $\beta$  is the indicator function of the subset  $[0,1]$  of  $[0, +\infty[$  and  $K = -C$ .

#### 4. g SUPERPOSITION OF RESISTANCE LAWS

It is usual to take into account at the same time several resistance laws in the same pair  $(\mathcal{V}^0, \mathcal{F})$  of linear spaces. Let  $\phi_1$  and  $\phi_2$  the respective pseudo-potentials of two such resistance laws. For every  $v$  in  $\mathcal{V}^0$ , the set of the possible values of the sum of the two forces is  $\partial \phi_1(v) + \partial \phi_2(v)$ . This is contained in  $\partial(\phi_1 + \phi_2)(v)$  and, in particular, if the functions  $\phi_1$  and  $\phi_2$  possess the additivity of the subdifferentials, the conjunction of the two resistance laws amounts exactly to the single following one

$$(4.13) \quad -f \in \partial(\phi_1 + \phi_2)(v) .$$

Suppose for instance  $\phi_1(v) = \psi_C^*(-v)$ , i.e. the resistance

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denote the resistance function of some viscosity law (cf. § 4. c : it is a nonnegative l.s.c. quadratic form on the space  $\mathcal{V}$  ) ; choose a strictly positive constant  $\lambda$  and take more generally

$$\phi_2(v) = \lambda q(v) = \frac{1}{\lambda} q(\lambda v) ,$$

so that  $\lambda$  may be interpreted as a viscosity coefficient. As a condition ensuring the additivity of subdifferentials make, for instance, the following assumption (cf. § 2. f) : the function  $\phi_1$  is continuous at the origin, at least for the Mackey topology  $\tau(\mathcal{V}, \mathcal{F})$ ; by § 2. c, 5°, this means the convex set  $C$  is compact for the weak topology  $\sigma(\mathcal{F}, \mathcal{V})$ . Then the resulting viscoplastic law may be expressed under the form (4.13).

Now the assumptions made imply, by § 2. d, that the polar function of  $\phi_1 + \phi_2$  is the infimal convolute  $\phi_1^* \nabla \phi_2^*$ . As already mentioned in Chapter 3, the dual  $q^*$  of the quadratic form  $q$  consists in a positive definite quadratic form, defined on some subspace of  $\mathcal{F}$ , and extended with the value  $+\infty$  outside of this subspace. By § 2.c, 2°, the dual of  $\phi_2$  is  $\frac{1}{\lambda} q^*$ . On the other hand, the dual  $\phi_1^*$  of  $\phi_1$  is the indicator function of the set  $-C$ . Thus using the equivalence between (2.6) and (2.7) (§ 2. e) the viscoplastic resistance law (4.13) amounts to

$$v \in \partial \left( \psi_{-C} \nabla \frac{1}{\lambda} q^* \right) (-f) ,$$

while the corresponding purely plastic resistance law would be written as



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$$v \in \partial \psi_{-C}(-f) .$$

By definition, for every  $y \in \mathcal{F}$  ,

$$\begin{aligned} (\psi_{-C} \nabla \frac{1}{\lambda} q^*)(y) &= \inf_{z \in \mathcal{F}} [\psi_{-C}(z) + \frac{1}{\lambda} q^*(y-z)] \\ &= \inf_{z \in -C} \frac{1}{\lambda} q^*(y-z) \end{aligned}$$

and, due to the assumed compactness of  $C$ , the infimum is a minimum.

Clearly this expression takes the value 0 for  $y \in -C$  and it takes

strictly positive values otherwise ; it may be said that  $\psi_{-C} \nabla \frac{1}{\lambda} q^*$

is a penalty function for the set  $-C$  and the penalty coefficient  $\frac{1}{\lambda}$

is the reciprocal of the viscosity coefficient (other remarks about pe-

nalty functions will be given, for the special case of Hilbert space,

in § 5. d).

Due to quadratic forms being even functions, one may equivalently speak of the set  $C$  instead of  $-C$  ; in short adding some viscosity effects to a plasticity law is equivalent to replacing the indicator function of the "rigidity set"  $C$ , by a penalty function of this set ; the smaller is the viscosity coefficient, the larger is the penalty coefficient.

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## 5. MOVING SETS

### 5. a HAUSDORFF DISTANCE AND VARIATION

Let  $t \rightarrow A(t)$  denote a multimapping or multifunction (i.e. a set-valued mapping) from the compact interval  $[0, T]$  into a metric space  $(E, d)$ . As in the following the real variable  $t$  will be interpreted as the time, we may refer to  $A$  as a moving set in  $E$ .

A natural way of formulating regularity assumptions about such a multimapping consists in using the Hausdorff distance between subsets of the metric space  $E$ .

If  $A$  and  $B$  are two subsets of  $E$ , we call the excess of  $A$  over  $B$  the expression

$$(5.1) \quad e(A, B) = \sup_{a \in A} d(a, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$$

The considered sets may be empty ; let us agree that "sup" and "inf" above are understood in the sense of the ordered set  $\bar{R}_+ = [0, +\infty]$  : the supremum of an empty collection of elements of this ordered set is 0 and the infimum is  $+\infty$ . Expression (5.1) defines a non symmetric écart ; it satisfies the triangle inequality. Clearly  $e(A, B) = 0$  if and only if  $A$  is contained in the closure  $\bar{B}$  of  $B$ .

The Hausdorff (improper) distance of  $A$  and  $B$  is then defined as the symmetric expression

$$h(A, B) = \max \{e(A, B), e(B, A)\}$$

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with value in  $\bar{R}_+$ . This is zero if and only if A and B have the same closure.

By means of Hausdorff distance, the classical concept of variation may be applied to moving sets. Let  $[s, t]$  be a compact subinterval of  $[0, T]$ ; for any finite subdivision of this interval, namely

$$S : s = \tau_0 \leq \tau_1 \leq \dots \leq \tau_n = t$$

put

$$V(S) = \sum_{i=1}^n h(A(\tau_{i-1}), A(\tau_i)) \in \bar{R}_+ .$$

The supremum of  $V(S)$  for  $S$  ranging over all the finite subdivisions of  $[s, t]$  is called the variation of A on this interval; notation  $\text{var}(A; s, t)$ . From  $h$  satisfying the triangle inequality one easily deduces that

$$(5.2) \quad s \leq t \leq u \Rightarrow \text{var}(A; s, u) = \text{var}(A; s, t) + \text{var}(A; t, u) .$$

In particular if  $\text{var}(A; 0, T) < +\infty$ , the variation is also finite on any subinterval of  $[0, T]$ ; in this case, introducing the non decreasing function  $v$  from  $[0, T]$  into  $R_+$

$$(5.3) \quad v(t) = \text{var}(A; 0, t)$$

yields

$$(5.4) \quad s \leq t \Rightarrow \text{var}(A; s, t) = v(t) - v(s) .$$

The numerical function  $v$  is Lipschitz with ratio  $\lambda$  if and only if the multimapping A satisfies itself the Lipschitz condition, with ratio  $\lambda$ , i.e., for any  $s$  and  $t$  in  $[0, T]$ ,

$$h(A(s), A(t)) \leq \lambda |t-s|$$

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The numerical function  $v$  is absolutely continuous on  $[0, T]$  if and only if the multimapping  $A$  possesses itself the absolute continuity, as formulated by means of Hausdorff distance, i.e. : for any  $\varepsilon > 0$ , there exists  $\eta > 0$  such that the implication

$$\sum_i |\tau_i - \sigma_i| < \eta \Rightarrow \sum_i h(A(\sigma_i), A(\tau_i)) < \varepsilon$$

holds for any finite family  $]\sigma_i, \tau_i[$  of non overlapping subintervals of  $[0, T]$ . In this case the numerical function  $v$  is almost everywhere differentiable ; the derivative, denoted by  $\dot{v}$ , is a nonnegative element of  $L^1(0, T ; \mathbb{R})$  which may be called the speed function of the moving set  $A$ . Clearly

$$(5.5) \quad s \leq t \Rightarrow h(A(s), A(t)) \leq \int_s^t \dot{v}(\tau) d\tau .$$

Let us restrict ourselves now to the case where, for any  $t$ , the set  $A(t)$  is closed ; then the non decreasing function  $v$  is constant over some subinterval of  $[0, T]$  if and only if the multimapping  $A$  is also constant over this subinterval. This implies the existence of a multimapping  $\mathcal{K}$  from  $[0, v(T)]$  into  $E$  yielding the factorization

$$A(t) = \mathcal{K}(v(t)) .$$

Evidently, for  $\sigma \leq \tau$  in  $[0, v(T)]$ , one has

$$\text{var}(\mathcal{K} ; s, \tau) = \tau - \sigma$$

so that  $\mathcal{K}$  is Lipschitz with ratio 1.

### 5. b THE CASE OF CONVEX SETS IN A NORMED SPACE

Let  $E$  denote a real normed linear space and  $F$  its

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topological dual endowed with the usual norm. This constitutes a dual pair as considered in Chapter 2 (keep in mind that the norm topology on E is compatible with the duality, but not the norm topology on F unless E is a reflexive Banach space).

Let C and C' be two non empty convex subsets of E ; as we are to deal with distances, it is immaterial to suppose these sets closed or not. Let  $\gamma$  and  $\gamma'$  be the respective support functions of C and C' which are positively homogeneous elements of  $\Gamma_0(F,E)$ , vanishing at the origin of F.

Denoting by B the closed unit ball of F , one finds

$$(5.6) \quad e(C,C') = \sup_{y \in B} (\gamma(y) - \gamma'(y))$$

(with the convention  $\infty - \infty = -\infty$ ).

This is easily proved by observing that, for  $\rho \in \mathbb{R}$ , the inequality  $\rho \geq e(C,C')$  means that, if  $\beta(\rho)$  denotes the closed ball centered at the origin with radius  $\rho$  , the set  $\overline{C'} + \beta(\rho)$  contains C ; express then this inclusion in terms of support functions. Another way of proof would start from the following formula giving the distance of a point a of E to the set C'

$$(5.7) \quad d(a,C') = \sup_{y \in B} [\langle a,y \rangle - \gamma'(y)]$$

In fact (cf. § 2.)

$$d(a,C') = (\psi_{C'} \vee |\cdot|)(a) .$$

Since the function  $|\cdot|$  is everywhere finite and continuous, since there

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exists at least one point where  $\psi_C$  takes a finite value (namely the value zero), and since both functions are convex, the inf-convolute  $\psi_C \nabla | \cdot |$  is convex, everywhere finite and continuous (cf. § 2) thus it equals its bipolar, i.e.

$$\begin{aligned} (\psi_C \nabla | \cdot |)(a) &= \sup_{y \in F} [\langle a, y \rangle - \gamma'(y) - \psi_B(y)] \\ &= \sup_{y \in B} [\langle a, y \rangle - \gamma'(y)] \end{aligned}$$

which is equality (5.7).

