



HAL
open science

Explicit and combined estimators for stable distributions parameters

Jacques Lévy Véhel, Anne Philippe, Caroline Robet

► **To cite this version:**

Jacques Lévy Véhel, Anne Philippe, Caroline Robet. Explicit and combined estimators for stable distributions parameters. 2018. hal-01791934v2

HAL Id: hal-01791934

<https://hal.science/hal-01791934v2>

Preprint submitted on 14 Nov 2018 (v2), last revised 12 Mar 2021 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Explicit and combined estimators for stable distributions parameters

Jacques Lévy Véhel¹ Anne Philippe² Caroline Robet ²

¹ INRIA Rennes Bretagne Atlantique & Case Law Analytics,

² Université de Nantes, Laboratoire de Mathématiques Jean Leray, UMR CNRS 6629

Abstract: This article focuses on the estimation of the stability index and scale parameter of stable random variables. While there is a sizable literature on this topic, no precise theoretical results seem available. We study an estimator based on log-moments, which always exist for such random variables. The main advantage of this estimator is that it has a simple closed form expression. This allows us to prove an almost sure convergence result as well as a central limit theorem. We show how to improve the accuracy of this estimator by combining it with previously defined ones. The closed form also enables us to consider the case of non identically distributed data, and we show that our results still hold provided deviations from stationarity are "small". Using a centro-symmetrization, we expand the previous estimators to skewed stable variables and we construct a test to check the skewness of the data. As applications, we show numerically that the stability index of multistable Lévy motion may be estimated accurately and consider a financial log, namely the S&P 500, where we find that the stability index evolves in time in a way that reflects with major financial events.

Keywords: averaging estimates; misspecified model; moment estimate; Monte Carlo approximation; stable distribution

1 Introduction

The class of α -stable distributions is ubiquitous in probability: such distributions appear as the limit of normalized sums of independent and identically distributed random variables. Their probability densities exist and are continuous but they are not known in closed form except for Gaussian distributions, Cauchy distributions, Lévy distributions and constants. Non-Gaussian stable distributions are a model of choice for real world phenomena exhibiting jumps. Indeed, for $\alpha < 2$, their density exhibit "heavy tails", resulting in a power-law decay of the probability of extreme events. They have been used extensively in recent years for modeling in domains such as biomedicine [13], geophysics [18], economy [11], finance [11], Internet traffic [1] and more. Non-Gaussian stable distributions are characterized by four parameters: a scale parameter usually denoted σ , (that

is proportional to variance in the Gaussian case), a stability parameter, denoted α , that governs the heaviness of the tail and that ranges in $(0, 2)$, a location parameter μ similar to the mean in the case of Gaussian distributions, and a skewness parameter β ranging in $[-1, 1]$. In order to use stable distributions for real world phenomena modeling, one needs to estimate these parameters. A number of estimators are of common use, such as the ones proposed by Fama-Roll [3], Mc Culloch [12] and Koutrouvelis [4, 5]. A difficulty with these estimators is that they do not possess a simple closed form expression. As a consequence, and to the best of our knowledge, no theoretical results are known about them, such as almost sure convergence and central limit theorems. Their asymptotic distributions as well as asymptotic variances are thus only accessible through numerical simulations. Another drawback of not having explicit and simple closed forms is that it is difficult to assess theoretically their performance in situations that slightly depart from the classical assumptions of identical and independent random samples. This is nevertheless desirable when one wishes to deal with real world data, which will often not verify these ideal hypotheses.

Our main aim in this work is to investigate the theoretical properties of a simple moment estimator. Since non-Gaussian stable random variables do not possess a finite variance, and, in some cases, a well defined mean, we shall consider in fact an estimator based on log-moments. This idea is not new, as it has long been remarked that log-moment always exist for stable random variables and are convenient to work with. For instance [10] considers the same estimator as the one we study and applies it to blind channel identification. Owing to its simple expression, we are able to prove almost sure convergence and a central limit theorem both in an identically and independent framework and in a case of slight deviation from stationarity. We compare the performance of our estimator with the Koutrouvelis regression method [4, 5]. The results depends on the value of α and on the size of the sample. We then combine these two estimators using a technique recently developed in [7] to enhance their performance, especially in the case of small samples. As applications, we show numerical experiments both on synthetic data (symmetric Lévy multistable motion) and on a financial log (S&P 500), which confirm our theoretical results that the estimator is able to track smooth enough variations of the stability index in time.

A stable distribution is characterized by four parameters. We write $X \sim S_\alpha(\sigma, \beta, \mu)$ to indicate that X has a stable distribution with the stability index $\alpha \in (0, 2]$, scale parameter $\sigma \in \mathbb{R}^+$, skewness $\beta \in [-1, 1]$ and location parameter $\mu \in \mathbb{R}$ (see [14]).

There are several parameterizations of stable distributions, each of which having advantages and drawbacks. The following one, with characteristic function ϕ , is probably the most popular:

$$\phi(t) = \begin{cases} \exp\left(-\sigma^\alpha |t|^\alpha \left(1 - i\beta \text{sign}(t) \tan\left(\frac{\pi\alpha}{2}\right)\right) + i\mu t\right), & \text{if } \alpha \neq 1, \\ \exp\left(-\sigma |t| \left(1 + i\beta \text{sign}(t) \frac{2 \log |t|}{\pi}\right) + i\mu t\right), & \text{if } \alpha = 1 \end{cases} \quad (1.1)$$

where $\text{sign}(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0 \end{cases}$

For a stable distribution, the density function is not known in closed form except in few cases (Gaussian distribution, Cauchy distribution, ...). This makes it difficult to use classical estimation methods based on the density function, such as maximum likelihood or Bayesian statistics: although the density may be estimated numerically, for instance through Fourier inversion, this procedure entails approximation errors that cannot be easily assessed.

Another difficulty is that, except in the Gaussian case, stable random variables have infinite moments of order at least the stability index. More precisely, for $X \sim S_\alpha(\sigma, \beta, \mu)$ with $0 < \alpha < 2$, $E[|X|^p] < \infty$ if and only if $0 < p < \alpha$ (see Prop 1.2.16 in [14]). This property implies that one cannot use a moment method for estimation. We study in the next section a simple log-moment estimator that allows one to make explicit computations.

In Section 2, we study estimators of α and σ for symmetric (that is, when $\mu = \beta = 0$) stable random variables: log-moments estimators and a combined estimator build with the Koutrouvelis ones. In Section 3, we expand the log-moment, Koutrouvelis and combined estimators to the skewed case studying two ways for the adaptation of the log-moments estimator. The properties of log-moment estimators also allow us to propose a method for testing the skewness of the data. In Section 4, we investigate the case of non-identically distributed where we prove robustness of the log-moments estimators under some conditions for the perturbations. In Section 5, we perform numerical experiments involving multistable Lévy motion and real data with the study of a financial index.

2 Estimation methods

We build an estimator for the parameters α and σ using a combined estimator whose general procedure of construction is described in [7]. In our special case, $\theta = (\alpha, \sigma)^\top$ are the parameters to estimate and we have access to p estimators for α and q estimators for σ .

We consider averaging estimators of θ of the form

$$\hat{\theta}_\lambda = \lambda^\top \begin{pmatrix} \hat{\alpha}_1 \\ \vdots \\ \hat{\alpha}_p \\ \hat{\sigma}_1 \\ \vdots \\ \hat{\sigma}_q \end{pmatrix}, \quad \lambda \in \Lambda, \quad (2.1)$$

where λ^\top denotes the transpose of λ and Λ is a given subset of $\mathcal{M}_{(p+q) \times 2}(\mathbb{R})$.

We measure the performance of $\widehat{\theta}_\lambda$ for $\lambda \in \Lambda$ using the mean square error (MSE). We select λ^* satisfying the conditions:

$$\lambda^* = \operatorname{argmin}_{\lambda \in \Lambda} E[\|\widehat{\theta}_\lambda - \theta\|^2].$$

In what follows, we present a classical estimator, namely the Koutrouvelis one ([4], [5]). The choice of this particular estimator is motivated by the results reported in [17] showing that it performs usually better than other methods such as the Fama-Roll and Mc Culloch ones. We want to combine the Koutrouvelis estimator described in Section 2.2 with the log-moments estimator (Section 2.1) to get better performance especially when $\alpha < 1$.

2.1 Symmetric case for log-moments

For a symmetric stable distribution closed form expressions are available for absolute log-moments [8], which allow one to derive expressions for estimating α and σ . First, note the following property:

Proposition 2.1. *Let $Z \sim S_\alpha(1, 0, 0)$, then*

$$E[|\log |Z||^p] < \infty \quad \text{for all } p > 0. \quad (2.2)$$

Proof. See Appendix. □

These expectations may be computed explicitly by remarking that:

$$E[(\log |Z|)^p] = \left. \frac{d^p E[|Z|^t]}{dt^p} \right|_{t=0}, \quad (2.3)$$

and by using the following result:

Proposition 2.2. *Let $Z \sim S_\alpha(1, 0, 0)$. Then, for all $0 < t < \min(\alpha, 1)$,*

$$E[|Z|^t] = \frac{\Gamma(1 - t/\alpha)}{\Gamma(1 - t) \cos(\pi t/2)}. \quad (2.4)$$

Proof. See Appendix. □

We deduce that $E[\log |Z|] = (\frac{1}{\alpha} - 1)\gamma$ and $\operatorname{Var}(\log |Z|) = \frac{\pi^2}{6\alpha^2} + \frac{\pi^2}{12}$, where γ is Euler constant ($\gamma = 0.57721\dots$).

Theorem 2.3. *Let (X_1, \dots, X_n) be a sequence of independent and identically distributed (iid) standard symmetric stable random variables $S_\alpha(1, 0, 0)$. Define*

$$\widehat{\alpha}_n(X_1, \dots, X_n) = \frac{\gamma}{\gamma + \frac{1}{n} \sum_{i=1}^n \log |X_i|}.$$

Then, $\widehat{\alpha}_n \xrightarrow{a.s.} \alpha$ when $n \rightarrow +\infty$. Moreover, with $f(x) = \frac{\pi^2}{6x^2} + \frac{\pi^2}{12}$,

$$\frac{\sqrt{n}(\widehat{\alpha}_n - \alpha)\gamma}{\widehat{\alpha}_n^2 \sqrt{f(\widehat{\alpha}_n)}} \xrightarrow{d} \mathcal{N}(0, 1). \quad (2.5)$$

Proof. The proof of this result may be found in the appendix. \square

In general, σ is unknown and we must use a joint estimation of both parameters. Let $W \sim S_\alpha(\sigma, 0, 0)$, then $\frac{W}{\sigma} \sim S_\alpha(1, 0, 0)$ and we deduce the log-moment of W by using the log-moment of Z . We get $E[\log |W|] = \left(\frac{1}{\alpha} - 1\right) \gamma + \log \sigma$ and $\text{Var}(\log |W|) = \frac{\pi^2}{6\alpha^2} + \frac{\pi^2}{12}$.

Theorem 2.4. *Let (X_1, \dots, X_n) be a sequence of independent and identically distributed (iid) symmetric stable random variables $S_\alpha(\sigma, 0, 0)$. Define*

$$\hat{\alpha}_{LOG}^{(n)}(X_1, \dots, X_n) = \left(\max \left(\frac{6}{\pi^2(n-1)} \sum_{i=1}^n \left[\log |X_i| - \frac{1}{n} \sum_{k=1}^n \log |X_k| \right]^2 - \frac{1}{2}, \frac{1}{4} \right) \right)^{-1/2}$$

and

$$\hat{\sigma}_{LOG}^{(n)}(X_1, \dots, X_n) = \exp \left(\frac{1}{n} \sum_{i=1}^n \log |X_i| - \left(\frac{1}{\hat{\alpha}_{LOG}^{(n)}} - 1 \right) \gamma \right).$$

Then,

$$(\hat{\alpha}_{LOG}^{(n)}, \hat{\sigma}_{LOG}^{(n)}) \xrightarrow{a.s.} (\alpha, \sigma) \text{ when } n \rightarrow +\infty. \quad (2.6)$$

Moreover,

$$\sqrt{n} \left(\begin{pmatrix} \hat{\alpha}_{LOG}^{(n)} \\ \hat{\sigma}_{LOG}^{(n)} \end{pmatrix} - \begin{pmatrix} \alpha \\ \sigma \end{pmatrix} \right) \xrightarrow{d} \mathcal{N}(0, F_{\alpha,\sigma} G_{\alpha,\sigma} \Sigma_{\alpha,\sigma} {}^t G_{\alpha,\sigma} {}^t F_{\alpha,\sigma}), \quad (2.7)$$

where

$$F_{\alpha,\sigma} = \begin{pmatrix} \sigma & \gamma\sigma/\alpha^2 \\ 0 & 1 \end{pmatrix}, \quad (2.8)$$

$$G_{\alpha,\sigma} = \begin{pmatrix} 1 & 0 \\ 6 \left(\left(\frac{1}{\alpha} - 1 \right) \gamma + \log \sigma \right) \alpha^3/\pi^2 & -3\alpha^3/\pi^2 \end{pmatrix}, \quad (2.9)$$

$$\Sigma_{\alpha,\sigma} = \begin{pmatrix} \text{Var}(\log |X_1|) & \text{Cov}(\log |X_1|, (\log |X_1|)^2) \\ \text{Cov}(\log |X_1|, (\log |X_1|)^2) & \text{Var}((\log |X_1|)^2) \end{pmatrix}. \quad (2.10)$$

