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Labeled Packing of Cycles and Circuits

Alice Joffard, Hamamache Kheddouci

Abstract

In 2013, Duchêne, Kheddouci, Nowakowski and Tahraoui [4, 9] introduced a labeled version of the graph packing problem. It led to the introduction of a new parameter for graphs, the k -labeled packing number λ^k . This parameter corresponds to the maximum number of labels we can assign to the vertices of the graph, such that we will be able to create a packing of k copies of the graph, while conserving the labels of the vertices.

The authors intensively studied the labeled packing of cycles, and, among other results, they conjectured that for every cycle C_n of order $n = 2k + x$, with $k \geq 2$ and $1 \leq x \leq 2k - 1$, the value of $\lambda^k(C_n)$ was 2 if x was 1 and k was even, and $x + 2$ otherwise.

In this paper, we disprove this conjecture by giving a counter example. We however prove that it gives a valid lower bound, and we give sufficient conditions for the upper bound to hold.

We then give some similar results for the labeled packing of circuits.

Key words: Packing of graphs, Labeled packing, Cycles, Circuits.

1 Definitions and notations

We will work with simple graphs, with no multiple edges nor loops. For an undirected graph $G = (V(G), E(G))$, $V(G)$ will represent the set of vertices of G , $E(G)$ will represent the set of edges of G . For a directed graph $G = (V(G), E(G))$, $V(G)$ will represent the set of vertices of G , $E(G)$ will represent the set of arcs of G . The *order* n of a graph G will represent its number of vertices, its *size* m will represent its number of edges or arcs, and $\Delta(G)$ will represent its maximal degree.

K_n will be the complete undirected graph of order n , $K_{p,q}$ the complete bipartite graph of partition sizes p and q , P_n the path of order n , C_n the cycle of order n , and S_n the star of order n . \overleftrightarrow{K}_n will be the *complete directed graph* of order n , meaning that its set of arcs

contains all the possible arcs from one of its vertices to another, and \vec{C}_n will be the *circuit* of order n , that is the cyclic orientation of the cycle C_n .

$G \cup H$, the *union* of G and H , will represent the graph $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$. The complement of G will be denoted by \bar{G} . The k_{th} power G^k of a undirected graph G will correspond to the graph G , with the addition of edges between any pair of vertices that have a path of length less or equal to k between them.

For p, q and r any integers, $p \bmod q$ will denote the remainder of the euclidean division of p by q , and $p \equiv r \bmod q$ is equivalent to $p \bmod q = r \bmod q$.

2 Introduction

The graph packing problem has been widely studied in the literature, in many different ways. The general idea of the problem is to find sufficient conditions on a set of graphs (H_1, H_2, \dots, H_k) or on a graph G , such that (H_1, H_2, \dots, H_k) will admit a packing in G , a packing being defined in the following way:

Definition 1. A packing of (H_1, H_2, \dots, H_k) in G is a set of injections $\alpha_i : V(H_i) \rightarrow V(G), i = 1, \dots, k$, such that $\alpha_i^*(E(H_i)) \cap \alpha_j^*(E(H_j)) = \emptyset$ for $i \neq j$ where $\alpha_i^* : E(H_i) \rightarrow E(G)$ is induced by α_i .

If for all $i \in [1, k], |V(H_i)| = |V(G)| = n$, the k injections are actually bijections. In the case of a k -placement, where the vertices of the same graph H are sent k times to the vertices of K_n , we can send the images of the vertices of H in the vertices of H instead of the ones of K_n . Thus, the bijections are actually permutations on the vertices of H . In this context, we call a *fixed point* any vertex $\alpha \in V(G)$ such that for all $i \in [1, k], \sigma_i(\alpha) = \alpha$.

The first results concerning the problem of graph packing focused on the packings into the complete graph K_n . In this case, the only conditions for the injections to form a packing is that the induced images of the edges should not intersect. Spencer and Sauer [8] got first interested in the packing of two graphs, and found two sufficient conditions on the graphs for them to admit a packing in K_n .

Theorem 1. [8] If $|E(H_1)| \cdot |E(H_2)| \leq \binom{n}{2}$, then (H_1, H_2) pack into K_n .

Theorem 2. [8] If $2\Delta(H_1)\Delta(H_2) < n$, then (H_1, H_2) pack into K_n .

Those two conditions are quite representative of two common approaches in graph packing theory, one that is interested in bounding the size of the graphs, and the other in bounding their maximal degree. Concerning the more general case of the packing of k graphs,

Bollobás and Eldridge formulated the following conjecture, still focusing on small sizes graphs:

Conjecture 1. [3] *If $|E(H_1)|, |E(H_2)|, \dots, |E(H_k)| \leq n - k$, then (H_1, H_2, \dots, H_k) pack in K_n .*

The conjecture has been proven for $k = 2$ [8] and $k = 3$ [6]. For $k \geq 4$, the conjecture remains open and is still the subject of many researches, as it is considered to be one of the most important open problems in graph packing theory. Another very important conjecture, about the packing of two graphs in K_n , is also interested on the maximal degree of two graphs:

Conjecture 2. [3] *If $(\Delta(H_1) + 1)(\Delta(H_2) + 1) \leq n + 1$, then (H_1, H_2) pack into K_n .*

Most of the studies concerning graph packing focused on the conditions on the graphs H_i to pack into the complete graph K_n . However, some authors chose a different point of view, which is to fix the graphs H_1, H_2, \dots, H_k and search for a minimal size graph G that would make the packing of the H_i in G possible. A great part of this work has focused on the cases where the H_i were trees.

Theorem 3. [7, 5] *For $n > 3$, T a tree of order n , $T \neq S_n$, there exists a 2-placement σ of T such that $\sigma(T) \subset T^3$.*

If the reader wants to go into more details concerning the packing of graphs, he is invited to refer to the numerous surveys on the subject [10, 11].

In 2013, Duchene, Kheddouci, Nowakowski, and Tahraoui introduced the notion of labeled packings of graphs. They gave the following definition of a p -labeled packing of k copies of a graph G :

Definition 2. [4, 9] *For $p \geq 1$, G a graph of order less or equal to n , and f a mapping from $V(K_n)$ to a set of labels of cardinality p , f is a p -labeled packing of k copies of G into K_n if and only if for every i in $\{1, 2, \dots, k\}$, there exists an injection $\sigma_i : V(G) \rightarrow V(K_n)$ such that:*

$$\begin{cases} \text{For every } i \neq j, \sigma_i^*(E(G)) \cap \sigma_j^*(E(G)) = \emptyset, \text{ where } \sigma_i^* : E(G) \rightarrow E(K_n) \text{ is induced by } \sigma_i \\ \text{For every } v \in V(G), f(\sigma_1(v)) = f(\sigma_2(v)) = \dots = f(\sigma_k(v)). \end{cases}$$

Differently said, a p -labeled packing of k copies of G is a labeling of the vertices of K_n with exactly p distinct labels, such that there exists a set of k injections from the vertices of G to the vertices of K_n such that the images of the edges of G never overlap in K_n , and for any vertex v , the k images of v have the same label. It can be seen as a generalization of the problem of graph packing, as for $p = 1$, all the vertices of G have the same label

and can thus be sent to any other vertex, as it is the case in the classic graph packing.

This definition allowed the authors to introduce a new parameter for a graph G , called the k -labeled packing number of G :

Definition 3. [4, 9] *The k -labeled packing number of G , denoted by $\lambda^k(G)$, is the largest p such that G admits a p -labeled packing of k copies of G .*

The authors linked this new parameter to the cycle decomposition of permutations in the following observation:

Observation 1. [4, 9] *The notion of k -labeled packing number is closely related to the cycle decomposition of the packings as permutations.*

Indeed, for $k = 2$, if (id, σ_2) is a packing of 2 copies of G decomposed into p disjoint cycles, then we know that in σ_2 , the vertices of G are sent to vertices that belong to the same cycle of the permutation, so that we can always associate one label to each of the p cycles. That way, we can obtain a valid p -labeled packing of 2 copies of G , so that we have $\lambda^2(G) \geq p$.

The authors, among other results, analyzed the value of λ^k for cycles, and proved the following:

Theorem 4. [4, 9] *Every cycle C_n of order $n \leq 2k$ does not admit any k -placement. For every cycle C_n of order $n = 2km + x$, $k, m \geq 2$, $x < 2k$, we have:*

$$\lambda^k(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } m = 2, k > 2 \text{ and } x \equiv 2 \pmod{4} \\ \lfloor \frac{n}{2} \rfloor + m + 1 & \text{if } x = 2k - 1 \\ \lfloor \frac{n}{2} \rfloor + m & \text{otherwise.} \end{cases}$$

Concerning the left values of n , they proposed the following conjecture:

Conjecture 3. [4, 9] *For every cycle C_n of order $n = 2k + x$, $k \geq 2$, $1 \leq x \leq 2k - 1$, we have:*

$$\lambda^k(C_n) = \begin{cases} 2 & \text{if } x = 1 \text{ and } k \text{ is even} \\ x + 2 & \text{otherwise.} \end{cases}$$

To support this conjecture, they gave the following results, that focus on some particular values of k and x :

Theorem 5. [4, 9] *Let C_n be the cycle of order $n = 2k + x$, $k \geq 2$, $2k - 3 \leq x \leq 2k - 1$ and $(k, n) \neq (2, 5)$. We have $\lambda^k(C_n) = x + 2$.*

Theorem 6. [4, 9] *Let C_n be the cycle of order $n = 2k + x$, k a power of 2, $x = 1$. We have $\lambda^k(C_n) = 2$.*

Theorem 7. [4, 9] Let C_n be the cycle of order $n = 2k + x$, k prime, $x = 1$. We have $\lambda^k(C_n) \leq x + 2$.

Theorem 8. [4, 9] Let C_n be the cycle of order $n = 2k + x$, k even, $x = 2$. We have $\lambda^k(C_n) \leq x + 2$.

3 Labeled packing of cycles

In this part, we start by disproving Conjecture 3. Indeed, we give a counter example to show that the upper bound it gives is not valid in general. However, we give some particular cases for which the upper bound is valid, and we prove that the lower bound always holds.

The counter example uses the values $k = 9$, $x = 3$, and therefore $n = 21$. We show that there exists a 7-labeled packing of 9 copies of C_{21} , so that $\lambda^9(C_{21}) \geq 7$, while Conjecture 3 would give $\lambda^9(C_{21}) = 3 + 2 = 5$. We take the 7-labeled packing of 9 copies of C_{21} presented in Figure 1.

Despite the fact that Conjecture 3 is now proven to be false, it gives a good lower bound for $\lambda^k(C_n)$, as expressed in the following theorem:

Theorem 9. For every cycle C_n of order $n = 2k + x$, $k \geq 2$, $1 \leq x \leq 2k - 1$, we have:

$$\lambda^k(C_n) \geq \begin{cases} 2 & \text{if } x = 1 \text{ and } k \text{ is even} \\ x + 2 & \text{otherwise.} \end{cases}$$

Proof. We split the proof into five cases, depending on the values of k and x , as follows:

1. k is odd
 - a) $1 \leq x \leq k - 1$
 - b) $k \leq x \leq 2k - 1$
2. k is even and $x = 1$
3. k is even and $x > 1$
 - a) $2 \leq x \leq k - 1$
 - b) $k \leq x \leq 2k - 1$

For each case, we construct a labeling f with the given number of labels, and give the associated packing $\sigma = \{\sigma_j, 1 \leq j \leq k\}$.