Equality (5.6) implies that the Hausdorff distance between the non empty convex sets  $C$  and  $C'$  is finite only if  $\text{dom } \gamma$  and  $\text{dom } \gamma'$  (i.e. the sets of the points of  $F$  where  $\gamma$  and  $\gamma'$  take finite values) consist in the same set denoted by  $D$  and then

$$(5.8) \quad h(C, C') = \sup_{y \in B \cap D} |\gamma(y) - \gamma'(y)|$$

Note that  $D$  is a conic convex subset of  $F$ ; its polar cone in  $E$  is the recession cone of  $C$  and  $C'$ . Recall that  $D$  equals the whole of  $F$  if and only if  $C$  and  $C'$  are bounded.

The expression (5.8) of the Hausdorff distance yields the following :

Let  $t \mapsto C(t)$  be a multimapping from  $[0, T]$  into the normed space  $E$ , with non empty convex values; denote by  $y \mapsto \gamma(t, y)$  the support function of  $C(t)$ . This multimapping is absolutely continuous (resp. Lipschitz with ratio  $\lambda$ ) if and only if the set  $D = \text{dom } \gamma(t, \cdot)$  is independant of  $t$ , with the existence of a finite non decreasing numerical

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function  $\rho : [0, T] \rightarrow \mathbb{R}$ , absolutely continuous (resp. Lipschitz with ratio  $\lambda$ ), such that for any  $y \in D$  and any subinterval  $[s, t]$  of  $[0, T]$  one has

$$|\gamma(t, y) - \gamma(s, y)| \leq |y| (\rho(t) - \rho(s))$$

( $|\cdot|$  denotes here the norm in  $F$ ).

Equivalently there exists  $\dot{\rho}$ , a nonnegative element of  $L^1(0, T; \mathbb{R})$  such that for any  $y$  in  $D$ , the numerical function  $t \mapsto \gamma(t, y)$  is absolutely continuous and its derivative  $\dot{\gamma}$  satisfies for almost every  $t$

$$(5.9) \quad |\dot{\gamma}(t, y)| \leq |y| \dot{\rho}(t)$$

(resp. the same inequality with  $\dot{\rho} = \lambda$ ). If such is the case one may take as  $\dot{\rho}$  the speed function of the moving set  $C$ .

Characterizing the regularity of the motion of a (closed) convex set  $t \mapsto C(t)$  by means of its support function  $\gamma(t, \cdot)$  is quite a natural procedure. In fact an essential feature in locally convex topological linear spaces is that a closed convex set equals the intersection of all the closed half-spaces containing it, or equivalently the intersection of the minimal ones among these half-spaces, i.e. the half-spaces which have in the present case the form  $\{x \in E : \langle x, y \rangle \leq \gamma(t, y)\}$ , with  $|y| = 1$ . Fixing here  $y$  yields a moving half space whose boundary hyperplane keeps a constant direction; the derivative  $\dot{\gamma}(t, y)$  may be interpreted as the speed of this moving hyperplane, or as the speed of the moving half-space itself. Then (5.9) expresses a uniform

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majoration of the speeds for the minimal half-spaces of all directions.

Example. Take as  $C(t)$  a convex set moving by translation, i.e.

$$C(t) = C_0 + w(t)$$

where  $C_0$  denotes a fixed convex set and  $w$  a fonction defined on  $[0, T]$  with values in  $E$ . Then, if  $\gamma_0$  is the support function of  $C_0$

$$\gamma(t, y) = \gamma_0(y) + \langle w(t), y \rangle .$$

One concludes that the multimapping is absolutely continuous if (and only if, in the case where  $C_0$  is bounded) the function  $t \mapsto w(t)$  is absolutely continuous. When  $E$  is a reflexive Banach space, the absolute continuity of  $w$  is known to imply for almost every  $t$  the existence of the strong derivative  $\dot{w}$  (cf. KOMURA [1]) and this yields for the speed  $\dot{v}$  of  $C$  the majoration

$$(5.10) \quad \dot{v} \leq |\dot{w}|$$

(equality when  $C_0$  is bounded).

## 5. c INTERSECTION OF TWO MOVING CONVEX SETS

The practical use of the preceding concepts requires some criteria of absolute continuity for multimappings. The object of this paragraph is to establish the following one (already published in MOREAU [22] or, for more details, [19]) :

PROPOSITION. Let  $t \mapsto A_t$  and  $t \mapsto B_t$  denote two multimappings from the compact interval  $[0, T]$  into the normed space  $E$ , with convex values.



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Suppose that for any  $t \in [0, T]$  the set  $A_t$  has a nonempty intersection with the interior of  $B_t$  and that the diameter of  $A_t \cap B_t$  is finite.  
Then if the two multimappings are absolutely continuous (resp. Lipschitz)  
such is also the multimapping  $t \mapsto A_t \cap B_t$ .

We shall decompose the proof into several lemmas which may be of use by themselves.

LEMMA 1. Let  $B_1, B_2$  denote two convex subsets of the normed space  $E$  and  $A_1, A_2$  two arbitrary subsets of  $E$  ; then (e denoting the "excess" as in § 5. a)

$$(5.11) \quad e(A_1, E \setminus B_1) \leq e(A_2, E \setminus B_2) + e(A_1, A_2) + e(B_1, B_2) .$$

Let us prove first that for any  $a \in E$

$$(5.12) \quad d(a, E \setminus B_1) \leq d(a, E \setminus B_2) + e(B_1, B_2) .$$

One makes calculation easier by performing a translation reducing to the case where  $a$  is the origin of  $E$ . Let  $g_1, g_2$  be the support functions of  $B_1$  and  $B_2$ , defined on the dual  $F$  of  $E$ . Let  $\rho$  be an arbitrary positive number satisfying the inequality  $\rho \leq d(0, E \setminus B_1)$ , which means that the open ball with center  $0$  and radius  $\rho$  is contained in  $B_1$ ; in terms of support functions this inclusion is equivalent to  $\rho \leq g_1(y)$  for any  $y$  belonging to  $\Sigma$ , the unit sphere of  $F$ . Now (5.6) implies

$$\forall y \in \Sigma : g_1(y) \leq g_2(y) + e(B_1, B_2) ;$$

therefore  $\rho - e(B_1, B_2) \leq g_2(y)$ ; inequality (5.12) (trivial if

$e(B_1, B_2) = +\infty$ ) follows. From it one obtains (5.11) by taking suprema

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for a ranging over  $A_1$ , then using the fact that the écart  $e$  satisfies the triangle inequality.

LEMMA 2. Let A and B denote two convex subsets of the normed space E ; suppose that B contains an open ball with radius  $\rho > 0$ , with center a belonging to A. Then

$$(5.13) \quad \forall x \in E : d(x, A \cap B) \leq \left(1 + \frac{|x-a|}{\rho}\right)(d(x,A) + d(x,B)).$$

Proof : Denote indifferently by  $|\cdot|$  the norm in E or the dual norm in F ; let f and g be the support functions of A and B. Similarly to (5.7) we have

$$d(x,A) = \sup \{ \langle x, u \rangle - f(u) : u \in F, |u| \leq 1 \}$$

and the corresponding expression for  $d(x,B)$ . Define a positively homogeneous function  $\phi$  on  $F \times F$  by

$$\phi(u,v) = \langle x, u + v \rangle - f(u) - g(v) .$$

For an arbitrarily chosen constant  $k > 0$  this yields

$$(5.14) \quad k(d(x,A) + d(x,B)) = \sup \{ \phi(u,v) : |u| \leq k, |v| \leq k \} .$$

The hypotheses in the Lemma to be proved imply, by elementary arguments, that the closure  $\overline{A \cap B}$  of  $A \cap B$  equals the intersection of the closures  $\bar{A}$  and  $\bar{B}$  of A and B. Then, the support function of  $A \cap B$  is the dual function of  $\psi_{\bar{A}} + \psi_{\bar{B}}$ , i.e. the  $\Gamma$ -hull of  $f \vee g$  ; by the facts summarized in § 2. d, this  $\Gamma$ -hull is the function  $f \vee g$  itself, i.e.

$$(f \vee g)(w) = \inf \{ f(u) + g(v) : u + v = w \} .$$

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Using again the expression (5.7) for the distance from a point to a convex set, this yields

$$\begin{aligned}
 (5.15) \quad d(x, A \cap B) &= \sup \{ \langle x, w \rangle - (f \vee g)(w) : |w| \leq 1 \} \\
 &= \sup \{ \phi(u, v) : |u + v| \leq 1 \} .
 \end{aligned}$$

Let us make calculation easier by supposing that a translation has been performed in  $E$  such that  $a = 0$  ; then the hypotheses made about  $A$  and  $B$  are expressed by  $f \geq 0$  and  $g \geq \rho | \cdot |$  , hence

$$|u + v| \leq 1 \Rightarrow \phi(u, v) \leq |x| - \rho |v| .$$

As  $\phi(0, 0) = 0$  and in view of (5.15) this implies

$$d(x, A \cap B) \leq \sup \{ \phi(u, v) : |v| \leq \frac{|x|}{\rho} , |u| \leq 1 + \frac{|x|}{\rho} \} .$$

After putting  $k = 1 + \frac{|x|}{\rho}$  in (5.14), the comparison of the sets over which the suprema are taken yields (5.13).

REMARK. In the case where  $E$  is a Hilbert space one may use trigonometry to establish a slightly better inequality ; see MOREAU [19] .

LEMMA 3. Let  $A$  and  $B$  denote two convex subsets of  $E$  ; take  $\alpha$  and  $\rho$  in  $]0, +\infty[$  such that  $\alpha < \rho < e(A, E \setminus B)$ . Then, for any  $x$  in  $E$  such that  $d(x, A) + d(x, B) \leq \alpha$  , one has

$$d(x, A \cap B) \leq \frac{\rho + \text{diam}(A \cap B)}{\rho - \alpha} (d(x, A) + d(x, B)) .$$

This results from (5.13) and from the inequality

$$|x - a| \leq \text{diam}(A \cap B) + d(x, A \cap B) .$$

Bringing together these lemmas one obtains easily :

LEMMA 4. Let  $T$  denote a topological space ; let  $t \mapsto A_t$  and  $t \mapsto B_t$

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be two multimappings from T into the normed space E, with convex variables. Let  $s \in T$  such that

$$\text{diam} (A_s \cap B_s) < +\infty ,$$

$$A_s \cap \text{int} B_s \neq \emptyset ,$$

$$\lim_{t \rightarrow s} e(A_t, A_s) = 0 \quad (\text{resp. } \lim_{t \rightarrow s} e(A_s, A_t) = 0) ,$$

$$\lim_{t \rightarrow s} e(B_t, B_s) = 0 \quad (\text{resp. } \lim_{t \rightarrow s} e(B_s, B_t) = 0) .$$

Then

$$\lim_{t \rightarrow s} e(A_t \cap B_t, A_s \cap B_s) = 0 \quad (\text{resp. } \lim_{t \rightarrow s} e(A_s \cap B_s, A_t \cap B_t) = 0)$$

and the two numerical functions  $t \mapsto \text{diam} (A_t \cap B_t)$  and  $t \mapsto e(A_t, E \setminus B_t)$  are upper semicontinuous (resp. lower semicontinuous) at the point  $s$ .