Proof. The proof of this result may be found in the appendix. \square

2.2 Numerical performance of the individual estimators

The Koutrouvelis estimator (see [4] and [5]) is based on exploiting the explicit expression of the iterated logarithm of the characteristic function ϕ . In the symmetric case, it takes the particularly simple form

$$\log(\log(|\phi(t)|^2)) = \log(2\sigma^\alpha) + \alpha \log |t|. \quad (2.11)$$

The empirical characteristic function given by $\hat{\phi}_n(t) = \frac{1}{n} \sum_{j=1}^n e^{itX_j}$ based on i.i.d observations (X_j) is a consistent estimator of ϕ . We estimate these parameters by regressing $y = \log(\log(|\hat{\phi}_n(t)|^2))$ on $w = \log |t|$ in the model

$$y_k = m + \alpha w_k + \epsilon_k \quad (2.12)$$

where $m = \log(2\sigma^\alpha)$, $t_k = \frac{\pi k}{25}$ for $k \in \{1, \dots, K\}$ with K depending on the parameter α and on the sample size, and ϵ_k denotes an error term. In our simulations, we use an easier version of the Koutrouvelis regression method which is more adapted for the symmetric case (see [17]). We describe the algorithm:

- Define an admissible error ($tol = 0.05$ in the simulation) and the maximum number of iterations ($iter_{max} = 10$).
- A regression applied to the Mc Culloch quantile method [12] provides initial estimates $\hat{\alpha}^{(n)}$ and $\hat{\sigma}^{(n)}$, where n denotes the length of the observed sample.
- While the number of iterations is less than $iter_{max}$ and $|\hat{s} - 1| > tol$ (starting from the second iteration):
 - find the number K of points in the regression depending on $\hat{\alpha}^{(n)}$ as in the classical Koutrouvelis regression,
 - define $w = (w_k)_{k \in \{1 \dots K\}}$ and $y = (y_k)_{k \in \{1 \dots K\}}$ by $w_k = \log |\pi k / 25|$,

$$y_k = \log\left(-\log\left(\left|\hat{\phi}_n\left(\frac{\pi k}{25\hat{\sigma}^{(n)}}\right)\right|^2\right)\right),$$

- compute the new $\hat{\alpha}^{(n)}$ given by

$$\hat{\alpha}^{(n)} = \min\left(\frac{\sum_{k=1}^K (w_k - \text{mean}(w))(y_k - \text{mean}(y))}{\sum_{k=1}^K (w_k - \text{mean}(w))^2}, 2\right)$$

- set $\hat{s} = \exp\left(\frac{\text{mean}(y - \hat{\alpha}^{(n)}w) - \log(2)}{\hat{\alpha}^{(n)}}\right)$
- set $\hat{\sigma}^{(n)} = \hat{\sigma}^{(n)}\hat{s}$

This modified version of Koutrouvelis gives performances (in terms of mean squared errors) similar to the original. However, it is much faster because this version does not necessitate the estimation of the parameters β and μ , which requires the numerical inversion of matrices of size $n \times n$.

In general, the Koutrouvelis regression method has a smaller mean squared error than other classical estimation methods based on empirical quantiles (Fama-Roll method [3], Mc Culloch's method [12]). A comparative of the performances of these classical methods is performed in [17]. For each pair of values (α, σ) , r independent samples of size n of iid stable random variables were generated. The empirical mean squared error of the sampling distribution of α and σ is given by

$$MSE_\alpha = \frac{1}{r} \sum_{i=1}^r (\hat{\alpha}_i - \alpha)^2, \quad MSE_\sigma = \frac{1}{r} \sum_{i=1}^r (\hat{\sigma}_i - \sigma)^2,$$

where $\hat{\alpha}$ (resp $\hat{\sigma}$) is an estimator of α (resp σ).

In the sequel, we use the abbreviation "KOUT" and "LOG" to refer respectively to the Koutrouvelis and log-moment estimator. For each α , the behaviors of $\hat{\alpha}_{KOUT}$ and $\hat{\alpha}_{LOG}$ are similar for all value of σ (Table 1) whereas, for each value of σ , $\hat{\sigma}_{KOUT}$ and $\hat{\sigma}_{LOG}$ improve when α is increasing (Table 2). Besides, when α is fixed, $\hat{\sigma}_{KOUT}$ and $\hat{\sigma}_{LOG}$ have the same behavior for all σ . For this reason, only the case where σ equals 1 is represented in Table 2.

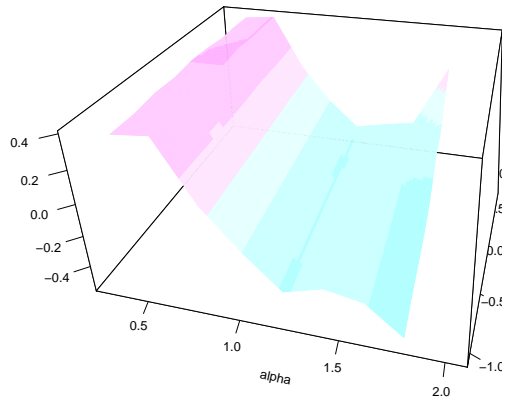
		$\alpha=0.2$	$\alpha=0.6$	$\alpha=1$	$\alpha=1.4$	$\alpha=1.8$
$\sigma=10$	LOG	$9.06 \cdot 10^{-5}$	$9.06 \cdot 10^{-4}$	$4.67 \cdot 10^{-3}$	$1.97 \cdot 10^{-2}$	$3.10 \cdot 10^{-2}$
	KOUT	$4.70 \cdot 10^{-4}$	$2.35 \cdot 10^{-3}$	$3.75 \cdot 10^{-3}$	$8.20 \cdot 10^{-3}$	$4.17 \cdot 10^{-3}$
$\sigma=1$	LOG	$8.07 \cdot 10^{-5}$	$1.06 \cdot 10^{-3}$	$4.47 \cdot 10^{-3}$	$2.20 \cdot 10^{-2}$	$3.19 \cdot 10^{-2}$
	KOUT	$4.27 \cdot 10^{-4}$	$2.11 \cdot 10^{-3}$	$3.91 \cdot 10^{-3}$	$7.58 \cdot 10^{-3}$	$4.28 \cdot 10^{-3}$
$\sigma=0.1$	LOG	$8.93 \cdot 10^{-5}$	$9.59 \cdot 10^{-4}$	$4.43 \cdot 10^{-3}$	$2.20 \cdot 10^{-2}$	$2.97 \cdot 10^{-2}$
	KOUT	$4.66 \cdot 10^{-4}$	$1.96 \cdot 10^{-3}$	$4.22 \cdot 10^{-3}$	$7.58 \cdot 10^{-3}$	$4.29 \cdot 10^{-3}$

Table 1: Mean squared error for $\hat{\alpha}_{LOG}$ and $\hat{\alpha}_{KOUT}$ ($r = 500$ and $n = 500$).

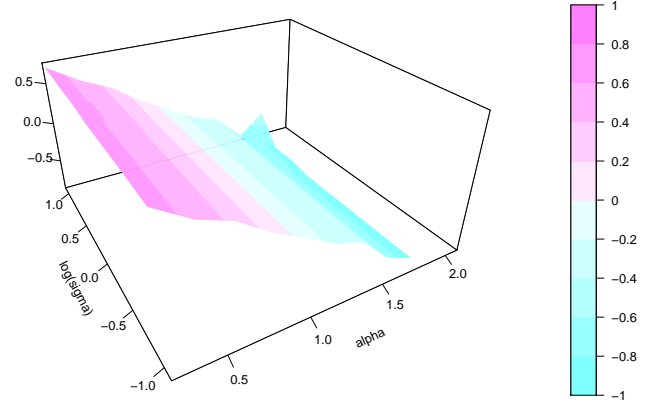
		$\alpha=0.2$	$\alpha=0.6$	$\alpha=1$	$\alpha=1.4$	$\alpha=1.8$
$\sigma=10$	LOG	7.10	$9.86 \cdot 10^{-1}$	$6.08 \cdot 10^{-1}$	$6.07 \cdot 10^{-1}$	$5.43 \cdot 10^{-1}$
	KOUT	11.5	$9.92 \cdot 10^{-1}$	$4.82 \cdot 10^{-1}$	$3.56 \cdot 10^{-1}$	$1.72 \cdot 10^{-1}$
$\sigma=1$	LOG	$8.20 \cdot 10^{-2}$	$9.45 \cdot 10^{-3}$	$6.52 \cdot 10^{-3}$	$6.98 \cdot 10^{-3}$	$5.77 \cdot 10^{-3}$
	KOUT	$1.18 \cdot 10^{-1}$	$9.82 \cdot 10^{-3}$	$4.44 \cdot 10^{-3}$	$3.83 \cdot 10^{-3}$	$1.56 \cdot 10^{-3}$
$\sigma=0.1$	LOG	$6.73 \cdot 10^{-4}$	$8.77 \cdot 10^{-5}$	$6.71 \cdot 10^{-5}$	$6.54 \cdot 10^{-5}$	$5.89 \cdot 10^{-5}$
	KOUT	$8.63 \cdot 10^{-4}$	$8.60 \cdot 10^{-5}$	$4.75 \cdot 10^{-5}$	$3.35 \cdot 10^{-5}$	$1.82 \cdot 10^{-5}$

Table 2: Mean squared error for $\hat{\sigma}_{LOG}$ and $\hat{\sigma}_{KOUT}$ ($r = 500$ and $n = 500$).

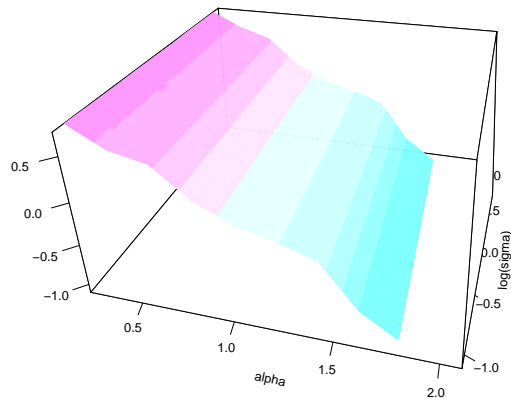
With a simulation study, we compare the empirical mean squared errors of α and σ for the methods introduced earlier. We represent the logarithmic ratio of mean squared errors on α and σ for the log-moment and the regression methods and several samples of size $r \times n$. For each sample of $r \times n$ observations, α and σ are fixed ($\alpha \in \{0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8\}$ and $\log(\sigma) \in \{-1, -0.75, -0.5, -0.25, 0, 0.25, 0.5, 0.75, 1\}$) (see Figure 1). Tables 1-2 and Figures 1-2 show that the log-moment estimator performs better than the Koutrouvelis one when $\alpha < 1$, while the converse is true for $\alpha > 1$, with the differences in performance increasing for extreme values of α .



(a)

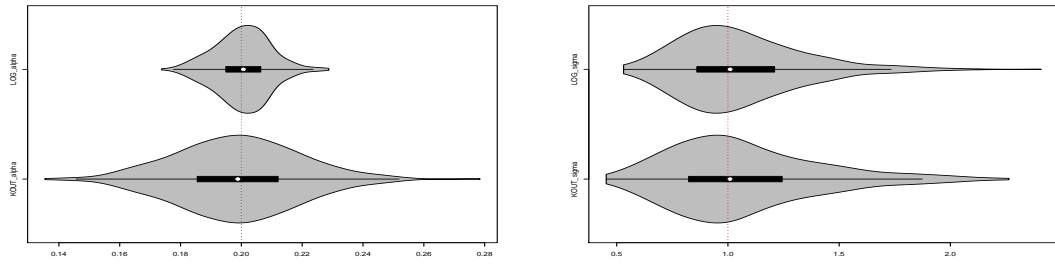


(b)

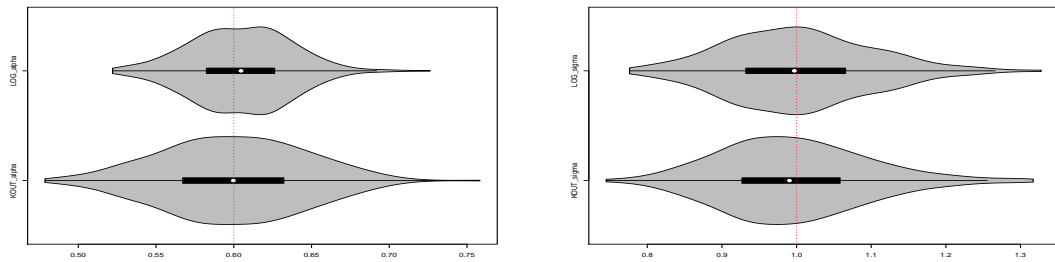


(c)

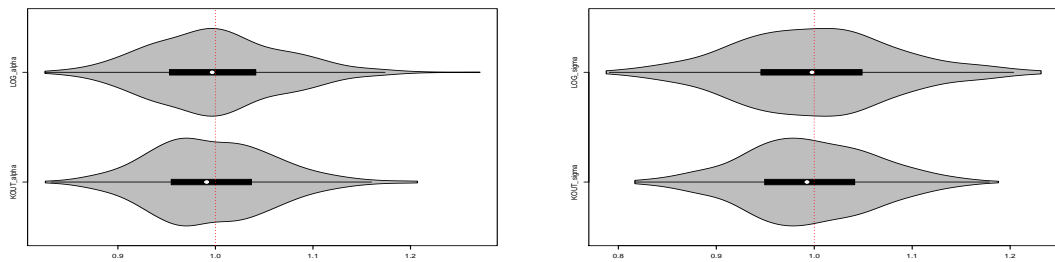
Figure 1: Log of the ratio between the α mean squared error for Koutrouvelis estimator and log estimator $r = 1000$ and $n = 100$ in Figure (a), $n = 500$ in Figure (b), $n = 2000$ in Figure (c).



