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On the Sendov conjecture for polynomials with simple zeros

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Abstract. The Sendov conjecture asserts that if all the zeros of a polynomial p lie in the closed unit disk then there must be a zero of p' within unit distance of each zero. In this paper we give a partial result when p has simple zeros.

Key words and phrases. Sendov's conjecture; polynomial; simple zeros; Walsh's Coincidence Theorem.

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1. Introduction

The well-known Sendov conjecture ([4] Problem 4.5) asserts that if $p(z) = \prod_{j=1}^n (z - z_j)$ is a polynomial with $|z_j| \leq 1$ ($1 \leq j \leq n$), then each disk $|z - z_j| \leq 1$ ($1 \leq j \leq n$) contains a zero of p' . Notice that by the Gauss-Lucas theorem the zeros w_k ($1 \leq k \leq n - 1$) of p' lie in the closed convex hull of the zeros of p , hence $|w_k| \leq 1$ for $1 \leq k \leq n - 1$.

This conjecture has been verified for polynomials of degree $n \leq 8$ or for arbitrary degree n if there are at most eight distinct roots: See Brown and Xiang [1] and the references therein. It is also true in general ($n \geq 2$) when $p(0) = 0$ ([7]). The Sendov conjecture is true with respect to the root z_j of p if $|z_j| = 1$ ([6]). Recently ([2]) it has been verified when n is larger than a fixed integer depending on the root z_j of p . We refer the reader to Marden [5] and Sendov [8] for further information and bibliographies.

In this paper we prove the following theorem.

Theorem 1 *Suppose that p has simple zeros z_j , $j = 1, \dots, n$ in the closed unit disk $\overline{D}(0, 1)$. Let*

$$r = \frac{1}{n} \min_{1 \leq j \leq n} \min_{i \neq j} |z_i - z_j|, \quad a_n = n^{\frac{1}{n-1}} - 1,$$

$$A_{2p} = 2p \left(\frac{r^2 a_{2p}}{24} \right)^p \quad \text{and} \quad A_{2p+1} = (2p+1) a_{2p+1}^{p+1} \left(\frac{r^2 \sqrt{3}}{72} \right)^p.$$

Then the Sendov conjecture holds for p when $|p(0)| \leq A_n$.

Our proof will make use of the the so-called Coincidence Theorem, a variant of Grace's Apolarity Theorem ([5]). We begin with the following definition.

Definition $\Phi(x_1, \dots, x_p)$ is a symmetric p -linear form of total degree p in the variables x_j ($1 \leq j \leq p$) belonging to \mathbb{C} if it is symmetric in these variables and if Φ is a polynomial of degree 1 in each x_j separately such that $\Phi(x, \dots, x)$ is a polynomial of degree p in x .

A circular region is an open or closed disk or halfplane in \mathbb{C} , or the complement of any such set. Now we recall Walsh's Coincidence Theorem ([5]).

Theorem 2 *Let Φ be a symmetric p -linear form of total degree p in x_1, \dots, x_p and let C be a circular region containing the p points $\alpha_1, \dots, \alpha_p$. Then in C there exists at least one point x such that $\Phi(\alpha_1, \dots, \alpha_p) = \Phi(x, \dots, x)$.*

Finally we also need the next result.

Theorem 3 ([3]) *Let*

$$q(z) = \prod_{j=1}^r (z - u_j)^{k_j}, \quad \sum_{j=1}^r k_j = m,$$

be a polynomial of degree m whose zeros u_1, \dots, u_r are distinct and have multiplicity k_1, \dots, k_r , respectively. For any zero u_j of q , let $M_j = \min_{i \neq j} |u_i - u_j|$, $j = 1, \dots, r$. Then q has no nontrivial critical point (the critical points which are not zeros of the polynomial) in

$$\bigcup_{j=1}^r \left\{ z \in \mathbb{C}; |z - u_j| < \frac{k_j}{m} M_j \right\}.$$

In Section 2 we give some preliminary results. Theorem 1 is proved in Section 3.

2. Preliminaries

We begin with the following lemma.

Lemma 1 *The Sendov conjecture is true with respect to the root z_j if $|z_j| \leq a_n$.*

Proof. Since $p'(z_j) = q(z_j)$, where $q(z) = p(z)/(z - z_j)$, we have

$$n \prod_{k=1}^{n-1} (z_j - w_k) = \prod_{k \neq j} (z_j - z_k).$$

Then

$$n \prod_{k=1}^{n-1} |z_j - w_k| = \prod_{k \neq j} |z_j - z_k| \leq (1 + |z_j|)^{n-1} \leq n,$$

and the lemma follows.

Distinguish one of the zeros of p , say z_n , and let $z_n = a$. By a rotation, if necessary, and using Lemma 1 we may suppose that $a_n < a < 1$. Let $0 < s \leq r$ and set $z_0 = \frac{a}{2} + \frac{i}{2}\sqrt{4 - a^2}$. Let $v_1(z_0, s), v_2(z_0, s) \in \partial D(0, 1) \cap \partial D(z_0, s)$ with $\operatorname{Re} v_1(z_0, s) < \operatorname{Re} v_2(z_0, s)$. We denote by $L(z_0, s)$ the line through $v_1(z_0, s)$ and $v_2(z_0, s)$ and by $H(z_0, s)$ the closed halfplane bounded by $L(z_0, s)$ such that $0 \notin H(z_0, s)$. We set $A(z_0, s) = \overline{D}(0, 1) \cap H(z_0, s)$. Finally let $v_3(z_0, s) \in L(z_0, s) \cap \partial D(a, 1)$ with $\operatorname{Re} v_3(z_0, s) < a/2$.