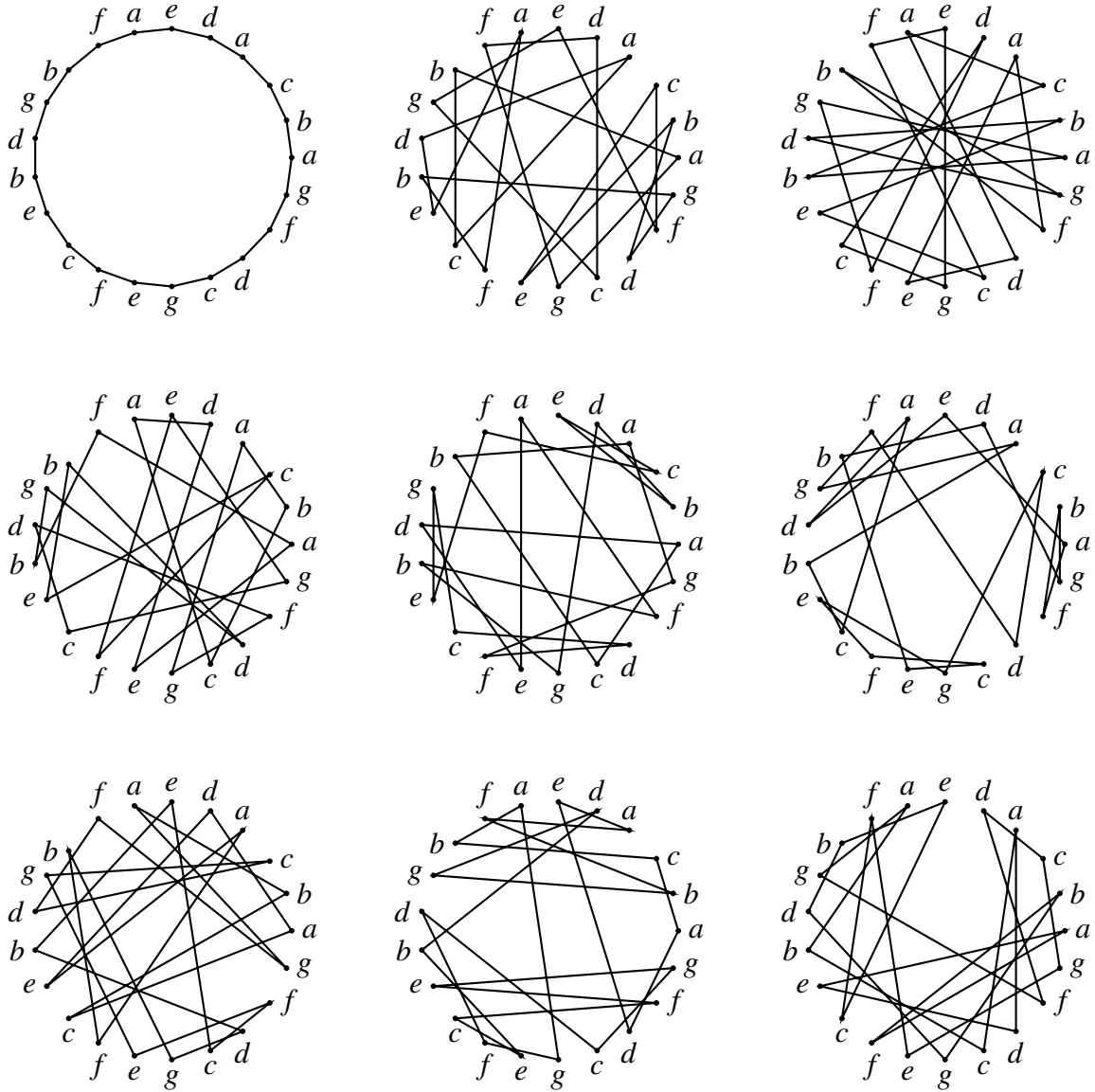


Figure 1: A 7-labeled packing of $k = 9$ copies of C_{21} . The labels are represented by the letters on the vertices.

Let $V(C_n) = \{v_0, \dots, v_n\}$ be the vertices of C_n . For each of those cases, we will partition $V(C_n)$ into a set F containing all the fixed points, and a set V_l for each other label l , containing all the vertices having this label:

$$F = \{v \in V(C_n) : \text{for all } j \in [1, k], \sigma_j(v) = v\}$$

$$V_l = \{v \in V(C_n) : f(v) = l\} \text{ for any } l.$$

We will also partition the set of edges $E(C_n)$ depending on the labels of their vertices:

$$E_F = \{\{u, v\} \in E(C_n) : u \in F \text{ or } v \in F\}$$

$$E_{p,q} = \{\{u, v\} \in E(C_n) : u \in V_p \text{ and } v \in V_q\}.$$

For f to be a labeled packing with respect to σ , we must have, for all $j \neq j'$ and $v \in V(C_n)$, $f(\sigma_j(v)) = f(\sigma_{j'}(v))$. In any case, for all $v \in F$, since $\sigma_j(v) = v$, we know that $f(\sigma_j(v)) = f(v) = f(\sigma_{j'}(v))$. Thus, we only have to show that for all $v \notin F$ and $j \neq j'$, $f(\sigma_j(v)) = f(\sigma_{j'}(v))$.

For σ to be a packing, we must have, for all $j \neq j'$, $\sigma_j^*(E(C_n)) \cap \sigma_{j'}^*(E(C_n)) = \emptyset$. Assuming that f is a labeled packing with respect to σ , we already know that for all labels p, q , if $X \neq E_{p,q}$, then $\sigma_j^*(E_{p,q}) \cap \sigma_{j'}^*(X) = \emptyset$.

We will show for each case that for all p, q and $\{u, v\}, \{u', v'\} \in E_{p,q}$, $\sigma_j(\{u, v\}) = \sigma_{j'}(\{u', v'\})$ implies $j = j'$, so that for all $j \neq j'$, $\sigma_j^*(E_{p,q}) \cap \sigma_{j'}^*(E_{p,q}) = \emptyset$.

To show that $\sigma_j^*(E_F) \cap \sigma_{j'}^*(E_F) = \emptyset$, since each edge of E_F contains one fixed point, we only have to show that its two neighbors u and v verify $\sigma_j(u) \neq \sigma_{j'}(u)$, $\sigma_j(v) \neq \sigma_{j'}(v)$, $\sigma_j(u) \neq \sigma_{j'}(v)$ and $\sigma_j(v) \neq \sigma_{j'}(u)$. For the two first conditions, we will always choose σ such that for all $v \notin F$ and $j \neq j'$, $\sigma_j(v) \neq \sigma_{j'}(v)$. The two last conditions are respected if $f(u) \neq f(v)$, assuming that f is a labeled packing with respect to σ . If $f(u) = f(v)$, we will show them.

For each of the five cases, we are thus going to give a construction and prove that:

- * For each $v \notin F$ and $j \neq j'$, $f(\sigma_j(v)) = f(\sigma_{j'}(v))$.
- * For any p, q and any $\{u, v\}, \{u', v'\} \in E_{p,q}$, if $\sigma_j(\{u, v\}) = \sigma_{j'}(\{u', v'\})$, then $j = j'$.
- * For any $v \notin F$ and $j \neq j'$, $\sigma_j(v) \neq \sigma_{j'}(v)$.
- * For each u, v such that there exists $x \in F$ such that $\{u, x\}, \{x, v\} \in E(C_n)$, if $f(u) = f(v)$, then, for all $j \neq j'$, $\sigma_j(u) \neq \sigma_{j'}(v)$ and $\sigma_j(v) \neq \sigma_{j'}(u)$.

1. k is odd:

a) $1 \leq x \leq k-1$:

In this case, the principle of the construction is the following: For $x = 1$, we label with 1 the vertex v_{2k} , with a every other vertex of even number, and with b every other vertex of odd number. Then, we create the first copy of C_n by joining the vertices in the following order:

$$v_0, v_{n-2}, v_1, v_{n-3}, v_2, v_{n-4}, \dots, v_{\frac{n-1}{2}-1}, v_{\frac{n-1}{2}}, v_{2k}, v_0.$$

We create each other copy by following the same order as the previous one but with every number but $2k$ added to 2 modulo $n - 1$. Note that, since k is odd, the obtained packing is exactly the decomposition of K_n given by Walecki [1]. For $x > 1$, we add fixed points to the previous construction, on $x - 1$ of the k edges that link a vertex u of label a to a vertex v of label b , cutting every edge into two edges.

To express it in a more formal way, we take the following labeling f , packing $\sigma = \{\sigma_j, 1 \leq j \leq k\}$, and vertices and edges partitions of C_n :

$$f(v_i) = \begin{cases} a & \text{if } 0 \leq i \leq 2k - 1 \text{ and } i \text{ is even} \\ b & \text{if } 0 \leq i \leq 2k - 1 \text{ and } i \text{ is odd} \\ i - 2k + 1 & \text{if } i \geq 2k. \end{cases}$$

$$\sigma_j(v_i) = \begin{cases} v_{i+2(j-1) \bmod 2k} & \text{if } i < 2k \\ v_i & \text{if } i \geq 2k. \end{cases}$$

$$F = \{v_i, 2k \leq i \leq n - 1\}$$

$$V_a = \{v_i, 0 \leq i \leq 2k - 1 \text{ and } i \equiv 0 \pmod{2}\}$$

$$V_b = \{v_i, 0 \leq i \leq 2k - 1 \text{ and } i \equiv 1 \pmod{2}\}.$$

$$E_F = \{\{v_{i-1}, v_{2k+i}\} \cup \{v_{2k-i}, v_{2k+i}\}, 1 \leq i \leq x - 1\} \cup \{\{v_0, v_{2k}\}, \{v_k, v_{2k}\}\}$$

$$E_{a,a} = \{\{v_i, v_{2k-i}\}, 1 \leq i \leq k - 1 \text{ and } i \equiv 0 \pmod{2}\}$$

$$E_{b,b} = \{\{v_i, v_{2k-i}\}, 1 \leq i \leq k - 1 \text{ and } i \equiv 1 \pmod{2}\}$$

$$E_{a,b} = \{\{v_i, v_{2k-1-i}\}, x - 1 \leq i \leq k - 1\}.$$

The obtained graph G is, as wanted, isomorphic to C_n .

Indeed, for $x = 1$, for H a graph, if $V(H) = \{v_1, v_2, \dots, v_n\}$ and

$E(H) = \{\{v_i, v_{i+1 \bmod n}\}, 1 \leq i \leq n\}$, we obviously have $H \simeq C_n$, and $G = \tau(H)$, with τ the permutation defined as follows:

$$\tau(v_i) = \begin{cases} v_{\frac{i}{2}} & \text{if } i < 2k \text{ and } i \equiv 0 \pmod{2} \\ v_{2k - \frac{i+1}{2}} & \text{if } i < 2k \text{ and } i \equiv 1 \pmod{2} \\ v_i & \text{if } i = 2k. \end{cases}$$

For $x > 1$, the extra fixed point are just inserted in some of the edges of C_{2k+1} , so that we always keep $G \simeq C_n$.

We now show that f and σ define a $(x+2)$ -labeled packing of k copies of C_n . As previously stated, for $x = 1$, since k is odd the construction gives us the same decomposition of K_n as the one presented by Walecki [1]. The idea of the decomposition is that all the edges that do not involve any fixed point are linking two vertices whose numbers have a unique difference. Thus, those edges do not overlap, and σ defines a packing. Moreover, as $n - 1$ is even, the parity of the indices of a vertex and its image are the same, as well as their labels. For $x > 1$, we know that the added fixed points will not create any superposition of edges, as their neighbors have different labels and are never sent to themselves. More rigorously, we have:

- * Since $2(j-1)$ and $2k$ are even, $i + 2(j-1) \pmod{2k}$ always has the same parity as i , and for any $v \notin F$ and $j \neq j'$, $f(\sigma_j(v)) = f(\sigma_{j'}(v))$.
- * For every p, q and every $\{u, v\}, \{u', v'\} \in E_{p,q}$, $\sigma_j(\{u, v\}) = \sigma_{j'}(\{u', v'\})$ implies $j = j'$.
Indeed, for $1 \leq i, i' \leq k-1$, if $\sigma_j^*(\{v_i, v_{2k-1-i}\}) = \sigma_{j'}^*(\{v_{i'}, v_{2k-1-i'}\})$, then:

$$\begin{cases} i + 2(j-1) = i' + 2(j'-1) \pmod{2k} \\ 2k-1-i + 2(j-1) = 2k-1-i' + 2(j'-1) \pmod{2k}. \end{cases}$$
 By adding the two equations, since k is odd, and $1 \leq j, j' \leq k$, we get $j = j'$.
Similarly, for $1 \leq i, i' \leq k-1$, if $\sigma_j^*(\{v_i, v_{2k-i}\}) = \sigma_{j'}^*(\{v_{i'}, v_{2k-i'}\})$, $j = j'$.
- * $i + 2(j-1) \pmod{2k} = i + 2(j'-1) \pmod{2k}$ implies $j = j'$, so that for any $v \notin F$ and $j \neq j'$, $\sigma_j(v) \neq \sigma_{j'}(v)$.
- * k and 0 have different parity, and so do $2k-i$ and $i-1$ for all i . Thus, for all u, v such that there exists $x \in F$ such that $\{u, x\}, \{x, v\} \in E(C_n)$, $f(u) \neq f(v)$.