Let us now complete the proof of the Proposition :

The hypotheses imply that the two multimappings  $t \mapsto A_t$  and  $t \mapsto B_t$  are continuous in the sense of Hausdorff distance. The finite numerical function  $t \mapsto \text{diam} (A_t \cap B_t)$  is continuous by Lemma 4 on the compact interval  $[0, T]$  , thus majorized by some constant  $R < +\infty$ . By the same lemma the numerical function  $t \mapsto e(A_t, E \setminus B_t)$  is continuous on  $[0, T]$ , with strictly positive values since  $A_t \cap \text{int} B_t \neq \emptyset$  , thus minorized by some constant  $\rho > 0$ . Choose  $\alpha \in ]0, \rho[$  ; the functions  $t \mapsto \text{var} (A ; 0, t)$  and  $t \mapsto \text{var} (B ; 0, t)$  being finite and continuous, there exists  $\delta > 0$  such that for  $\sigma$  and  $\tau$  in  $[0, T]$ , the condition  $|\sigma - \tau| < \delta$  ensures that  $h(A_\sigma, A_\tau)$  and  $h(B_\sigma, B_\tau)$  are less than  $\frac{\alpha}{2}$ .

Then Lemma 3 implies

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$$h(A_\sigma \cap B_\sigma, A_\tau \cap B_\tau) \leq \frac{\rho + R}{\rho - \alpha} (h(A_\sigma, A_\tau) + h(B_\sigma, B_\tau))$$

which yields the expected majorations.

5. d DISTANCE AND PENALTY FUNCTION IN A HILBERT SPACE

Let  $H$  be a real Hilbert space ; denote by  $(.|.)$  the scalar product in it and by  $|\cdot|$  the norm. By means of this scalar product,  $H$  may be identified by its dual ; in other words  $(.|.)$  is a bilinear form on  $H \times H$  which places  $H$  in duality with itself and the norm-topology is compatible with this duality.

Easy computation yields that the function

$$Q : x \mapsto \frac{1}{2} |x|^2$$

which clearly belongs to  $\Gamma_0(H,H)$  equals its own dual (actually it can be proved that  $Q$  is the only function equal to its dual).

Let  $C$  be a non empty closed convex subset of  $H$  ; denote by  $q$  the numerical function defined on  $H$  by

$$q(x) = \frac{1}{2} [d(x,C)]^2 = (\psi_C \nabla Q)(x).$$

Elementarily this function is convex, everywhere finite, continuous, Fréchet-differentiable with gradient

$$(5.16) \quad \text{grad } q(x) = x - \text{proj}_C x ,$$

where  $\text{proj}_C x$  denotes the nearest point to  $x$  in  $C$ . (All this is a special case of a theory in which the indicator function  $\psi_C$  is replaced by an arbitrary element of  $\Gamma_0(H,H)$  ; see MOREAU [6].)

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Choose a strictly positive constant  $\lambda$  ; then  $x \mapsto \frac{1}{\lambda} q(x)$  defines what is commonly called a penalty function of the set  $C$ , i.e. a finite function which takes the value 0 when  $x \in C$  and rapidly growing positive values when the distance from  $x$  to  $C$  increases. So to speak, the smaller is the constant  $\lambda$ , the greater is the penalty for  $x$  of lying at a distance from  $C$ . The penalty function may be considered as an approximation of  $\psi_C$  in a sense which concerns also the subdifferentials as follows : Denote by  $A$  the multimapping  $x \mapsto \partial \psi_C(x)$  from  $H$  into itself, which constitutes a special case of maximal monotone operator. In general, for a chosen  $\lambda > 0$ , the single valued, everywhere defined mapping

$$(5.17) \quad A_\lambda = \frac{I - (I + \lambda A)^{-1}}{\lambda} ,$$

where  $I$  denotes identity, is classically called a Yosida approximation, or Yosida regularization, of  $A$  ; it is Lipschitz with ratio  $\frac{1}{\lambda}$ . Here  $A_\lambda$  may easily be explicited ; by definition the equality

$y = (I + \lambda A)^{-1}(x)$  means  $x \in (I + \lambda A)(y)$  or equivalently  $x - y \in \partial \psi(y)$  for  $\partial \psi(y)$  is a cone so that the factor  $\lambda$  may be omitted. This is well known to characterize  $y$  as equal to  $\text{proj}_C x$  ; hence (5.17) becomes

$$(5.18) \quad A_\lambda(x) = \frac{1}{\lambda} (x - \text{proj}_C x) = \frac{1}{\lambda} \text{grad } q(x) .$$

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5. e MOVING CONVEX SET IN A HILBERT SPACE

With the same notations as in the preceding paragraph, suppose  $t \mapsto C(t)$  is an absolutely continuous multimapping from  $[0, T]$  into  $H$ , with non empty closed convex values ; put

$$q(t, x) = \frac{1}{2} [d(x, C(t))]^2 .$$

Let  $t \mapsto z(t)$  be an absolutely continuous mapping from  $[0, T]$  into  $H$ .

Classically the continuity of  $t \mapsto C(t)$  in the sense of Hausdorff distance and the continuity of  $t \mapsto z(t)$  imply the continuity of the mapping

$$t \mapsto \text{proj} (z(t), C(t)) .$$

The proof of it is based on some majoration of the square of the displacement of the projection which implies nothing about the absolute continuity of this mapping ; however :

LEMMA 1. If  $t \mapsto C(t)$  and  $t \mapsto z(t)$  are absolutely continuous, on  $[0, T]$  so is the numerical function  $k : t \mapsto d(z(t), C(t))$ .

In fact, with the notation  $e$  of § 5. a, one has

$$d(z, C) = e(\{z\}, C)$$

so that, using the triangle inequality concerning the écart  $e$ , one obtains finally, for arbitrary  $\sigma$  and  $\tau$  in  $[0, T]$ ,

$$(5.19) \quad |d(z(\sigma), C(\sigma)) - d(z(\tau), C(\tau))| \\ \leq d(z(\sigma), z(\tau)) + h(C(\sigma), C(\tau))$$

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It just remains to apply the definition of absolute continuity.

This lemma implies that the function  $k$  possesses for almost every  $t$  a derivative denoted by  $\dot{k}(t)$ ; thus the function

$$t \mapsto \frac{1}{2} (k(t))^2 = q(t, z(t))$$

possesses, for the same values of  $t$ , a derivative equal to  $k(t) \dot{k}(t)$ .

The absolute continuity of the multimapping  $C$  means that its variation function  $v : t \mapsto \text{var} (C ; 0, t)$  is absolutely continuous, thus possesses a derivative  $\dot{v}(t)$  for almost every  $t$ . Similarly the absolute continuity of the vector function  $t \mapsto z(t)$  implies the existence of its strong derivative  $\dot{z}(t)$  for almost every  $t$  (by virtue of  $H$  being a reflexive Banach space ; see KOMURA [1]).

Let us prove now the following, which will be of use in next paragraph :

LEMMA 2. For any  $t$  in  $[0, T]$  such that the derivatives  $\dot{z}(t)$ ,  $\dot{v}(t)$ ,  $\dot{k}(t)$  exist, one has

$$(5.20) \quad |k(t) \dot{k}(t) - (\dot{z}(t) | \text{grad } q(t, z(t)))| \leq k(t) \dot{v}(t) .$$

In fact for such a value of  $t$  the hypotheses imply the existence of

$$\lim_{s \rightarrow t} \frac{q(s, z(s)) - q(t, z(t))}{s - t} = k(t) \dot{k}(t) .$$

Now

$$(5.21) \quad \frac{q(s, z(s)) - q(t, z(t))}{s - t} = \frac{q(s, z(s)) - q(s, z(t))}{s - t} + \frac{q(s, z(t)) - q(t, z(t))}{s - t} .$$



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As the numerical function  $x \mapsto q(s, x)$  is convex on  $H$ , its gradient at some point is also a subgradient ; this yields

$$\begin{aligned} (z(s) - z(t) | \text{grad } q(s, z(t))) &\leq q(s, z(s)) - q(s, z(t)) \\ &\leq (z(s) - z(t) | \text{grad } q(s, z(s))) . \end{aligned}$$

The mapping  $s \mapsto \text{proj}(x, C(s))$  is continuous, the mapping  $x \mapsto \text{proj}(x, C(s))$  is nonexpanding, thus the mapping

$$(s, x) \mapsto \text{grad } q(s, x) = x - \text{proj}(x, C(s))$$

from  $[0, T] \times H$  into  $H$  is continuous ; hence one obtains the existence of

$$\lim_{s \rightarrow t} \frac{q(s, z(s)) - q(s, z(t))}{s - t} = (z(t) | \text{grad } q(t, z(t))) .$$

Therefore the last term in (5.21) possesses also a limit which may be interpreted as the derivative at the point  $t$  for the function

$$(5.22) \quad s \mapsto q(s, z(t)) = \frac{1}{2} [d(z(t), C(s))]^2 .$$

Writing the same inequality as in (5.19), but with constant  $z$ , yields

$$\begin{aligned} |d(z(t), C(s)) - d(z(t), C(t))| &\leq h(C(s), C(t)) \\ &\leq |v(s) - v(t)| \end{aligned}$$

so that the derivative of the function (5.22) has its absolute value majorized by  $k(t) \dot{v}(t)$  ; this completes the proof of (5.20).

## 5. f THE SWEEPING PROCESS

Suppose given an absolutely continuous multimapping  $t \mapsto C(t)$  from  $[0, T]$  into the real Hilbert space  $H$ , with nonempty closed convex

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values ; denote by  $x \mapsto \psi(t,x)$  the indicator function of  $C(t)$ .

