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# Explicit and combined estimators for symmetric stable distributions parameters

Jacques Lévy Véhel<sup>1</sup>      Anne Philippe<sup>2</sup>      Caroline Robet <sup>2</sup>

<sup>1</sup> INRIA Rennes Bretagne Atlantique & Case Law Analytics,

<sup>2</sup> Université de Nantes, Laboratoire de Mathématiques Jean Leray, UMR CNRS 6629

**Abstract:** This article focuses on the estimation of the stability index and scale parameter of symmetric stable random variables. While there is a sizable literature on this topic, no precise theoretical results seem available. We study an estimator based on log-moments, which always exist for such random variables. The main advantage of this estimator is that it has a simple closed form expression. This allows us to prove an almost sure convergence result as well as a central limit theorem. We show how to improve the accuracy of this estimator by combining it with previously defined ones. The closed form also enables us to consider the case of non identically distributed data, and we show that our results still hold provided deviations from stationarity are "small". As applications, we show numerically that the stability index of multistable Lévy motion may be estimated accurately and consider a financial log, namely the S&P 500, where we find that the stability index evolves in time in a consistent with major financial events.

*Keywords:* averaging estimates; misspecified model; moment estimate; Monte Carlo approximation; stable distribution

## 1 Introduction

The class of  $\alpha$ -stable distributions is ubiquitous in probability: such distributions appear as the limit of normalized sums of independent and identically distributed random variables. Their probability densities exist and are continuous but they are not known in closed form except for Gaussian distributions, Cauchy distributions, Lévy distributions and constants. Non-Gaussian stable distributions are a model of choice for real world phenomena exhibiting jumps. Indeed, for  $\alpha < 2$ , their density exhibit "heavy tails", resulting in a power-law decay of the probability of extreme events. They have been used extensively in recent years for modeling in domains such as biomedicine [13], geophysics [18], economy [10], finance [10], Internet traffic [1] and more. Non-Gaussian stable distributions are characterized by four parameters: a scale parameter usually denoted  $\sigma$ , (that reduces to variance in the Gaussian case), a stability parameter, denoted  $\alpha$ , that governs the heaviness of the tail and that ranges in  $(0, 2)$ , a location parameter  $\mu$  similar

to the mean in the case of Gaussian distributions, and an asymmetry parameter  $\beta$  ranging in  $[-1, 1]$ . In order to use stable distributions for real world phenomena modeling, one needs to estimate these parameters. A number of estimators are of common use, such as the ones proposed by Fama-Roll [3], Mc Culloch [11] and Koutrouvelis [4, 5]. A difficulty with these estimators is that they do not possess a simple closed form expression. As a consequence, and to the best of our knowledge, no theoretical results are known about them, such as almost sure convergence and central limit theorems. Their asymptotic distributions as well as asymptotic variances are thus only known numerical simulations. Another drawback of not having explicit and simple closed forms is that it is difficult to assess theoretically their performance in situations that slightly depart from the classical assumptions of identical and independent random samples. This is nevertheless desirable when one wishes to deal with real world data, which will often not verify these ideal hypotheses.

Our main aim in this work is to investigate the theoretical properties of a simple moment estimator. Since non-Gaussian stable random variables do not possess finite variance, and, in some cases, a well defined mean, we shall consider in fact an estimator based on log-moments. This idea is not new, as it has long been remarked that log-moment always exist for stable random variables and are convenient to work with. For instance [9] considers the same estimator as the one we study and applies it to blind channel identification. Owing to its simple expression, we are able to prove almost sure convergence and a central limit theorem both in an identically and independent framework and in a case of slight deviation from stationarity. We compare the performance of our estimator with the Koutrouvelis regression method [4, 5]. The results depends on the value of  $\alpha$  and on the size of the sample. We then combine these two estimators using a technique recently developed in [6] to enhance their performance, especially in the case of small samples. As applications, we show numerical experiments both on synthetic data (symmetric Lévy multistable motion) and on a financial log (S&P 500), which confirm our theoretical results that the estimator is able to track smooth enough variations of the stability index in time.

## 2 $\alpha$ -stable Distribution

There are several parameterizations of stable distributions, each of which having advantages and drawbacks. The following one is probably the most popular:

**Definition 2.1.** *A random variable  $X$  is said to have a stable distribution if there are parameters  $0 < \alpha \leq 2$ ,  $\sigma \geq 0$ ,  $-1 \leq \beta \leq 1$  and  $\mu \in \mathbb{R}$  such that its characteristic function has the following form:*

$$E[e^{itX}] = \begin{cases} \exp\left(-\sigma^\alpha |t|^\alpha \left(1 - i\beta \text{sign}(t) \tan\left(\frac{\pi\alpha}{2}\right)\right) + i\mu t\right), & \text{if } \alpha \neq 1, \\ \exp\left(-\sigma |t| \left(1 + i\beta \text{sign}(t) \frac{2 \log |t|}{\pi}\right) + i\mu t\right), & \text{if } \alpha = 1 \end{cases} \quad (2.1)$$

$$\text{where } \text{sign}(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0 \end{cases}$$

A stable distribution is therefore characterized by four parameters. We write

$$X \sim S_\alpha(\sigma, \beta, \mu)$$

to indicate that  $X$  has a stable distribution with the stability index  $\alpha$ , scale parameter  $\sigma$ , skewness  $\beta$  and location parameter  $\mu$  (see [14]). A drawback of this representation is that it is not continuous with respect to the parameters.

A continuous parameterization which is often used takes the following form [12]:

$$E[e^{itX}] = \begin{cases} \exp\left(-\sigma^\alpha |t|^\alpha \left(1 + i\beta \text{sign}(t) \tan\left(\frac{\pi\alpha}{2}\right) (\sigma^{1-\alpha} |t|^{1-\alpha} - 1)\right) + i\mu t\right), & \text{if } \alpha \neq 1, \\ \exp\left(-\sigma |t| \left(1 + i\beta \text{sign}(t) \frac{2(\log |t| + \log \sigma)}{\pi}\right) + i\mu t\right), & \text{if } \alpha = 1 \end{cases} \quad (2.2)$$

In the following sections, we deal exclusively with stable random variables such that  $\mu = \beta = 0$ . In this case, both representations coincide.

For a stable distribution, the density function is not known in closed form except in few cases (Gaussian distribution, Cauchy distribution, ...). This makes it difficult to use classical estimation methods based on the density function, such as maximum likelihood or Bayesian statistics: although the density may be estimated numerically, for instance through Fourier inversion, this procedure entails approximation errors that cannot be easily assessed.

Another difficulty is that, except in the Gaussian case, stable random variables have infinite moments of order at least the stability index. More precisely,

**Proposition 2.2.** (see Prop 1.2.16 in [14]) *Let  $X \sim S_\alpha(\sigma, \beta, \mu; 1)$  with  $0 < \alpha < 2$ . We have*

$$\begin{cases} E[|X|^p] < \infty & \text{for any } 0 < p < \alpha, \\ E[|X|^p] = \infty & \text{for any } p \geq \alpha. \end{cases} \quad (2.3)$$

This property implies that one cannot use a moment method for estimation. We study in the next section a simple log-moment estimator that allows one to make explicit computations.

### 3 Log-moment estimation method

For a symmetric stable distribution closed form expressions are available for absolute log-moments [7], which allow one to derive expressions for estimating  $\alpha$  and  $\sigma$ . First, note the following property:

**Proposition 3.1.** *Let  $Z \sim S_\alpha(1, 0, 0)$ , then*

$$E[|\log |Z||^p] < \infty \quad \text{for all } p > 0. \quad (3.1)$$

*Proof.* See Appendix. □

These expectations may be computed explicitly by remarking that:

$$E[(\log |Z|)^p] = \left. \frac{d^p E[|Z|^t]}{dt^p} \right|_{t=0}, \quad (3.2)$$

and by using the following result:

**Proposition 3.2.** *Let  $Z \sim S_\alpha(1, 0, 0)$ . Then, for all  $0 < t < \min(\alpha, 1)$ ,*

$$E[|Z|^t] = \frac{\Gamma(1 - t/\alpha)}{\Gamma(1 - t) \cos(\pi t/2)}. \quad (3.3)$$

*Proof.* See Appendix. □

We deduce that  $E[\log |Z|] = (\frac{1}{\alpha} - 1) \gamma$  and  $\text{Var}(\log |Z|) = \frac{\pi^2}{6\alpha^2} + \frac{\pi^2}{12}$ , where  $\gamma$  is Euler constant ( $\gamma = 0.57721\dots$ ).

**Theorem 3.3.** *Let  $(X_1, \dots, X_n)$  be a sequence of independent and identically distributed (iid) standard symmetric stable random variables  $S_\alpha(1, 0, 0)$ . Define*

$$\hat{\alpha}_n(X_1, \dots, X_n) = \frac{\gamma}{\gamma + \frac{1}{n} \sum_{i=1}^n \log |X_i|}.$$

*Then,  $\hat{\alpha}_n \xrightarrow{a.s.} \alpha$  when  $n \rightarrow +\infty$ . Moreover, with  $f(x) = \frac{\pi^2}{6x^2} + \frac{\pi^2}{12}$ ,*

$$\frac{\sqrt{n}(\hat{\alpha}_n - \alpha)\gamma}{\hat{\alpha}_n^2 \sqrt{f(\hat{\alpha}_n)}} \xrightarrow{d} \mathcal{N}(0, 1). \quad (3.4)$$

*Proof.* The proof of this result may be found in the appendix. □

In general,  $\sigma$  is unknown and we must use a joint estimation of both parameters. Let  $W \sim S_\alpha(\sigma, 0, 0)$ , then  $\frac{W}{\sigma} \sim S_\alpha(1, 0, 0)$  and we deduce the log-moment of  $W$  by using the log-moment of  $Z$ . We get  $E[\log |W|] = (\frac{1}{\alpha} - 1) \gamma + \log \sigma$  and  $\text{Var}(\log |W|) = \frac{\pi^2}{6\alpha^2} + \frac{\pi^2}{12}$ .