In the following example, we present the packing of $k = 3$ copies of C_8 with 4 labels.

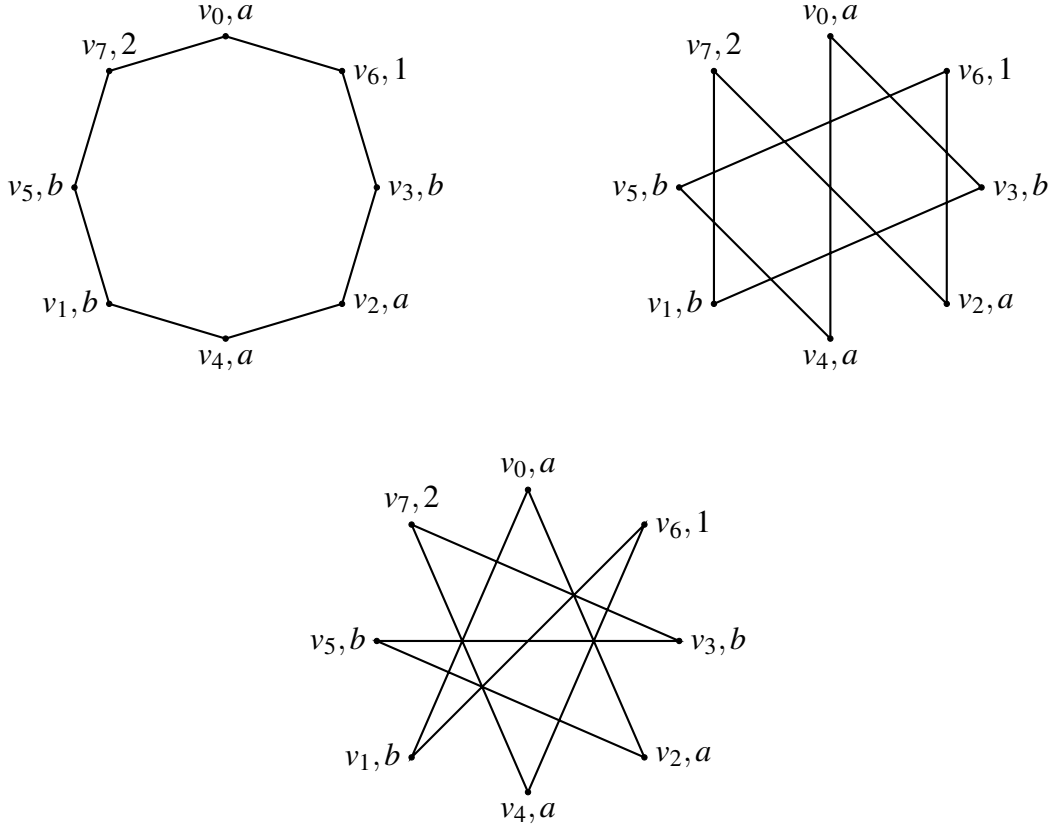


Figure 2: A 4-labeled packing of $k = 3$ copies of C_8 .

b) $k \leq x \leq 2k - 1$:

Here, for $x = k$ and thus $n = 3k$, we construct the first copy of C_n by linking successively a fixed point, a vertex of label a , and a vertex of label b . Then, we construct the $k - 1$ other copies by rotating j times clockwise, $1 \leq j \leq k - 1$, the positions of the vertices of label a , and j times anti-clockwise the positions of the vertices of label b . For $x > k$, we add fixed points to the previous construction on $x - k$ of the k edges that link a vertex of label a to a vertex of label b . We thus take :

$$f(v_i) = \begin{cases} \frac{i}{3} + 1 & \text{if } i \leq 3k \text{ and } i \equiv 0 \pmod{3} \\ a & \text{if } i \leq 3k \text{ and } i \equiv 1 \pmod{3} \\ b & \text{if } i \leq 3k \text{ and } i \equiv 2 \pmod{3} \\ i - 2k + 1 & \text{if } i > 3k. \end{cases}$$

$$\sigma_j(v_i) = \begin{cases} v_i & \text{if } i \equiv 0 \pmod{3} \text{ or } i > 3k \\ v_{i+3(j-1) \pmod{3k}} & \text{if } i \leq 3k \text{ and } i \equiv 1 \pmod{3} \\ v_{i-3(j-1) \pmod{3k}} & \text{if } i \leq 3k \text{ and } i \equiv 2 \pmod{3}. \end{cases}$$

$$\begin{aligned} F &= \{v_i, i \equiv 0 \pmod{3} \text{ or } i > 3k\} \\ V_a &= \{v_i, i \leq 3k \text{ and } i \equiv 1 \pmod{3}\} \\ V_b &= \{v_i, i \leq 3k \text{ and } i \equiv 2 \pmod{3}\}. \end{aligned}$$

$$\begin{aligned} E_F &= \{\{v_{i-1 \pmod{3k}}, v_i\} \cup \{v_i, v_{i+1 \pmod{3k}}\}, 0 \leq i < 3k \text{ and } i \equiv 0 \pmod{3}\} \cup \\ &\quad \{\{v_{3i-9k+1}, v_i\} \cup \{v_i, v_{3i-9k+2}\}, 3k \leq i < n\} \\ E_{a,a} &= E_{b,b} = \emptyset \\ E_{a,b} &= \{\{v_{3i+1}, v_{3i+2}\}, x-k \leq i \leq k-1\}. \end{aligned}$$

For $x = k$, we have $E(G) = \{\{v_i, v_{i+1 \pmod{n}}\}, 0 \leq i \leq n-1\}$, so that $G \simeq C_n$. For $x > k$, the extra fixed points are inserted in some of the edges of C_{3k} , and we keep $G \simeq C_n$.

With the way we defined σ and f , we obviously have that all the vertices are always sent to vertices that have the same label. Now, to show that σ defines a packing, the idea is that the edges involving fixed points will never create any overlap since their neighbors have different labels and are never sent to themselves, and the other edges either, precisely because k is odd. The full proof is given next:

- * Since $3(j-1)$ and $3k$ are multiples of 3, $i + 3(j-1) \pmod{3k}$ always has the same remainder in the euclidean division by 3 as i , and so does $i - 3(j-1) \pmod{3k}$. Thus, for all $v \notin F$ and $j \neq j'$, $f(\sigma_j(v)) = f(\sigma_{j'}(v))$.
- * For any $\{u, v\}, \{u', v'\} \in E_{a,b}$, $\sigma_j(\{u, v\}) = \sigma_{j'}(\{u', v'\})$ implies $j = j'$.
Indeed, for $x-k \leq i \leq k-1$, if $\sigma_j^*(\{v_{3i+1}, v_{3i+2}\}) = \sigma_{j'}^*(\{v_{3i'+1}, v_{3i'+2}\})$, then:

$$\begin{cases} 3i+1+3(j-1) \pmod{3k} = 3i'+1+3(j'-1) \pmod{3k} \\ 3i+2-3(j-1) \pmod{3k} = 3i'+2-3(j'-1) \pmod{3k}. \end{cases}$$
 By subtracting the two equations, since k is odd, and $1 \leq j, j' \leq k$, we get $j = j'$.
- * $i + 3(j-1) \pmod{3k} = i + 3(j'-1) \pmod{3k}$ implies $j = j'$, and
 $i - 3(j-1) \pmod{3k} = i - 3(j'-1) \pmod{3k}$ implies $j = j'$.
 Thus, for all $v \notin F$ and $j \neq j'$, $\sigma_j(v) \neq \sigma_{j'}(v)$.
- * Since for all i , $i-1$ and $i+1$ have a different remainder in the euclidean division by 3, and so do $3i-9k+1$ and $3i-9k+2$, we have that for all u, v such that there exists $x \in F$ such that $\{u, x\}, \{x, v\} \in E(C_n)$, $f(u) \neq f(v)$.

In the following figure, we give an example of the construction for $k = 3$ and $n = 11$.

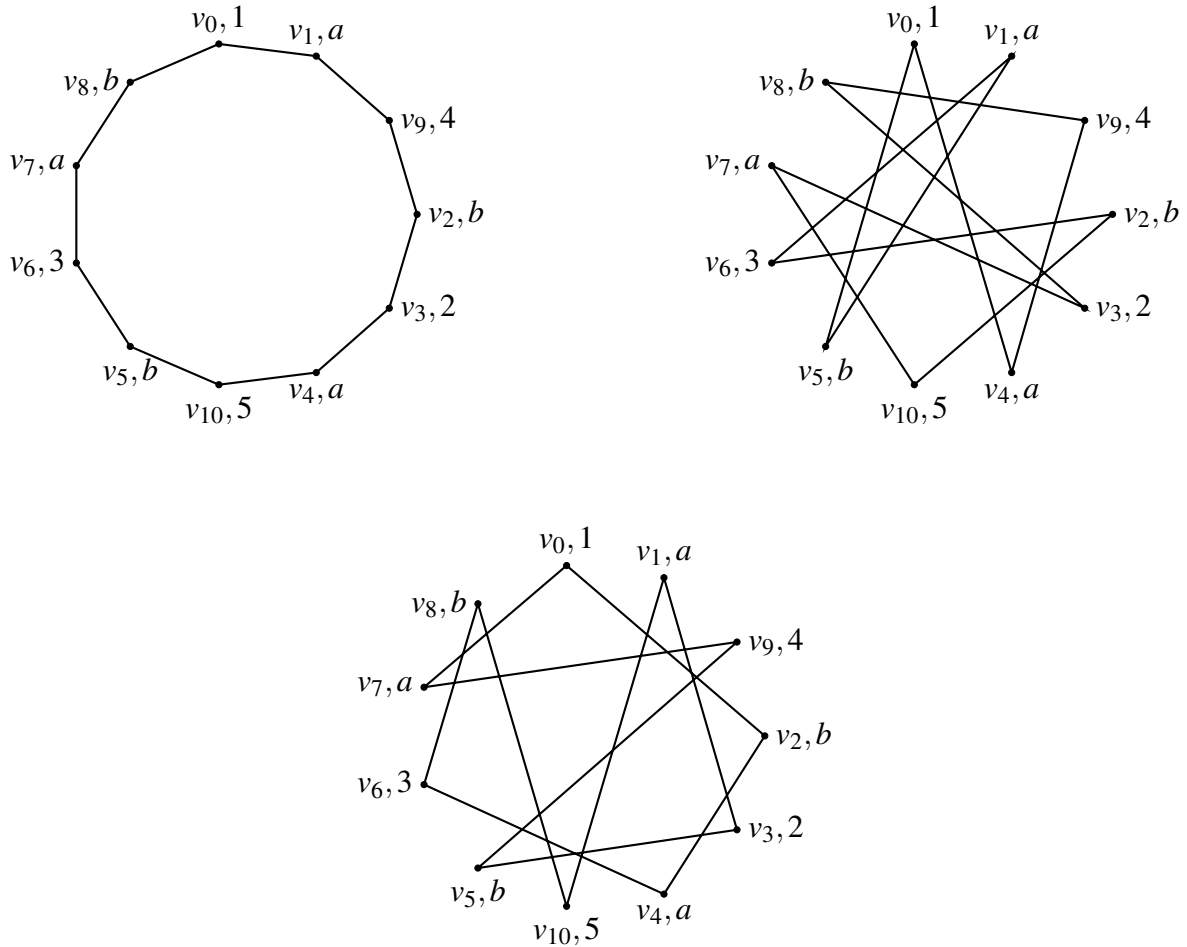


Figure 3: A 7-labeled packing of $k = 3$ copies of C_{11} .