We put the problem of finding an absolutely continuous (single valued) mapping  $u : [0,T] \rightarrow H$  agreeing with some initial condition

$$(5.23) \quad u(0) = a, \text{ given in } C(0)$$

and whose derivative  $u$  satisfies for almost every  $t$  in  $[0,T]$

$$(5.24) \quad -\dot{u}(t) \in \partial \psi(t, u(t)) .$$

Interpreting  $u$  as a moving point in  $H$ , we call it a solution of the sweeping process by the moving convex set  $C$ . The reason of this name lies in the following mechanical image of condition (5.24) :

As  $\partial \psi(t,x)$  is empty when  $x \notin C$ , this condition implies  $u(t) \in C(t)$  for almost every  $t$ , thus for every  $t$ , by virtue of our continuity assumptions. Suppose, to make things clearer, that the moving convex set  $C$  possesses a nonempty interior. As long as the point  $u(t)$  lies in this interior, the subdifferential  $\partial \psi(t,u(t))$ , i.e. the cone of normal outward vectors at the point  $u(t)$  of the convex set (cf. § 2. e) reduces to the single element  $0$  ; then (5.24) implies that the moving point  $u$  remains at rest. It is only when  $u$  is caught up with by the boundary of  $C$  that it may take a nonzero velocity, so as to go on belonging to  $C$ , and by (5.24) this velocity possesses an inward normal direction with regard to  $C$ . In other words, condition (5.24) governs the quasistatic evolution of a material point  $u$  subject to the following mechanical actions :

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1° some resistance acting along the line of its velocity and opposite in direction ;

2° the moving perfect constraint whose geometric condition is  $u \in C(t)$  (cf. § 3. d).

Elementarily the initial value problem formulated above possesses at most one solution. Such uniqueness property holds more generally with "evolution equations" of the form

$$(5.25) \quad -\dot{u}(t) \in A(t, u(t)) \quad ,$$

where  $A(t, \cdot)$  denotes, for each  $t \in [0, T]$ , a monotone multimapping (or multivalued operator) from  $H$  into itself. In fact, monotonicity immediately implies that if  $u_1, u_2$ , absolutely continuous, are solution of (5.25), the function

$$t \mapsto |u_1(t) - u_2(t)|$$

is non increasing ; therefore these two solutions are equal if they agree with the same initial value.

Equations such as (5.25) have already been studied, but mainly under hypotheses involving that the set

$$\text{dom } A(t, \cdot) = \{x \in H : A(t, x) \neq \emptyset\}$$

is independant of  $t$  ; see references in BREZIS [1]. Here, on the contrary, the problem becomes trivial if  $\text{dom } A(t, \cdot)$ , namely  $C(t)$ , is constant ; thus the simple equation (5.24) furnishes the occasion of focusing upon the difficulties which arise from the variation of the

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domain. In the same line must be quoted :

1° H. BREZIS [2] who studied by a "double regularization" technique the case

$$A(t, \cdot) = \partial \phi(\cdot) + \partial \psi(t, \cdot)$$

with  $\phi \in \Gamma_0(H, H)$  independent of  $t$  and under some hypotheses involving the projection mapping  $x \mapsto \text{proj}(x, C(t))$  ; they do not seem directly comparable with our absolute continuity assumption.

2° C. PERALBA [1], [2] who succeeded in generalizing to the case  $A = \partial \phi$ , with  $\phi \in \Gamma_0(H, H)$  depending on time in a suitable way, the author's regularization method (see MOREAU [17]).

Because of its insertion in this context we also choose a regularization technique, i.e. the use of penalty functions, to prove, in next paragraph, an existence theorem. Another advantage of doing so refers to the application of equation (5.24) to elastoplastic mechanical systems, developed in Chapter 6 below : as explained in § 4. g; when the considered convex is the rigidity set defining a law of plasticity, the replacement of its indicator function by some penalty function comes to take into account some additional viscosity. The reasoning used below could then be adapted to prove that the solution of an elasto-viscoplastic problem tends to the solution of the elastoplastic problem when viscosity tends to zero. From the physical standpoint this may be as important as the existence question itself.

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The existence theorem obtained will supply the needs of Chapter 6. Actually a deeper insight into the sweeping process can be gained from a discretization method (published as multigraph in MOREAU [18]) which consists in proving first the convergence of the "catching up algorithm" (cf. § 5. h below) ; this method permits weaker hypotheses, by replacing the concept of the variation of a multimapping by that of retraction : use instead of Hausdorff distance the "unilateral" écart  $e$ . On the other hand, a generalization of the process can be defined in this line for the case of a possibly discontinuous moving convex set  $C$ , provided its variation (resp. retraction) is finite.

On the application of the discretization method to equations of the form (5.25), with  $A(t,.) = A_0(.) - f(t)$  see J. NECAS [1] .

#### 5. g EXISTENCE THEOREM

The study of equation (5.24) is made greatly easier by the following remark : the sweeping process associates the chain of the positions of the moving point  $u$  to the chain of the positions of the moving set  $C$  in a way which does not depend on the timing. More precisely, the change of variable in Lebesgue integral, along with the fact that the set  $\partial \psi$  is a cone, i.e. the multiplication by a nonnegative scalar sends it into itself, implies : Let  $\pi$  denote a non decreasing absolutely continuous mapping from  $[0, T]$  onto an interval  $[0, T']$ ;

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suppose  $C = C' \circ \pi$ , i. e.

$$(5.26) \quad \forall t \in [0, T] : C(t) = C'(\pi(t))$$

where  $C'$  is an absolutely continuous multimapping from  $[0, T']$  into  
 $H$ , with nonempty closed convex values; let  $u' : [0, T'] \rightarrow H$  be a solu-  
tion of the sweeping process for  $C'$ ; then the mapping  $u = u' \circ \pi$  is  
a solution of the sweeping process for  $C$ .

As explained in § 5. a, taking for  $\pi$  the variation function  $v$  of the given multimapping  $C$  yields a factorization of the form (5.26), with  $C'$  Lipschitz with ratio 1. This reduces the existential study of the sweeping problem to the Lipschitz case, i.e. the case where the speed function of the moving convex set belong to  $L^\infty(0, T; R)$ , or even is merely a constant.

Let us now proceed to establish :

PROPOSITION. For any  $a$  in  $C(0)$  the sweeping problem, as formulated in the preceding paragraph, possesses a (unique) solution.

Let  $n$  be positive integer. Denote by  $u_n : [0, T] \rightarrow H$  the solution of the differential equation

$$(5.27) \quad -\dot{u}_n = n \operatorname{grad} q(t, u_n(t))$$

for the initial condition

$$(5.2P) \quad u_n(0) = a .$$

In fact the expression (5.16) of  $\operatorname{grad} q$  implies, under the hypotheses made concerning  $t \mapsto C(t)$ , that the mapping  $(t, x) \mapsto n \operatorname{grad} q(t, x)$  is

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continuous relatively to  $t$  and is Lipschitz with ratio  $n$  relatively to  $x$  ; hence classically the existence and the uniqueness of  $u_n$  which is a continuously differentiable function from  $[0, T]$  into  $H$ .

Observe that the construction of the ordinary differential equation (5.27) consists in replacing the right member  $A = \partial \psi$  of (5.24) by its Yosida regularization (5.18), with  $\lambda = \frac{1}{n}$  ; equivalently, the indicator function of  $C$  is replaced by the penalty function  $nq$  : thus the moving point  $u_n(t)$  is allowed to not belong to  $C(t)$  but then, in view of the expression (5.14) of  $\text{grad } q$  , it must have a velocity directed toward its projection on  $C(t)$  ; the magnitude of this velocity is proportional to the distance from  $u_n(t)$  to  $C(t)$  and proportional to the penalty coefficient  $n$  .

LEMMA 1. If the speed function  $\dot{v}$  of the moving set  $C$  belongs to  $L^2(0, T ; R)$ , the sequence of the derivatives  $\dot{u}_n$  is bounded in  $L^2(0, T ; H)$ .

Denote by  $h_n(t)$  the common value of

$$\frac{1}{n} |\dot{u}_n(t)| = |\text{grad } q(t, u_n(t))| = d(u_n(t), C(t)) .$$

Inequality (5.20) (§ 5. e, Lemma 2) yields, for almost every  $t$  ,

$$|h_n(t) \dot{h}_n(t) - (\dot{u}_n(t) | \text{grad } q(t, u_n(t)))| \leq h_n(t) \dot{v}(t)$$

hence, due to (5.27),

$$(5.29) \quad h_n(t) \dot{h}_n(t) + n (h_n(t))^2 \leq h_n(t) \dot{v}(t)$$

As  $a \in C(0)$ , one has  $h_n(0) = 0$ , thus, by integration over  $[0, T]$  ,

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$$\frac{1}{2} (h_n(T))^2 + n \int_0^T (h_n(t))^2 dt \leq \int_0^T h_n(t) v(t) dt$$

Denoting by  $\|\cdot\|$  the norm in  $L^2(0, T; R)$  as well as the norm in  $L^2(0, T; H)$ , this yields

$$(5.30) \quad \|\dot{u}_n\| \leq \|\dot{v}\|$$

which proves the lemma.

REMARK. More may be obtained from inequality (5.29). Suppose only the absolute continuity of  $t \rightarrow C(t)$  so that the derivatives  $\dot{v}(t)$  and  $\dot{h}_n(t)$  exist for almost every  $t$ . For the values of  $t$  such that  $h_n(t) \neq 0$ , inequality (5.29) implies

$$\dot{h}_n(t) + n h_n(t) \leq \dot{v}(t)$$

and this is also true when  $h_n(t) = 0$  (then  $\dot{h}_n(t) = 0$  since zero is the minimal value of  $h_n$ ). The elementary treatment of this differential inequality, with the initial condition  $h_n(0) = 0$ , yields :

$$(5.31) \quad |\dot{u}_n(t)| \leq \dot{v}(t)$$

for almost every  $t$ .

In particular, if  $v \in L^p(0, T; R)$ , with  $1 \leq p \leq +\infty$ , the same inequality as (5.30) holds for  $L^p$  norms.

From such majorations, there are many ways of establishing the convergence of the sequence  $u_n$  to a function which is a solution of the sweeping process. In view of our  $L^2$  framework, the most efficient seems to make use of the following elementary property of Hilbert spaces, due to M. CRANDALL and A. PAZY [1] :



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Consider a real Hilbert space with scalar product noted  $\langle . | . \rangle$  and norm noted  $\| . \|$  . Let  $(r_n)$  be a sequence of positive real numbers ; let  $(z_n)$  be a sequence of elements of this Hilbert space such that

$$\forall n, \forall m : \langle z_n - z_m | r_n z_n - r_m z_m \rangle \leq 0$$

Then :

If  $r_n$  is strictly increasing in  $n$  ,  $\|z_n\|$  is decreasing and  $\lim_{n \rightarrow \infty} z_n$  exists .