**Theorem 3.4.** *Let  $(X_1, \dots, X_n)$  be a sequence of independent and identically distributed (iid) symmetric stable random variables  $S_\alpha(\sigma, 0, 0)$ . Define*

$$\hat{\alpha}_{LOG}^{(n)}(X_1, \dots, X_n) = \frac{\pi}{\sqrt{\max(\frac{6}{n-1} \sum_{i=1}^n (\log |X_i| - \frac{1}{n} \sum_{k=1}^n \log |X_k|)^2 - \frac{\pi^2}{2}, \frac{\pi^2}{4})}}$$

and

$$\hat{\sigma}_{LOG}^{(n)}(X_1, \dots, X_n) = \exp \left( \frac{1}{n} \sum_{i=1}^n \log |X_i| - \left( \frac{1}{\hat{\alpha}_{LOG}^{(n)}} - 1 \right) \gamma \right).$$

*Then,*

$$(\hat{\alpha}_{LOG}^{(n)}, \hat{\sigma}_{LOG}^{(n)}) \xrightarrow{a.s.} (\alpha, \sigma) \text{ when } n \rightarrow +\infty. \quad (3.5)$$

Moreover,

$$\sqrt{n} \left( \begin{pmatrix} \widehat{\alpha}_{LOG}^{(n)} \\ \widehat{\sigma}_{LOG}^{(n)} \end{pmatrix} - \begin{pmatrix} \alpha \\ \sigma \end{pmatrix} \right) \xrightarrow{d} \mathcal{N}(0, F_{\alpha,\sigma} G_{\alpha,\sigma} \Sigma_{\alpha,\sigma} {}^t G_{\alpha,\sigma} {}^t F_{\alpha,\sigma}), \quad (3.6)$$

where

$$F_{\alpha,\sigma} = \begin{pmatrix} \sigma & \gamma\sigma/\alpha^2 \\ 0 & 1 \end{pmatrix}, \quad (3.7)$$

$$G_{\alpha,\sigma} = \begin{pmatrix} 1 & 0 \\ 6 \left( \left( \frac{1}{\alpha} - 1 \right) \gamma + \log \sigma \right) \alpha^3 / \pi^2 & -3\alpha^3 / \pi^2 \end{pmatrix}, \quad (3.8)$$

$$\Sigma_{\alpha,\sigma} = \begin{pmatrix} \text{Var}(\log |X_1|) & \text{Cov}(\log |X_1|, (\log |X_1|)^2) \\ \text{Cov}(\log |X_1|, (\log |X_1|)^2) & \text{Var}((\log |X_1|)^2) \end{pmatrix}. \quad (3.9)$$

*Proof.* The proof of this result may be found in the appendix.  $\square$

## 4 Numerical experiments and comparison with Koutrouvelis estimator

We conduct in this section extensive numerical experiments to assess the performance of the log-moment estimator on synthetic data. We also present comparisons with a classical estimator, namely the Koutrouvelis one. The choice of this particular estimator is motivated by the results reported in [17] showing that it performs usually better than other methods such as the Fama-Roll and Mc Culloch ones.

### 4.1 Koutrouvelis Regression Method

The Koutrouvelis estimator (see [4] and [5]) is based on exploiting the explicit expression of the iterated logarithm of the characteristic function  $\phi$ . In the symmetric case, it takes the particularly simple form

$$\log(\log(|\phi(t)|^2)) = \log(2\sigma^\alpha) + \alpha \log |t|. \quad (4.1)$$

The empirical characteristic function given by  $\widehat{\phi}_n(t) = \frac{1}{n} \sum_{j=1}^n e^{itX_j}$  based on i.i.d observations  $(X_j)$  is a consistent estimator of  $\phi$ . We estimate these parameters by regressing  $y = \log(\log(|\widehat{\phi}_n(t)|^2))$  on  $w = \log |t|$  in the model

$$y_k = m + \alpha w_k + \epsilon_k \quad (4.2)$$

where  $m = \log(2\sigma^\alpha)$ ,  $t_k = \frac{\pi k}{25}$  for  $k \in \{1, \dots, K\}$  with  $K$  depending on the parameter  $\alpha$  and on the sample size, and  $\epsilon_k$  denotes an error term. In our simulations, we use an easier version of the Koutrouvelis regression method which is more adapted for the symmetric case (see [17]). We describe the algorithm:

- Define an admissible error ( $tol = 0.05$  in the simulation) and the maximum number of iterations ( $iter_{max} = 10$ ).

- A regression applied to the Mc Culloch quantile method [11] provides initial estimates  $\hat{\alpha}^{(n)}$  and  $\hat{\sigma}^{(n)}$ , where  $n$  denotes the length of the observed sample.
- While the number of iterations is less than  $iter_{max}$  and  $|\hat{s} - 1| > tol$  (starting from the second iteration):
  - find the number  $K$  of points in the regression depending on  $\hat{\alpha}^{(n)}$  as in the classical Koutrouvelis regression,
  - define  $w = (w_k)_{k \in \{1 \dots K\}}$  and  $y = (y_k)_{k \in \{1 \dots K\}}$  by  $w_k = \log |\pi k / 25|$ ,

$$y_k = \log(-\log \left( \left| \hat{\phi}_n \left( \frac{\pi k}{25 \hat{\sigma}^{(n)}} \right) \right|^2 \right)),$$

- compute the new  $\hat{\alpha}^{(n)}$  given by

$$\hat{\alpha}^{(n)} = \min \left( \frac{\sum_{k=1}^K (w_k - \text{mean}(w))(y_k - \text{mean}(y))}{\sum_{k=1}^K (w_k - \text{mean}(w))^2}, 2 \right)$$

- set  $\hat{s} = \exp \left( \frac{\text{mean}(y - \hat{\alpha}^{(n)} w) - \log(2)}{\hat{\alpha}^{(n)}} \right)$
- set  $\hat{\sigma}^{(n)} = \hat{\sigma}^{(n)} \hat{s}$

This modified version of Koutrouvelis gives performances (in terms of mean squared errors) similar to the original. However, it is much faster because this version does not necessitate the estimation of the parameters  $\beta$  and  $\mu$ , which requires the numerical inversion of matrices of size  $n \times n$ .

## 4.2 Performance of the estimators

In general, the Koutrouvelis regression method has a smaller mean squared error than other classical estimation methods based on empirical quantiles (Fama-Roll method [3], Mc Culloch's method [11]). A comparative of the performances of these classical methods is performed in [17]. For each pair of values  $(\alpha, \sigma)$ ,  $r$  independent samples of size  $n$  of iid stable random variables were generated. The empirical mean squared error of the sampling distribution of  $\alpha$  and  $\sigma$  is given by

$$MSE_{\alpha} = \frac{1}{r} \sum_{i=1}^r (\hat{\alpha}_i - \alpha)^2, \quad MSE_{\sigma} = \frac{1}{r} \sum_{i=1}^r (\hat{\sigma}_i - \sigma)^2,$$

where  $\hat{\alpha}$  (resp  $\hat{\sigma}$ ) is an estimator of  $\alpha$  (resp  $\sigma$ ).

In the sequel, we use the abbreviation "KOUT" and "LOG" to refer respectively to the Koutrouvelis and log-moment estimator. For each  $\alpha$ , the behaviors of  $\hat{\alpha}_{KOUT}$  and  $\hat{\alpha}_{LOG}$  are similar for all value of  $\sigma$  (Table 1) whereas, for each value of  $\sigma$ ,  $\hat{\sigma}_{KOUT}$  and  $\hat{\sigma}_{LOG}$  improve when  $\alpha$  is increasing (Table 2). Besides, when  $\alpha$  is fixed,  $\hat{\sigma}_{KOUT}$  and

$\hat{\sigma}_{LOG}$  have the same behavior for all  $\sigma$ . For this reason, only the case where  $\sigma$  equals 1 is represented in Table 2.

	$\alpha = 0.2$		$\alpha = 0.6$		$\alpha = 1$		$\alpha = 1.4$		$\alpha = 1.8$	
	LOG	KOUT	LOG	KOUT	LOG	KOUT	LOG	KOUT	LOG	KOUT
$\sigma = 10$	0.0906	0.470	0.906	2.35	4.67	3.75	19.7	8.20	31.0	4.17
$\sigma = 1$	0.0807	0.427	1.06	2.11	4.47	3.91	22.0	7.58	31.9	4.28
$\sigma = 0.1$	0.0893	0.466	0.959	1.96	4.43	4.22	22.0	7.58	29.7	4.29

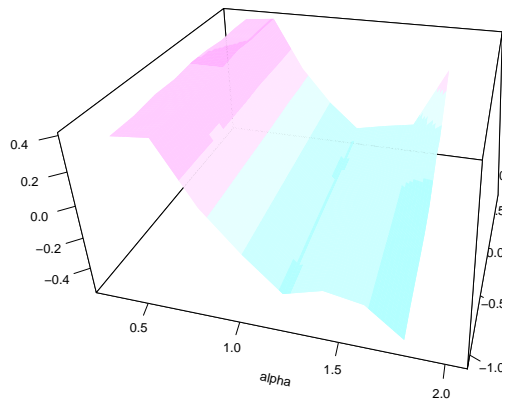
Table 1: Mean squared error for  $\hat{\alpha}_{KOUT}$  and  $\hat{\alpha}_{LOG}$  ( $r = 500$  and  $n = 500$ ). Each entry has been multiplied by 1000 for the sake of readability.

		$\alpha = 0.2$	$\alpha = 0.6$	$\alpha = 1$	$\alpha = 1.4$	$\alpha = 1.8$
$\sigma = 10$	LOG	7.101	$9.862e^{-01}$	$6.082e^{-01}$	$6.066e^{-01}$	$5.428e^{-01}$
	KOUT	11.501	$9.919e^{-01}$	$4.819e^{-01}$	$3.556e^{-01}$	$1.717e^{-01}$
$\sigma = 1$	LOG	$8.20e^{-02}$	$9.45e^{-03}$	$6.52e^{-03}$	$6.98e^{-03}$	$5.77e^{-03}$
	KOUT	$1.18e^{-01}$	$9.82e^{-03}$	$4.44e^{-03}$	$3.83e^{-03}$	$1.56e^{-03}$
$\sigma = 0.1$	LOG	$6.73e^{-04}$	$8.77e^{-05}$	$6.71e^{-05}$	$6.54e^{-05}$	$5.89e^{-05}$
	KOUT	$8.63e^{-04}$	$8.60e^{-05}$	$4.75e^{-05}$	$3.35e^{-05}$	$1.82e^{-05}$

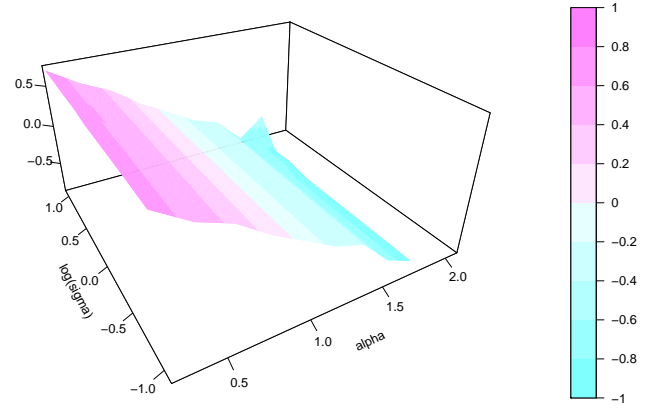
Table 2: Mean squared error for  $\hat{\sigma}_{KOUT}$  and  $\hat{\sigma}_{LOG}$  ( $r = 500$  and  $n = 500$ ).