This concludes the case where k is odd.

2. k is even and $x = 1$:

In this case, because $x = 1$, n is odd, and we can use a labeling of Walecki's decomposition of K_n [1]: We label v_{2k} , the fixed point, with label 1, and every other vertex with

label b . Then, we create the first copy of C_n by joining the vertices in the following order:

$$v_0, v_{n-2}, v_1, v_{n-3}, v_2, v_{n-4}, \dots, v_{\frac{n-1}{2}-1}, v_{\frac{n-1}{2}}, v_{2k}, v_0.$$

We construct the other copies by following the same order but with every number added each time to 1 modulo $n - 1$. The corresponding expressions of f , σ , and the corresponding partitions are:

$$f(v_i) = \begin{cases} 1 & \text{if } i = 2k \\ a & \text{otherwise.} \end{cases}$$

$$\sigma_j(v_i) = \begin{cases} v_i & \text{if } i = 2k \\ v_{i+j-1 \pmod{2k}} & \text{otherwise.} \end{cases}$$

$$F = \{v_{2k}\}$$

$$V_a = \{v_i, 0 \leq i \leq 2k - 1\}.$$

$$E_F = \{\{v_0, v_{2k}\} \cup \{v_k, v_{2k}\}\}$$

$$E_{a,a} = \{\{v_i, v_{2k-i}\}, 1 \leq i \leq k - 1\} \cup \{\{v_i, v_{2k-1-i}\}, 0 \leq i \leq k - 1\}.$$

The obtained graph G is isomorphic to C_n . Indeed, $G = \tau(H)$, with τ the permutation defined as follows:

$$\tau(v_i) = \begin{cases} v_{\frac{i}{2}} & \text{if } i < 2k \text{ and } i \equiv 0 \pmod{2} \\ v_{2k - \frac{i+1}{2}} & \text{if } i < 2k \text{ and } i \equiv 1 \pmod{2} \\ v_i & \text{if } i = 2k. \end{cases}$$

We now show that f and σ define a 2-labeled packing of k copies of C_n . Intuitively, the construction obviously gives us the same decomposition of K_n as the one presented by Walecki [1], and the labels are all the same except for the fixed point. Still, we give a more rigorous proof:

- * Since $i + j - 1 \pmod{2k} < 2k$, we have, for all $v \notin F$, $f(\sigma_j(v)) = a = f(\sigma_{j'}(v))$.
- * Similarly to the case 1.a), for all p, q and $\{u, v\}, \{u', v'\} \in E_{p,q}$, $\sigma_j(\{u, v\}) = \sigma_{j'}(\{u', v'\})$ implies $j = j'$.
- * Similarly to 1.a), for all $v \notin F$ and $j \neq j'$, $\sigma_j(v) \neq \sigma_{j'}(v)$.

* For all $j \neq j'$, with $1 \leq j, j' \leq k$, $0 + j - 1 \pmod{2k} \neq k + j' - 1 \pmod{2k}$. Thus, for all u, v such that there exists $x \in F$ such that $\{u, x\}, \{x, v\} \in E(C_n)$, if $f(u) = f(v)$, then, $\sigma_j(u) \neq \sigma_{j'}(v)$ and $\sigma_j(v) \neq \sigma_{j'}(u)$.

The following example, for $k = 4$, gives an idea of the general construction for the case where k is even and $x = 1$. We can clearly see here how it corresponds to a labelling of Walecki's construction [1].

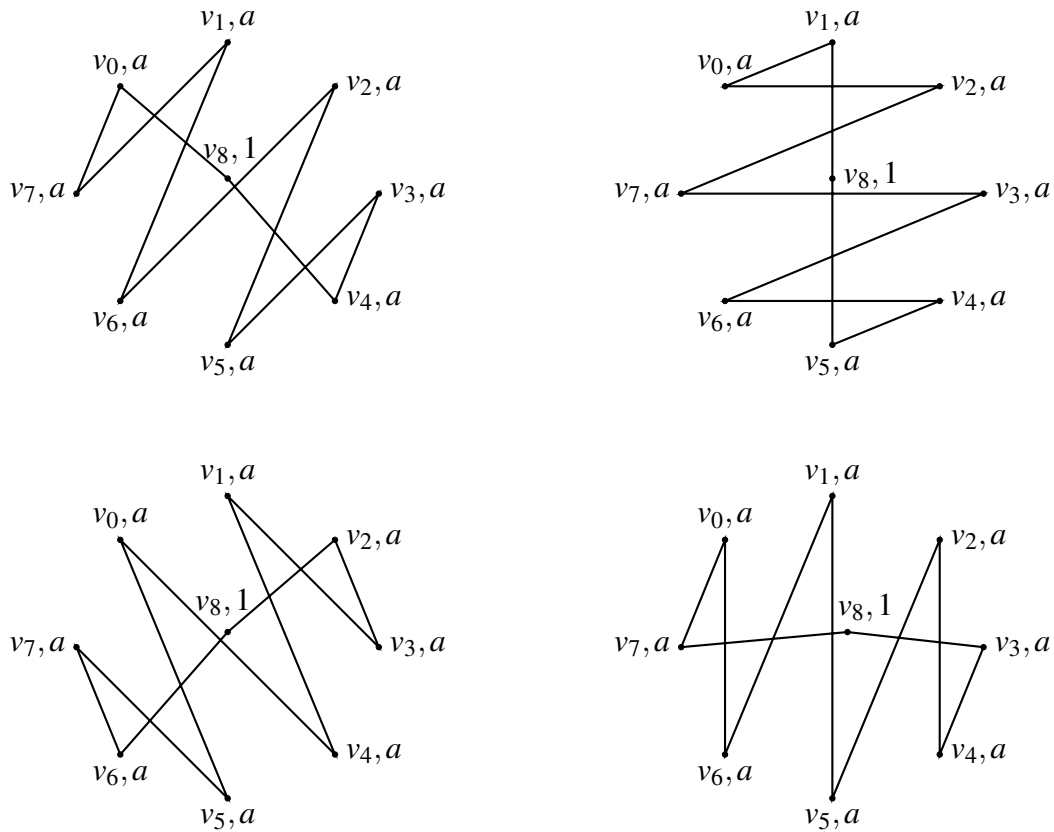


Figure 4: A 2-labeled packing of $k = 4$ copies of C_9 .

3. k is even and $x > 1$:

a) $2 \leq x \leq k - 1$:

The construction for this case seems more complicated, but it was actually inspired by the ones following Walecki's construction [1]. For $x = 2$, the first copy of C_n follows the following sequence of vertices:

$$C_n = \begin{cases} (v_0, v_2, v_{n-4}, v_4, v_{n-6}, v_6 \dots v_{\frac{k}{2}}, v_{n-2}, v_1, v_3, v_{n-3}, v_5, v_{n-5} \dots \\ v_{\frac{k}{2}+1}, v_{\frac{3k}{2}}, v_{\frac{k}{2}+3}, v_{\frac{3k}{2}-2}, v_{\frac{k}{2}+5}, v_{\frac{3k}{2}-4} \dots v_{\frac{3k}{2}+1}, v_{n-1}, v_0) & \text{if } k \equiv 0 \pmod{4} \\ (v_0, v_2, v_{n-4}, v_4, v_{n-6}, v_6 \dots v_{\frac{3k}{2}+1}, v_{n-2}, v_{n-3}, v_1, v_{n-5}, v_3 \dots \\ v_{\frac{3k}{2}}, v_{\frac{k}{2}+1}, v_{\frac{3k}{2}-2}, v_{\frac{k}{2}+3}, v_{\frac{3k}{2}-4}, v_{\frac{k}{2}+5} \dots v_{\frac{k}{2}}, v_{n-1}, v_0) & \text{if } k \equiv 2 \pmod{4}. \end{cases}$$

We label v_{2k} and v_{2k+1} , the fixed points, with respective labels $2k$ and $2k + 1$, and the other vertices with a if their number is even and b if it is odd. Then, we create the other copies by adding each time 2 modulo $n - 2$ to every number but $2k$ and $2k + 1$ in this sequence. For $x > 2$, we add extra fixed points in some edges that are linking a vertex of label a and a vertex of label b . Thus, $G \simeq C_n$. More formally, we take:

$$f(v_i) = \begin{cases} a & \text{if } 0 \leq i \leq 2k - 1 \text{ and } i \text{ is even} \\ b & \text{if } 0 \leq i \leq 2k - 1 \text{ and } i \text{ is odd} \\ i - 2k + 1 & \text{if } i \geq 2k. \end{cases}$$

$$\sigma_j(v_i) = \begin{cases} v_{i+2(j-1) \pmod{2k}} & \text{if } i < 2k \\ v_i & \text{if } i \geq 2k. \end{cases}$$

$$F = \{v_i, 2k \leq i \leq n - 1\}$$

$$V_a = \{v_i, 0 \leq i \leq 2k - 1 \text{ and } i \equiv 0 \pmod{2}\}$$

$$V_b = \{v_i, 0 \leq i \leq 2k - 1 \text{ and } i \equiv 1 \pmod{2}\}.$$

$$E_F = \{\{v_{\alpha(k,i)}, v_{k+3+i}\} \cup \{v_{k+3+i}, v_{\alpha(k,i+1)}\}, k - 1 \leq i \leq k + x - 4\} \cup \\ \{\{v_{\beta(k,1)}, v_{2k}\}, \{v_{(-1)^{\frac{k}{2}} \pmod{2k}}, v_{2k}\}\} \cup \{\{v_0, v_{2k+1}\}, \{v_{\beta(k,0)}, v_{2k+1}\}\}$$

$$E_{a,a} = \{\{v_{\gamma(k,i)}, v_{\gamma(k,i+1)}\}, 0 \leq i \leq \frac{k}{2} - 2\}$$

$$E_{b,b} = \{\{v_{(-1)^{\frac{k}{2}+\gamma(k,i)} \pmod{2k}}, v_{(-1)^{\frac{k}{2}+\gamma(k,i+1)} \pmod{2k}}\}, 0 \leq i \leq \frac{k}{2} - 2\}$$

$$E_{a,b} = \{\{v_{\alpha(k,i)}, v_{\alpha(k,i+1)}\}, k + x - 3 \leq i \leq 2k - 2\}.$$

With:

$$\alpha(k, i) = (-1)^{\frac{k}{2}} (1 + 2 \lceil \frac{k-1}{2} \rceil + (-1)^{\frac{(-1)^{i+1}}{2}} \cdot 2 \lfloor \frac{i-k+1}{2} \rfloor) + \frac{(-1)^{i+1}}{2} \cdot (k + (-1)^{\frac{k}{2}-1}) \pmod{2k}$$

$$\beta(k, i) = \frac{k}{2} + \frac{(-1)^{\frac{k}{2}+i+1}}{2} (k + 1) \pmod{2k}$$

$$\gamma(k, i) = 2 \cdot (-1)^i \lceil \frac{i+1}{2} \rceil \pmod{2k}.$$

The obtained graph G is isomorphic to C_n , as for $x = 2$, $G = \tau(H)$, with τ the permutation defined as follows:

$$\tau(v_i) = \begin{cases} v_{2 \cdot (-1)^{i+1} \lceil \frac{i}{2} \rceil} \pmod{2k} & \text{if } 0 \leq i \leq \frac{k}{2} - 1 \\ v_{(-1)^{\frac{k}{2} + 2} \cdot (-1)^{i+1 - \frac{k}{2}} \lceil \frac{i - \frac{k}{2}}{2} \rceil} \pmod{2k} & \text{if } \frac{k}{2} \leq i \leq k - 1 \\ v_{\alpha(k,i)} & \text{if } k \leq i \leq 2k - 1 \\ v_{\frac{k}{2}} & \text{if } i = 2k \\ v_i & \text{if } i = 2k + 1. \end{cases}$$

For $x > 2$, the extra fixed points are inserted in some edges of C_{2k+2} who are linking a vertex of label a and a vertex of label b . Thus, $G \simeq C_n$.