If  $r_n$  is strictly decreasing,  $\|z_n\|$  is increasing ; if in addition  $\|z_n\|$  is bounded,  $\lim_{n \rightarrow \infty} z_n$  exists .

From this we are to prove :

LEMMA 2. If  $\dot{v} \in L^2(0, T ; R)$  the sequence  $\dot{u}_n$  is strongly convergent in  $L^2(0, T ; H)$ .

In fact, let  $m$  and  $n$  be two positive integers ; for any  $t$  in  $[0, T]$  , the values of the functions  $u_m, \dot{u}_m, u_n, \dot{u}_n$  satisfy

$$(5.32) \quad \frac{d}{dt} |u_m - u_n|^2 = 2(u_m - u_n | \dot{u}_m - \dot{u}_n) .$$

Denote by  $p_m, p_n$  the respective projections of  $u_m(t)$  and  $u_n(t)$  on  $C(t)$  ; by (5.16) and (5.27) one has

$$-\dot{u}_m = m(u_m - p_m) \in \partial \psi(t, p_m)$$

and the same for  $n$  ; due to the monotonicity of  $\partial \psi$  , this yields by easy calculation

$$(u_m - u_n | \dot{u}_m - \dot{u}_n) \leq -\left(\frac{1}{m} \dot{u}_m - \frac{1}{n} \dot{u}_n | \dot{u}_m - \dot{u}_n\right)$$

Recall that  $u_m(0) = u_n(0) = a$  , integrate (5.32) over  $[0, T]$  , denote

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by  $\langle . | . \rangle$  the scalar product of the Hilbert space  $L^2(0, T; H)$  and by  $\| . \|$  its norm ; this inequality implies

$$0 \leq \frac{1}{2} |u_m(T) - u_n(T)|^2 \leq - \langle \frac{1}{m} u_m - \frac{1}{n} u_n | u_m - u_n \rangle .$$

The sequence  $r_n = \frac{1}{n}$  is strictly decreasing ; the sequence  $\|u_n\|$  is bounded according to Lemma 1 ; apply CRANDALL and PAZY's result in  $L^2(0, T; H)$ .

Next :

LEMMA 3. If  $\dot{v} \in L^2(0, T; R)$  the sequence of functions  $u_n$  converges uniformly on  $[0, T]$  to an absolutely continuous function  $u$  whose derivative is the  $L^2$  - limit of the sequence  $\dot{u}_n$  ; this function is solution of the sweeping process for the initial condition  $u(0) = a$ .

Furthermore, for almost every  $t$ ,

$$(5.33) \quad |\dot{u}(t)| \leq \dot{v}(t) .$$

In fact, denote by  $\dot{u}$  the limit of  $\dot{u}_n$  in  $L^2(0, T; H)$  and define  $u : [0, T] \rightarrow H$  by

$$u(t) = a + \int_0^t \dot{u}(s) ds ,$$

so that  $u$  is absolutely continuous with a strong derivative equal to  $\dot{u}$  almost everywhere. Still denoting by  $\| . \|$  the norm in  $L^2(0, T; H)$ , the inequality

$$|u(t) - u_n(t)| = \left| \int_0^t (\dot{u}(s) - \dot{u}_n(s)) ds \right| \leq \sqrt{t} \|\dot{u} - \dot{u}_n\|$$

shows that  $u$  is the uniform limit of  $u_n$ .

It remains to prove that  $u$  and  $\dot{u}$  verify (5.24) almost

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everywhere. Put

$$p_n(t) = \text{proj} (u_n(t), C(t)) .$$

Then, in view of (5.16) and (5.27)

$$\begin{aligned}
(5.34) \quad & u_n(t) - p_n(t) = \text{grad } q(t, u_n(t)) = -\frac{1}{n} \dot{u}_n(t) \\
& -\dot{u}_n(t) \in \partial \psi(t, p_n(t))
\end{aligned}$$

and, in view of (5.30), the functions  $p_n$  converge to  $u$  in  $L^2(0, T ; H)$ .

The convergences in  $L^2(0, T ; H)$  imply the existence of  $N'$ , an infinite subset of  $N$ , such that for any  $t$  which does not belong to a certain subset  $\omega$  of  $[0, T]$  with zero measure, the limit of  $p_v(t)$  in  $H$ , for  $v$  tending to infinity in  $N'$ , is  $u(t)$  and the limit of  $\dot{u}_v(t)$  in  $H$  is  $\dot{u}(t)$ . As the graph of the multimapping  $x \mapsto \partial \psi(t, x)$  is closed in  $H \times H$ , (5.34) implies that (5.16) holds for any  $t \notin \omega$ . On the other hand (5.33) follows from (5.31).

From this lemma, the proof of the formulated Proposition is completed, by performing an absolutely continuous change a variable reducing to the case  $\dot{v} \in L^\infty(0, T ; R)$ , which a fortiori implies  $\dot{v} \in L^2(0, T ; R)$ .

REMARK. Inequality (5.33) is clearly preserved by such a change of variable, so that in general for any solution  $u$  of the sweeping process

$$|\dot{u}(t)| \leq \dot{v}(t) .$$

By integration, this yields that the length of the path traveled by the moving point  $u$  during an interval of time  $[t_1, t_2]$  is majorized by

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var  $(C ; t_1, t_2)$ . This property becomes specially suggestive in the special case where  $C$  moves by translation i.e.

$$C(t) = C_0 + w(t) ,$$

with  $w$  absolutely continuous. Then, in view of § 5. b, example,

$$(5.35) \quad \int_{t_1}^{t_2} |\dot{u}(t)| dt \leq \int_{t_1}^{t_2} |\dot{w}(t)| dt .$$

The association of the function  $u$ , a solution of the sweeping process, with the given function  $w$  defining the translation imposed to  $C$ , may be visualized as a driving affected with play; (5.35) expresses that such a play makes the driven point travel a path which cannot be longer than the path traveled by the driving device.

#### 5. h DISCRETIZATION ALGORITHM

A method of "time discretization" for the approximate solution of the preceding problem consists in choosing a subdivision of  $[0, T]$ , namely  $0 = t_0 < t_1 < \dots < t_n = T$  and constructing a sequence  $x_0, x_1, \dots, x_n$  of points of  $H$  such that  $x_i$  constitutes an approximation of  $u(t_i)$ . Adopting  $\frac{1}{t_i - t_{i-1}} (x_i - x_{i-1})$  as an approximation of  $\dot{u}(t_i)$  induces to replace (5.24) by

$$(5.36) \quad x_{i-1} - x_i \in (t_i - t_{i-1}) \partial \psi(t_i, x_i)$$

which is a recurrence condition of "implicit" type concerning the desired sequence (an "explicit" method would consist in interpreting the same quotient as an approximation of  $\dot{u}(t_{i-1})$ ); but this yields an unworkable

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recurrence condition). As  $\partial \psi(t_i, x_i)$  is a cone, the strictly positive factor  $t_i - t_{i-1}$  in the right member of (5.36) may be omitted and this condition equivalently amounts to

$$(5.37) \quad x_i = \text{proj}(x_{i-1}, C(t_i)) .$$

Thus, starting with  $x_0 = a$ , the point sequence  $(x_i)$  is constructed by successive projections on the sequence of closed convex sets  $C(t_i)$ . It is as if the moving point  $u$ , instead of being swept along with the moving set  $C$  was left behind except that, from time to time, it catches up with this set intantaneously, by the shortest way. We propose to call this the catching up algorithm.

The question is wether the step function  $x : [0, T] \rightarrow H$  defined from this sequence by

$$(5.38) \quad x(t) = x_i \quad \text{for } t \in ]t_{i-1}, t_i] ,$$

converges to the solution  $u$  of the sweeping process, for the same initial value  $a$ , when finer and finer subdivisions of  $[0, T]$  are considered.

A direct proof of the convergence of this family of step functions may be given, yielding another way to establish the existence of the solution  $u$  itself (cf. MOREAU [17], [18]). As this existence has been obtained above by a regularization, or penalty, technique we think it interesting and unusual to study also the discretization algorithm by some extension of the penalty method : the trick consists in making the

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penalty coefficient vary with  $t$  (cf. MOREAU [17]).

PROPOSITION. For any  $\varepsilon > 0$  there exists  $\eta > 0$  such that the majoration

$$\sup_i (t_i - t_{i-1}) < \eta$$

(resp. there exists  $\eta' > 0$  such that the majoration

$$\sup_i \text{var} (C ; t_{i-1}, t_i) < \eta')$$

ensures

$$\forall t \in [0, T] : |u(t) - x(t)| < \varepsilon .$$

Let  $\rho : [0, T] \rightarrow \mathbb{R}_+$  a nonnegative ruled function (actually it will suffice in the following to take as  $\rho$  a step function). The classical theory of differential equations ensures the existence of

$u_\rho : [0, T] \rightarrow H$ , solution of

$$-\dot{u}_\rho(t) = \rho(t) \text{grad } q(t, u_\rho(t))$$

agreeing with the initial condition  $u_\rho(0) = a$ . Denote by  $h_\rho$  the absolutely continuous numerical function

$$h_\rho(t) = d(u_\rho(t), C(t)) = |\text{grad } q(t, u_\rho(t))| .$$

The same calculation as in § 5. g, proof of Lemma 1, yields the differential inequality

$$(5.39) \quad \dot{h}_\rho + \rho h_\rho \leq \dot{v} ,$$

from which elementary techniques leads to :

LEMMA 1. If the speed function  $\dot{v}$  of  $C$  is majorized by some constant  $M \geq 0$ , the function  $h_\rho$  is majorized by the constant  $M J(\rho)$ , where

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$J(\rho)$  denotes the supremum over  $[0, T]$  of the numerical function  $k$  defined on this interval by the differential equation  $\dot{k} + \rho k = 1$  with the initial condition  $k(0) = 0$ .