With a simulation study, we compare the empirical mean squared errors of  $\alpha$  and  $\sigma$  for the methods introduced earlier. We represent the logarithmic ratio of mean squared errors on  $\alpha$  and  $\sigma$  for the log-moment and the regression methods and several samples of size  $r \times n$ . For each sample of  $r \times n$  observations,  $\alpha$  and  $\sigma$  are fixed ( $\alpha \in \{0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8\}$  and  $\log(\sigma) \in \{-1, -0.75, -0.5, -0.25, 0, 0.25, 0.5, 0.75, 1\}$ ) (see Figure 1). Tables 1-2 and Figures 1-2 show that the log-moment estimator performs better than the Koutrouvelis one when  $\alpha < 1$ , while the converse is true for  $\alpha > 1$ , with the differences in performance increasing for extreme values of  $\alpha$ .

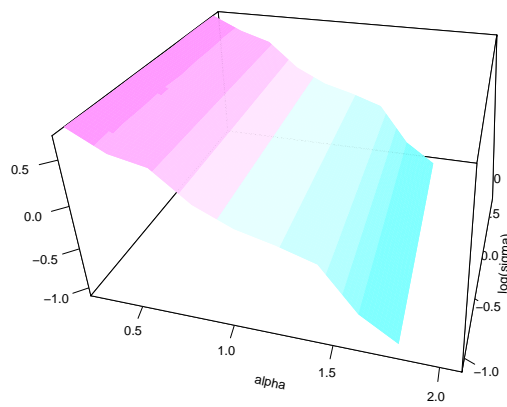




(a)

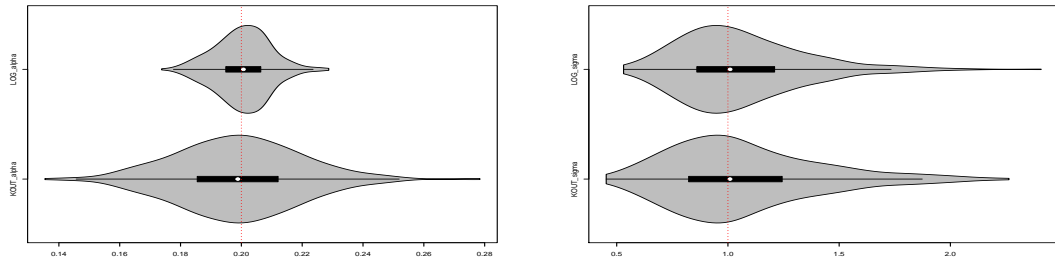


(b)

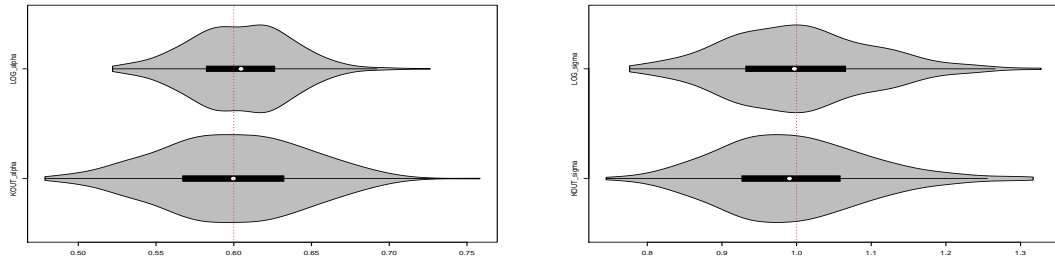


(c)

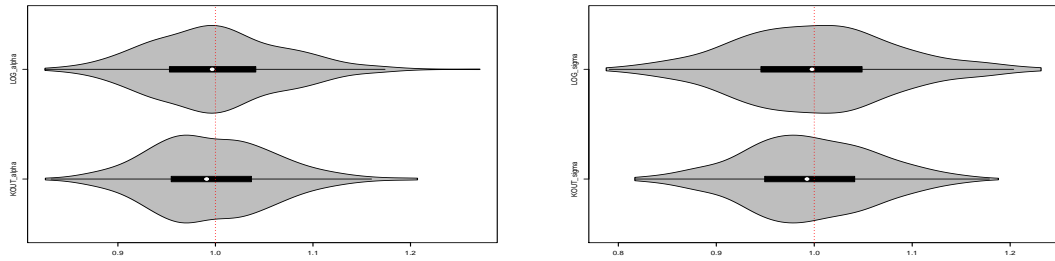
Figure 1: Log of the ratio between the  $\alpha$  mean squared error for Koutrouvelis estimator and log estimator  $r = 1000$  and  $n = 100$  in Figure (a),  $n = 500$  in Figure (b),  $n = 2000$  in Figure (c).



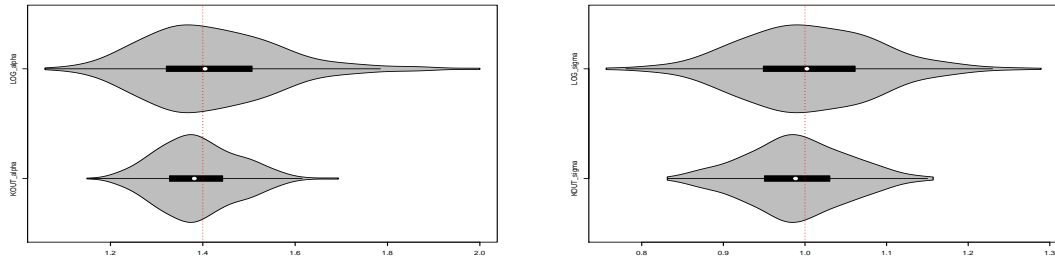
$\alpha=0.2, \sigma=1$



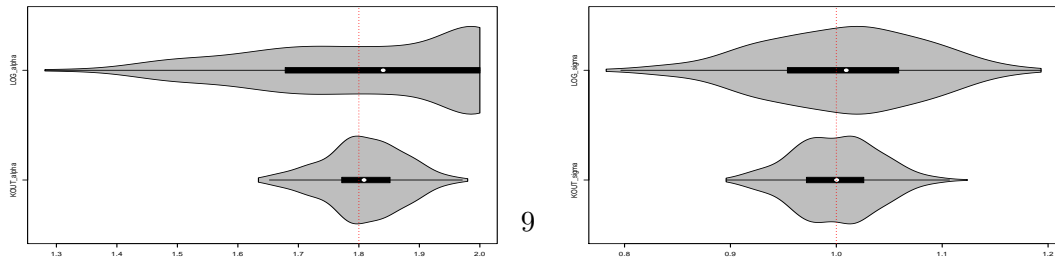
$\alpha=0.6, \sigma=1$



$\alpha=1, \sigma=1$



$\alpha=1.4, \sigma=1$



$\alpha=1.8, \sigma=1$

Figure 2: Empirical density functions for Log-moment (LOG) and Koutrouvelis regression (KOUT) estimators of  $\alpha$  in the first column and  $\sigma$  in the second column for  $r = 500$  and  $n = 500$ .

## 5 A combined estimator

As seen in the previous section, the Koutrouvelis regression estimator is preferred when  $\alpha > 1$  whereas the log-estimator becomes significantly better than the Koutrouvelis estimator, as  $\alpha (\leq 1)$  decreases. We want to construct an estimator of  $\alpha$  which will be at least as good as the best estimator, for each  $\alpha$ , for small samples ( $n \approx 100$ ). To do this, we use a combined estimator whose general procedure of construction is described in [6]. In our special case,  $\theta = (\alpha, \sigma)^\top$  are the parameters to estimate and we have access to the estimators  $\hat{\alpha}_{KOUT}$ ,  $\hat{\alpha}_{LOG}$  and  $\hat{\sigma}_{KOUT}$ <sup>1</sup>.

We consider averaging estimators of  $\theta$  of the form

$$\hat{\theta}_\lambda = \lambda^\top \begin{pmatrix} \hat{\alpha}_{KOUT} \\ \hat{\alpha}_{LOG} \\ \hat{\sigma}_{KOUT} \end{pmatrix}, \quad \lambda \in \Lambda, \quad (5.1)$$

where  $\lambda^\top$  denotes the transpose of  $\lambda$  and  $\Lambda$  is a given subset of  $\mathbb{R}^{3 \times 2}$ .

A convenient way to measure the performance of  $\hat{\theta}_\lambda$  is to compare it to  $\hat{\theta}^*$ , defined as the best linear combination  $\hat{\theta}_\lambda$  obtained for a non-random vector  $\lambda \in \Lambda$ . Specifically,  $\hat{\theta}^*$  is the linear combination  $\lambda^{*\top} \begin{pmatrix} \hat{\alpha}_{KOUT} \\ \hat{\alpha}_{LOG} \\ \hat{\sigma}_{KOUT} \end{pmatrix}$  minimizing the mean square error (MSE), i.e.  $\lambda^* = \underset{\lambda \in \Lambda}{\operatorname{argmin}} E[\|\hat{\theta}_\lambda - \theta\|^2]$ .

Clearly, the larger the set  $\Lambda$  is, the better it will be. However, choosing the whole space  $\Lambda = \mathbb{R}^{3 \times 2}$  is generally not exploitable. We must impose some conditions on the set  $\Lambda$  in order to have an explicit form for  $\lambda^*$ .