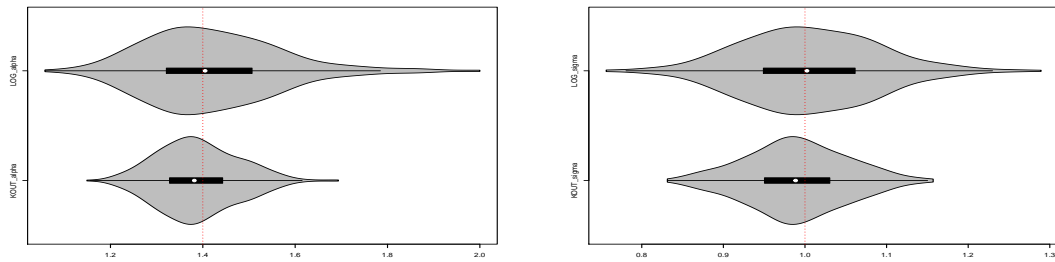
$\alpha=0.2, \sigma=1$



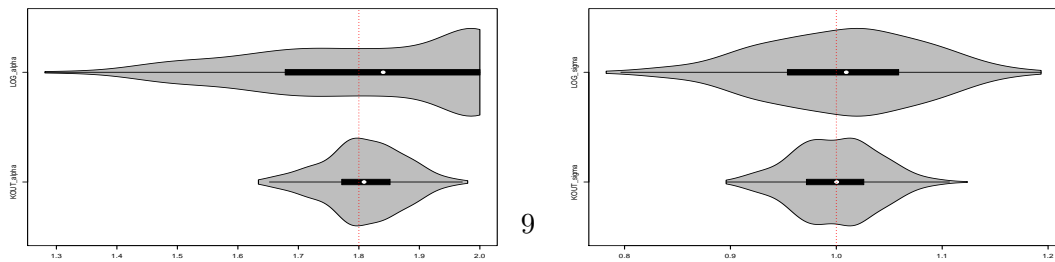
$\alpha=0.6, \sigma=1$



$\alpha=1, \sigma=1$



$\alpha=1.4, \sigma=1$



$\alpha=1.8, \sigma=1$

Figure 2: Empirical density functions for Log-moment (LOG) and Koutrouvelis regression (KOUT) estimators of α in the first column and σ in the second column for $r = 500$ and $n = 500$.

2.3 Combined estimator

As seen previously, the Koutrouvelis regression estimator is preferred when $\alpha > 1$ whereas the log-estimator becomes significantly better than the Koutrouvelis estimator, as $\alpha (\leq 1)$ decreases. We want to construct an estimator of α which will be at least as good as the best estimator, for each α , for small samples ($n \approx 100$). To do this, we use a combined estimator whose general procedure of construction is described in [7]. In our special case, $\theta = (\alpha, \sigma)^\top$ are the parameters to estimate and we have access to the estimators $\hat{\alpha}_{KOUT}$, $\hat{\alpha}_{LOG}$ and $\hat{\sigma}_{KOUT}$ ¹.

We consider averaging estimators of θ of the form

$$\hat{\theta}_\lambda = \lambda^\top \begin{pmatrix} \hat{\alpha}_{KOUT} \\ \hat{\alpha}_{LOG} \\ \hat{\sigma}_{KOUT} \end{pmatrix}, \quad \lambda \in \Lambda, \quad (2.13)$$

where λ^\top denotes the transpose of λ and $\Lambda \subset \mathcal{M}_{3 \times 2}(\mathbb{R})$.

A convenient way to measure the performance of $\hat{\theta}_\lambda$ is to compare it to $\hat{\theta}^*$, defined as the best linear combination $\hat{\theta}_\lambda$ obtained for a non-random vector $\lambda \in \Lambda$. Specifically, $\hat{\theta}^*$ is the linear combination $\lambda^{*\top} \begin{pmatrix} \hat{\alpha}_{KOUT} \\ \hat{\alpha}_{LOG} \\ \hat{\sigma}_{KOUT} \end{pmatrix}$ minimizing the mean square error (MSE),

i.e. $\lambda^* = \underset{\lambda \in \Lambda}{\operatorname{argmin}} E[\|\hat{\theta}_\lambda - \theta\|^2]$.

Clearly, the larger the set Λ is, the better it will be. However, choosing the whole space $\Lambda = \mathbb{R}^{3 \times 2}$ is generally not exploitable. We must impose some conditions on the set Λ in order to have an explicit form for λ^* .

Define

$$J = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and consider the maximal constraint set

$$\Lambda_{\max} = \{\lambda \in \mathbb{R}^{3 \times 2} / \lambda^\top J = I_2\} = \left\{ \begin{pmatrix} a & b \\ 1-a & -b \\ 0 & 1 \end{pmatrix} / (a, b) \in \mathbb{R}^2 \right\}$$

with I_2 the identity matrix. The mean squared error $E[\|\hat{\theta}_\lambda - \theta\|^2]$ is minimized on the set Λ_{\max} for a unique solution

$$\lambda^* = \Sigma^{-1} J (J^\top \Sigma^{-1} J)^{-1}, \quad (2.14)$$

where Σ is the Gram matrix

$$\Sigma = E \left(\begin{pmatrix} \hat{\alpha}_{KOUT} - \alpha \\ \hat{\alpha}_{LOG} - \alpha \\ \hat{\sigma}_{KOUT} - \sigma \end{pmatrix} \begin{pmatrix} \hat{\alpha}_{KOUT} - \alpha \\ \hat{\alpha}_{LOG} - \alpha \\ \hat{\sigma}_{KOUT} - \sigma \end{pmatrix}^\top \right). \quad (2.15)$$

¹Since $\hat{\sigma}_{KOUT}$ and $\hat{\sigma}_{LOG}$ have similar performances, we use only one in the combination.

Since the matrix Σ is unknown, the averaging estimator $\hat{\theta}_{\max}$ is obtained by replacing Σ by its estimation $\hat{\Sigma}$:

$$\begin{cases} \hat{\lambda}_{\max} = \hat{\Sigma}^{-1} J (J^{\top} \hat{\Sigma}^{-1} J)^{-1} \\ \hat{\theta}_{\max} = \hat{\lambda}_{\max}^{\top} \begin{pmatrix} \hat{\alpha}_{KOUT} \\ \hat{\alpha}_{LOG} \\ \hat{\sigma}_{KOUT} \end{pmatrix} \end{cases} \quad (2.16)$$

A way to estimate Σ is to use Monte Carlo simulations. Let us compute a first estimation of the parameters by

$$\begin{cases} \hat{\alpha}_0 = \frac{\hat{\alpha}_{KOUT} + \hat{\alpha}_{LOG}}{2} \\ \hat{\sigma}_0 = \hat{\sigma}_{KOUT} \end{cases} \quad (2.17)$$

We simulate B samples of size n of a symmetric stable distribution with parameters $\hat{\alpha}_0$ and $\hat{\sigma}_0$. Then, the three estimators are computed, which gives $\hat{\alpha}_{KOUT}^{(b)}$, $\hat{\alpha}_{LOG}^{(b)}$ and $\hat{\sigma}_{KOUT}^{(b)}$ for $b = 1 \dots B$, and each entry of Σ is estimated by its empirical counterpart. Experiments show that errors entailed by the estimation of Σ are negligible compared to the advantage of having several estimators for small samples. Note that similar estimators could be built by combining more than 2 estimators for α . For example, it would be possible to add the Mc Culloch quantile estimator. Thus would increase the size of the covariance matrix whose estimation will be worse and entail the risk of constructing a combined estimator never better than each individual ones. The weight for the log estimator in the combination is represented in Figure 3. We represent in Table 3 the mean squared errors for several values of α . For each value, we remark that the combination between Koutrouvelis and log estimators is always better than each estimator separately. This is confirmed by the plots in Figure 4 comparing the empirical distributions of each estimator.

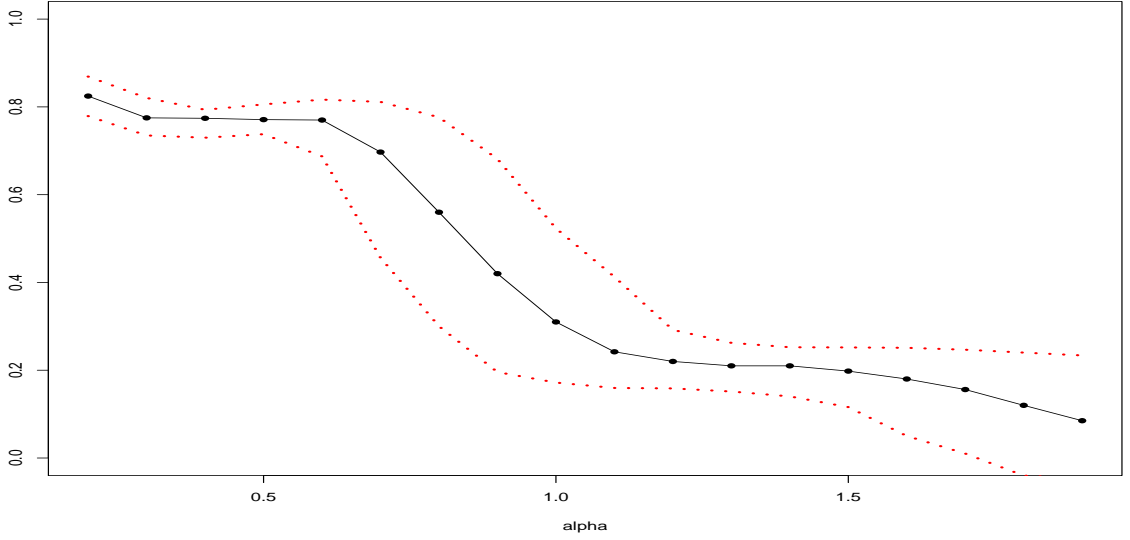


Figure 3: Weight (average) for the LOG-moment estimator $\hat{\alpha}_{LOG}$ in the combined estimator depending on α . The upper (resp. lower) bound of the interval correspond to the 95% quantile (resp. 5%) for $r = 500$ replications of the combination with $n = 100$ and $B = 1000$.

α	0.2	0.3	0.4	0.5	0.6	0.7
KOUT	$2.4 \cdot 10^{-3}$	$3.1 \cdot 10^{-3}$	$5.0 \cdot 10^{-3}$	$8.3 \cdot 10^{-3}$	$1.3 \cdot 10^{-2}$	$1.5 \cdot 10^{-2}$
LOG	$5.0 \cdot 10^{-4}$	$9.1 \cdot 10^{-4}$	$1.9 \cdot 10^{-3}$	$3.4 \cdot 10^{-3}$	$5.6 \cdot 10^{-3}$	$7.6 \cdot 10^{-3}$
COMB	$4.0 \cdot 10^{-6}$	$7.2 \cdot 10^{-4}$	$1.5 \cdot 10^{-3}$	$2.9 \cdot 10^{-3}$	$5.2 \cdot 10^{-3}$	$6.8 \cdot 10^{-3}$
α	0.8	0.9	1	1.1	1.2	1.3
KOUT	$1.4 \cdot 10^{-2}$	$1.5 \cdot 10^{-2}$	$1.8 \cdot 10^{-2}$	$1.9 \cdot 10^{-2}$	$2.8 \cdot 10^{-2}$	$3.4 \cdot 10^{-2}$
LOG	$1.2 \cdot 10^{-3}$	$2.0 \cdot 10^{-2}$	$3.3 \cdot 10^{-2}$	$4.6 \cdot 10^{-2}$	$5.9 \cdot 10^{-2}$	$7.8 \cdot 10^{-2}$
COMB	$8.3 \cdot 10^{-3}$	$1.1 \cdot 10^{-2}$	$1.5 \cdot 10^{-2}$	$1.8 \cdot 10^{-2}$	$2.6 \cdot 10^{-2}$	$3.1 \cdot 10^{-2}$
α	1.4	1.5	1.6	1.7	1.8	1.9
KOUT	$4.0 \cdot 10^{-2}$	$4.1 \cdot 10^{-2}$	$4.1 \cdot 10^{-2}$	$2.8 \cdot 10^{-2}$	$1.9 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$
LOG	$8.5 \cdot 10^{-2}$	$8.8 \cdot 10^{-2}$	$9.9 \cdot 10^{-2}$	$8.7 \cdot 10^{-2}$	$7.4 \cdot 10^{-2}$	$7.6 \cdot 10^{-2}$
COMB	$3.3 \cdot 10^{-2}$	$3.4 \cdot 10^{-2}$	$3.5 \cdot 10^{-2}$	$2.5 \cdot 10^{-2}$	$1.9 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$

Table 3: Mean squared errors for Koutrouvelis regression (KOUT), Log-moment (LOG) and the combined (COMB) estimators of α for $r = 500$, $n = 100$, $B = 1000$ and $\sigma = 1$.