Since $2k$ is even, all the vertices are always sent to vertices that have the same label. The neighbors of the fixed points always have different labels and are never sent to themselves, so that their edges will not create any problem for the packing. To see that the other edges do not overlap either, the principle is similar to the one of Walecki's construction [1]. Indeed, for every type of edge, or differently said, for every couple of labels an edge can link, we chose the difference of the numbers of the vertices to be unique modulo $2k$ in the sequence, so that no other edge can overlap it. To be more precise, we show that the four previously mentioned properties hold:

- * Similarly to the case 1.a), for all $v \notin F$ and $j \neq j'$, $f(\sigma_j(v)) = f(\sigma_{j'}(v))$.
- * For all p, q and for all $\{u, v\}, \{u', v'\} \in E_{p,q}$, $\sigma_j(\{u, v\}) = \sigma_{j'}(\{u', v'\})$ implies $j = j'$.
Indeed, for $p = q = a$, for $0 \leq i \leq \frac{k}{2} - 2$, suppose that:

$$\sigma_j^*(\{v_{\gamma(k,i)}, v_{\gamma(k,i+1)}\}) = \sigma_{j'}^*(\{v_{\gamma(k,i')}, v_{\gamma(k,i'+1)}\}).$$

Then, by adding the two equations resulting from the vertices, we get:

$$2((-1)^{i+1} \lceil \frac{i}{2} \rceil + (-1)^i \lceil \frac{i+1}{2} \rceil) + 2j \equiv 2((-1)^{i'+1} \lceil \frac{i'}{2} \rceil + (-1)^{i'} \lceil \frac{i'+1}{2} \rceil) + 2j' \pmod{2k}.$$

Since k is even, this gives $j = j'$. The case where $p = q = b$ is similar. For $p = a$ and $q = b$, for $k + x - 3 \leq i \leq 2k - 2$, suppose that

$$\sigma_j^*(\{v_{\alpha(k,i)}, v_{\alpha(k,i+1)}\}) = \sigma_{j'}^*(\{v_{\alpha(k,i')}, v_{\alpha(k,i'+1)}\}).$$

Then, by adding the two equations, we get:

$$2.((-1)^{\frac{(-1)^{i+1}}{2}} \lfloor \frac{i-k+1}{2} \rfloor + (-1)^{\frac{(-1)^{i+1}+1}{2}} \lfloor \frac{i-k+2}{2} \rfloor) + 4j \equiv$$

$$2.((-1)^{\frac{(-1)^{j'+1}}{2}} \lfloor \frac{j'-k+1}{2} \rfloor + (-1)^{\frac{(-1)^{j'+1}+1}{2}} \lfloor \frac{j'-k+2}{2} \rfloor) + 4j' \pmod{2k}.$$

Since k is even, this also gives $j = j'$.

- * Similarly to 1.a), for all $v \notin F$ and $j \neq j'$, $\sigma_j(v) \neq \sigma_{j'}(v)$.
- * Since $k + (-1)^{\frac{k}{2}-1}$ is odd, $\alpha(k, i)$ and $\alpha(k, i + 1)$ have different parity for all i such that $k - 1 \leq i \leq k + x - 4$. Plus, $\beta(k, 1)$ is even while $(-1)^{\frac{k}{2}} \pmod{2k}$ is odd, and 0 is even while $\beta(k, 0)$ is odd. Thus, for all u, v such that there exists $x \in F$ such that $\{u, x\}, \{x, v\} \in E(C_n)$, $f(u) \neq f(v)$.

This construction is illustrated in the following figure, for $k = 4$ and $n = 10$.

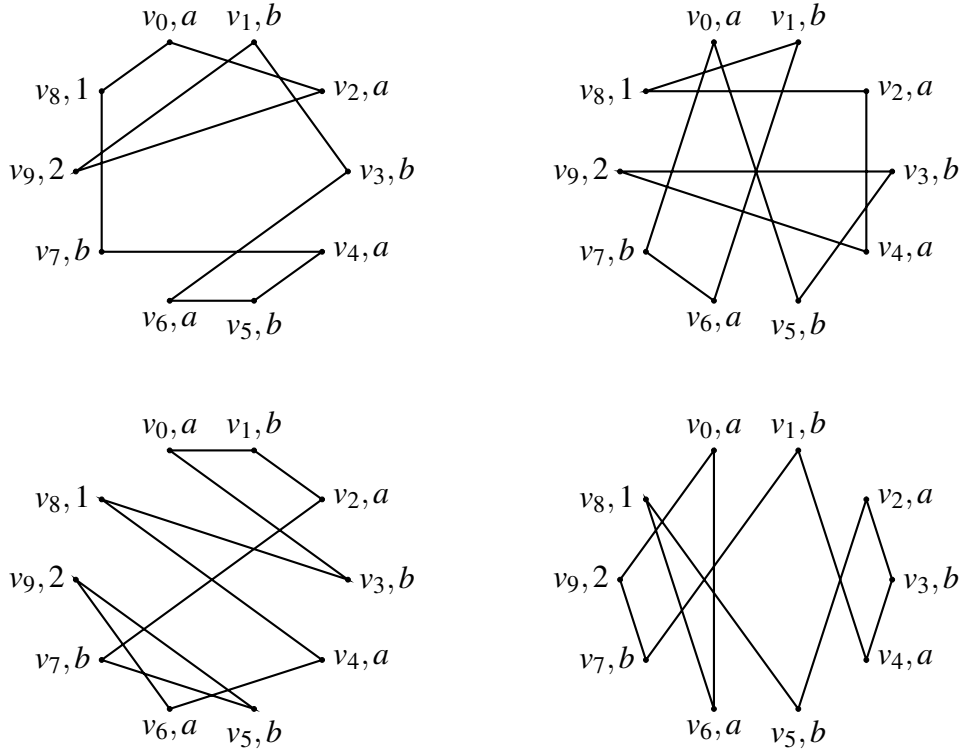


Figure 5: A 4-labeled packing of $k = 4$ copies of C_{10} .

b) $k \leq x \leq 2k - 1$:

For this case, we also got inspired by Walecki's construction [1], but with a simpler construction. For $x = k$, we build a $(k + 2)$ -labeled packing of G in the following way: We label the vertices of number $i \leq 2k$ by their number, and the other vertices with a if their number is even and b if their number is odd. We create the first copy of C_n by joining the vertices in the following order:

$$v_{2k}, v_0, v_{2k+1}, v_1, \dots, v_{3k-1}, v_{k-1}, v_{2k-2}, v_{k+1}, v_{2k-4}, v_{k+3}, v_{2k-6}, \dots, v_{2k-k}, v_{k+(k-1)}, v_{2k}.$$

The other copies are created by adding 2 modulo $n - 2$ to the numbers of the a and b label vertices of this sequence. For $x > k$, we add fixed points on the edges linking a vertex of label a to a vertex of label b . We have:

$$f(v_i) = \begin{cases} a & \text{if } 0 \leq i \leq 2k - 1 \text{ and } i \text{ is even} \\ b & \text{if } 0 \leq i \leq 2k - 1 \text{ and } i \text{ is odd} \\ i - 2k + 1 & \text{if } i \geq 2k. \end{cases}$$

$$\sigma_j(v_i) = \begin{cases} v_{i+2(j-1) \bmod 2k} & \text{if } i < 2k \\ v_i & \text{if } i \geq 2k. \end{cases}$$

$$F = \{v_i, 2k \leq i \leq n - 1\}$$

$$V_a = \{v_i, 0 \leq i \leq 2k - 1 \text{ and } i \equiv 0 \pmod{2}\}$$

$$V_b = \{v_i, 0 \leq i \leq 2k - 1 \text{ and } i \equiv 1 \pmod{2}\}.$$

$$\begin{aligned} E_F &= \{\{v_{2k+i}, v_{i-1 \bmod 2k}\} \cup \{v_{2k+i}, v_i\}, 0 \leq i \leq k - 1\} \cup \\ &\{\{v_{4k-2\lfloor \frac{i}{2} \rfloor - 2}, v_{k+i}\}, \{v_{k+i}, v_{2\lfloor \frac{i}{2} \rfloor + (-1)^{i+1-k}}\}, 2k \leq i \leq k + x - 1\} \\ E_{a,a} &= E_{b,b} = \emptyset \\ E_{a,b} &= \{\{v_{4k-2\lfloor \frac{i}{2} \rfloor - 2}, v_{2\lfloor \frac{i}{2} \rfloor + (-1)^{i+1-k}}\}, k + x \leq i \leq 3k - 1\}. \end{aligned}$$

For $x = k$, $G = \tau(H)$, with τ the permutation defined as follows, so that G is isomorphic to C_n .

$$\tau(v_i) = \begin{cases} v_{2k+\frac{i}{2}} & \text{if } 0 \leq i \leq 2k - 1, i \equiv 0 \pmod{2} \\ v_{\frac{i-1}{2}} & \text{if } 0 \leq i \leq 2k - 1, i \equiv 1 \pmod{2} \\ v_{4k-i-2} & \text{if } 2k \leq i \leq 3k - 1, i \equiv 0 \pmod{2} \\ v_{i-k} & \text{if } 2k \leq i \leq 3k - 1, i \equiv 1 \pmod{2}. \end{cases}$$

For $x > 2$, the extra fixed points are inserted in some edges of C_{3k} .

In this construction, all the fixed points have one neighbor with label a and one neighbor with label b , that are never sent to themselves or each other. The edges involving them will thus never overlap. The other edges link a vertex of label a with a vertex of label b , and the differences between their numbers are all distinct. Plus, as $2k$ is even, the parity of the vertices remains the same in all the copies, so do their labels. To be more precise, we have:

- * Similarly to the case 1.a), for all $v \notin F$ and $j \neq j'$, $f(\sigma_j(v)) = f(\sigma_{j'}(v))$.
- * For all p, q and for all $\{u, v\}, \{u', v'\} \in E_{p, q}$, $\sigma_j(\{u, v\}) = \sigma_{j'}(\{u', v'\})$ implies $j = j'$.
Indeed, suppose, for a given i , $k + x \leq i \leq 3k - 1$, that:

$$\sigma_j^*(\{v_{4k-2\lfloor \frac{i}{2} \rfloor - 2}, v_{2\lfloor \frac{i}{2} \rfloor + (-1)^{i+1} - k}\}) = \sigma_{j'}^*(\{v_{4k-2\lfloor \frac{i'}{2} \rfloor - 2}, v_{2\lfloor \frac{i'}{2} \rfloor + (-1)^{i'+1} - k}\}).$$

Then, by adding the two resulting equations, we get:

$$4j + (-1)^{i+1} \pmod{2k} = 4j' + (-1)^{i'+1} \pmod{2k}.$$

Since the parity of both sides of this equality must be the same, this gives $j = j'$.

- * Similarly to the case 1.a), for all $v \notin F$ and $j \neq j'$, $\sigma_j(v) \neq \sigma_{j'}(v)$.
- * Since for all i with $2k \leq i \leq k + x - 1$, $4k - 2\lfloor \frac{i}{2} \rfloor - 2$ is even, and $2\lfloor \frac{i}{2} \rfloor + (-1)^{i+1} - k$ is odd, they have different parity, and so do $i - 1 \pmod{2k}$ and i for all i with $0 \leq i \leq k - 1$. We therefore have, for all u, v such that there exists $x \in F$ such that $\{u, x\}, \{x, v\} \in E(C_n)$, $f(u) \neq f(v)$.