Define

$$J = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and consider the maximal constraint set

$$\Lambda_{\max} = \{\lambda \in \mathbb{R}^{3 \times 2} / \lambda^\top J = I_2\} = \left\{ \begin{pmatrix} a & b \\ 1-a & b \\ 0 & 1 \end{pmatrix} / (a, b) \in \mathbb{R}^2 \right\}$$

with  $I_2$  the identity matrix. The mean squared error  $E[\|\hat{\theta}_\lambda - \theta\|^2]$  is minimized on the set  $\Lambda_{\max}$  for a unique solution

$$\lambda^* = \Sigma^{-1} J (J^\top \Sigma^{-1} J)^{-1}, \quad (5.2)$$

where  $\Sigma$  is the Gram matrix

$$\Sigma = E \left( \begin{pmatrix} \hat{\alpha}_{KOUT} - \alpha \\ \hat{\alpha}_{LOG} - \alpha \\ \hat{\sigma}_{KOUT} - \sigma \end{pmatrix} \begin{pmatrix} \hat{\alpha}_{KOUT} - \alpha \\ \hat{\alpha}_{LOG} - \alpha \\ \hat{\sigma}_{KOUT} - \sigma \end{pmatrix}^\top \right). \quad (5.3)$$

---

<sup>1</sup>Since  $\hat{\sigma}_{KOUT}$  and  $\hat{\sigma}_{LOG}$  have similar performances, we use only one in the combination.

Since the matrix  $\Sigma$  is unknown, the averaging estimator  $\hat{\theta}_{\max}$  is obtained by replacing  $\Sigma$  by its estimation  $\hat{\Sigma}$ :

$$\begin{cases} \hat{\lambda}_{\max} = \hat{\Sigma}^{-1} J (J^{\top} \hat{\Sigma}^{-1} J)^{-1} \\ \hat{\theta}_{\max} = \hat{\lambda}_{\max}^{\top} \begin{pmatrix} \hat{\alpha}_{KOUT} \\ \hat{\alpha}_{LOG} \\ \hat{\sigma}_{KOUT} \end{pmatrix} \end{cases} \quad (5.4)$$

A way to estimate  $\Sigma$  is to use Monte Carlo simulations. Let us compute a first estimation of the parameters by

$$\begin{cases} \hat{\alpha}_0 = \frac{\hat{\alpha}_{KOUT} + \hat{\alpha}_{LOG}}{2} \\ \hat{\sigma}_0 = \hat{\sigma}_{KOUT} \end{cases} \quad (5.5)$$

We simulate  $B$  samples of size  $n$  of a symmetric stable distribution with parameters  $\hat{\alpha}_0$  and  $\hat{\sigma}_0$ . Then, the three estimators are computed, which gives  $\hat{\alpha}_{KOUT}^{(b)}$ ,  $\hat{\alpha}_{LOG}^{(b)}$  and  $\hat{\sigma}_{KOUT}^{(b)}$  for  $b = 1 \dots B$ , and each entry of  $\Sigma$  is estimated by its empirical counterpart. Experiments show that errors entailed by the estimation of  $\Sigma$  are negligible compared to the advantage of having several estimators for small samples. Note that similar estimators could be built by combining more than 2 estimators for  $\alpha$ . For example, it would be possible to add the Mc Culloch quantile estimator. Thus would increases the size of the covariance matrix whose estimation will be worse and entail the risk of constructing a combined estimator never better than each individual ones. The weight for the log estimator in the combination is represented in Figure 3. We represent in Table 3 the mean squared errors for several values of  $\alpha$ . For each value, we remark that the combination between Koutrouvelis and log estimators is always better than each estimator separately. This is confirmed by the plots in Figure 4 comparing the empirical distributions of each estimator.

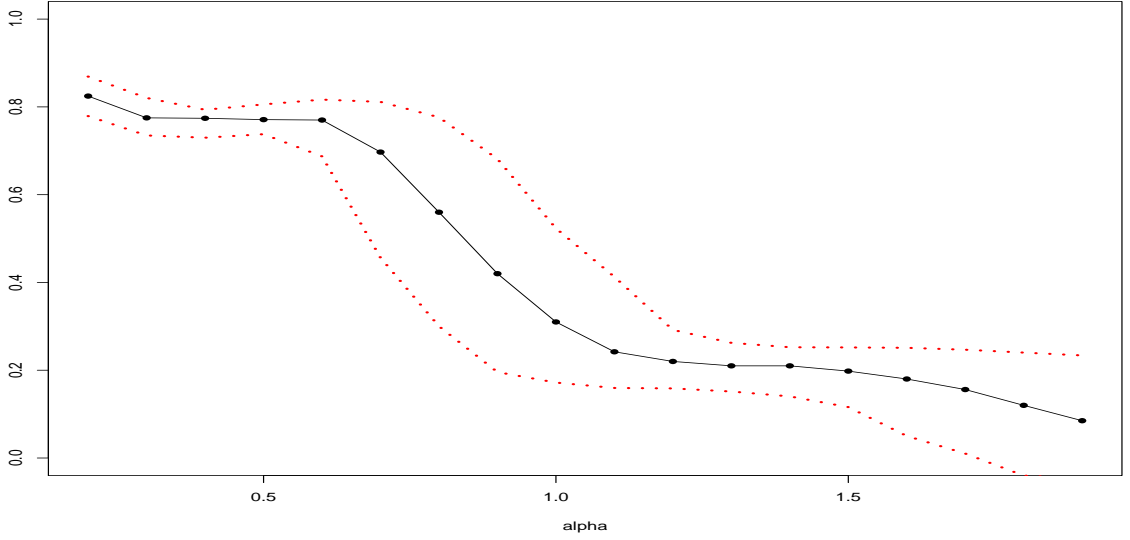


Figure 3: Weight (average) for the LOG-moment estimator  $\hat{\alpha}_{LOG}$  in the combined estimator depending on  $\alpha$ . The upper (resp. lower) bound of the interval correspond to the 95% quantile (resp. 5%) for  $r = 500$  replications of the combination with  $n = 100$  and  $B = 1000$ .

$\alpha$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
KOUT	2.4	3.1	5.0	8.3	13	15.1	14.1	15.0	18.1
LOG	0.5	0.9	1.9	3.4	5.6	7.6	12.0	19.9	33.1
COMB	0.004	0.7	1.5	2.9	5.2	6.8	8.3	10.8	15.1
$\alpha$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
KOUT	19.5	27.7	34.3	40.2	41.7	41.1	28.4	19.6	11.3
LOG	46.3	58.7	78.4	85.4	88.2	98.8	87.3	73.8	75.9
COMB	17.6	25.6	31.2	33.6	34.5	35.4	25.5	19.1	11.1

Table 3: Mean squared errors ( $\times 1000$ ) for Koutrouvelis regression (KOUT), Log-moment (LOG) and the combined (COMB) estimators of  $\alpha$  for  $r = 500$ ,  $n = 100$ ,  $B = 1000$  and  $\sigma = 1$ .

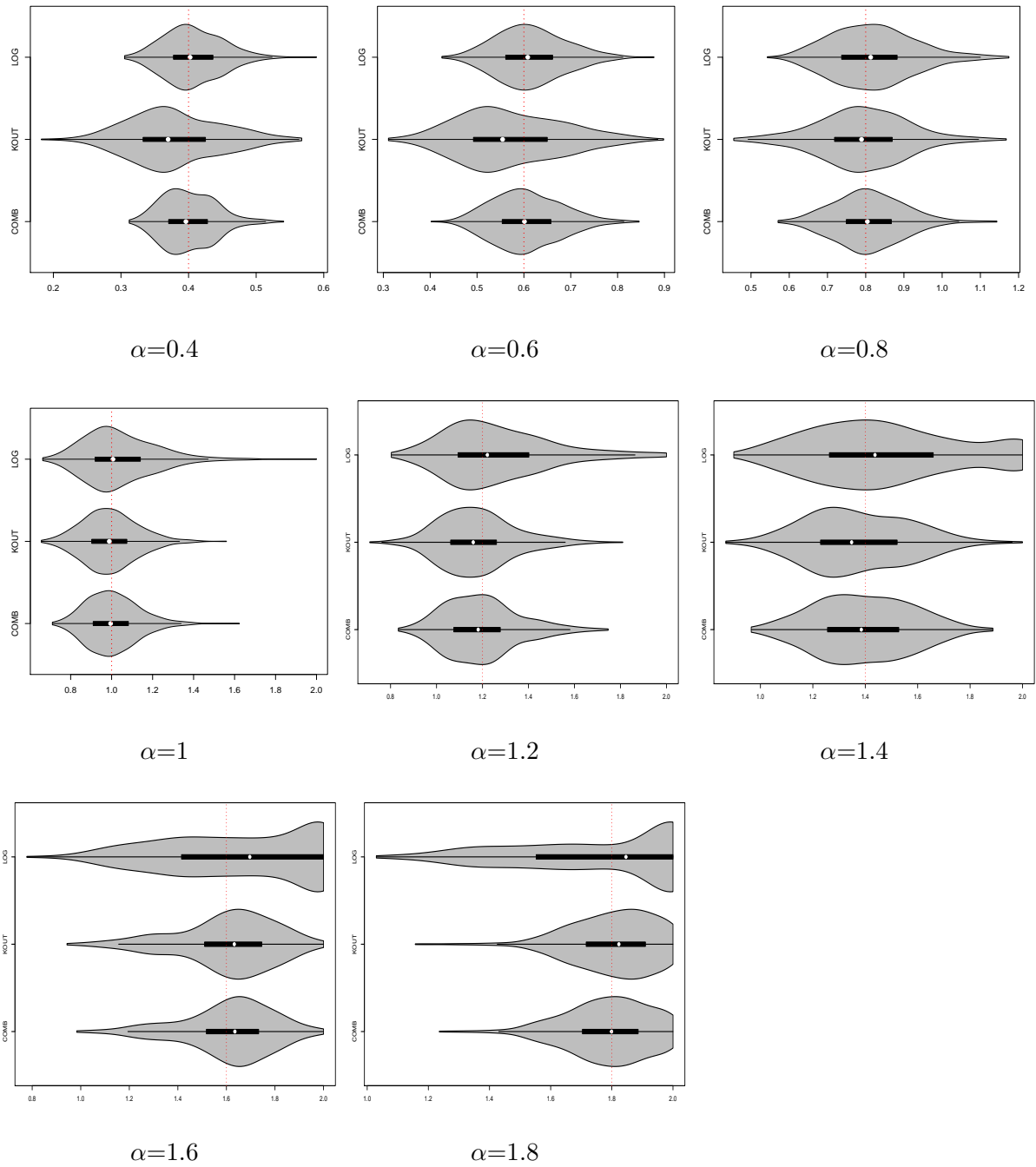


Figure 4: Empirical density functions for Koutrouvelis regression (KOUT), Log-moment (LOG) and the combined (COMB) estimators of  $\alpha$  for  $r = 500$ ,  $n = 100$ ,  $B = 1000$  and  $\sigma = 1$ .