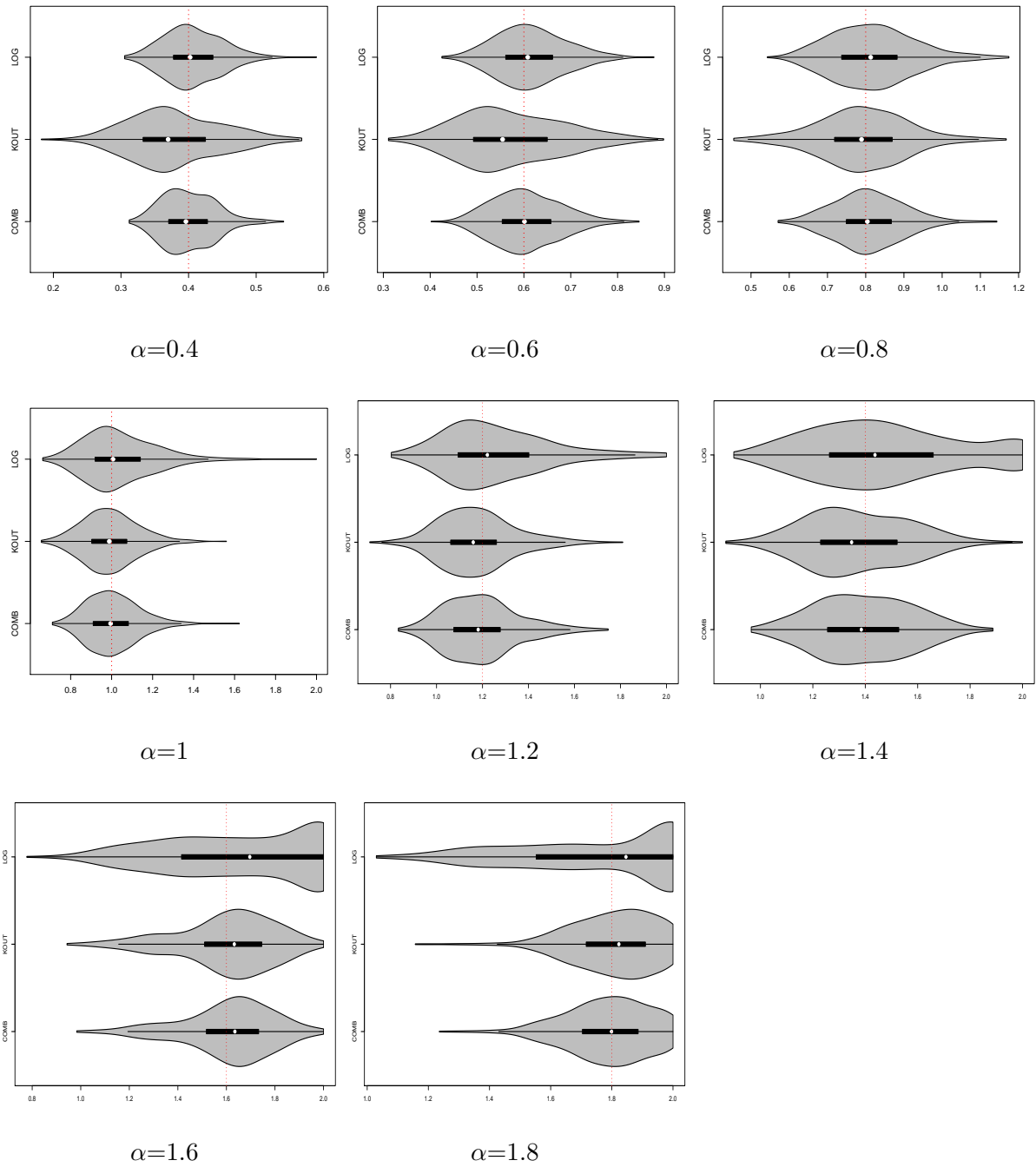


Figure 4: Empirical density functions for Log-moment (LOG), Koutrouvelis regression (KOUT) and the combined (COMB) estimators of α for $r = 500$, $n = 100$, $B = 1000$ and $\sigma = 1$.

3 Skewed stable distributions

3.1 Adaptation of estimators for the skewed case

In the case $X \sim S_\alpha(\sigma, \beta, 0)$, we have

$$E[\log |X|] = \left(\frac{1}{\alpha} - 1\right) \gamma + \log \sigma - \frac{\log |\cos \theta|}{\alpha}$$

$$E[(\log |X| - E[\log |X|])^2] = \text{Var}(\log |X|) = \frac{\pi^2}{6\alpha^2} + \frac{\pi^2}{12} - \frac{\theta^2}{\alpha^2}$$

where γ is Euler constant and $\theta = \arctan(\beta \tan \frac{\alpha\pi}{2})$. (see [6] prop. 4).

Let (X_1, \dots, X_{2n}) be a sequence of $2n$ independent and identically distributed (iid) stable random variables $S_\alpha(\sigma, \beta, 0)$. We use the centro-symmetrization introduced in [6] to the observed data to obtain n iid symmetric stable random variables $S_\alpha(2\sigma, 0, 0)$ $(X_{2k} - X_{2k-1})_{k \in \{1, \dots, n\}}$. Then, we estimate α by taking then $\hat{\alpha}_{LOG}^{(n)}(X_2 - X_1, \dots, X_{2n} - X_{2n-1})$, where $\hat{\alpha}_{LOG}^{(n)}$ is introduced in Theorem 2.4. Note that we can also estimate β using $\text{Var}[\log |X|]$.

Another way to estimate α is to use the $(2n - 1)$ random variables $(X_k - X_{k-1})_{k \in \{2, \dots, 2n\}}$. In this case, the drawback is the loss of the independence but we preserve the same sample size. The mean squared errors of the first method do not depend on β since $(X_{2m} - X_{2m-1})$ are symmetric and independent. Numerically, we observe that the properties of second estimate also does not depend on β . Moreover numerical results given Table 4 show that the two estimates have similar performance for $\alpha \leq 1$.

$2n$		$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
100	n obs. i.i.d	$1.01 \cdot 10^{-3}$	$4.44 \cdot 10^{-3}$	$1.27 \cdot 10^{-2}$	$3.30 \cdot 10^{-2}$	$10.7 \cdot 10^{-2}$
	$2n - 1$ obs. dep.	$0.970 \cdot 10^{-3}$	$4.14 \cdot 10^{-3}$	$1.08 \cdot 10^{-2}$	$2.44 \cdot 10^{-2}$	$6.08 \cdot 10^{-2}$
500	n obs. i.i.d	$1.80 \cdot 10^{-4}$	$7.90 \cdot 10^{-4}$	$2.10 \cdot 10^{-3}$	$4.68 \cdot 10^{-3}$	$1.01 \cdot 10^{-2}$
	$2n - 1$ obs. dep.	$1.75 \cdot 10^{-4}$	$7.58 \cdot 10^{-4}$	$1.86 \cdot 10^{-3}$	$3.81 \cdot 10^{-3}$	$7.33 \cdot 10^{-3}$
1000	n obs. i.i.d	$9.12 \cdot 10^{-5}$	$3.86 \cdot 10^{-4}$	$9.90 \cdot 10^{-4}$	$2.23 \cdot 10^{-3}$	$4.79 \cdot 10^{-3}$
	$2n - 1$ obs. dep.	$8.94 \cdot 10^{-5}$	$3.69 \cdot 10^{-4}$	$8.88 \cdot 10^{-4}$	$1.84 \cdot 10^{-3}$	$3.48 \cdot 10^{-3}$

Table 4: Mean squared errors for α using log-moments for $2n$ random variables i.i.d. $S_\alpha(1, \beta, 0)$.

Koutrouvelis method does not vary with skewed distributions because the modulus of the characteristic function depends only on α and σ (see performances in Tables 1 and 3). We combine the log moment after symmetrization (named after LOG sym.) with the Koutrouvelis estimator in the same way that with symmetric variables (see Section 2.3) to obtain a new estimator whose numerical performances are reported on Table 5.

α	0.2	0.4	0.6	0.8	1
COMB	$2.39 \cdot 10^{-3}$	$2.21 \cdot 10^{-3}$	$5.88 \cdot 10^{-3}$	$9.08 \cdot 10^{-3}$	$1.67 \cdot 10^{-2}$
KOUT	$6.14 \cdot 10^{-3}$	$4.92 \cdot 10^{-3}$	$1.10 \cdot 10^{-2}$	$1.36 \cdot 10^{-2}$	$1.72 \cdot 10^{-2}$
LOG sym.	$1.90 \cdot 10^{-3}$	$4.36 \cdot 10^{-3}$	$1.34 \cdot 10^{-2}$	$2.67 \cdot 10^{-2}$	$7.29 \cdot 10^{-2}$
α	1.2	1.4	1.6	1.8	
COMB	$2.70 \cdot 10^{-2}$	$3.53 \cdot 10^{-2}$	$3.54 \cdot 10^{-2}$	$2.20 \cdot 10^{-2}$	
KOUT	$2.71 \cdot 10^{-2}$	$4.15 \cdot 10^{-2}$	$4.16 \cdot 10^{-2}$	$2.41 \cdot 10^{-2}$	
LOG sym.	$1.24 \cdot 10^{-1}$	$1.38 \cdot 10^{-1}$	$1.29 \cdot 10^{-1}$	$1.10 \cdot 10^{-1}$	

Table 5: Mean squared errors for the combined (COMB), Koutrouvelis regression (KOUT) and Log-moment (LOG) estimators of α for $r = 500$, $n = 100$, $B = 1000$, $\beta = 0.6$ and $\sigma = 1$.

For skewed data (see Table 5), the combined estimators still have good performance but we loose in term of mean square errors comparing to the symmetric case.

3.2 Test of symmetry

We propose a test for checking the skewness of dataset. We want to test $H_0 : \beta = 0$ against $H_1 : \beta \neq 0$ using the properties of the estimators studied in previous section

Let (X_1, \dots, X_{2n}) be a sequence of $2n$ independent and identically distributed (iid) stable random variables $S_\alpha(\sigma, \beta, 0)$. Under the null hypothesis H_0 , both estimates $\widehat{\alpha}_{LOG}((X_{2k})_k)$ and $\widehat{\alpha}_{LOG}((X_{2k} - X_{2k-1})_k)$ are consistent, and so the difference between this estimators tends to zero. For skewed variables, this convergence does not occur since $\widehat{\alpha}_{LOG}((X_{2k})_k)$ does not converge any more to α . These facts suggest to construct a test based on the difference of these estimators. Denote

$$L_1 := E[\log |Z|] = \left(\frac{1}{\alpha} - 1\right) \gamma + \log \sigma$$

$$L_2 := E[(\log |Z| - E[\log |Z|])^2] = \frac{\pi^2}{6\alpha^2} + \frac{\pi^2}{12}$$

$$L_3 := E[(\log |Z| - E[\log |Z|])^3] = 2\zeta(3) \left(\frac{1}{\alpha^3} - 1\right)$$

$$L_4 := E[(\log |Z| - E[\log |Z|])^4] = \pi^4 \left(\frac{3}{20\alpha^4} + \frac{1}{12\alpha^2} + \frac{19}{240}\right)$$

$$C := Cov((\log |X_2| - E[\log |X_2|])^2, (\log |X_2 - X_1| - E[\log |X_2 - X_1|])^2) \\ = E[(\log |X_2| - E[\log |X_2|])^2 (\log |X_2 - X_1| - E[\log |X_2 - X_1|])^2] - L_2^2$$

where Z is an $S_\alpha(\sigma, 0, 0)$ random variable, ζ is the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\zeta(3) = 1.2020569 \dots$

Proposition 3.1. For $w \in (0, 1)$, define the critical region

$$R_w = \left\{ n \frac{(\widehat{\alpha}_{LOG}(\{X_{2k}\}_k) - \widehat{\alpha}_{LOG}(\{X_{2k} - X_{2k-1}\}_k))^2}{\frac{18\widehat{\alpha}_{LOG}^6}{\pi^4} (\widehat{L}_4 - \widehat{L}_2^2 - \widehat{C})} > t_w \right\}$$

where t_w is the $1 - w$ quantile of the Chi-squared distribution with 1 degree of freedom, and where \widehat{L}_4 , \widehat{L}_2 and \widehat{C} are respectively the empirical moments of L_4 , L_2 and C . We decide to reject the null hypothesis if $(X_1, \dots, X_{2n}) \in R_w$. The test has an asymptotically significance level equal to w and is asymptotically consistent under H_1 .