Consider now another function  $\overset{\sigma}{\wedge}$  similar to  $\rho$  and the corresponding  $u_\sigma$  and  $h_\sigma$ . The same inequality as in § 5. b, proof of Lemma 2, yields, for any  $t$  in  $[0, T]$ ,

$$\frac{1}{2} |u_\rho(t) - u_\sigma(t)|^2 \leq \int_0^t -(\text{grad } q(s, u_\rho(s)) - \text{grad } q(s, u_\sigma(s))) \rho(s) \text{grad } q(s, u_\rho(s)) - \sigma(s) \text{grad } q(s, u_\sigma(s)) ds.$$

The integrand is a scalar product in  $H$ , majorized by

$$(h_\rho + h_\sigma)(\rho h_\rho + \sigma h_\sigma) = \rho h_\rho^2 + \sigma h_\sigma^2 + (\rho + \sigma) h_\rho h_\sigma.$$

Now from Lemma 1 and inequality (5.39) one obtains

$$h_\rho \dot{h}_\rho + \rho h_\rho^2 \leq M J(\rho)$$
$$h_\sigma \dot{h}_\sigma + \rho h_\sigma h_\rho \leq M J(\sigma)$$

and two symmetrical inequalities. Adding them together and integrating gives the proof of the following :

LEMMA 2. If the function  $\dot{v}$  is majorized by some constant  $M \geq 0$  one has, for every  $t$  in  $[0, T]$  ,

$$(5.40) \quad |u_\rho(t) - u_\sigma(t)|^2 \leq 4t M^2 (J(\rho) + J(\sigma)) .$$

If, in particular,  $\sigma$  is a constant  $m$

$$J(\sigma) = \frac{1}{m} (1 - e^{-mT}) \leq \frac{1}{m} .$$

By § 5. g, the solution  $u$  of the sweeping process is the limit of the corresponding  $u_\sigma$  when  $m$  (for instance an integer) tends to infinity ;

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thus (5.40) implies

$$(5.41) \quad |u_\rho(t) - u(t)|^2 \leq 4t M^2 J(\rho).$$

For the continuation take as  $\rho$  the step function associated with the subdivision

$$0 = t_0 < t_1 < \dots < t_n = T$$

as follows : denoting by  $m_i$  the middle point of the interval

$[t_i, t_{i+1}]$ , put

$$(5.42) \quad \rho(t) = \begin{cases} A & \text{if } t_i < t < m_i \\ 0 & \text{if } m_i \leq t \leq t_{i+1} \end{cases},$$

where  $A$  is a constant independent of  $i$ .

Denote by  $p$  the supremum of the  $t_{i+1} - t_i$ ; studying the function  $k$  associated with  $\rho$  as in Lemma 1 yields :

LEMMA 3. If  $\rho$  is defined by (5.42) and  $A \geq 4$  one has

$$J(\rho) \leq \frac{1}{\sqrt{A}} + \frac{p}{2}.$$

Hint : the function  $K : t \mapsto \max \left\{ \frac{1}{\sqrt{A}}, k(t) \right\}$  possesses for almost every

$t$  a derivative  $\dot{K}(t)$ . When  $t \in ]t_i, m_i[$  one has

$$\begin{aligned} \dot{K}(t) &\leq -1 && \text{if } K(t) > \frac{1}{\sqrt{A}} \\ \dot{K}(t) &= 0 && \text{if } K(t) = \frac{1}{\sqrt{A}}. \end{aligned}$$

When  $t \in [m_i, t_{i+1}]$  one has

$$\begin{aligned} \dot{K}(t) &= 1 && \text{if } K(t) > \frac{1}{\sqrt{A}} \\ \dot{K}(t) &= 0 && \text{if } K(t) = \frac{1}{\sqrt{A}}. \end{aligned}$$

From these lemmas we can proceed to the proof of the Proposition.



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Observe first that the two alternative statements of this Proposition are equivalent since the variation function  $v$  of  $C$  is continuous on  $[0, T]$ , thus uniformly continuous.

The statement concerning variations is visibly indifferent to any (absolutely continuous) non decreasing change of variable ; we take profit of this fact in supposing that a change of variable has been performed reducing to the case where the speed function  $\dot{v}$  of  $C$  is the constant 1 (see § 5. a).

First step. Denote by  $\pi$  the following absolutely continuous non decreasing mapping from the interval  $[0, T]$  onto itself ( $m_i$  denotes as before the middle point of  $[t_i, t_{i+1}]$ )

$$\pi(t) = \begin{cases} t_i & \text{if } t_i \leq t \leq m_i \\ 2t - t_{i+1} & \text{if } m_i \leq t \leq t_{i+1} \end{cases}$$

and put

$$C(\pi(t)) = C'(t) .$$

In other words, on each interval of the form  $[t_i, m_i]$  the convex set  $C'$  remains fixed, equal to  $C(t_i)$  ; on the next interval  $[m_i, t_{i+1}]$ , it runs through the same chain of configurations as  $C$  on  $[t_i, t_{i+1}]$ , with a timing adjusted in such a way that  $C'$  catches up with  $C$  at the instant  $t_{i+1}$ . Call  $u'$  the solution of the sweeping process for the moving convex set  $C'$  and the same initial value  $a$  as  $u$  ; in view of the change of variable one has

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$$u'(t) = u(\pi(t)) .$$

By virtue of (5.31), the function  $u$  is Lipschitz with ratio 1 ; thus, for any  $t \in [0, T]$ ,

$$(5.43) \quad |u(t) - u'(t)| \leq \frac{p}{2} .$$

Second step. Put

$$q'(t, x) = \frac{1}{2} (d(x, C'(t)))^2 .$$

Defining  $\rho$  by (5.42), denote by  $u'_\rho$  the solution of

$$(5.44) \quad -\dot{u}'_\rho(t) = \rho(t) \text{grad } q'(t, u'_\rho(t))$$

agreeing with the initial condition  $u'_\rho(0) = a$ . The integration of this differential equation may be explicited : On each interval of the form

$[m_i, t_{i+1}]$  the function  $\rho$  vanishes, so that

$$(5.45) \quad t \in [m_i, t_{i+1}] \Rightarrow u'_\rho(t) = u'_\rho(m_i) .$$

For  $t$  ranging over an interval of the form  $]t_i, m_i[$ ,  $\rho$  takes the constant value  $A$  and the function  $x \mapsto q'(t, x)$  is independent of  $t$ , with

$$\text{grad } q'(t, x) = x - \text{proj}(x, C(t_i))$$

so that, on this interval

$$(5.46) \quad u'_\rho(t) = u'_\rho(t_i) + [y_i - u'_\rho(t_i)] [1 - \exp A(t_i - t)]$$

where

$$y_i = \text{proj}(u'_\rho(t_i), C(t_i)) .$$

Supposing  $A \geq 4$ , it results from (5.41) and from Lemma 3

that, for any  $t \in [0, T]$

$$(5.47) \quad |u'_\rho(t) - u'(t)|^2 \leq 16 t \left( \frac{1}{\sqrt{A}} + \frac{p}{2} \right) .$$

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Note that (5.45) and (5.46) yield

$$(5.48) \quad u'_\rho(t_{i+1}) = u'_\rho(m_i) = u'_\rho(t_i) + [y_i - u'_\rho(t_i)] \left[ 1 - \exp \frac{A(t_i - t_{i+1})}{2} \right]$$

Third step. Let  $A$  tend to  $+\infty$ ; as all the  $t_i - t_{i+1}$  are  $< 0$ , (5.48) shows that, for each  $i < n$ , the difference  $u'_\rho(t_{i+1}) - y_i$  tends to zero in  $H$ . As the mapping  $\text{proj}(\cdot, C(t_i))$  used in the definition of  $y_i$  is continuous, this proves by iteration that, for each  $i < n$ , the value  $u'_\rho(t_{i+1})$  tends to  $x(t_{i+1})$  as defined by (5.38). Then (5.46) shows that  $u'_\rho(t)$  tends to  $x(t)$  for any  $t$  in  $]t_i, m_i]$  and finally also for any  $t$  in  $[m_i, t_{i+1}]$  by virtue of (5.46).

In view of (5.47) this pointwise convergence yields, for any  $t \in [0, T]$ ,

$$|x(t) - u'(t)| \leq \sqrt{8 T p}$$

which proves the Proposition, by comparing with (5.43).

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## 6 QUASI-STATIC EVOLUTION OF AN ELASTOPLASTIC SYSTEM

### 6. a FORMULATION OF THE PROBLEM

The framework in all this Chapter is that of a configuration space  $\mathcal{U}$  endowed with a linear space structure ; thus the practical applications of the following mainly concern systems whose displacements are treated as "infinitely small".

According to the usual conception of elastoplasticity, every state of the system is represented by two components which both are elements of  $\mathcal{U}$  :

The visible (or "exposed") component, denoted by  $x$  ; it is the part of the system which undergoes external forces, called loads, and may also be submitted to constraints.

The hidden (or "plastic") component denoted by  $p$ .

Strictly speaking, the configuration space of the system is then the product space  $\mathcal{U} \times \mathcal{U}$ .

The difference  $x - p = e \in \mathcal{U}$  will be called the elastic deviation.

Let us denote as before by  $\mathcal{F}$  the linear space of forces, placed in separating duality with  $\mathcal{U}$  ; the forces experienced by the component  $p$  are :

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1° The force  $s \in \mathcal{F}$  of "elastic restoring toward  $x$ " related to  $e$  by

$$(6.1) \quad s = A(e) \quad ,$$

where  $A$  denotes a given selfadjoint nonnegative linear mapping from  $\mathcal{U}$  into  $\mathcal{F}$ .

2° The force of "plastic resistance"  $f \in \mathcal{F}$  related to the velocity  $\dot{p}$  (at any instant where this velocity exists) by the resistance law studied in § 4. e.

$$(6.2) \quad \dot{p} \in \partial \psi_C(-f) \quad ,$$

where  $C$  denotes a fixed nonempty closed convex subset of  $\mathcal{F}$ .

The forces experienced by the component  $x$  are

1° The reaction  $r \in \mathcal{F}$  of a perfect affine constraint (cf. § 3. c); this constraint maintains  $x$  at every instant in an affine manifold which moves in a given way, say

$$(6.3) \quad \mathcal{L} = U + g(t)$$

where  $U$  denotes a fixed closed linear subspace of  $\mathcal{U}$  and  $t \mapsto g(t)$  is a given function of time, with values in  $\mathcal{U}$ , which may be called the guiding (or "driving"). Such a constraint constitutes the statical law

$$(6.4) \quad -r \in \partial \psi_{\mathcal{L}}(x) \quad .$$

2° The load  $c(t)$ , a given time-dependent element of  $\mathcal{F}$ .

3° The force  $-s$  of "elastic restoring toward  $p$ ". Supposing in this way that the elastic force acting on  $x$  is the negative of the elastic

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force acting on  $p$  merely means that the total power of the elastic forces vanishes in any evolution which preserves the elastic deviation  $x - p$ ; in other words the elastic energy depends on this deviation only.

The problem is that of determining the evolution of  $x$  and  $p$  in  $\mathcal{U}$ , under the hypothesis that the motion is sufficiently slow for inertia to be negligible.