## 6 Case of non-identically distributed stable variables

In applications, it may be the case that one needs to analyze non-stationary phenomena. For instance, it seems plausible that financial logs which display jumps will see the intensity of these jump depend on external events, such as crises (see next section for an illustration on the S&P 500). Sometimes, the variation of  $\alpha$  will be slow, and it is of interest to investigate under which conditions our estimator still behaves correctly in situations where the data at hand deviate slightly from the iid hypothesis. In the sequel, we examine two cases: deterministic and random small perturbations of  $\alpha$ , leading to random variables which are not identically distributed. We do not dispense here with the independence assumption, although this would be a desirable extension. This generalization will be the topic of a future work.

### 6.1 Deterministic perturbations

Let  $(X_i)$  be a sequence of independent variables and  $X$  random variable independent of  $(X_i)$  such that

$$X_i \sim S_{\alpha_i}(\sigma_i, 0, 0) \text{ and } X \sim S_{\alpha}(\sigma, 0, 0).$$

We denote  $Y_i = \log |X_i|$ ,  $Y = \log |X|$ . Assume that, for each integer  $i$ ,  $\alpha_i = \alpha + \varepsilon_i$  and  $\sigma_i = \sigma + \eta_i$  with  $\varepsilon_i$  and  $\eta_i$  deterministic such that

$$\frac{|\varepsilon_i|}{\alpha} \leq c_{\alpha} < 1 \text{ and } \frac{|\eta_i|}{\sigma} \leq c_{\sigma} < 1.$$

**Proposition 6.1.** *Under the conditions*

$$\frac{1}{n} \sum_{i=1}^n |\varepsilon_i| \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n |\eta_i| \xrightarrow[n \rightarrow \infty]{} 0,$$

one has

$$\widehat{\alpha}_{LOG}^{(n)} \xrightarrow[n \rightarrow \infty]{a.s.} \alpha.$$

where  $\widehat{\alpha}_{LOG}^{(n)}$  is defined in Theorem 3.4.

*Proof.* See Appendix. □

**Proposition 6.2.** *Let  $\Sigma_{\alpha, \sigma}$  be the covariance matrix between  $Y$  and  $Y^2$ :*

$$\Sigma_{\alpha, \sigma} = \begin{pmatrix} \text{Var}(Y) & \text{Cov}(Y, Y^2) \\ \text{Cov}(Y, Y^2) & \text{Var}(Y^2) \end{pmatrix} \quad (6.1)$$

and set

$$H_{\alpha, \sigma} = \begin{pmatrix} 6\left(\frac{1}{\alpha} - 1\right)\gamma + \log \sigma & \alpha^3 / \pi^2 \\ -3\alpha^3 / \pi^2 & \end{pmatrix} \quad (6.2)$$

With the conditions

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n |\varepsilon_i| \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{and} \quad \frac{1}{\sqrt{n}} \sum_{i=1}^n |\eta_i| \xrightarrow[n \rightarrow \infty]{} 0,$$

the following central limit theorem holds for  $\hat{\alpha}_{LOG}^{(n)}$ :

$$\sqrt{n} \left( \hat{\alpha}_{LOG}^{(n)} - \alpha \right) \xrightarrow{d} \mathcal{N}(0, {}^t H_{\alpha,\sigma} \Sigma_{\alpha,\sigma} H_{\alpha,\sigma}) \quad (6.3)$$

*Proof.* See Appendix. □

## 6.2 Random perturbations

Let  $(X_i)$  be a sequence of independent variables and  $X$  random variable independent of  $(X_i)$  such that  $X_i \sim S_{\alpha_i}(\sigma, 0, 0)$  and  $X \sim S_{\alpha}(\sigma, 0, 0)$ . Assume that, for each integer  $i$ ,  $\alpha_i = \alpha + \varepsilon_i$ , with  $\varepsilon_i$  a random variable, and that there is a constant  $c_{\alpha} < 1$  such that  $\mathbb{P}(\alpha_i \in [\alpha(1 - c_{\alpha}), \alpha(1 + c_{\alpha})]) = 1$ . We denote  $Y_i = \log |X_i|$ ,  $Y = \log |X|$ .

**Proposition 6.3.** *Under the conditions*

$$\frac{1}{n} \sum_{i=1}^n E[|\varepsilon_i|] \xrightarrow{n \rightarrow \infty} 0$$

we have

$$\hat{\alpha}_{LOG}^{(n)} \xrightarrow[n \rightarrow \infty]{a.s.} \alpha.$$

If, in addition,  $\frac{1}{\sqrt{n}} \sum_{i=1}^n E[|\varepsilon_i|] \xrightarrow{n \rightarrow \infty} 0$ , then the following central limit theorem holds:

$$\sqrt{n} \left( \hat{\alpha}_{LOG}^{(n)} - \alpha \right) \xrightarrow{d} \mathcal{N}(0, H_{\alpha,\sigma} \Sigma_{\alpha,\sigma} {}^t H_{\alpha,\sigma})$$

where  $\Sigma_{\alpha,\sigma}$  and  $H_{\alpha,\sigma}$  are defined in (6.1) and (6.2).

*Proof.* See Appendix. □

## 7 Numerical results on synthetic data: multistable Lévy motion

We now put our log-moment and combined estimators to use in the case of the multistable Lévy motion defined in [2] (see also [8] for further properties of this process). The basic idea is to allow the stability index to evolve with time, so that the jump intensity, which is governed by  $\alpha$ , varies along a trajectory. Such a feature is commonly encountered in time series observed in fields such as finance or biomedicine. Let us briefly recall the definition of such processes.

Let  $\alpha : [0, 1] \rightarrow (0, 2)$  be continuously differentiable. We note  $r^{<s>} = \text{sign}(r)|r|^s$  for  $r \in \mathbb{R}$  and  $s \in \mathbb{R}$ . Symmetric multistable Lévy motion is defined by

$$M_{\alpha}(t) = C_{\alpha}(t) \sum_{(X,Y) \in \Pi} 1_{(0,t]}(X) Y^{<-1/\alpha(t)>} \quad (7.1)$$



where  $C_\theta = \left( \int_0^\infty u^{-\theta} \sin(u) du \right)^{-1/\theta}$  and  $\Pi$  is a Poisson point process on  $\mathbb{R}^+ \times \mathbb{R}$  with plane Lebesgue measure  $\mathcal{L}^2$  as mean measure. This process is simulated by using the field

$$X(t, u) = C_{\alpha(u)} \sum_{(X,Y) \in \Pi} 1_{(0,t]}(X) Y^{<-1/\alpha(u)>}.$$

For each  $u \in (0, 1)$ ,  $X(\cdot, u)$  is an  $\alpha(u)$ -stable process with independent increments which can be implemented using the RSTAB program available in [16] or in [14]. The interval  $[0, 1]$  is discretised in  $N$  equal parts and  $X(\cdot, u)$  is implemented by the cumulative sum of  $N$  independent stable variables with  $\alpha(u)$  as characteristic exponent.

In Figure 5, we display sample paths of multistable processes for several  $\alpha$  functions. Then, we estimate these functions at all point  $t_0$  thanks to the combined estimator with a window of  $n$  observations around  $t_0$ .

In Figure 6, we iterate 100 times the simulation and the estimation for a multistable process with  $\alpha(t) = 1.5 - 0.48 \sin(2\pi(t + 1/4))$ . For each point where the function  $\alpha$  is estimated, we obtain the empirical distribution of the combined estimator. This procedure is repeated for several sizes of window (100, 200, 1000 and 2000). We observe that the standard error which correspond to the standard deviation for the combined estimator is decreasing when the size of the window  $n$  increases whereas the bias is increasing for  $n$  large. Finally, the mean squared error is decreasing when  $n$  increases until  $n = 1000$  and then increases for larger value. Figure 7 represents the bias, standard error and mean square error as function of  $t$  for various values of the window size.

The mean squared error as a function of  $\alpha$  is reported in Figure 8. The mean squared error does not vary much according to the value of  $\alpha$  in  $[1, 2]$  when  $n$  is fixed.

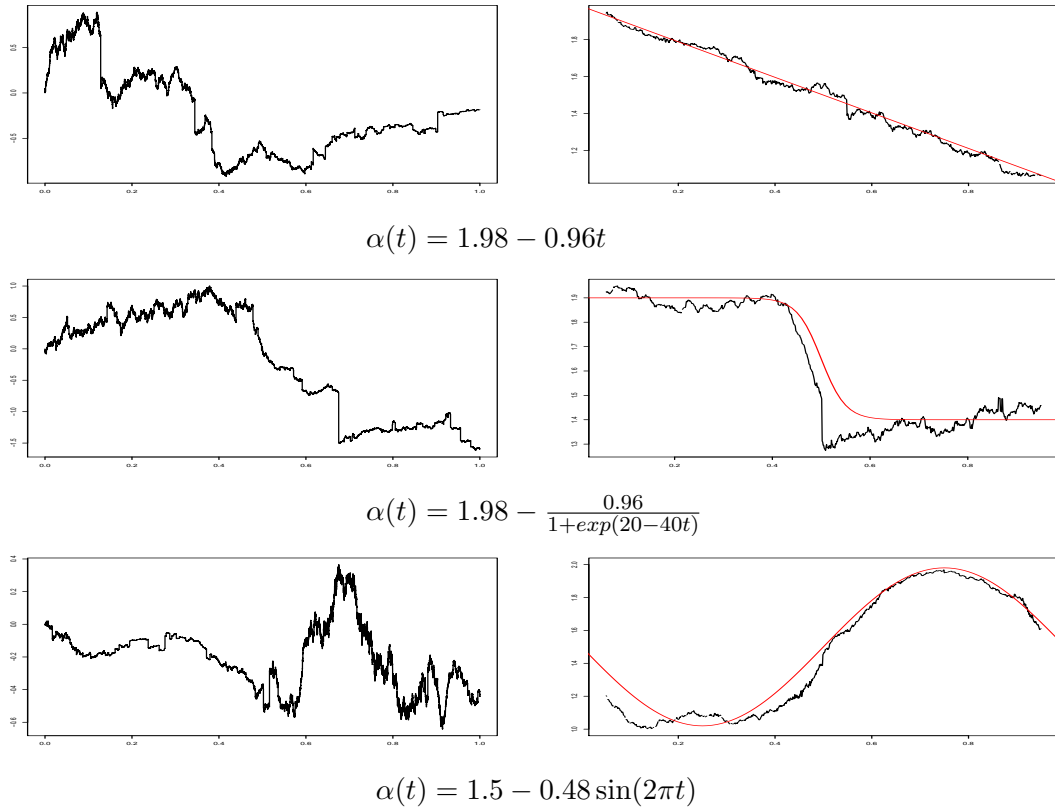
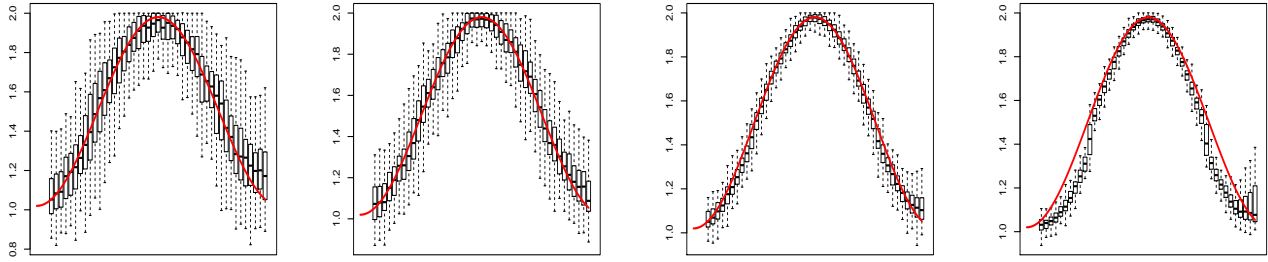