Proof. Under the null hypothesis,

$$\sqrt{n} \left(\frac{1}{n} \sum_{k=1}^n \begin{pmatrix} (Y_k - \overline{Y}_n)^2 \\ (Z_k - \overline{Z}_n)^2 \end{pmatrix} - \begin{pmatrix} L_2 \\ L_2 \end{pmatrix} \right) \xrightarrow{d} \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} L_4 - L_2^2 & C \\ C & L_4 - L_2^2 \end{pmatrix} \right)$$

Then by multidimensional delta method, we get

$$\sqrt{n} \left(\begin{pmatrix} \widehat{\alpha}_{LOG}((X_{2k})_k) \\ \widehat{\alpha}_{LOG}((X_{2k} - X_{2k-1})_k) \end{pmatrix} - \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \right) \xrightarrow{d} \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \frac{9\alpha^6}{\pi^4} \begin{pmatrix} L_4 - L_2^2 & C \\ C & L_4 - L_2^2 \end{pmatrix} \right)$$

and

$$\frac{\sqrt{n} \widehat{\alpha}_{LOG}((X_{2k})_k) - \widehat{\alpha}_{LOG}((X_{2k} - X_{2k-1})_k)}{\sqrt{\frac{18\alpha^6}{\pi^4} (L_4 - L_2^2 - C)}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Finally, applying Slutsky Theorem and the consistency of \widehat{L}_4 , \widehat{L}_2 and \widehat{C} , we obtain that the asymptotical significance level is equal to w .

Under H_1 , we have

$$\widehat{\alpha}_{LOG}((X_{2k})_k) - \widehat{\alpha}_{LOG}((X_{2k} - X_{2k-1})_k) \xrightarrow{a.s.} \frac{\pi\alpha}{\sqrt{\pi^2 - 6\theta^2}} - \alpha \neq 0$$

with $\theta = \arctan(\beta \tan \frac{\alpha\pi}{2})$. Then, under the alternative $\beta \neq 0$ we have a consistent test, $P(R_w) \xrightarrow[n \rightarrow \infty]{} 1$. \square

In Table 6, we use Monte Carlo experiment to evaluate the probability to reject the null hypothesis H_0 . For small values of α , the empirical significance level converges slowly to w : this is due to the form of the density that is concentrated around zero, and the poor quality of the estimation of the coefficient $L_4 - L_2^2 - C$. The estimation converges rather slowly to this coefficient (which increases when α decreases).

	$\alpha = 0.2$	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8
$2n = 200$	0.065	0.074	0.084	0.087	0.088	0.097	0.088	0.06	0.042
$2n = 1000$	0.0953	0.1019	0.0971	0.89	0.074	0.065	0.067	0.061	0.035
$2n = 10^4$	0.18	0.16	0.10	0.071	0.051	0.050	0.048	0.051	0.053

Table 6: Probabilities to reject the null hypothesis under H_0 for several sizes of samples and different values of α . The significance level is $w = 5\%$.

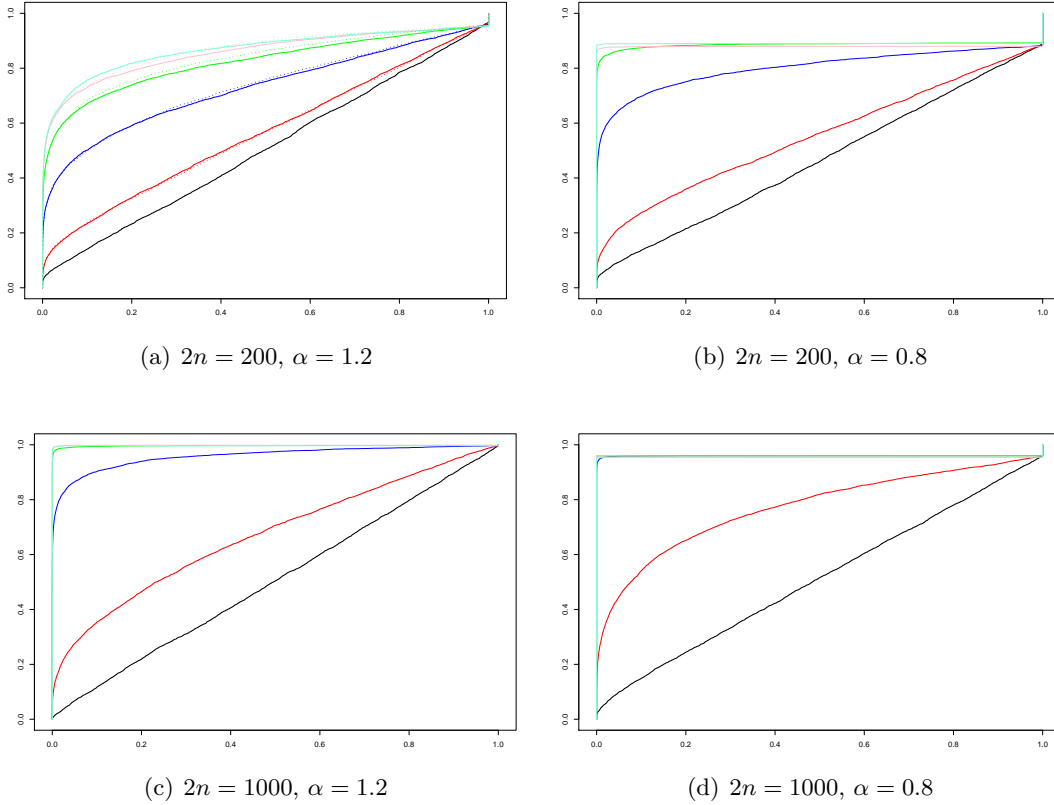


Figure 5: Representation of the empirical cumulative distribution function of the p-value for different values of β , $\beta = 0$ (black), $\beta = 0.2$ (red), $\beta = 0.4$ (dark blue), $\beta = 0.6$ (green), $\beta = 0.8$ (pink) and $\beta = 1$ (light blue). In (a), we add in dotted lines the value for $\beta \in \{-1, -0.8, -0.6, -0.4, -0.2\}$ which correspond exactly to the positive ones.

In Figure 5, we can see that the power increases when β goes away from 0 or when the sample sizes increases. This convergence under the alternative hypothesis depends on the values of α and β . We observe that the p-value take the value 1 with non-null probability for small size of samples. This jump due to the truncation in the log-moment estimator defined in Theorem 2.4 asymptotically disappears.

4 Case of non-identically distributed stable variables

In applications, it may be the case that one needs to analyze non-stationnary phenomena. For instance, it seems plausible that financial logs which display jumps will see the intensity of these jump depend on external events, such as crises (see next section for an illustration on the S&P 500). Sometimes, the variation of α will be slow, and it is of interest to investigate under which conditions our estimator still behaves correctly in

situations where the data at hand deviate slightly from the iid hypothesis. In the sequel, we examine two cases: deterministic and random small perturbations of α , leading to random variables which are not identically distributed. We do not dispense here with the independence assumption, although this would be a desirable extension. This generalization will be the topic of a future work.

4.1 Deterministic perturbations

Let (X_i) be a sequence of independent variables and X random variable independent of (X_i) such that

$$X_i \sim S_{\alpha_i}(\sigma_i, 0, 0) \text{ and } X \sim S_{\alpha}(\sigma, 0, 0).$$

We denote $Y_i = \log |X_i|$, $Y = \log |X|$. Assume that, for each integer i , $\alpha_i = \alpha + \varepsilon_i$ and $\sigma_i = \sigma + \eta_i$ with ε_i and η_i deterministic such that

$$\frac{|\varepsilon_i|}{\alpha} \leq c_{\alpha} < 1 \text{ and } \frac{|\eta_i|}{\sigma} \leq c_{\sigma} < 1.$$

Proposition 4.1. *Under the conditions*

$$\frac{1}{n} \sum_{i=1}^n |\varepsilon_i| \xrightarrow[n \rightarrow \infty]{} 0 \text{ and } \frac{1}{n} \sum_{i=1}^n |\eta_i| \xrightarrow[n \rightarrow \infty]{} 0,$$

one has

$$\widehat{\alpha}_{LOG}^{(n)} \xrightarrow[n \rightarrow \infty]{a.s.} \alpha.$$

where $\widehat{\alpha}_{LOG}^{(n)}$ is defined in Theorem 2.4.

Proof. See Appendix. □

Proposition 4.2. *Let $\Sigma_{\alpha, \sigma}$ be the covariance matrix between Y and Y^2 :*

$$\Sigma_{\alpha, \sigma} = \begin{pmatrix} \text{Var}(Y) & \text{Cov}(Y, Y^2) \\ \text{Cov}(Y, Y^2) & \text{Var}(Y^2) \end{pmatrix} \quad (4.1)$$

and set

$$H_{\alpha, \sigma} = \begin{pmatrix} 6\left(\left(\frac{1}{\alpha} - 1\right)\gamma + \log \sigma\right)\alpha^3 / \pi^2 \\ -3\alpha^3 / \pi^2 \end{pmatrix} \quad (4.2)$$

With the conditions

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n |\varepsilon_i| \xrightarrow[n \rightarrow \infty]{} 0 \text{ and } \frac{1}{\sqrt{n}} \sum_{i=1}^n |\eta_i| \xrightarrow[n \rightarrow \infty]{} 0,$$

the following central limit theorem holds for $\widehat{\alpha}_{LOG}^{(n)}$:

$$\sqrt{n} \left(\widehat{\alpha}_{LOG}^{(n)} - \alpha \right) \xrightarrow{d} \mathcal{N}(0, {}^t H_{\alpha, \sigma} \Sigma_{\alpha, \sigma} H_{\alpha, \sigma}) \quad (4.3)$$

Proof. See Appendix. □

4.2 Random perturbations

Let (X_i) be a sequence of independent variables and X random variable independent of (X_i) such that $X_i \sim S_{\alpha_i}(\sigma, 0, 0)$ and $X \sim S_{\alpha}(\sigma, 0, 0)$. Assume that, for each integer i , $\alpha_i = \alpha + \varepsilon_i$, with ε_i a random variable, and that there is a constant $c_{\alpha} < 1$ such that $\mathbb{P}(\alpha_i \in [\alpha(1 - c_{\alpha}), \alpha(1 + c_{\alpha})]) = 1$. We denote $Y_i = \log |X_i|$, $Y = \log |X|$.

Proposition 4.3. *Under the conditions*

$$\frac{1}{n} \sum_{i=1}^n E[|\varepsilon_i|] \xrightarrow{n \rightarrow \infty} 0$$

we have

$$\widehat{\alpha}_{LOG}^{(n)} \xrightarrow[n \rightarrow \infty]{a.s.} \alpha.$$

If, in addition, $\frac{1}{\sqrt{n}} \sum_{i=1}^n E[|\varepsilon_i|] \xrightarrow{n \rightarrow \infty} 0$, then the following central limit theorem holds:

$$\sqrt{n} \left(\widehat{\alpha}_{LOG}^{(n)} - \alpha \right) \xrightarrow{d} \mathcal{N}(0, H_{\alpha, \sigma} \Sigma_{\alpha, \sigma} {}^t H_{\alpha, \sigma})$$

where $\Sigma_{\alpha, \sigma}$ and $H_{\alpha, \sigma}$ are defined in (4.1) and (4.2).

Proof. See Appendix. □

5 Some applications for the combined estimator

5.1 Numerical results on synthetic data: multistable Lévy motion

We now put our log-moment and combined estimators to use in the case of the multistable Lévy motion defined in [2] (see also [9] for further properties of this process). The basic idea is to allow the stability index evolve with time, so that the jump intensity, which is governed by α , varies along a trajectory. Such a feature is commonly encountered in times series observed in fields such as finance or biomedicine. Let us briefly recall the definition of such processes.

Let $\alpha : [0, 1] \rightarrow (0, 2)$ be continuously differentiable. We note $r^{<s>} = \text{sign}(r)|r|^s$ for $r \in \mathbb{R}$ and $s \in \mathbb{R}$. Symmetric multistable Lévy motion is defined by

$$M_{\alpha}(t) = C_{\alpha(t)} \sum_{(X, Y) \in \Pi} 1_{(0, t]}(X) Y^{<-1/\alpha(t)>} \quad (5.1)$$

where $C_{\theta} = \left(\int_0^{\infty} u^{-\theta} \sin(u) du \right)^{-1/\theta}$ and Π is a Poisson point process on $\mathbb{R}^+ \times \mathbb{R}$ with plane Lebesgue measure \mathcal{L}^2 as mean measure. This process is simulated by using the field

$$X(t, u) = C_{\alpha(u)} \sum_{(X, Y) \in \Pi} 1_{(0, t]}(X) Y^{<-1/\alpha(u)>}$$

For each $u \in (0, 1)$, $X(., u)$ is an $\alpha(u)$ -stable process with independent increments which can be implemented using the RSTAB program available in [16] or in [14]. The interval $[0,1]$ is discretised in N equal parts and $X(., u)$ is implemented by the cumulative sum of N independent stable variables with $\alpha(u)$ as characteristic exponent.

In Figure 6, we display sample paths of multistable processes for several α functions. Then, we estimate these functions at all point t_0 thanks to the combined estimator with a window of n observations around t_0 .