We present this last construction in the following figure, that takes the example of $k = 4$ and $n = 12$

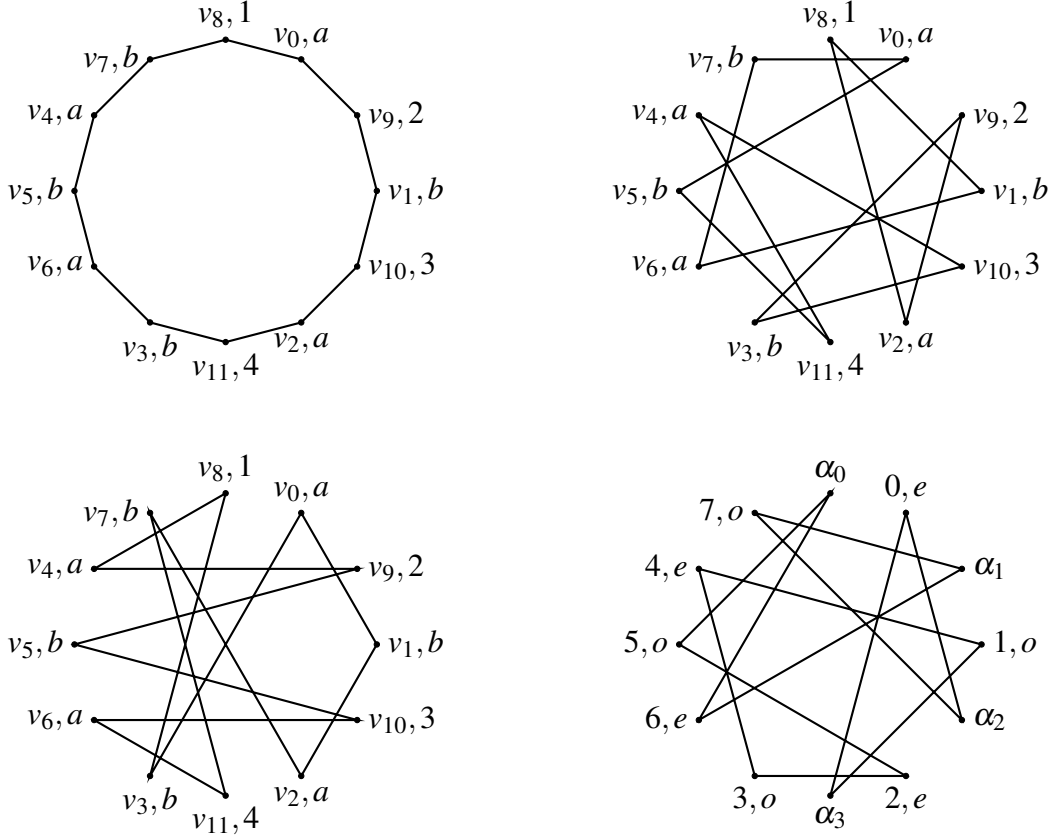


Figure 6: A 6-labeled packing of $k = 4$ copies of C_{12} .

This concludes case 3), and thus the entire proof of Theorem 2. \square

Concerning the upper bound given by Conjecture 3, we have already seen, with a counter-example, that it did not hold in general. That being said, we still found some sufficient conditions for it to hold. The following lemma helped in the process of finding those conditions:

Lemma 1. *Let G be the cycle C_n of order $n = 2k + x$, $k \geq 2$, $1 \leq x \leq 2k - 1$. Let f be a p -labeling of k copies of G . Let $q = \min_{i \in \{1, \dots, p\}} |\{v \in V(C_n) : f(v) = i\}|$ be the minimum, over all p labels, of the number of vertices that have this label. Then, there exists in G a set of at least $2k$ vertices that are associated, together, to at most $2q$ labels.*

Proof. Let j be one of the labels associated to exactly q vertices, and V_j be the set of vertices of G with label j . All the vertices of V_j have two neighbors. Let N_j be the set

of neighbors, and $M_j = N_j - (N_j \cap V_j)$ be the set of neighbors that are not in V_j . Let L be the number of different labels l_1, l_2, \dots, l_L associated to the vertices of M_j , $V(l_1), V(l_2), \dots, V(l_L)$ be the sets of vertices of G associated to those respective labels, and $U = V(l_1) \cup V(l_2) \cup \dots \cup V(l_L)$. Let e be the number of edges linking two vertices from V_j in G , and, for all $1 \leq i \leq L$, e_i be the number of edges linking a vertex from V_j to a vertex from $V(l_i)$ in G . To have a valid packing, we must have, for every i with $1 \leq i \leq L$, $ke_i \leq q|V(l_i)|$. Therefore, we have:

$$\sum_{i=1}^L e_i \leq \frac{q}{k} \sum_{i=1}^L |V(l_i)|.$$

But $\sum_{i=1}^L e_i = 2q - 2e$, and, since the $V(l_i)$ are pairwise disjoint, $\sum_{i=1}^L |V(l_i)| = |U|$. Therefore,

$$|U| \geq \frac{k}{q}(2q - 2e).$$

If $e = 0$, we obtain $|U| \geq 2k$. Moreover, at best, in M_j , we have $2q$ distinct vertices that all have distinct labels, so that $L \leq 2q$. By taking the set U , we obtain a set of at least $2k$ vertices in G associated to at most $2q$ labels. If $e \geq 1$, to have a valid packing, we must have $ke \leq \frac{q(q-1)}{2}$, we thus have $|U| \geq 2k - (q-1)$. Plus, at best, in M_j , we have $2q - 2$ distinct vertices that all have distinct labels, so that $L \leq 2q - 2$. Therefore, by taking the set $U \cup V_j$, we obtain a set of at least $2k + 1$ vertices in G associated to at most $2q - 1$ labels. Those two cases give the result. \square

Lemma 1 led, in particular, to the following theorem, that gives a sufficient condition for the upper bound given by Conjecture 3 to hold.

Theorem 10. *Let G be the cycle C_n of order $n = 2k + x$, $k \geq 2$, $1 \leq x \leq 2k - 1$. Let f be a p -labeling of k copies of G . Let $q = \min_{i \in \{1, \dots, p\}} |\{v \in V(C_n) : f(v) = i\}|$ be the minimum, over all p labels, of the number of vertices that have this label. If $q = 1$, meaning that f has fixed points, or $q \leq \frac{x}{2}$, then $p \leq x + 2$.*

Proof. We know by Lemma 1 that there exists in G a group of at least $2k$ vertices associated to at most $2q$ labels, and the rest of the labels are represented at least q times. Therefore, we have:

$$p \leq 2q + \lfloor \frac{n-2k}{q} \rfloor \leq 2q + \frac{x}{q}.$$

But, for all $q \geq 1$, $2q + \frac{x}{q} \leq x + 2$.

Indeed, for $q=1$, we have $2q + \frac{x}{q} = x + 2$.

For $q > 1$, $2q + \frac{x}{q} \leq x + 2 \leftrightarrow q \leq \frac{x}{2}$, which is true by hypothesis. Therefore, we have $p \leq x + 2$. \square

The disadvantage of Theorem 10 is that the conditions it gives are on a preexisting packing, while we want the upper bound to hold for all packings, with conditions on k and x only. Such conditions are thus given in the following theorem:

Theorem 11. *Let C_n be the cycle of order $n = 2k + x$, $k \geq 2$, $1 \leq x \leq 2k - 1$. If $x \geq \sqrt{4k - 2}$, then $\lambda^k(C_n) \leq x + 2$.*

Proof. Let us suppose that $\lambda^k(C_n) \geq x + 3$. Then, there exists a $(x + 3)$ -labeling f of k copies of G . Let q be the minimum, over all p labels, of the number of vertices that have this label. Then, by Theorem 10, we have $q \geq \frac{x+1}{2}$. We have therefore at least $x + 3$ labels, each represented by at least $\frac{x+1}{2}$ vertices, so that we have:

$$\frac{(x+1)(x+3)}{2} \leq 2k+x$$

Or

$$x^2 + 2x + (3 - 4k) \leq 0.$$

For $x \geq \sqrt{4k - 2}$, this last inequality is false and we obtain a contradiction. \square

We now have a satisfying sufficient condition for the upper bound of $x + 2$ to be valid, even more satisfying when taking into account the fact that the proportion of cases it covers grows with n . That being said, this bound is not the one given in Conjecture 3 for the particular case where k is even and $x = 1$. The following theorem thus gives a sufficient condition for this case, that depends on the considered packing:

Theorem 12. *Let C_n be the cycle of order $n = 2k + 1$, $k \geq 2$, k even. Let f be a p -labeling of k copies of C_n . Let $q = \min_{i \in \{1, \dots, p\}} |\{v \in V(C_n) : f(v) = i\}|$ be the minimum, over all p labels, of the number of vertices that have this label. If $q = 1$, then $p \leq 2$.*

Proof. From Theorem 10, since $q = 1$, we already have $\lambda^k(C_n) \leq 3$. Let us assume that $\lambda^k(C_n) = 3$. Then, there exists a 3-labeled packing of k copies of C_n . By Lemma 1, we know that there exists in this packing a set of $2k$ vertices associated to exactly 2 labels, called a and b , and one fixed point.

We have $k(2k + 1) = \frac{(2k+1)(2k+1-1)}{2}$, so that every edge between any two vertices is going to belong to exactly one of k the copies of G . In particular, every edge between the fixed point and the other vertices is going to belong to a copy, so that the fixed point has to be linked to exactly one vertex with label a , and one vertex with label b . Therefore, there has to be k vertices with label a and k vertices with label b . The number of edges linking two vertices of label a in G is therefore $\frac{k(k-1)}{2k} = \frac{k-1}{2}$, which means that k is odd, and contradicts the hypothesis. Therefore, $\lambda^k(C_n) \neq 3$, and $\lambda^k(C_n) \leq 2$. \square

For the cases where the previously seen conditions are not verified, the following theorem gives a necessary condition on p for a p -labeled packing of C_{2k+x} to exist and thus provides a different upper bound on $\lambda^k(C_{2k+x})$:

Theorem 13. *Let C_n be the cycle of order $n = 2k + x$. For $p \in \mathbb{N}^*$, if $p \leq \lambda^k(C_n)$, there exists a partition n_1, n_2, \dots, n_p of n into p parts such that $\sum_{i=1}^p \lfloor \frac{n_i(n_i-1)}{2k} \rfloor + \sum_{i=1}^p \sum_{j=i+1}^p \lfloor \frac{n_i n_j}{k} \rfloor \geq n$.*

Proof. Since $p \leq \lambda^k(C_n)$, there exists a p -labeled packing f of k copies of C_n into K_n . Let $1, 2, \dots, p$ be the name of the labels of f , and n_1, n_2, \dots, n_p be the number of vertices respectively associated to those labels. Obviously, n_1, n_2, \dots, n_p is a partition of n into p parts.