Figure 5: Trajectories of multistable processes on  $(0,1)$  with  $N = 20000$  points in the first column. The functions  $\alpha(t)$  (red) and  $\widehat{\alpha}_{COMB}(t)$  (black) are represented in the second column with  $n = 2000$ .



(a)  $n = 100$

(b)  $n = 200$

(c)  $n = 1000$

(d)  $n = 2000$

Figure 6: Box-plots of the estimator  $\widehat{\alpha}_{COMB}^{(n)}$  for 100 replications of a multistable process with characteristic exponent  $\alpha(t) = 1.5 - 0.48 \sin(2\pi(t + 1/4))$ , represented in red. The box-plots represent the behavior of the estimator for several sizes  $n$  of window.

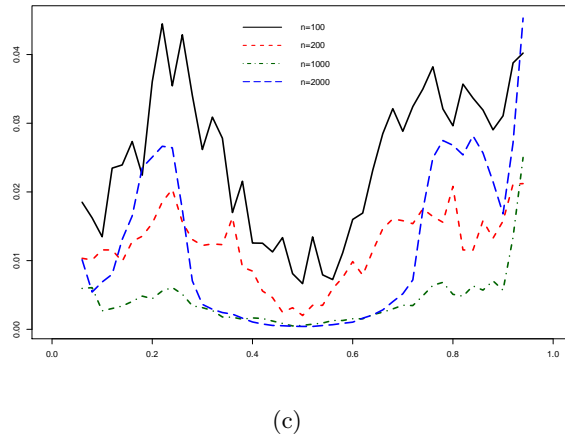
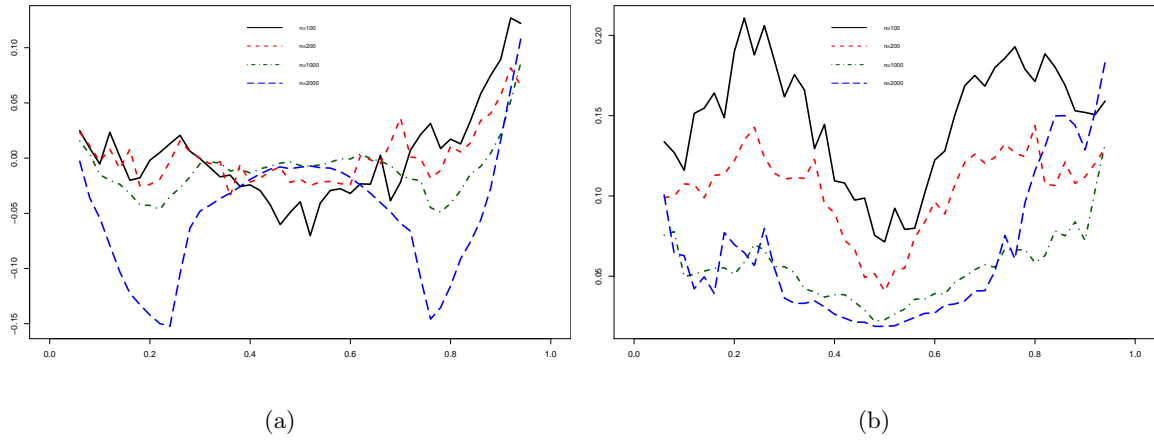


Figure 7: Representation of the bias (a), standard error (b) and mean square error (c) as function of  $t$  for  $n = 100$  (black solid line),  $n = 200$  (red dashed),  $n = 1000$  (green dotted) and  $n = 2000$  (blue dotted and dashed mix.). The statistics are evaluated on the same trajectories in Figure 6.

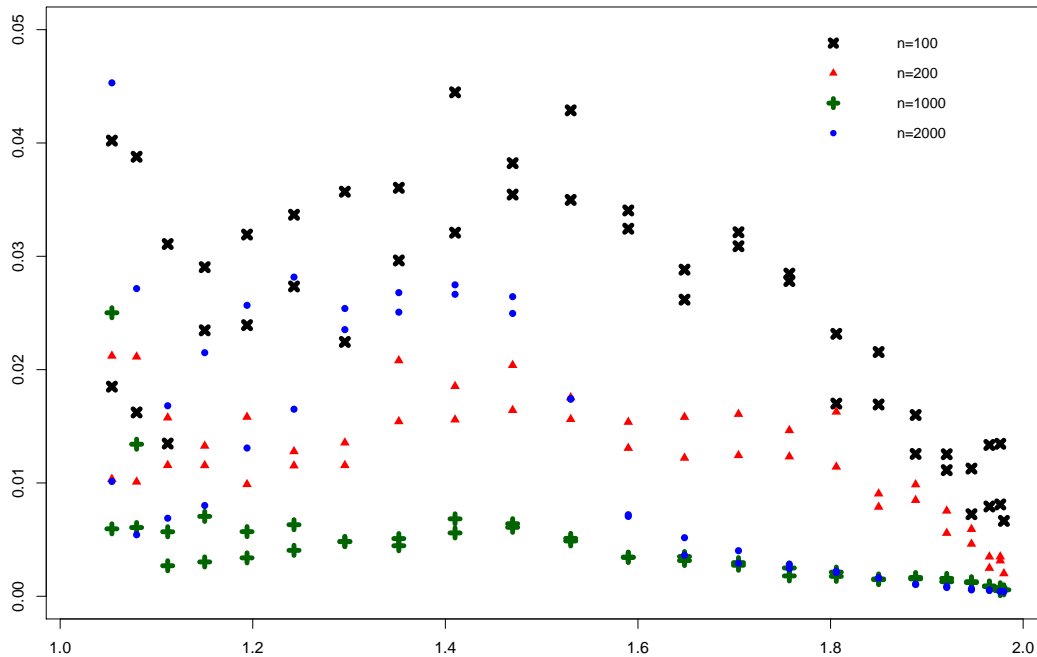


Figure 8: Mean squared error according to the value of  $\alpha$  of the combined estimation for a multistable process with  $\alpha(t) = 1.5 - 0.48 \sin(2\pi(t + 1/4))$ . The statistics are evaluated on the same trajectories as Figure 6.

As these figures show, reasonable estimates are obtained on these experiments, due to the fact that the variations of  $\alpha$  are "slow" compared to the sampling frequency: this feature ensures that centering a window around any given  $t_0$  and treating all points inside this window as having the same  $\alpha$  value is an acceptable approximation as far as estimation is concerned.

## 8 Application on financial logs

This last section deals with real data. We apply the combined estimator to estimate the characteristic exponent of the financial index Standard & Poor's 500 (abbreviated as the *S&P 500*, see Figure 9). This is an American stock market index based on the 500 companies having largest capitalization. The stock market returns of the financial index *S&P 500*, which correspond to the renormalized growth rate  $((Y_{t+1} - Y_t)/Y_t)_t$ , are supposed to be independent stable random variables. The parameter is estimated by applying the estimator defined in Theorem 3.4 in sliding window of several sizes during the period 1996-2017 (see Figure 10). In Figure 11, the estimation of the characteristic

exponent is done between 1929 and 1936. A sudden drop is observed at the end of 1929. This change corresponds to the Wall Street financial crash of 1929.