In Figure 7, we iterate 100 times the simulation and the estimation for a multistable process with $\alpha(t) = 1.5 - 0.48 \sin(2\pi(t + 1/4))$. For each point where the function α is estimated, we obtain the empirical distribution of the combined estimator. This procedure is repeated for several sizes of window (100, 200, 1000 and 2000). We observe that the standard error which correspond to the standard deviation for the combined estimator is decreasing when the size of the window n increases whereas the bias is increasing for n large. Finally, the mean squared error is decreasing when n increases until $n = 1000$ and then increases for larger value. Figure 8 represents the bias, standard error and mean square error as function of t for various values of the window size.

The mean squared error as a function of α is reported in Figure 9. The mean squared error does not vary much according to the value of α in $[1, 2]$ when n is fixed.

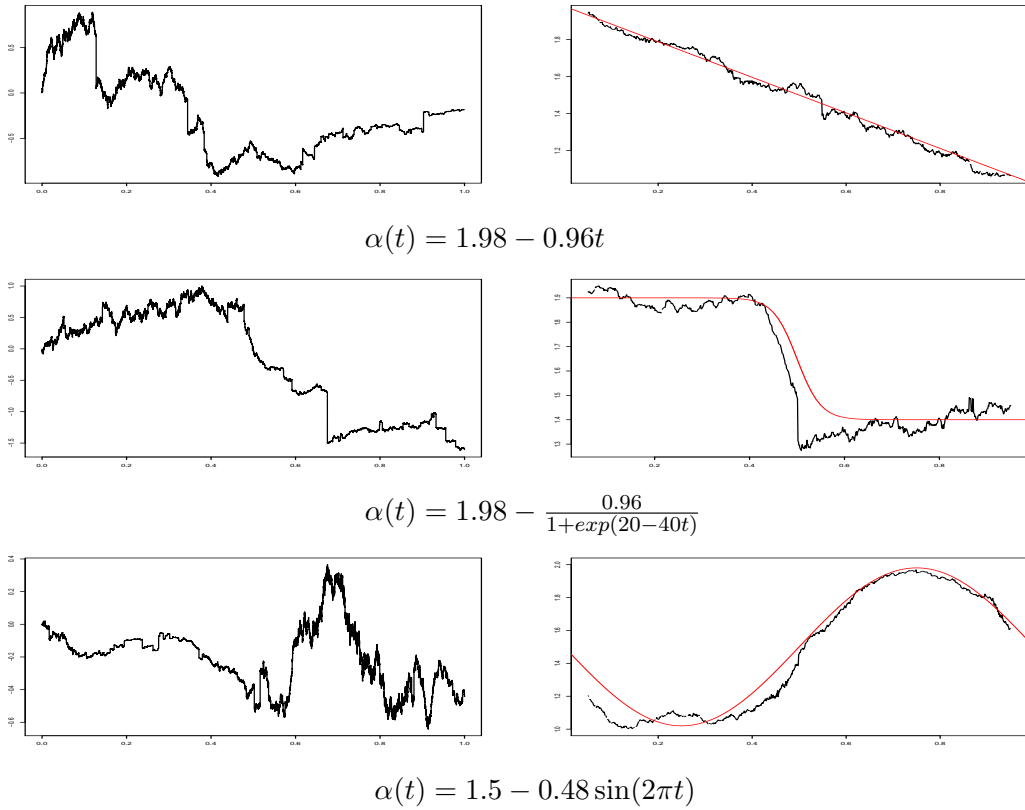
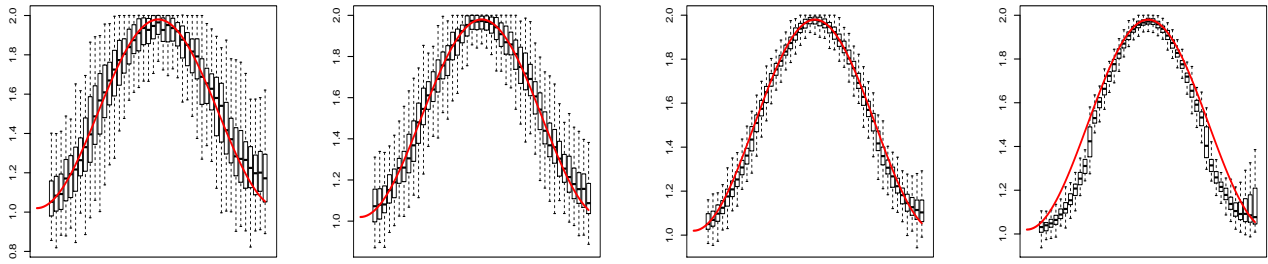


Figure 6: Trajectories of multistable processes on $(0,1)$ with $N = 20000$ points in the first column. The functions $\alpha(t)$ (red) and $\widehat{\alpha}_{COMB}(t)$ (black) are represented in the second column with $n = 2000$.



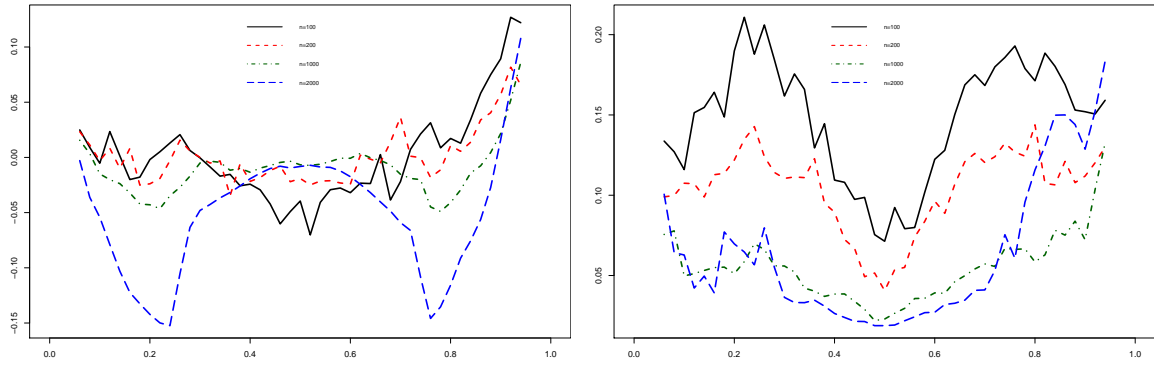
(a) $n = 100$

(b) $n = 200$

(c) $n = 1000$

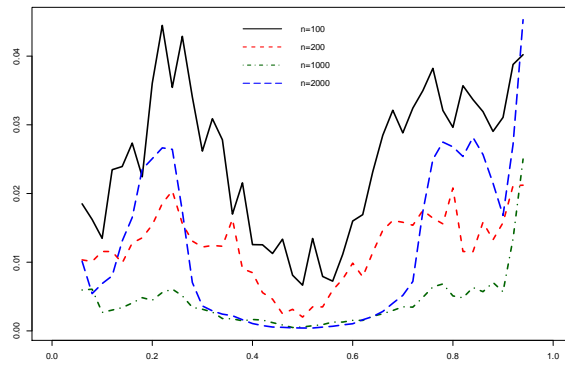
(d) $n = 2000$

Figure 7: Box-plots of the estimator $\widehat{\alpha}_{COMB}^{(n)}$ for 100 replications of a multistable process with characteristic exponent $\alpha(t) = 1.5 - 0.48 \sin(2\pi(t + 1/4))$, represented in red. The box-plots represent the behavior of the estimator for several sizes n of window.



(a)

(b)



(c)

Figure 8: Representation of the bias (a), standard error (b) and mean square error (c) as function of t for $n = 100$ (black solid line), $n = 200$ (red dashed), $n = 1000$ (green dotted) and $n = 2000$ (blue dotted and dashed mix.). The statistics are evaluated on the same trajectories in Figure 7.

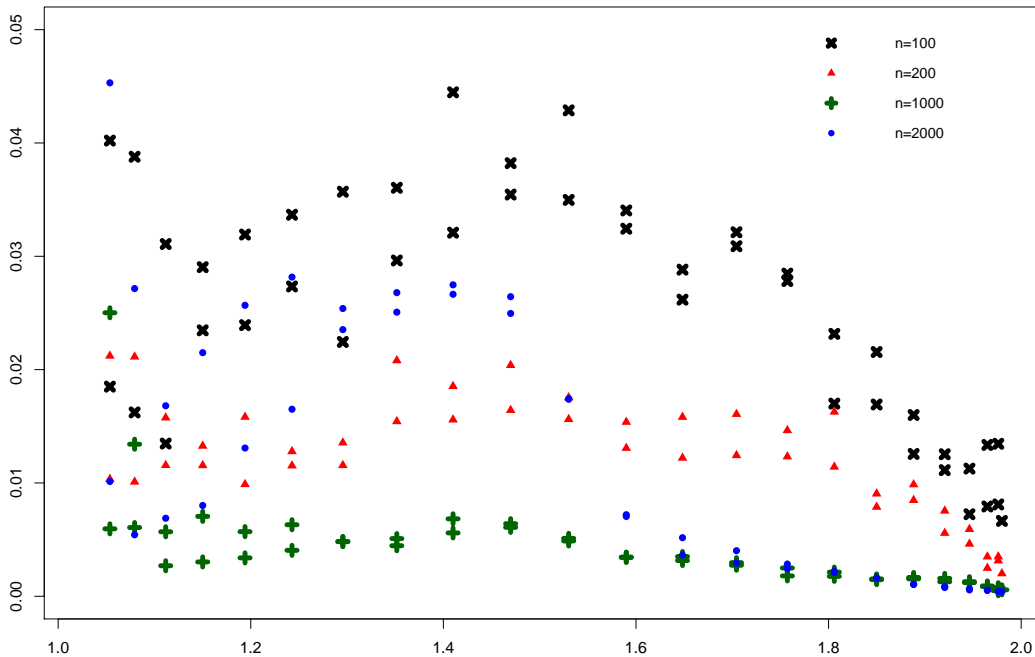


Figure 9: Mean squared error according to the value of α of the combined estimation for a multistable process with $\alpha(t) = 1.5 - 0.48 \sin(2\pi(t + 1/4))$. The statistics are evaluated on the same trajectories as Figure 7.

As these figures show, reasonable estimates are obtained on these experiments, due to the fact that the variations of α are "slow" compared to the sampling frequency: this feature ensures that centering a window around any given t_0 and treating all points inside this window as having the same α value is an acceptable approximation as far as estimation is concerned.

5.2 Application on financial logs

This last section deals with real data. We want to apply the combined estimator to estimate the characteristic exponent of the financial index Standard & Poor's 500 (abbreviated as the *S&P 500*, see Figure 10). This is an American stock market index based on the 500 companies having largest capitalization. The stock market returns of the financial index *S&P 500*, which correspond to the renormalized growth rate $((Y_{t+1} - Y_t)/Y_t)_t$, are supposed to be independent stable random variables. In Figure 11, we first test the symmetry of the data in sliding window of size 1000 using the test defined in Proposition 3.1. We represent the empirical distribution function for the p -value of S&P 500 returns

since 1996. The cumulative distribution function is very close to the uniform one. Then, the null hypothesis of symmetric data is not reject for S&P 500 returns since 1996. As a consequence, the parameter is estimated by applying the estimator defined in Theorem 2.4 in sliding window of several sizes during the period 1996-2017 (see Figure 12).

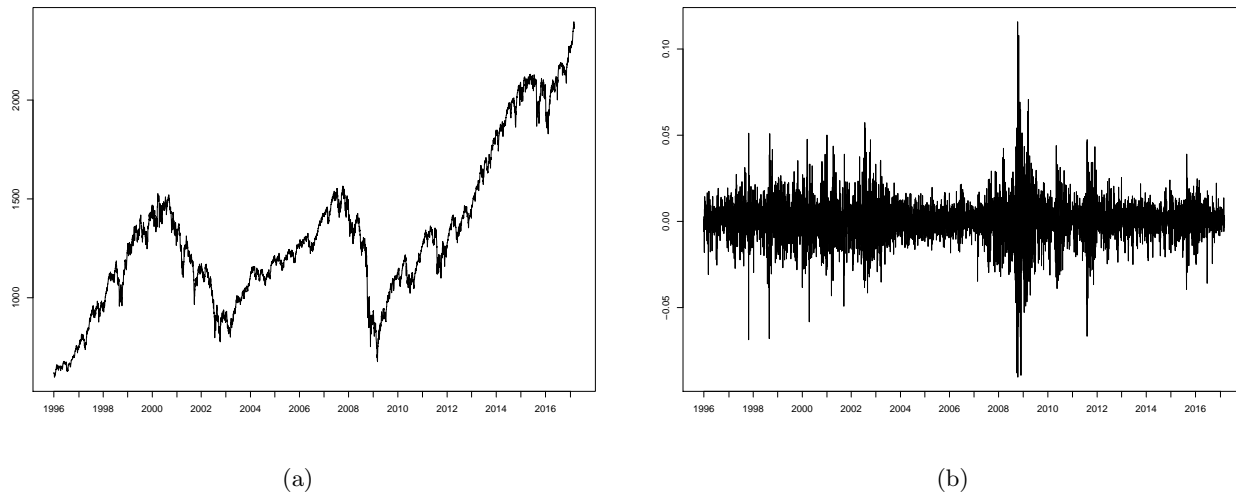


Figure 10: Evolution of the financial index S&P 500 as function of t (a) and its return (b), between 1996 and 2017.