For all $i \in [1, p]$, $j \in [i+1, p]$, let m_i be the number of edges of C_n in f linking two vertices of label i , and let $m_{i,j}$ be the number of edges of C_n in f linking a vertex of label i and a vertex of label j .

$$\text{We have } \sum_{i=1}^p m_i + \sum_{i=1}^p \sum_{j=i+1}^p m_{i,j} = n.$$

For f to be a valid packing, we must have, for all $i \in [1, p]$, $m_i \leq \lfloor \frac{n_i(n_i-1)}{2k} \rfloor$, and, for all $j \in [i+1, p]$, $m_{i,j} \leq \sum_{j=i+1}^p \lfloor \frac{n_i n_j}{k} \rfloor$.

$$\text{Thus, } \sum_{i=1}^p \lfloor \frac{n_i(n_i-1)}{2k} \rfloor + \sum_{i=1}^p \sum_{j=i+1}^p \lfloor \frac{n_i n_j}{k} \rfloor \geq n. \quad \square$$

By regrouping the sufficient conditions of Theorems 8, 10 and 11, we can already restrict the possible values of k and x for which the second inequality of Conjecture 3 might not hold. Table 1 lists those values and gives the upper bound given by Theorem 13 for each of those, until $k = 35$.

| k | x | n | maximal possible value of $\lambda^k(C_n)$ |
|-----|-----|-----|--|
| 9 | 3 | 21 | $x + 4$ |
| 15 | 6 | 36 | $x + 3$ |
| 16 | 4 | 36 | $x + 5$ |
| 16 | 5 | 37 | $x + 3$ |
| 24 | 7 | 55 | $x + 4$ |
| 25 | 5 | 55 | $x + 6$ |
| 25 | 6 | 56 | $x + 4$ |
| 27 | 3 | 57 | $x + 4$ |
| 28 | 4 | 60 | $x + 3$ |
| 33 | 4 | 70 | $x + 3$ |
| 33 | 5 | 71 | $x + 3$ |
| 34 | 10 | 78 | $x + 3$ |
| 35 | 4 | 74 | $x + 3$ |
| 35 | 8 | 78 | $x + 5$ |
| 35 | 9 | 79 | $x + 3$ |

Table 1: Values of k and x , until $k = 35$, for which the second inequality of Conjecture 3 is unproven. The upper bound given by Theorem 13 is also expressed, depending on x .

4 Labeled Packing of Circuits

In this part, we study the directed version of the previous problem, that is the labeled packing of circuits, where the circuit \vec{C}_n of order n is the directed graph defined by $V(\vec{C}_n) = \{v_i, 0 \leq i \leq n-1\}$ and $E(\vec{C}_n) = \{(v_i, v_{i+1 \pmod n}), 0 \leq i \leq n-1\}$.

We define a packing of directed graphs (H_1, H_2, \dots, H_k) in the directed graph G the same way as a packing of graphs, the only difference being that for all i , $E(H_i)$ stands for the arcs of H_i , and $E(G)$ for the arcs of G , so that the induced images of the arcs of the H_i must go into the arcs of G without intersecting. We focus on the packings of k copies of H into the complete digraph \vec{K}_n , the digraph of order n with all possible arcs. Given those modifications, the definitions of a labeled packing and of $\lambda^k(G)$ for G a digraph are direct.

The results we present are very similar to the ones presented for cycles, with some adaptations. We first have the following conditions for the k -placement to exist:

Theorem 14. \vec{C}_{k+x} admits a k -placement if and only if $x \geq 1$, and $(x, n) \notin \{(1, 4), (1, 6)\}$.

Proof. First, for a k -placement of \vec{C}_n to exist, we must have $k|E(\vec{C}_n)| \leq |E(\vec{K}_n)|$, so that $n \geq k + 1$. Plus, if $x = 1$, the k -placement would actually be a decomposition of \vec{K}_n , and we know from Theorem 1.1 in [2] that such a decomposition exists if and only if $n \neq 4$, and $n \neq 6$. Now, since the existence of a packing of k copies of \vec{C}_{k+x} implies the existence of a packing of $k - 1$ copies of $\vec{C}_{(k-1)+(x+1)}$, and since we have the following packings of $k = 2$ copies of \vec{C}_4 and $k = 4$ copies of \vec{C}_6 , this gives the result:



Figure 7: A packing of $k = 2$ copies of \vec{C}_4 .

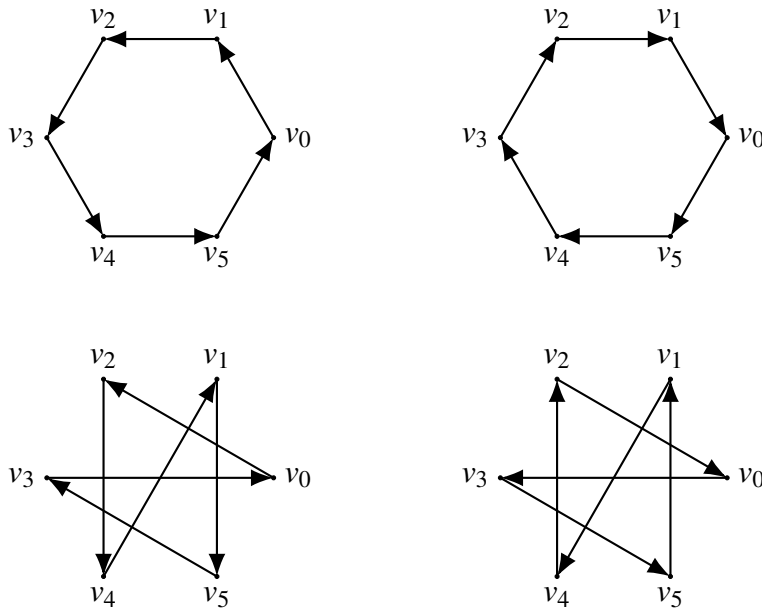


Figure 8: A packing of $k = 4$ copies of \vec{C}_6 .

□

When those conditions are satisfied, we can study the value of $\lambda^k(\vec{C}_n)$. We first have the following lemma, that is an extension of Duchêne, Kheddouci, Nowakowski and Tahraoui's Lemma 7 [4] to the case of circuits, following the same proof:

Lemma 2. *For every circuit \vec{C}_n of order $n > k$, with $(x, n) \notin \{(1, 4), (1, 6)\}$, we have $\lambda^k(\vec{C}_n) \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{\lceil \frac{n}{2} \rceil}{k} \rfloor$.*

When $n \geq 2k$, the value of $\lambda^k(\vec{C}_n)$ can be exactly found with the following theorem:

Theorem 15. *For every circuit \vec{C}_n of order $n = 2km + x$, $k, m \geq 1$, $x < 2k$, and $(x, n) \notin \{(1, 4), (1, 6)\}$ we have $\lambda^k(\vec{C}_n) = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{\lceil \frac{n}{2} \rceil}{k} \rfloor$.*

Proof. From Lemma 2, we get the upper bound.

For the lower bound, we give a construction of the corresponding labeled packing of \vec{C}_n . The idea is to put a fixed point one vertex out of two on the first copy of \vec{C}_n , and to have at least k vertices of any other label so that they will never be sent to themselves and thus never create superpositions of arcs.

Let $V(C_n) = \{v_0, \dots, v_n\}$ be the set of vertices of \vec{C}_n and $E(C_n) = \{(v_i, v_{i+1 \bmod n}), 0 \leq i \leq n-1\}$ be its set of arcs.

We label the vertices with the labeling f :

$$f(v_i) = \begin{cases} v_{\frac{i}{2}+1} & \text{if } 0 \leq i \leq 2(\lfloor \frac{n}{2} \rfloor - 1) \text{ and } i \text{ is even} \\ v_{\lfloor \frac{i}{2k} \rfloor + \lfloor \frac{n}{2} \rfloor + 1} & \text{if } 1 \leq i \leq 2k \lfloor \frac{\lceil \frac{n}{2} \rceil}{k} \rfloor - 1 \text{ and } i \text{ is odd} \\ v_{\lfloor \frac{i}{2k} \rfloor + \lfloor \frac{n}{2} \rfloor} & \text{otherwise.} \end{cases}$$

For each label l , we rename the L vertices that have label l , following the increasing order of their number, into l_0, l_1, \dots, l_L , and we associate to the labeling the set of permutations $\sigma = \{\sigma_j, 1 \leq j \leq k\}$:

$$\begin{cases} \sigma_j(l_i) = l_{i+1 \bmod L} & \text{for all } l \\ \sigma_j(v_i) = v_i & \text{if } i \text{ is even.} \end{cases}$$

For any j , a vertex and its image by σ_j obviously have the same label. Plus, since for all i with $0 \leq i \leq n-1$ and i even, v_i is a fixed points, and for all i odd and, $j \neq j'$, $\sigma_j(v_i) \neq \sigma_{j'}(v_i)$, σ is a packing.

An example of the construction is given in the following figure, for $k = 3$ and $n = 7$:

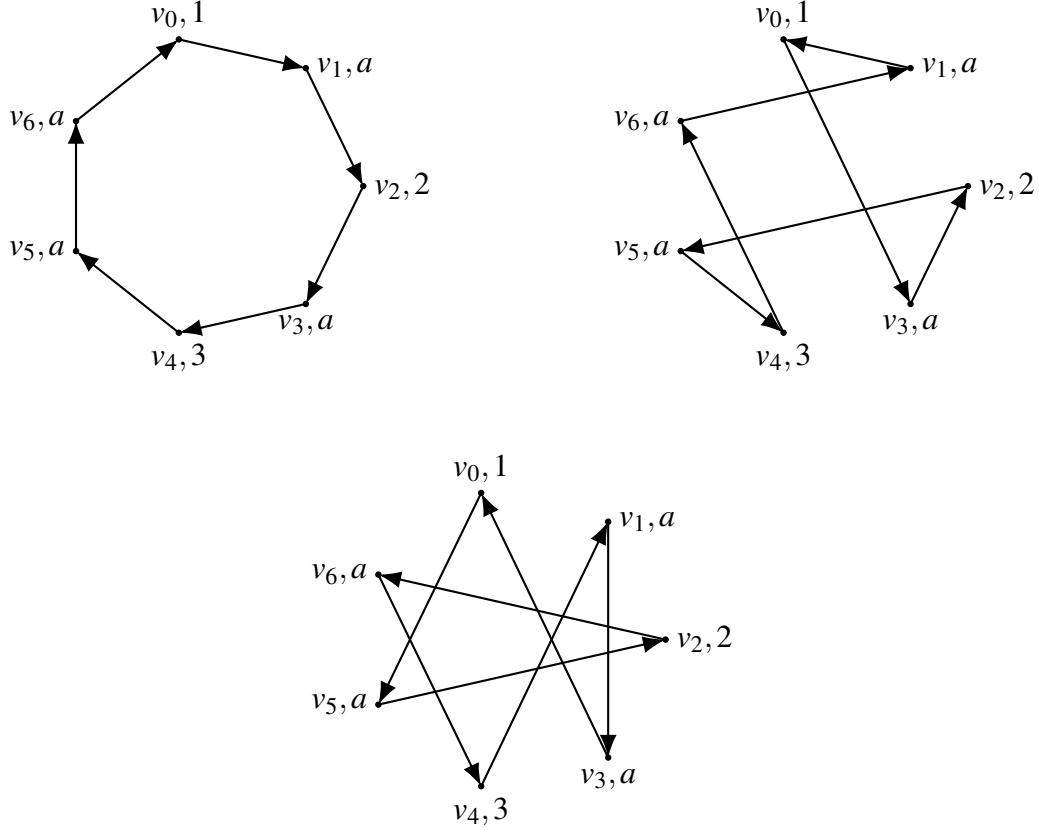


Figure 9: A 4-labeled packing of $k = 3$ copies of \vec{C}_7 .

□

The case where $k + 1 \leq n \leq 2k - 1$ is more complicated. We present here two theorems for this case, one for k even, the second for k odd:

Theorem 16. For every circuit \vec{C}_n of order $n = k + x$, $k \geq 2$, $1 \leq x \leq k - 1$, k even, $(x, n) \notin \{(1, 4), (1, 6)\}$, we have $\lambda^k(\vec{C}_n) \geq x + 1$.