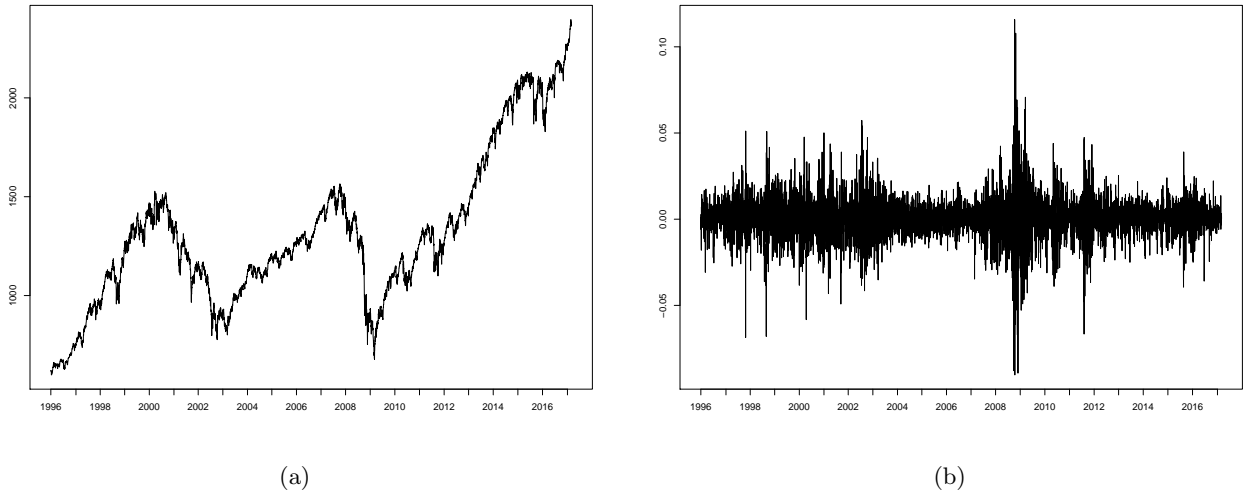
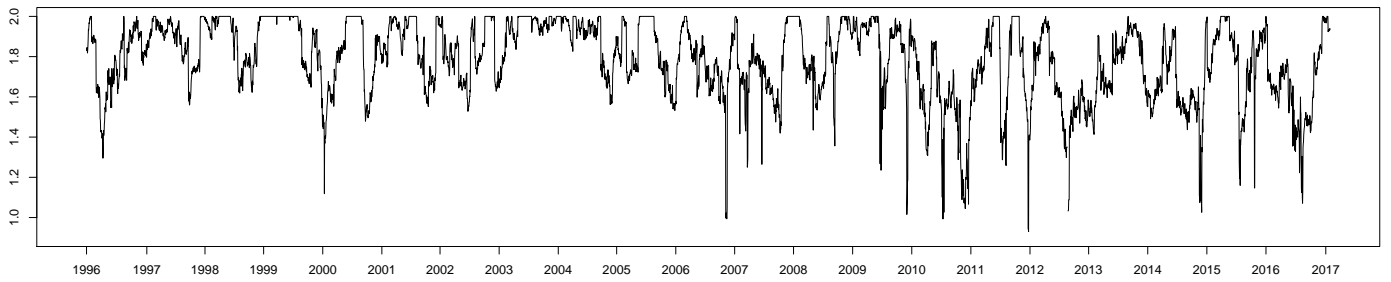
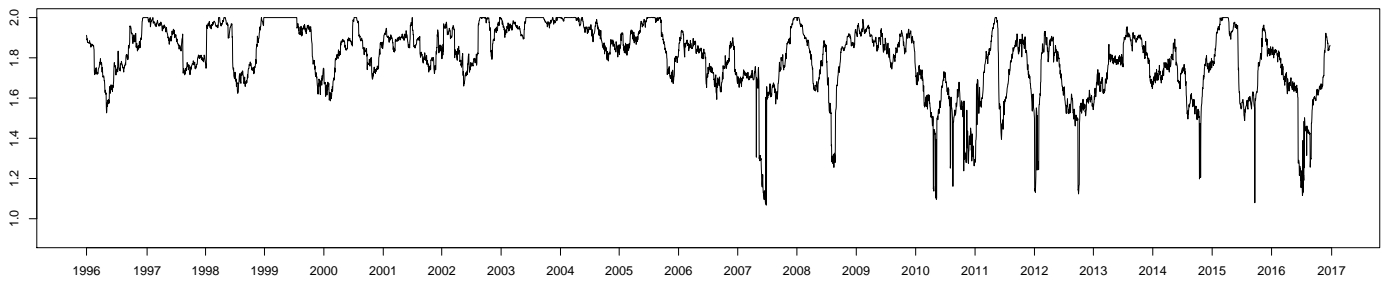


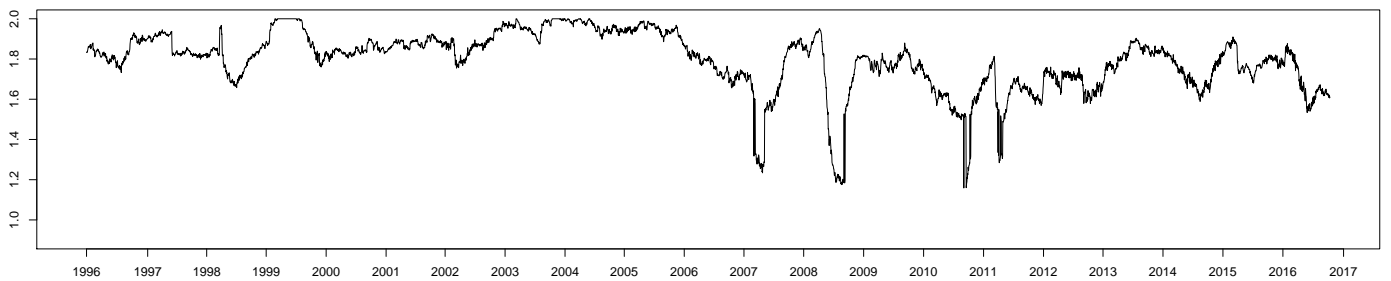
Figure 9: Evolution of the financial index S&P 500 as function of  $t$  (a) and its return (b), between 1996 and 2017.



(a)  $n=50$



(b)  $n=100$



(c)  $n=200$

Figure 10: Values of the combined estimator  $\widehat{\alpha}_{COMB}^{(n)}$  for the S&P 500 characteristic exponent  $\alpha$  in sliding window of several sizes  $n$  for working days between 1996 and 2017.



Figure 11: Values of the combined estimator  $\widehat{\alpha}_{COMB}^{(n)}$  for the S&P 500 characteristic exponent  $\alpha$  around the Wall Street financial crash of 1929 (sliding window of size  $n = 200$  observations).

## 9 Appendix

### Proof of Proposition 3.1

Let  $Z$  be a stable random variable  $\sim S_\alpha(1, 0, 0)$ . Since:

- $Z$  has bounded density,
- $\lim_{\lambda \rightarrow +\infty} \lambda^\alpha P(Z > \lambda) = C_\alpha$ ,
- $\lim_{\lambda \rightarrow +\infty} \lambda^\alpha P(Z < -\lambda) = C_\alpha$ ,

(see [14], Property 1.2.15), we deduce for all  $p > 0$ ,

$$E[|\log |Z||^p] = \int_0^\infty \mathbb{P}(|\log |Z||^p > x) dx < \infty.$$

### Proof of Proposition 3.2

We use the formula of Property 1.2.17 and 1.2.15 in [14] to compute  $E[|Z|^t]$  for  $0 < t < \alpha$ :

$$E[|Z|^t] = \frac{2^{t-1} \Gamma(1 - t/\alpha)}{t \int_0^{+\infty} \frac{\sin^2(u)}{u^{t+1}} du} = \frac{2^{t-1} \Gamma(1 - t/\alpha)}{\int_0^{+\infty} \frac{\sin(2u)}{u^t} du} = \frac{(1-t) \Gamma(1 - t/\alpha)}{\Gamma(2-t) \cos(\pi t/2)}$$

Furthermore, if  $0 < t < 1$ , we have:

$$E[|Z|^t] = \frac{\Gamma(1 - t/\alpha)}{\Gamma(1-t) \cos(\pi t/2)}$$

### Proof of Theorem 3.3

By the strong law of large numbers for the random variables  $(\log |X_i|)_i$  with finite expectation (Proposition 3.1), and the continuous mapping theorem for  $g(x) = \frac{\gamma}{\gamma+x}$ , we



get  $\hat{\alpha}_n \xrightarrow[n \rightarrow \infty]{a.s.} \alpha$ . Then we apply the central limit theorem and the delta method with the function  $f(x) = \frac{\pi^2}{6x^2} + \frac{\pi^2}{12}$  to obtain

$$\frac{\sqrt{n}(\hat{\alpha}_n - \alpha)\gamma}{\hat{\alpha}_n^2 \sqrt{f(\hat{\alpha}_n)}} \xrightarrow{d} \mathcal{N}(0, 1)$$

### Proof of Theorem 3.4

We prove this theorem by using the strong law of large numbers, continuous mapping theorem, multidimensional central limit theorem and delta method. We need the third and the fourth log-moment of a stable law  $Z \sim S_\alpha(1, 0, 0)$  for the covariance matrix. We have

$$E \left[ (\log |Z| - E[\log |Z|])^3 \right] = 2\zeta(3) \left( \frac{1}{\alpha^3} - 1 \right)$$

and we get

$$E[(\log |Z|)^3] = \frac{4\gamma^3 + 2\gamma\pi^2 + 8\zeta(3)}{4\alpha^3} + \frac{-12\gamma^3 - 2\gamma\pi^2}{4\alpha^2} + \frac{12\gamma^3 + \gamma\pi^2}{4\alpha} + \frac{-4\gamma^3 - \gamma\pi^2 - 8\zeta(3)}{4}$$

and

$$E \left[ (\log |Z| - E[\log |Z|])^4 \right] = \pi^4 \left( \frac{3}{20\alpha^4} + \frac{1}{12\alpha^2} + \frac{19}{240} \right)$$

$$E[(\log |Z|)^4] = \frac{240\gamma^4 + 240\gamma^2\pi^2 + 36\pi^4 + 1920\zeta(3)\gamma}{240\alpha^4} + \frac{-960\gamma^4 - 480\gamma^2\pi^2 - 1920\zeta(3)\gamma}{240\alpha^3} \\ + \frac{1440\gamma^4 + 360\gamma^2\pi^2 + 20\pi^4}{240\alpha^2} + \frac{-960\gamma^4 - 240\gamma^2\pi^2 - 1920\zeta(3)\gamma}{240\alpha} + \frac{240\gamma^4 + 120\gamma^2\pi^2 + 19\pi^4 + 1920\zeta(3)\gamma}{240}$$

where  $\zeta$  is the Riemann zeta function  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$  and  $\zeta(3) = 1.2020569 \dots$