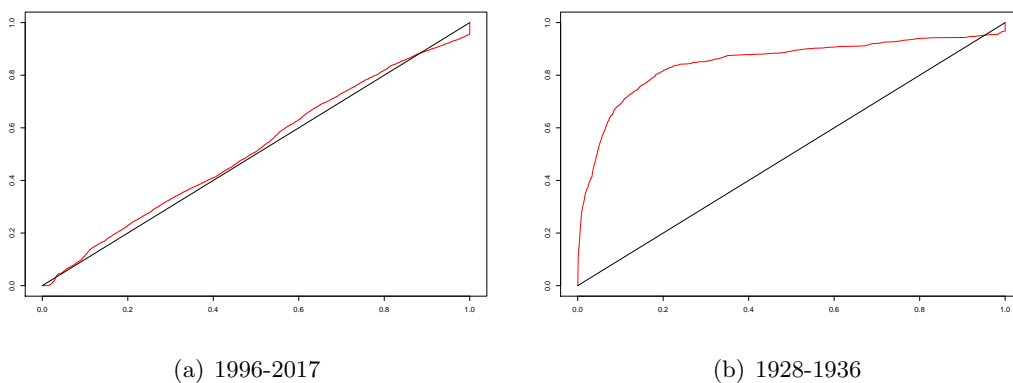
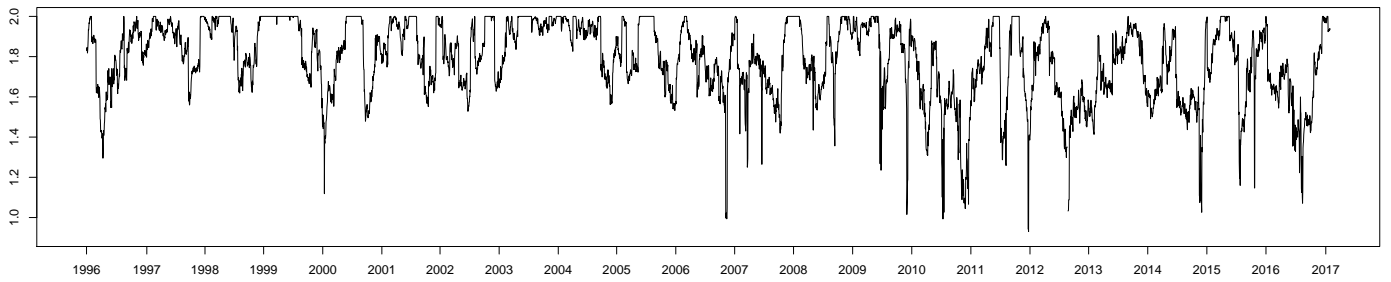
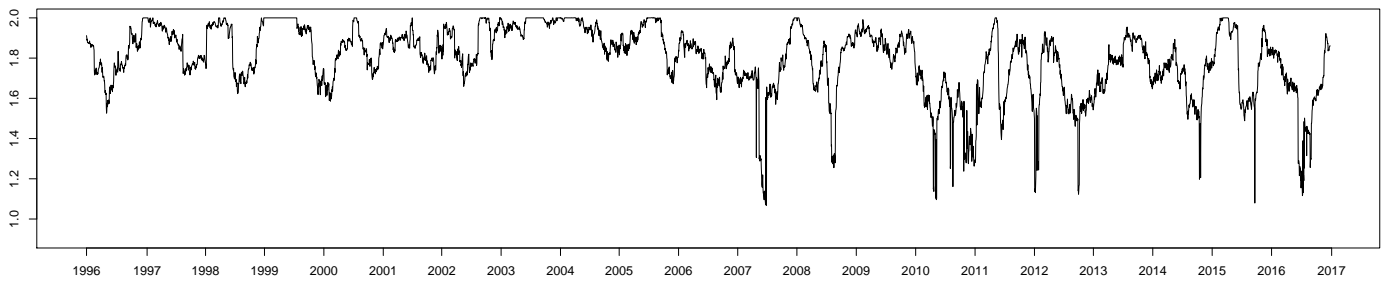


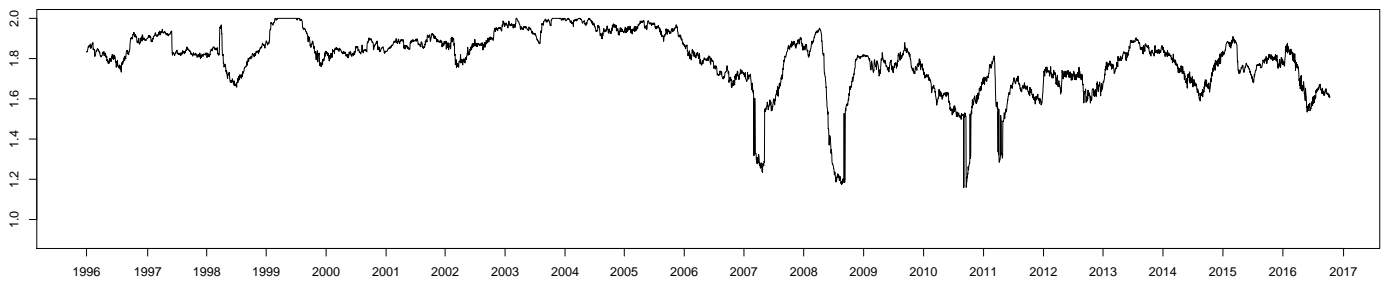
Figure 11: Representation of the empirical cumulative distribution function of the p-value for S&P 500 returns between 1996 and 2017 (Figure (a)) and between 1928 and 1936 (Figure (b)) with sliding window of size 1000.



(a) $n=50$



(b) $n=100$



(c) $n=200$

Figure 12: Values of the combined estimator $\widehat{\alpha}_{COMB}^{(n)}$ for the S&P 500 characteristic exponent α in sliding window of several sizes n for working days between 1996 and 2017.

We reject the hypothesis of symmetry between 1929 and 1936 (see Figure 11). In Figure 13, the estimation of the characteristic exponent is done between 1929 and 1936

using the skewed combined estimator defined in Section 3. A sudden drop is observed at the end of 1929. This change corresponds to the Wall Street financial crash of 1929. The estimation for symmetric data (in red) is added in the figure to see the difference between the two estimations, particularly during the crisis.

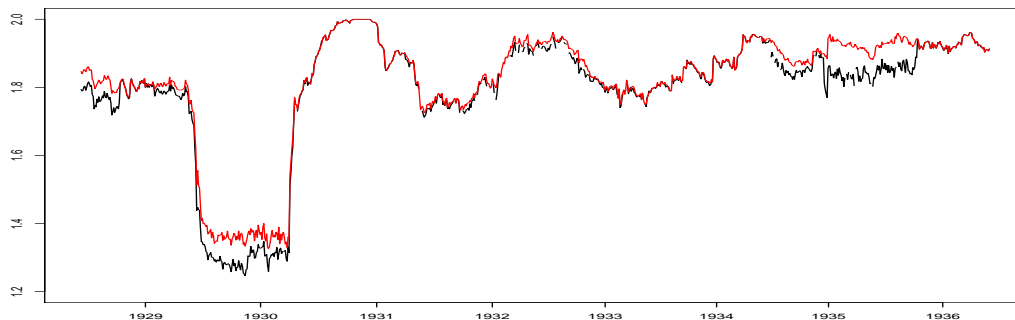


Figure 13: Values of the skewed combined estimator $\widehat{\alpha}_{COMB}^{(n)}$ (in black) and the symmetric combined estimator (in red) for the S&P 500 characteristic exponent α around the Wall Street financial crash of 1929 (sliding window of size $n = 200$ observations).

6 Appendix

Proof of Proposition 2.1

Let Z be a stable random variable $\sim S_\alpha(1, 0, 0)$. Since:

- Z has bounded density,
- $\lim_{\lambda \rightarrow +\infty} \lambda^\alpha P(Z > \lambda) = C_\alpha$,
- $\lim_{\lambda \rightarrow +\infty} \lambda^\alpha P(Z < -\lambda) = C_\alpha$,

(see [14], Property 1.2.15), we deduce for all $p > 0$,

$$E[|\log |Z||^p] = \int_0^\infty \mathbb{P}(|\log |Z||^p > x) dx < \infty.$$

Proof of Proposition 2.2

We use the formula of Property 1.2.17 and 1.2.15 in [14] to compute $E[|Z|^t]$ for $0 < t < \alpha$:

$$E[|Z|^t] = \frac{2^{t-1}\Gamma(1-t/\alpha)}{t \int_0^{+\infty} \frac{\sin^2(u)}{u^{t+1}} du} = \frac{2^{t-1}\Gamma(1-t/\alpha)}{\int_0^{+\infty} \frac{\sin(2u)}{u^t} du} = \frac{(1-t)\Gamma(1-t/\alpha)}{\Gamma(2-t) \cos(\pi t/2)}$$

Furthermore, if $0 < t < 1$, we have:

$$E[|Z|^t] = \frac{\Gamma(1-t/\alpha)}{\Gamma(1-t) \cos(\pi t/2)}$$

Proof of Theorem 2.3

By the strong law of large numbers for the random variables $(\log |X_i|)_i$ with finite expectation (Proposition 2.1), and the continuous mapping theorem for $g(x) = \frac{\gamma}{\gamma+x}$, we get $\hat{\alpha}_n \xrightarrow[n \rightarrow \infty]{a.s.} \alpha$. Then we apply the central limit theorem and the delta method with the function $f(x) = \frac{\pi^2}{6x^2} + \frac{\pi^2}{12}$ to obtain

$$\frac{\sqrt{n}(\hat{\alpha}_n - \alpha)\gamma}{\hat{\alpha}_n^2 \sqrt{f(\hat{\alpha}_n)}} \xrightarrow{d} \mathcal{N}(0, 1)$$

Proof of Theorem 2.4

We prove this theorem by using the strong law of large numbers, continuous mapping theorem, multidimensional central limit theorem and delta method. We need the third and the fourth log-moment of a stable law $Z \sim S_\alpha(1, 0, 0)$ for the covariance matrix. We have

$$E \left[(\log |Z| - E[\log |Z|])^3 \right] = 2\zeta(3) \left(\frac{1}{\alpha^3} - 1 \right)$$

and we get

$$E[(\log |Z|)^3] = \frac{4\gamma^3 + 2\gamma\pi^2 + 8\zeta(3)}{4\alpha^3} + \frac{-12\gamma^3 - 2\gamma\pi^2}{4\alpha^2} + \frac{12\gamma^3 + \gamma\pi^2}{4\alpha} + \frac{-4\gamma^3 - \gamma\pi^2 - 8\zeta(3)}{4}$$

and

$$\begin{aligned} E \left[(\log |Z| - E[\log |Z|])^4 \right] &= \pi^4 \left(\frac{3}{20\alpha^4} + \frac{1}{12\alpha^2} + \frac{19}{240} \right) \\ E[(\log |Z|)^4] &= \frac{240\gamma^4 + 240\gamma^2\pi^2 + 36\pi^4 + 1920\zeta(3)\gamma}{240\alpha^4} + \frac{-960\gamma^4 - 480\gamma^2\pi^2 - 1920\zeta(3)\gamma}{240\alpha^3} \\ &\quad + \frac{1440\gamma^4 + 360\gamma^2\pi^2 + 20\pi^4}{240\alpha^2} + \frac{-960\gamma^4 - 240\gamma^2\pi^2 - 1920\zeta(3)\gamma}{240\alpha} \\ &\quad + \frac{240\gamma^4 + 120\gamma^2\pi^2 + 19\pi^4 + 1920\zeta(3)\gamma}{240} \end{aligned}$$

where ζ is the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and $\zeta(3) = 1.2020569\dots$