Proof. We give the associated construction, inspired by Walecki's [1]. Let $\{v_0, v_1, \dots, v_{n-1}\}$ be the vertices of \vec{C}_n . We label them with f :

$$f(v_i) = \begin{cases} i - k + 1 & \text{if } i \geq k \\ a & \text{otherwise.} \end{cases}$$

We associate to it the set of permutations $\sigma = \{\sigma_j, 1 \leq j \leq k\}$:

$$\sigma_j(v_i) = \begin{cases} v_i & \text{if } i \geq k \\ v_{i+j-1 \bmod k} & \text{otherwise.} \end{cases}$$

We partition the set of vertices $V(\vec{C}_n)$ into a set F containing all the fixed points, and a set V_a containing all the vertices of label a . We partition the set of arcs $E(\vec{C}_n)$ into two sets E_F and $E_{a,a}$ defined as follows:

$$E_F = \{(v_{i-1}, v_{k+i}) \cup (v_{k+i}, v_{k-i}), 1 \leq i \leq x-1\} \cup \{(v_k, v_0), (v_{\frac{k}{2}}, v_k)\}$$

$$E_{a,a} = \{(v_{k-i}, v_i), 1 \leq i \leq \frac{k}{2} - 1\} \cup \{(v_i, v_{k-1-i}), x-1 \leq i \leq \frac{k}{2} - 1\}.$$

The obtained graph G is isomorphic to \vec{C}_n . Indeed, for $x = 1$, if $H = (V(H) = \{v_1, v_2, \dots, v_n\}, E(H) = \{(v_i, v_{i+1 \bmod n}), 1 \leq i \leq n\})$, we have $H \simeq \vec{C}_n$, and $G = \tau(H)$, with:

$$\tau(v_i) = \begin{cases} v_{\frac{i}{2}} & \text{if } i < k, i \equiv 0 \pmod{2} \\ v_{k-\frac{i+1}{2}} & \text{if } i < k, i \equiv 1 \pmod{2} \\ v_i & \text{if } i = k. \end{cases}$$

For $x > 1$, the extra fixed point are inserted in some of the edges of \vec{C}_{k+1} , so that $G \simeq \vec{C}_n$.

For all $v \in F$, since $\sigma_j(v) = v$, we know that $f(\sigma_j(v)) = f(v) = f(\sigma_{j'}(v))$. Since $i + j - 1 \bmod k < k$, for any $v \notin F$, $f(\sigma_j(v)) = a = f(\sigma_{j'}(v))$, and f is a valid labeled packing with respect to σ .

Thus, in particular, for all $j, j', \sigma_j^*(E_F) \cap \sigma_{j'}^*(E_{a,a}) = \emptyset$.

For all $\{u, v\}, \{u', v'\} \in E_{a,a}$, $\sigma_j(\{u, v\}) = \sigma_{j'}(\{u', v'\})$ implies $j = j'$. Indeed, for $1 \leq i \leq \frac{k}{2} - 1$, if $\sigma_j^*((v_{k-i}, v_i)) = \sigma_{j'}^*((v_{k-i'}, v_{i'}))$, then:

$$\begin{cases} i + j - 1 = i' + j' - 1 \pmod{k} \\ k - i + j - 1 = k - i' + j' - 1 \pmod{k}. \end{cases}$$

By adding the two equations, since k is even, and $1 \leq j, j' \leq k$, we get $j = j'$, or $j' = j + \frac{k}{2}$. But injecting $j' = j + \frac{k}{2}$ into the first equation gives $i \equiv i' + \frac{k}{2} \pmod{k}$, which is a contradiction, as $1 \leq i' \leq \frac{k}{2} - 1$.

Similarly, for $x - 1 \leq i, i' \leq k - 1$, if $\sigma_j^*((v_i, v_{k-1-i})) = \sigma_{j'}^*((v_{i'}, v_{k-1-i'}))$, $j = j'$.

For $1 \leq i \leq k - 1$ and $x - 1 \leq i' \leq k - 1$, $\sigma_j^*((v_{k-i}, v_i)) = \sigma_{j'}^*((v_{i'}, v_{k-1-i'}))$ is impossible, as the sum of both equations gives an equality between an odd number and an even number.

Thus, for all $j \neq j'$, $\sigma_j^*(E_{a,a}) \cap \sigma_{j'}^*(E_{a,a}) = \emptyset$.

To show that $\sigma_j^*(E_F) \cap \sigma_{j'}^*(E_F) = \emptyset$, since each edge of E_F contains one fixed point, we only have to show that its ingoing neighbor u and outgoing neighbor v verify $\sigma_j(u) \neq \sigma_{j'}(u)$ and $\sigma_j(v) \neq \sigma_{j'}(v)$. But since $i + j - 1 \pmod k = i + j' - 1 \pmod k$ implies $j = j'$, for all $v \notin F$ and $j \neq j'$, $\sigma_j(v) \neq \sigma_{j'}(v)$.

An example of the construction is given in the following figure, for $k = 4$ and $x = 3$:

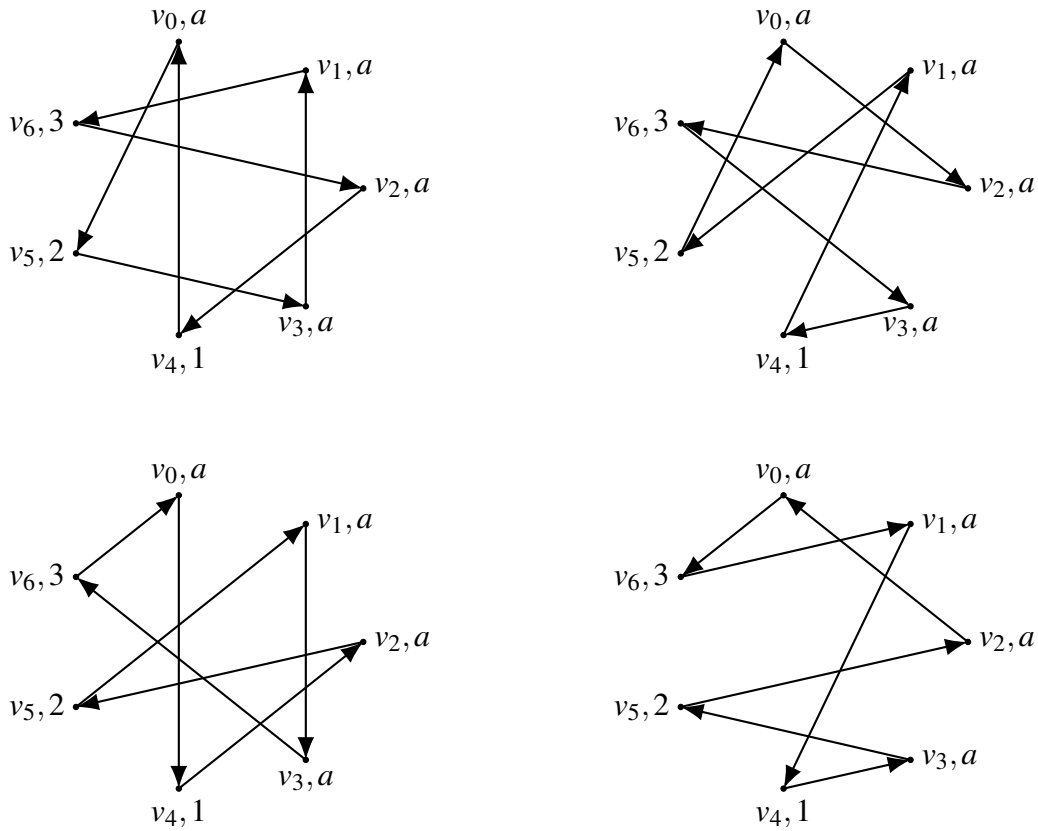


Figure 10: A 4-labeled packing of $k = 4$ copies of \vec{C}_7 .

This concludes the proof of the theorem. □

The following result, for the case where k is odd, can be deduced from the previous one:

Theorem 17. For every circuit \vec{C}_n of order $n = k + x$, $k \geq 2$, $2 \leq x \leq k - 1$, k odd, $(x, n) \notin \{(1, 4), (1, 6)\}$, we have $\lambda^k(\vec{C}_n) \geq x$.

Proof. We have $n = (k + 1) + (x - 1)$ with $k + 1$ even, and $x - 1 \geq 1$. Thus, by Theorem 15, there exists a x -labeled packing of $k + 1$ copies of \vec{C}_n . \square

The next results are interested in finding an upper bound. The proofs are not given as they are similar to the ones for cycles.

The first result is the analogue of Lemma 1 for the case of circuits. The proof is identical, with the exception that we only count the outgoing neighbors and outgoing arcs of the vertices that have one of the labels that are represented by q vertices.

Lemma 3. Let \vec{C}_n be the circuit of order $n = k + x$, $k \geq 2$, $1 \leq x \leq k - 1$, $(x, n) \notin \{(1, 4), (1, 6)\}$. Let f be a p -labeling of k copies of \vec{C}_n . Let $q = \min_{i \in \{1, \dots, p\}} |\{v \in V(C_n) \text{ such that } f(v) = i\}|$ be the minimum, over all p labels, of the number of vertices that have this label. Then, there exists in \vec{C}_n a set of at least k vertices that are associated, together, to at most q labels.

This lemma leads to the two following theorems:

Theorem 18. Let \vec{C}_n be the circuit of order $n = k + x$, $k \geq 2$, $1 \leq x \leq k - 1$, $(x, n) \notin \{(1, 4), (1, 6)\}$. Let f be a p -labeling of k copies of \vec{C}_n . Let $q = \min_{i \in \{1, \dots, p\}} |\{v \in V(\vec{C}_n) : f(v) = i\}|$ be the minimum, over all p labels, of the number of vertices that have this label. If $q \leq x$, then $p \leq x + 2$.

As it was the case for cycles, the previous theorem gives conditions on the considered packing, while we want conditions on k and x only. Thus, we give the following result:

Theorem 19. Let \vec{C}_n be the circuit of order $n = k + x$, $k \geq 2$, $1 \leq x \leq k - 1$, $(x, n) \notin \{(1, 4), (1, 6)\}$. If $x \geq \sqrt{k - 1}$, then $\lambda^k(\vec{C}_n) \leq x + 2$.

Finally, considering necessary conditions on the number of arcs between each pair of labels gives an analogue of Theorem 13 for circuits:

Theorem 20. Let \vec{C}_n be the circuit of order $n = k + x$, $(x, n) \notin \{(1, 4), (1, 6)\}$. For $p \in \mathbb{N}^*$, if $p \leq \lambda^k(\vec{C}_n)$, there exists a partition n_1, n_2, \dots, n_p of n into p parts such that $\sum_{i=1}^p \lfloor \frac{n_i(n_i-1)}{k} \rfloor + \sum_{i=1}^p \sum_{j=i+1}^p \lfloor \frac{2n_i n_j}{k} \rfloor \geq n$.

References

- [1] B. Alspach. The wonderful walecki construction. *Bull. Inst. Combin. Appl*, 52:7–20, 2008.
- [2] B. Alspach, H. Gavlas, M. Šajna, and H. Verrall. Cycle decompositions iv: complete directed graphs and fixed length directed cycles. *Journal of Combinatorial Theory, Series A*, 103(1):165–208, 2003.
- [3] B. Bollobás and S. E. Eldridge. Packings of graphs and applications to computational complexity. *Journal of Combinatorial Theory, Series B*, 25(2):105–124, 1978.
- [4] E. Duchene, H. Kheddouci, R. J. Nowakowski, and M. A. Tahraoui. Labeled packing of graphs. *Australasian Journal of Combinatorics*, 57:109–126, 2013.
- [5] A. Kaneko and K. Suzuki. Packing two copies of a tree into its third power. *Discrete mathematics*, 306(21):2779–2785, 2006.
- [6] H. Kheddouci, S. Marshall, J.-F. Saclé, and M. Woźniak. On the packing of three graphs. *Discrete Mathematics*, 236(1-3):197–225, 2001.
- [7] H. Kheddouci, J.-F. Saclé, and M. Woźniak. Packing two copies of a tree into its fourth power. *Discrete Mathematics*, 213(1-3):169–178, 2000.
- [8] N. Sauer and J. Spencer. Edge disjoint placement of graphs. *Journal of Combinatorial Theory, Series B*, 25(3):295–302, 1978.
- [9] M. A. Tahraoui. *Coloring, packing and embedding of graphs*. PhD thesis, Université Claude Bernard-Lyon I, 2012.
- [10] M. Woźniak. Packing of graphs and permutations—a survey. *Discrete Mathematics*, 276(1-3):379–391, 2004.
- [11] H. Yap. Packing of graphs—a survey. In *Annals of Discrete Mathematics*, volume 38, pages 395–404. Elsevier, 1988.