### Proof of Proposition 6.1

By the Kolmogorov strong law of large numbers for non-identically distributed random variables (Theorem 2.3.10 in [15]), we get

$$\frac{1}{n} \sum_{i=1}^n Y_i^2 - \left( \frac{1}{n} \sum_{i=1}^n Y_i \right)^2 - \left[ \frac{1}{n} \sum_{i=1}^n E[Y_i^2] - \left( \frac{1}{n} \sum_{i=1}^n E[Y_i] \right)^2 \right] \xrightarrow{a.s.} 0$$

as

$$\frac{1}{n} \sum_{i=1}^n E[Y_i] - E[Y] = \frac{\gamma}{n} \sum_{i=1}^n \left( \frac{1}{\alpha_i} - \frac{1}{\alpha} \right) + \frac{1}{n} \sum_{i=1}^n (\log \sigma_i - \log \sigma) \\ = \frac{\gamma}{n} \sum_{i=1}^n \frac{-\varepsilon_i}{(\alpha_i^*)^2} + \frac{1}{n} \sum_{i=1}^n \frac{\eta_i}{\sigma_i^*}$$

and

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n E[Y_i^2] - E[Y^2] \\
&= \frac{\gamma^2 + \pi^2/6}{n} \sum_{i=1}^n \left( \frac{1}{\alpha_i^2} - \frac{1}{\alpha^2} \right) - \frac{2\gamma^2}{n} \sum_{i=1}^n \left( \frac{1}{\alpha_i} - \frac{1}{\alpha} \right) + \frac{2\gamma}{n} \sum_{i=1}^n \left( \left( \frac{1}{\alpha_i} - 1 \right) \log \sigma_i - \left( \frac{1}{\alpha} - 1 \right) \log \sigma \right) \\
&\quad + \frac{1}{n} \sum_{i=1}^n \left( (\log \sigma_i)^2 - (\log \sigma)^2 \right) \\
&= \frac{\gamma^2 + \pi^2/6}{n} \sum_{i=1}^n \frac{-2\varepsilon_i}{\tilde{\alpha}_i^3} - \frac{2\gamma^2}{n} \sum_{i=1}^n \frac{-\varepsilon_i}{(\alpha_i^*)^2} + \frac{2\gamma}{n} \sum_{i=1}^n \left( \frac{1}{\check{\sigma}_i} \left( \frac{1}{\check{\alpha}_i} - 1 \right) \eta_i - \frac{\log(\check{\sigma}_i)}{(\check{\alpha}_i)^2} \varepsilon_i \right) \\
&\quad + \frac{1}{n} \sum_{i=1}^n \frac{2 \log \tilde{\sigma}_i}{\tilde{\sigma}_i} \eta_i
\end{aligned}$$

where  $\tilde{\alpha}_i$ ,  $\alpha_i^*$ ,  $\check{\alpha}_i$  (respectively  $\tilde{\sigma}_i$ ,  $\sigma_i^*$ ,  $\check{\sigma}_i$ ) have ranged between  $\alpha$  and  $\alpha_i$  (respectively  $\sigma$  and  $\sigma_i$ ).

$$\begin{aligned}
& \left| \frac{1}{n} \sum_{i=1}^n E[Y_i] - E[Y] \right| \leq \frac{\gamma}{\alpha^2(1-c_\alpha)^2} \frac{1}{n} \sum_{i=1}^n |\varepsilon_i| + \frac{1}{\sigma(1-c_\sigma)} \frac{1}{n} \sum_{i=1}^n |\eta_i| \\
& \left| \frac{1}{n} \sum_{i=1}^n E[Y_i^2] - E[Y^2] \right| \\
& \leq \left( \frac{2\gamma^2 + \pi^2/3}{\alpha^3(1-c_\alpha)^3} + \frac{2\gamma^2}{\alpha^2(1-c_\alpha)^2} + \frac{2\gamma \max(|\log(\sigma - \sigma c_\sigma)|, |\log(\sigma + \sigma c_\sigma)|)}{\alpha^2(1-c_\alpha)^2} \right) \frac{1}{n} \sum_{i=1}^n |\varepsilon_i| \\
& \quad + \left( \frac{2\gamma}{\sigma(1-c_\sigma)} \left( \frac{1}{\alpha(1-c_\alpha)} + 1 \right) + \frac{2 \max(|\log(\sigma - \sigma c_\sigma)|, |\log(\sigma + \sigma c_\sigma)|)}{\sigma(1-c_\sigma)} \right) \frac{1}{n} \sum_{i=1}^n |\eta_i|
\end{aligned}$$

Under the conditions:

- $\frac{1}{n} \sum_{i=1}^n |\varepsilon_i| \rightarrow 0$
- $\frac{1}{n} \sum_{i=1}^n |\eta_i| \rightarrow 0$

we have

$$\frac{1}{n} \sum_{i=1}^n E[Y_i^2] \xrightarrow{n \rightarrow \infty} E[Y^2] \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n E[Y_i] \xrightarrow{n \rightarrow \infty} E[Y].$$

By the continuous mapping theorem with  $g(x, y) = \frac{\pi}{\sqrt{\max(6(y-x^2) - \frac{\pi^2}{2}, \frac{\pi^2}{4})}}$ , we obtain

$$\hat{\alpha}_n - \alpha \xrightarrow{a.s.} 0.$$

**Proof of Proposition 6.2**

Define the covariance-matrix  $\Sigma_{\alpha,\sigma}^{(n)}$  as follows:

$$\Sigma_{\alpha,\sigma}^{(n)} := \begin{pmatrix} \frac{1}{n} \sum_{i=1}^n \text{Var}(Y_i) & \frac{1}{n} \sum_{i=1}^n \text{Cov}(Y_i, Y_i^2) \\ \frac{1}{n} \sum_{i=1}^n \text{Cov}(Y_i, Y_i^2) & \frac{1}{n} \sum_{i=1}^n \text{Var}(Y_i^2) \end{pmatrix}$$

. With the conditions of Proposition 6.1, we have

$$\Sigma_{\alpha,\sigma}^{(n)} \xrightarrow{n \rightarrow \infty} \Sigma_{\alpha,\sigma} := \begin{pmatrix} \text{Var}(Y) & \text{Cov}(Y, Y^2) \\ \text{Cov}(Y, Y^2) & \text{Var}(Y^2) \end{pmatrix}$$

By the central limit theorem for non-identically distributed random variables (Theorem 3.3.9 in [15]), as  $\sup_k E \left[ \left| \begin{matrix} Y_k \\ (Y_k)^2 \end{matrix} \right|^4 \right] < \infty$ , we have,

$$\sqrt{n} \left( \begin{pmatrix} \frac{1}{n} \sum_{i=1}^n Y_i \\ \frac{1}{n} \sum_{i=1}^n Y_i^2 \end{pmatrix} - \begin{pmatrix} \frac{1}{n} \sum_{i=1}^n E[Y_i] \\ \frac{1}{n} \sum_{i=1}^n E[Y_i^2] \end{pmatrix} \right) \xrightarrow{d} \mathcal{N}(0, \Sigma_{\alpha,\sigma})$$

Under conditions of Proposition 6.2, we get

$$\sqrt{n} \left( \frac{1}{n} \sum_{i=1}^n E[Y_i^2] - E[Y^2] \right) \xrightarrow{n \rightarrow \infty} 0$$

and

$$\sqrt{n} \left( \frac{1}{n} \sum_{i=1}^n E[Y_i] - E[Y] \right) \xrightarrow{n \rightarrow \infty} 0,$$

then

$$\sqrt{n} \left( \begin{pmatrix} \frac{1}{n} \sum_{i=1}^n Y_i \\ \frac{1}{n} \sum_{i=1}^n Y_i^2 \end{pmatrix} - \begin{pmatrix} E[Y] \\ E[Y^2] \end{pmatrix} \right) \xrightarrow{d} \mathcal{N}(0, \Sigma_{\alpha,\sigma}).$$

By applying the delta-method with  $g(x, y) = \frac{\pi}{\sqrt{\max(6(y-x^2) - \frac{\pi^2}{2}, \frac{\pi^2}{4})}}$ , we obtain the result.

**Proof of Proposition 6.3**

By the strong law of large numbers for non-identically distributed random variables (Theorem 2.3.10 in [15]), we get

$$\frac{1}{n} \sum_{i=1}^n Y_i^2 - \left( \frac{1}{n} \sum_{i=1}^n Y_i \right)^2 - \left[ \frac{1}{n} \sum_{i=1}^n E[Y_i^2] - \left( \frac{1}{n} \sum_{i=1}^n E[Y_i] \right)^2 \right] \xrightarrow{a.s.} 0$$

With the same calculation, we have

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n E[Y_i^2] - E[Y^2] &= \frac{\gamma^2 + \pi^2/6}{n} \sum_{i=1}^n E \left[ \frac{1}{\alpha_i^2} - \frac{1}{\alpha^2} \right] + \frac{-2\gamma^2 + 2 \log(\sigma)}{n} \sum_{i=1}^n E \left[ \frac{1}{\alpha_i} - \frac{1}{\alpha} \right] \\ &= \frac{\gamma^2 + \pi^2/6}{n} \sum_{i=1}^n E \left[ \frac{-2\varepsilon_i}{(\tilde{\alpha}_i)^3} \right] + \frac{-2\gamma^2 + 2 \log(\sigma)}{n} \sum_{i=1}^n E \left[ \frac{-\varepsilon_i}{(\alpha_i^*)^2} \right] \end{aligned}$$

and

$$\frac{1}{n} \sum_{i=1}^n E[Y_i] - E[Y] = \frac{\gamma}{n} \sum_{i=1}^n E \left[ \frac{1}{\alpha_i} - \frac{1}{\alpha} \right] = \frac{\gamma}{n} \sum_{i=1}^n E \left[ \frac{-\varepsilon_i}{(\alpha_i^*)^2} \right]$$

with  $\alpha_i^*$  and  $\tilde{\alpha}_i \in (\min(\alpha, \alpha_i), \max(\alpha, \alpha_i))$ . As  $\alpha_i^*$  and  $\tilde{\alpha}_i$  are almost surely included in  $[\alpha(1 - c_\alpha), \alpha(1 + c_\alpha)]$ , we get

$$\left| \frac{1}{n} \sum_{i=1}^n E[Y_i^2] - E[Y^2] \right| \leq \left( \frac{\gamma^2 + \pi^2/6}{(\alpha(1 - c_\alpha))^3} + \frac{|-2\gamma^2 + 2 \log(\sigma)|}{(\alpha(1 - c_\alpha))^2} \right) \frac{1}{n} \sum_{i=1}^n E[|\varepsilon_i|]$$

and

$$\left| \frac{1}{n} \sum_{i=1}^n E[Y_i] - E[Y] \right| \leq \frac{\gamma}{(\alpha(1 - c_\alpha))^2} \frac{1}{n} \sum_{i=1}^n E[|\varepsilon_i|]$$

if we suppose  $\frac{1}{n} \sum_{i=1}^n E[|\varepsilon_i|] \rightarrow 0$ , we get

$$\frac{1}{n} \sum_{i=1}^n E[Y_i^2] \xrightarrow{n \rightarrow \infty} E[Y^2] \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n E[Y_i] \xrightarrow{n \rightarrow \infty} E[Y].$$

The central limit theorem is proved as Prop. 6.2.

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