Proof of Proposition 4.1

By the Kolmogorov strong law of large numbers for non-identically distributed random variables (Theorem 2.3.10 in [15]), we get

$$\frac{1}{n} \sum_{i=1}^n Y_i^2 - \left(\frac{1}{n} \sum_{i=1}^n Y_i \right)^2 - \left[\frac{1}{n} \sum_{i=1}^n E[Y_i^2] - \left(\frac{1}{n} \sum_{i=1}^n E[Y_i] \right)^2 \right] \xrightarrow{a.s.} 0$$

as

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n E[Y_i] - E[Y] &= \frac{\gamma}{n} \sum_{i=1}^n \left(\frac{1}{\alpha_i} - \frac{1}{\alpha} \right) + \frac{1}{n} \sum_{i=1}^n (\log \sigma_i - \log \sigma) \\ &= \frac{\gamma}{n} \sum_{i=1}^n \frac{-\varepsilon_i}{(\alpha_i^*)^2} + \frac{1}{n} \sum_{i=1}^n \frac{\eta_i}{\sigma_i^*} \end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n E[Y_i^2] - E[Y^2] \\
&= \frac{\gamma^2 + \pi^2/6}{n} \sum_{i=1}^n \left(\frac{1}{\alpha_i^2} - \frac{1}{\alpha^2} \right) - \frac{2\gamma^2}{n} \sum_{i=1}^n \left(\frac{1}{\alpha_i} - \frac{1}{\alpha} \right) \\
&\quad + \frac{2\gamma}{n} \sum_{i=1}^n \left(\left(\frac{1}{\alpha_i} - 1 \right) \log \sigma_i - \left(\frac{1}{\alpha} - 1 \right) \log \sigma \right) + \frac{1}{n} \sum_{i=1}^n ((\log \sigma_i)^2 - (\log \sigma)^2) \\
&= \frac{\gamma^2 + \pi^2/6}{n} \sum_{i=1}^n \frac{-2\varepsilon_i}{\tilde{\alpha}_i^3} - \frac{2\gamma^2}{n} \sum_{i=1}^n \frac{-\varepsilon_i}{(\alpha_i^*)^2} + \frac{2\gamma}{n} \sum_{i=1}^n \left(\frac{1}{\tilde{\sigma}_i} \left(\frac{1}{\tilde{\alpha}_i} - 1 \right) \eta_i - \frac{\log(\tilde{\sigma}_i)}{(\tilde{\alpha}_i)^2} \varepsilon_i \right) \\
&\quad + \frac{1}{n} \sum_{i=1}^n \frac{2 \log \tilde{\sigma}_i}{\tilde{\sigma}_i} \eta_i
\end{aligned}$$

where $\tilde{\alpha}_i$, α_i^* , $\tilde{\alpha}_i$ (respectively $\tilde{\sigma}_i$, σ_i^* , $\tilde{\sigma}_i$) have ranged between α and α_i (respectively σ and σ_i).

$$\begin{aligned}
& \left| \frac{1}{n} \sum_{i=1}^n E[Y_i] - E[Y] \right| \leq \frac{\gamma}{\alpha^2(1-c_\alpha)^2} \frac{1}{n} \sum_{i=1}^n |\varepsilon_i| + \frac{1}{\sigma(1-c_\sigma)} \frac{1}{n} \sum_{i=1}^n |\eta_i| \\
& \left| \frac{1}{n} \sum_{i=1}^n E[Y_i^2] - E[Y^2] \right| \\
& \leq \left(\frac{2\gamma^2 + \pi^2/3}{\alpha^3(1-c_\alpha)^3} + \frac{2\gamma^2}{\alpha^2(1-c_\alpha)^2} + \frac{2\gamma \max(|\log(\sigma - \sigma c_\sigma)|, |\log(\sigma + \sigma c_\sigma)|)}{\alpha^2(1-c_\alpha)^2} \right) \frac{1}{n} \sum_{i=1}^n |\varepsilon_i| \\
& \quad + \left(\frac{2\gamma}{\sigma(1-c_\sigma)} \left(\frac{1}{\alpha(1-c_\alpha)} + 1 \right) + \frac{2 \max(|\log(\sigma - \sigma c_\sigma)|, |\log(\sigma + \sigma c_\sigma)|)}{\sigma(1-c_\sigma)} \right) \frac{1}{n} \sum_{i=1}^n |\eta_i|
\end{aligned}$$

Under the conditions:

- $\frac{1}{n} \sum_{i=1}^n |\varepsilon_i| \rightarrow 0$
- $\frac{1}{n} \sum_{i=1}^n |\eta_i| \rightarrow 0$

we have

$$\frac{1}{n} \sum_{i=1}^n E[Y_i^2] \xrightarrow{n \rightarrow \infty} E[Y^2] \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n E[Y_i] \xrightarrow{n \rightarrow \infty} E[Y].$$

By the continuous mapping theorem with $g(x, y) = \frac{\pi}{\sqrt{\max(6(y-x^2) - \frac{\pi^2}{2}, \frac{\pi^2}{4})}}$, we obtain

$$\hat{\alpha}_n - \alpha \xrightarrow{a.s.} 0.$$

Proof of Proposition 4.2

Define the covariance-matrix $\Sigma_{\alpha,\sigma}^{(n)}$ as follows:

$$\Sigma_{\alpha,\sigma}^{(n)} := \begin{pmatrix} \frac{1}{n} \sum_{i=1}^n \text{Var}(Y_i) & \frac{1}{n} \sum_{i=1}^n \text{Cov}(Y_i, Y_i^2) \\ \frac{1}{n} \sum_{i=1}^n \text{Cov}(Y_i, Y_i^2) & \frac{1}{n} \sum_{i=1}^n \text{Var}(Y_i^2) \end{pmatrix}$$

. With the conditions of Proposition 4.1, we have

$$\Sigma_{\alpha,\sigma}^{(n)} \xrightarrow[n \rightarrow \infty]{} \Sigma_{\alpha,\sigma} := \begin{pmatrix} \text{Var}(Y) & \text{Cov}(Y, Y^2) \\ \text{Cov}(Y, Y^2) & \text{Var}(Y^2) \end{pmatrix}$$

By the central limit theorem for non-identically distributed random variables (Theorem 3.3.9 in [15]), as $\sup_k E \left[\left| \begin{matrix} Y_k \\ (Y_k)^2 \end{matrix} \right|^4 \right] < \infty$, we have,

$$\sqrt{n} \left(\begin{pmatrix} \frac{1}{n} \sum_{i=1}^n Y_i \\ \frac{1}{n} \sum_{i=1}^n Y_i^2 \end{pmatrix} - \begin{pmatrix} \frac{1}{n} \sum_{i=1}^n E[Y_i] \\ \frac{1}{n} \sum_{i=1}^n E[Y_i^2] \end{pmatrix} \right) \xrightarrow{d} \mathcal{N}(0, \Sigma_{\alpha,\sigma})$$

Under conditions of Proposition 4.2, we get

$$\sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n E[Y_i^2] - E[Y^2] \right) \xrightarrow[n \rightarrow \infty]{} 0$$

and

$$\sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n E[Y_i] - E[Y] \right) \xrightarrow[n \rightarrow \infty]{} 0,$$

then

$$\sqrt{n} \left(\begin{pmatrix} \frac{1}{n} \sum_{i=1}^n Y_i \\ \frac{1}{n} \sum_{i=1}^n Y_i^2 \end{pmatrix} - \begin{pmatrix} E[Y] \\ E[Y^2] \end{pmatrix} \right) \xrightarrow{d} \mathcal{N}(0, \Sigma_{\alpha,\sigma}).$$

By applying the delta-method with $g(x, y) = \frac{\pi}{\sqrt{\max(6(y-x^2) - \frac{\pi^2}{2}, \frac{\pi^2}{4})}}$, we obtain the result.

Proof of Proposition 4.3

By the strong law of large numbers for non-identically distributed random variables (Theorem 2.3.10 in [15]), we get

$$\frac{1}{n} \sum_{i=1}^n Y_i^2 - \left(\frac{1}{n} \sum_{i=1}^n Y_i \right)^2 - \left[\frac{1}{n} \sum_{i=1}^n E[Y_i^2] - \left(\frac{1}{n} \sum_{i=1}^n E[Y_i] \right)^2 \right] \xrightarrow{a.s.} 0$$

With the same calculation, we have

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n E[Y_i^2] - E[Y^2] &= \frac{\gamma^2 + \pi^2/6}{n} \sum_{i=1}^n E \left[\frac{1}{\alpha_i^2} - \frac{1}{\alpha^2} \right] + \frac{-2\gamma^2 + 2 \log(\sigma)}{n} \sum_{i=1}^n E \left[\frac{1}{\alpha_i} - \frac{1}{\alpha} \right] \\ &= \frac{\gamma^2 + \pi^2/6}{n} \sum_{i=1}^n E \left[\frac{-2\varepsilon_i}{(\tilde{\alpha}_i)^3} \right] + \frac{-2\gamma^2 + 2 \log(\sigma)}{n} \sum_{i=1}^n E \left[\frac{-\varepsilon_i}{(\alpha_i^*)^2} \right] \end{aligned}$$

and

$$\frac{1}{n} \sum_{i=1}^n E[Y_i] - E[Y] = \frac{\gamma}{n} \sum_{i=1}^n E \left[\frac{1}{\alpha_i} - \frac{1}{\alpha} \right] = \frac{\gamma}{n} \sum_{i=1}^n E \left[\frac{-\varepsilon_i}{(\alpha_i^*)^2} \right]$$

with α_i^* and $\tilde{\alpha}_i \in (\min(\alpha, \alpha_i), \max(\alpha, \alpha_i))$. As α_i^* and $\tilde{\alpha}_i$ are almost surely included in $[\alpha(1 - c_\alpha), \alpha(1 + c_\alpha)]$, we get

$$\left| \frac{1}{n} \sum_{i=1}^n E[Y_i^2] - E[Y^2] \right| \leq \left(\frac{\gamma^2 + \pi^2/6}{(\alpha(1 - c_\alpha))^3} + \frac{|-2\gamma^2 + 2\log(\sigma)|}{(\alpha(1 - c_\alpha))^2} \right) \frac{1}{n} \sum_{i=1}^n E[|\varepsilon_i|]$$

and

$$\left| \frac{1}{n} \sum_{i=1}^n E[Y_i] - E[Y] \right| \leq \frac{\gamma}{(\alpha(1 - c_\alpha))^2} \frac{1}{n} \sum_{i=1}^n E[|\varepsilon_i|]$$

if we suppose $\frac{1}{n} \sum_{i=1}^n E[|\varepsilon_i|] \rightarrow 0$, we get

$$\frac{1}{n} \sum_{i=1}^n E[Y_i^2] \xrightarrow{n \rightarrow \infty} E[Y^2] \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n E[Y_i] \xrightarrow{n \rightarrow \infty} E[Y].$$

The central limit theorem is proved as Prop. 4.2.

Acknowledgement

J. Lévy Véhel gratefully acknowledges financial support from SMABTP.

References

- [1] I. A. Dimitriadis, C. Alberola-López, M. Martín-Fernández, P. C. de-la Higuera, F. Simmross-Wattenberg, and J. I. Asensio-Pérez. Anomaly detection in network traffic based on statistical inference and α -stable modeling. *IEEE Transactions on Dependable and Secure Computing*, 8:494–509, 02 2011.
- [2] K. J. Falconer and J. Lévy Véhel. Multifractional, multistable, and other processes with prescribed local form. *Journal of Theoretical Probability*, 22(2):375–401, Jun 2009.
- [3] E. F. Fama and R. Roll. Parameter estimates for symmetric stable distributions (ref: V63 p817-36). *Journal of the American Statistical Association*, 66:331–338, 1971.
- [4] I. A. Koutrouvelis. Regression-type estimation of the parameters of stable laws. *J. Amer. Statist. Assoc.*, 75(372):918–928, 1980.
- [5] I. A. Koutrouvelis. An iterative procedure for the estimation of the parameters of stable laws. *Comm. Statist. B—Simulation Comput.*, 10(1):17–28, 1981.
- [6] E. E. Kuruoglu. Density parameter estimation of skewed alpha-stable distributions. *IEEE Transactions on Signal Processing*, 49(10):2192–2201, Oct 2001.
- [7] F. Lavancier and P. Rochet. A general procedure to combine estimators. *Comput. Statist. Data Anal.*, 94:175–192, 2016.
- [8] R. Le Guével. An estimation of the stability and the localisability functions of multistable processes. *Electron. J. Stat.*, 7:1129–1166, 2013.
- [9] R. Le Guével and J. Lévy Véhel. A Ferguson - Klass - LePage series representation of multistable multifractional processes and related processes. *Bernoulli*, 18(4):1099–1127, 2012.
- [10] X. Ma and C. L. Nikias. Parameter estimation and blind channel identification in impulsive signal environments. *IEEE transactions on signal processing*, 43(12):2884–2897, 1995.
- [11] B. B. Mandelbrot. *Fractals and Scaling In Finance: Discontinuity, Concentration, Risk*. Springer Publishing Company, Incorporated, 1st edition, 1997.
- [12] J. H. McCulloch. Simple consistent estimators of stable distribution parameters. *Comm. Statist. B—Simulation Comput.*, 15(4):1109–1136, 1986.
- [13] D. Salas-Gonzalez, J. M. Górriz, J. Ramírez, I. A. Illán, and E. W. Lang. Linear intensity normalization of fp-cit spect brain images using the α -stable distribution. *NeuroImage*, 65:449 – 455, 2013.

- [14] G. Samorodnitsky and M. S. Taqqu. *Stable non-Gaussian random processes*. Stochastic Modeling. Chapman & Hall, New York, 1994. Stochastic models with infinite variance.
- [15] P. Sen and J. Singer. *Large Sample Methods in Statistics: An Introduction with Applications*. Chapman & Hall/CRC Texts in Statistical Science. Taylor & Francis, 1994.
- [16] S. Stoev and M. Taqqu. Simulation methods for linear fractional stable motion and farima using the fast fourier transform. *Fractals*, 12(1):95–121, 3 2004.
- [17] R. Weron. Performance of the estimators of stable law parameters. Technical report, 1995.
- [18] C. Yang, K. Hsu, and K. Chen. The use of the levy-stable distribution for geophysical data analysis. *Hydrogeology Journal*, 17(5):1265–1273, 7 2009.