Therefore, the dynamical equations amount to express the quasi-equilibrium of  $x$ , namely

$$(6.5) \quad r + c - s = 0$$

and the quasi-equilibrium of  $p$ , namely

$$(6.6) \quad s + f = 0 .$$

To illustrate the preceding formulation by a practical example, the reader may take back the situation of a lattice of bars, presented in § 3. i, j. If the behavior of each bar is elastoplastic, the  $\frac{1}{2} n(n-1)$ -uple of their respective elongations, namely the element  $e \in E$ , has to be written as a sum, say  $e' + p$ ; here  $e'$  denotes the "elastic part" of  $e$ , related to the tension  $s \in S$  by a linear elasticity law such as (6.1);  $p$  denotes the "plastic part" of  $e$ : its "velocity"  $\dot{p}$  is related to  $s$  by relations of the form (6.2), (6.6). At this stage one may avoid the explicit consideration of the linear mappings  $D$  and  $D^*$  by using the third procedure of § 3. j, namely the elimination of  $(X, Y)$ :

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then the sum  $e' + p$  is interpreted as the "visible" configuration, to be denoted here by  $x$  ; finally write simply  $e$  instead of  $e'$ .

The same pattern applies to an elastoplastic continuous medium, occupying a domain  $\Omega$  of the physical space. Then elements  $e, e', p, s$  are some tensor fields defined on  $\Omega$  ; the spaces  $E$  and  $S$  are some function spaces. The corresponding quasistatic evolution problem may be treated in the line of the following paragraphs, but with some complications which will not be investigated in this lectures ; the difficulty arises from the fact that, with regard to the Hilbert norm defined by means of the elastic energy (see § 6. b) the convex  $C$  possesses an empty interior. Then the theorem on the absolute continuity of intersections (§ 5. c) will be applied relatively to some  $L^\infty$ -norm ; the absolute continuity of the considered intersection will finally hold with regard to the Hilbert norm too, as this latter is majorized by the  $L^\infty$ -norm (multiplied by a constant).

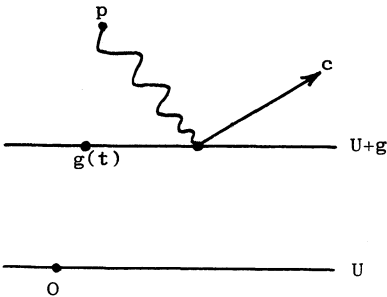
Observe that the continuous medium problem is studied by G. DUVAUT and J.L. LIONS, [1], Chap. 5 . Their method is that of vanishing viscosity, basically similar to the regularization technique we used in § 5. g ; but they must restrict themselves to the special case where the "load", denoted here by  $c$ , is identically zero ; thus the motion is only caused by the "guiding"  $g$ . Paragraph 6. c below explains why this special case is more tractable : it corresponds to a set

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$(C-c-g) \cap V$  which moves by translation, so that the intersection theorem is not required for proving its absolute continuity (cf. § 5. b).

We shall not deal in the present lectures with systems governed by behavioral laws of Hencky's type ; the reader will refer to H. Lanchon's lectures on this subject. Hencky's law is also studied in the book of DUVAUT and LIONS, by methods involving the duality of convex functionals.

In order to help the reader to visualize the formulated problem let us finally present a very simple model in which the dimension of  $\mathcal{U}$  equals 2. The considered system consists of two particles  $x$  and  $p$  moving in the plane  $\mathcal{U}$ . The particle  $x$  is guided without friction on the material straight line  $U + g(t)$ , a line which remains parallel to



the fixed line  $U$  and moves in a given way. The particle  $p$  will be visualized as a plot, whose contact with the plane  $\mathcal{U}$  is affected by a given friction. The two particles are connected by a spring whose length in the state

of zero tension is zero. In addition, a given force  $c(t)$  is applied to  $x$ . One studies motions during which the various forces equilibrate each



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other at any instant ; in particular the friction resistance undergone by  $p$  must exactly counterbalance the spring tension.

Investigating this elementary model raises an important observation : though the friction between  $p$  and the underlying plane has the characteristics of perfect plasticity, the behavior of the component  $x$  exhibits strain hardening. In fact suppose the line  $U + g$  is fixed, for instance with  $g$  identically zero ; suppose the friction of  $p$  is isotropic, i.e. it obeys elementary Coulomb's law. Clearly any motion during which the spring is strained enough for the point  $p$  to yield (this imposes a definite value for the distance between  $x$  and  $p$ ) necessarily brings this point closer to the line. Therefore this evolution leaves the system in a state for which the elastic domain, i.e. the set of the values of the load  $\lambda$  which may be applied without causing yield, is larger than before.

Such an example suggests that strain hardening can be described, in practical situations, by including in the definition of the hidden component  $p$  a sufficient number of internal state variables and postulating that the behavior of such a  $p$  is governed by a law similar to that of perfect plasticity. This has been developed, in our framework of convex pseudo-potentials, by Q.S. NGUYEN [1] (see also, for the use of internal state variables without convexity, J. KRATOCHVIL and J. NECAS [1]).

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6. b THE HILBERT SPACE NOTATION

Let us restrict ourselves for sake of simplicity to the usual case where the self-adjoint linear mapping  $A : \mathcal{U} \rightarrow \mathcal{F}$  introduced by the elasticity law (6.1) is one-to-one. Then one makes the treatment of the problem much easier by the notation trick which consists in interpreting the one-to-one mapping  $A$  as an identification of the spaces  $\mathcal{U}$  and  $\mathcal{F}$ . Denote by  $H$  this single space ; the symmetric bilinear form defined on  $\mathcal{U} \times \mathcal{U}$  by

$$(u, u') \mapsto \langle u, A u' \rangle = \langle u', A u \rangle$$

becomes an inner product in  $H$ , which will be denoted as  $(u' | u)$ . As the quadratic form

$$u \mapsto \frac{1}{2} \langle u, A u \rangle = \frac{1}{2} (u | u)$$

represents the elastic energy, it is nonnegative, thus positive definite due to  $A$  being one-to-one. This means that a pre-Hilbert norm  $|\cdot|$  is defined on  $H$  by

$$|u| = \sqrt{(u | u)} .$$

Let us make the assumption that  $H$  is complete relatively to this norm, i.e. it is a Hilbert space.

This of course is automatically satisfied in finite dimensional cases. In the case of continuous media also, one is accustomed to formulate the problems in suitable function spaces for this assumption to hold.

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Observe that the inner product  $(\cdot|\cdot)$  in  $H$  and the identification map  $A : \mathcal{U} \rightarrow \mathcal{F}$  are connected in such a way that subdifferential relations of the form  $-f \in \partial \phi(u)$  may equivalently be understood in the sense of the duality  $(\mathcal{U}, \mathcal{F} ; \langle \cdot, \cdot \rangle)$ , with  $u \in \mathcal{U}$  and  $f \in \mathcal{F}$ , or in the sense of the duality  $(H, H ; (\cdot|\cdot))$  with  $u$  and  $f$  elements of  $H$ .

Let us write the formulation of the problem in these notations. Denote by  $V$  the subspace of  $H$  orthogonal to  $U$ ; observe that (6.1) becomes  $s = e$ ; eliminate  $r$  by (6.5) and  $f$  by (6.6); the preceding conditions take the equivalent form

$$(6.7) \quad x \in U + g$$

$$(6.8) \quad s \in V + c$$

$$(6.9) \quad x = p + s$$

$$(6.10) \quad \dot{p} \in \partial \psi_C(s) .$$

Given the compact time interval  $[0, T]$ , the problem is that of determining the three functions  $t \mapsto x$ ,  $t \mapsto p$ ,  $t \mapsto s$ , with values in  $H$ , absolutely continuous on this interval (this makes the derivative  $\dot{p}$  exist for almost every  $t$ ) satisfying conditions (6.7) to (6.10) for almost every  $t$ ; and some initial conditions

$$(6.11) \quad x(0) = x_0, \quad s(0) = s_0 .$$

Let us make now some assumptions about the data.

ASSUMPTION 1. The given functions  $t \mapsto g$  and  $t \mapsto c$  are absolutely

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continuous on  $[0, T]$  . In addition, we visibly lose no generality in sup-  
posing that  $c$  takes its values in  $U$  and that  $g$  takes its values  
in  $V$  .

ASSUMPTION 2 . The initial data  $x_0$  and  $s_0$  satisfy the conditions

$$x_0 \in U + g(0) \quad , \quad s_0 \in V + c(0)$$

evidently required by (6.7) and (6.8), and the condition

$$(6.12) \quad s_0 \in C$$

required by (6.10). In fact (6.10) makes that for almost every  $t$ , the  
set  $\partial \psi_C(t)$  is non empty, thus  $s(t) \in C$ , and the latter must also be  
true for every  $t$  in  $[0, T]$ , by continuity.

Observe also that (6.8) with (6.10) requires the moving affine  
manifold  $V + c$  to meet the convex set  $C$  for almost every  $t$ , thus for  
every  $t$  by the continuity of  $c$ . This may equivalently be written as

$$(6.13) \quad c \in \text{proj}_U C \quad .$$

The mechanical meaning of this necessary condition is clear : a load  $c$   
(recall that we supposed  $c \in U$ ) which does not satisfy it cannot be  
counterbalanced by the forces  $r \in V$  (the reaction of the affine perfect  
constraint) and  $s \in C$ . As the law of plastic resistance (6.10) only per-  
mits  $s \in C$ , this means that if, starting from a configuration defined by  
some values of  $x$  and  $p$ , the system experiences a load  $c$  which does  
not verify (6.13), its evolution cannot be quasi-static. Of course, there

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are in this situation other necessary conditions, namely  $x-p \in V + c$ , a consequence of (6.8) and (6.9).

For mathematical convenience, we shall suppose that the set  $C$  possesses a nonempty interior ; then let us agree to replace (6.13) by the stronger following condition.

ASSUMPTION 3. For any  $t$  in  $[0, T]$  the affine manifold  $V + c(t)$  intersects the interior of  $C$ .

Without discussing here the physical meaning of this assumption, let us call it the "safe load hypothesis".

In addition, we shall avoid some technical job of covering the interval  $[0, T]$  and piecing together local solutions, by making also a last inessential hypothesis :

ASSUMPTION 4. The set  $C$  is bounded.

Then :

LEMMA. Assumptions 3 and 4 and the absolute continuity of the function  $t \mapsto c$  imply the following : there exists a strictly positive real constant  $\rho$  and an absolutely continuous mapping  $h : [0, T] \rightarrow H$  such that, for every  $t \in [0, T]$ , one has  $h(t) \in C \cap (V + c(t))$  and the closed ball with center  $h(t)$  and radius  $\rho$  is contained in  $C$ .

Outlined proof : Using the notation  $e$  of § 5. a, arguments similar to that of § 5. c prove that the numerical function

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$$t \mapsto e(V + c(t), H \setminus C)$$

is continuous on  $[0, T]$ , with strictly positive values, thus strictly minorized by some constant  $\rho > 0$ . The set

$$C_\rho = \{x \in H : d(x, H \setminus C) \geq \rho\}$$

is closed and convex, with nonempty interior. For every  $t$  in  $[0, T]$ , the affine manifold  $V + c(t)$  intersects the interior of  $C_\rho$ . The multi-mapping  $t \mapsto V + c(t)$  is absolutely continuous, implying by § 5. c the absolute continuity of the moving convex set  $t \mapsto C_\rho \cap (V + c(t))$ . Take as  $h$  a solution of the sweeping process by this moving non empty closed convex set (cf. § 5. f).

6. c NEW UNKNOWN FUNCTIONS

Conditions (6.7), (6.8), (6.9) may be written as

$$x - g \in U$$

$$c - s \in C$$

$$(x - g) + (c - s) = p + c - g ;$$

this may equivalently be expressed by means of the orthogonal projectors relative to the complementary orthogonal subspaces  $U$  and  $V$

$$x - g = \text{proj}_U (p + c - g)$$

$$c - s = \text{proj}_V (p + c - g)$$

or, as we have supposed  $c \in U$  and  $g \in V$ ,

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$$\text{proj}_U p = x - c - g$$

$$\text{proj}_V p = c + g - s$$

Let us define two new unknowns  $y$  and  $z$  by

$$(6.14) \quad y = s - c - g = -\text{proj}_V p$$

$$(6.15) \quad z = x - c - g = \text{proj}_U p$$

which implies

$$(6.16) \quad p = z - y$$

Due to Assumption 1, the functions  $t \mapsto y$  and  $t \mapsto z$  are absolutely continuous if and only if such are  $t \mapsto s$  and  $t \mapsto p$ .

Under this change of unknowns, conditions (6.7) to (6.10) equivalently amount to

$$(6.17) \quad \left\{ \begin{array}{l} \dot{z} - \dot{y} \in \partial \psi_C(y+c+g) \\ z \in U, \quad y \in V \end{array} \right.$$

to be satisfied for almost every  $t$  in  $[0, T]$ .

Let us first draw a consequence of (6.17).

**PROPOSITION.** If conditions (6.17) are verified for almost every  $t$ , the function  $t \mapsto y$  satisfies for these values of  $t$

$$(6.18) \quad -\dot{y} \in \partial \psi_{(C-c-g) \cap V}(y) ;$$

in other words this function is a solution of the sweeping process by the non empty closed convex moving set  $t \mapsto (C-c(t) - g(t)) \cap V$ .

In fact the second line of (6.17) implies  $-\dot{z} \in U$ , thus

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-  $\dot{z} \in \partial \psi_V(y)$ . Elementary calculation concerning translation in the space  $H$  yields

$$\partial \psi_C(y+c+g) = \partial \psi_{C-c-g}(y) .$$

On the other hand

$$\psi_{(C-c-g) \cap V} = \psi_{C-c-g} + \psi_V ,$$

thus

$$\partial \psi_{C-c-g}(y) + \partial \psi_V(y) \subset \partial \psi_{(C-c-g) \cap V}(y) .$$

Therefore (6.18) follows from the first line of (6.17).

REMARK. As  $y$  and  $\dot{y}$  essentially belong to  $V$ , it is indifferent to understand the subdifferential in (6.18) in the sense of the duality between  $H$  and itself or in the sense of the duality between the Hilbert subspaces  $V$  and itself.

COROLLARY 1. If two solutions of (6.17) agree with the same initial condition  $y(0) = y_0$  they coincide in what concerns the function  $t \mapsto y$ .

As explained in § 5. f, this uniqueness property follows from the multimapping  $\partial \psi_{(C-c-g) \cap V}$  being monotone.

In view of the definition (6.14) of  $y$  this Corollary is equivalent to

COROLLARY 2. If two solutions of the system of conditions (6.7) to (6.10) agree with the same initial condition  $s(0) = s_0$ , these two solutions coincide in what concerns the function  $t \mapsto s$ .



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By the way, (6.18) implies under Assumption 1 that the function  $t \mapsto s$  related to  $y$  by (6.14) verifies, for almost every  $t$ ,

$$(6.19) \quad -\dot{s} \in -\dot{g} + \partial \psi_{C \cap (V + c)}(s) ,$$

an evolution "equation" analogous to that of the sweeping process. An algorithm of time discretization would also be available for the numerical solution of it.

6. d EXISTENCE THEOREM

Let us proceed to the proof of :

PROPOSITION. Under Assumptions 1, 3, 4, whichever is  $y_0$  in  $V \cap (C - c(0) - g(0))$ , whichever is  $z_0$  in  $U$ , there exists at least one pair of functions  $t \mapsto y$  and  $t \mapsto z$ , absolutely continuous from  $[0, T]$  into  $H$ , satisfying (6.17) for almost every  $t$  and the initial conditions  $y(0) = y_0$ ,  $z(0) = z_0$ .

First step. Under the hypotheses made there exists an absolutely continuous function, let us already denote it by  $t \mapsto y$ , satisfying (6.18) for almost every  $t$  and the initial condition  $y(0) = y_0$ . In fact this function is the solution of the sweeping process, for this initial condition, by the moving convex set  $t \mapsto (C - c(t) - g(t)) \cap V$ . The existence theorem of § 5. g apply because  $t \mapsto C - c(t) - g(t)$  is absolutely continuous (see § 5. b about a translating convex set), thus the considered

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intersection is also absolutely continuous, by virtue of Assumptions 3 and 4 and the intersection theorem of § 5. c. Defining  $y$  in this way, one has  $y(t) \in V$  for every  $t$ , thus  $\partial \psi_V(y) = U$ . The additivity of the subdifferentials holds for the functions  $\psi_{C-c-g}$  and  $\psi_V$  since, by Assumption 3,  $V + c + g$  intersects the interior of  $C$  (recall that  $g \in V$ ) so that there exists a point at which both functions are finite and the function  $\psi_{C-c-g}$  is continuous ; then (6.18) implies for almost every  $t$  the existence of at least one element of  $U$ , which will be already denoted as  $\dot{z}(t)$ , such that

$$(6.20) \quad \dot{z}(t) - \dot{y}(t) \in \partial \psi_{C-c-g}(y(t)) = \partial \psi_C(y(t) + c(t) + g(t)).$$

This is the first of conditions (6.17).

Second step. For a value of  $t$  such that (6.20) holds the point  $\dot{z} - \dot{y}$  is a conjugate of the point  $y + c + g$  relatively to the pair of dual functions  $\gamma$ , namely the support function of  $C$ , and  $\psi_C$  (see § 2. e, Example). This may be written as

$$(6.21) \quad \gamma(\dot{z} - \dot{y}) - (\dot{z} - \dot{y} | y + c + g) = 0$$

which implies that for almost every  $t$ , the closed convex set

$$(6.22) \quad \Phi(t) = \{w \in H : \gamma(w) - (w | y + c + g) = 0\} \\ = \{w \in H : \gamma(w) - (w | y + c + g) \leq 0\}$$

possesses a nonempty intersection with the affine manifold  $U - \dot{y}(t)$ . As

$\gamma$  is a numerical function independent of  $t$  and as  $t \mapsto y + c + g$  is a

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continuous mapping from  $[0, T]$  into  $H$  one observes that  $t \mapsto \Phi(t)$  is a measurable multimapping from  $[0, T]$  into  $H$  (the measurability theory of multimappings is due for a part to C. CASTAING ; see his lectures ; see also, for an exposition of some basic facts in the case of a separable space, R.T. ROCKAFELLAR [4]). Such is also the multimapping  $t \mapsto U - \dot{y}(t)$ , as the function  $t \mapsto \dot{y}$  belongs to  $L^1(0, T ; H)$  ; thus the intersection of the two multimappings is measurable too. Since for almost every  $t$  this intersection is nonempty, it possesses a dense collection of measurable selectors. Denote by  $t \mapsto \dot{p}(t)$  one of these selectors ; as  $\dot{p}(t) \in U - \dot{y}(t)$ , by putting  $\dot{z}(t) = \dot{p}(t) + \dot{y}(t)$  one has  $\dot{z}(t) \in U$  and (6.20) holds for almost every  $t$ . If we succeed in proving that  $\dot{p}$ , thus  $\dot{z}$ , belong to  $L^1(0, T ; H)$ , the primitive  $z$  of  $\dot{z}$  adjusted to the initial value  $z(0) = z_0$ , will constitute with the function  $y$  determined above one of the desired solutions of (6.17).

Third step. As  $t \mapsto \dot{p}(t)$  is measurable it just remains to prove that the numerical function  $t \mapsto |\dot{p}(t)|$  is majorized by an element of  $L^1(0, T ; \mathbb{R})$ . By the lemma of § 6. b there exists a strictly positive constant  $\rho$  and a continuous function  $h : [0, T] \rightarrow H$  such that for every  $t$  one has  $h(t) \in V + c(t)$  and the ball with center  $h(t)$  and radius  $\rho$  is contained in  $C$ . This inclusion of convex sets is equivalent to the following inequality between their support functions

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$$(6.23) \quad \forall w \in H : \rho |w| + (h(t)|w) \leq \gamma(w).$$

The definition (6.22) of  $\Phi(t)$  may be transformed by writing

$$(w|y+c+g) = (w|h) + (w+\dot{y}|y+c+g-h) - (\dot{y}|y+c+g-h).$$

Recall that  $\dot{p} + \dot{y} \in U$ , that  $c - h \in V$ , that  $g \in V$ , that  $\dot{y} \in V$ , that  $c \in U$ ; then

$$(\dot{p}|y+c+g) = (\dot{p}|h) - (\dot{y}|y+g-h).$$

Therefore, in view of (6.23),  $\dot{p} \in \Phi(t)$  implies

$$|\dot{p}| \leq \frac{1}{\rho} (\dot{y}|y+g-h) \leq \frac{1}{\rho} |\dot{y}| |y+g-h| \leq \frac{M}{\rho} |\dot{y}|$$

where  $M$  denotes a majorant of the continuous functions  $t \mapsto |y+g-h|$  over the compact interval  $[0, T]$ . As a solution of the sweeping process, the function  $t \mapsto y$  is absolutely continuous, thus the function  $t \mapsto \dot{y}$  belongs to  $L^1(0, T; H)$ ; this completes the proof.

By the definitions of  $y$  and  $z$ , it follows :

COROLLARY. Under Assumptions 1, 2, 3, 4 the evolution problem for the considered elastoplastic system possesses at least one solution ; this solution is unique in what concerns the function  $t \mapsto s$ .

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