Lemma 2 *With the above notations we have*

$$\operatorname{Re} v_1(z_0, s) = \frac{1}{4}(a(2 - s^2) - s((4 - a^2)(4 - s^2))^{\frac{1}{2}}),$$

$$\operatorname{Im} v_1(z_0, s) = \frac{-a}{4\sqrt{4 - a^2}}(a(2 - s^2) - s((4 - a^2)(4 - s^2))^{\frac{1}{2}}) + \frac{2 - s^2}{\sqrt{4 - a^2}},$$

$$\operatorname{Re} v_2(z_0, s) = \frac{1}{4}(a(2 - s^2) + s((4 - a^2)(4 - s^2))^{\frac{1}{2}}),$$

$$\operatorname{Im} v_2(z_0, s) = \frac{-a}{4\sqrt{4 - a^2}}(a(2 - s^2) + s((4 - a^2)(4 - s^2))^{\frac{1}{2}}) + \frac{2 - s^2}{\sqrt{4 - a^2}},$$

$$L(z_0, s) = \{c + id; c, d \in \mathbb{R} \text{ and } d\sqrt{4 - a^2} = -ac + 2 - s^2\},$$

and

$$\operatorname{Re} v_3(z_0, s) = \frac{1}{4}(a(6 - a^2 - s^2) - ((4 - a^2)(4 - a^2 - s^2)(a^2 + s^2))^{\frac{1}{2}}).$$

Moreover

$$\frac{a}{2} - \operatorname{Re} v_3(z_0, s) > \frac{s^2}{6}.$$

Proof. We only prove the inequality since all the other formulas follow from elementary computations. Set

$$\Delta = (4 - a^2)(4 - a^2 - s^2)(a^2 + s^2).$$

We can write

$$\begin{aligned} \frac{a}{2} - \operatorname{Re} v_3(z_0, s) &= \frac{a}{2} + \frac{1}{4}(\sqrt{\Delta} - a(6 - a^2 - s^2)) \\ &= \frac{a}{2} + \frac{1}{4} \frac{\Delta - a^2(6 - a^2 - s^2)^2}{\sqrt{\Delta} + a(6 - a^2 - s^2)} \\ &= \frac{1}{4} \frac{2a\sqrt{\Delta} + 2a^2(6 - a^2 - s^2) + \Delta - a^2(6 - a^2 - s^2)^2}{\sqrt{\Delta} + a(6 - a^2 - s^2)} \\ &= \frac{1}{4} \frac{N}{D}. \end{aligned}$$

Since

$$\sqrt{\Delta} > a((4 - a^2)(4 - a^2 - s^2))^{\frac{1}{2}} > a(4 - a^2 - s^2),$$

we get

$$\begin{aligned} N &= 2a\sqrt{\Delta} + 2a^2(6 - a^2 - s^2) + \Delta - a^2(6 - a^2 - s^2)^2 \\ &= 2a\sqrt{\Delta} + \Delta - a^2(6 - a^2 - s^2)(4 - a^2 - s^2) \\ &> 2a^2(4 - a^2 - s^2) + s^2(4 - a^2)(4 - a^2 - s^2) \\ &\quad - a^2(4 - a^2 - s^2)(2 - s^2) \\ &= 4s^2(4 - a^2 - s^2) > 8s^2. \end{aligned}$$

Since $D < \sqrt{32} + 6 < 12$, we get

$$\frac{a}{2} - \operatorname{Re} v_3(z_0, s) > \frac{s^2}{6}.$$

Lemma 3 Suppose that there exists $k \in \{1, \dots, n-1\}$ such that $\operatorname{Re} w_k \leq a/2$ and $\operatorname{Im} w_k \geq \operatorname{Im} v_3(z_0, \frac{r}{2})$. Then $\operatorname{Re} w_k < \operatorname{Re} v_3(z_0, \frac{r}{2})$.

Proof. We claim that p' has no critical point in $A(z_0, \frac{r}{2})$. Then the lemma follows. To verify this claim suppose first that $p(z_0) = 0$. Since p has simple zeros Theorem 3 implies that $p'(z) \neq 0$ for all $z \in D(z_0, r)$ and the claim is true in this case. Now assume that $p(z_0) \neq 0$. If there exists $z_j \in \overline{D}(z_0, \frac{r}{2}) \cap \overline{D}(0, 1)$ such that $p(z_j) = 0$ for some $j \in \{1, \dots, n-1\}$, again Theorem 3 implies that p' has no critical point in $D(z_j, r)$ and the claim is true. Finally, if $p(z) \neq 0$ for $z \in \overline{D}(z_0, \frac{r}{2}) \cap \overline{D}(0, 1)$, the Gauss-Lucas Theorem implies that $p'(z) \neq 0$ for $z \in A(z_0, \frac{r}{2})$.

Remark 1. Let $z'_0 = \frac{a}{2} - \frac{i}{2}\sqrt{4-a^2}$. Lemmas similar to Lemma 2 and Lemma 3 hold if we consider the point z'_0 instead of z_0 .

Lemma 4 Let

$$\alpha_j = \frac{\sin(2j\pi)/n}{1 - \cos(2j\pi)/n}, \quad 1 \leq j \leq n-1.$$

Then $\alpha_1 \cdots \alpha_{p-1} = 1$ when $n = 2p$ and $\alpha_1 \cdots \alpha_p \geq (\sqrt{3}/3)^p$ when $n = 2p+1$.

Proof. Let $n = 2p$. For $j \in \{1, \dots, p-1\}$ we have

$$\alpha_{p-j} = \frac{\sin(j\pi/p)}{1 + \cos(j\pi/p)} = \frac{1 - \cos(j\pi/p)}{\sin(j\pi/p)} = \frac{1}{\alpha_j},$$

and the result follows.

Now let $n = 2p+1$ with $p \geq 2$. We can write

$$\alpha_1 \cdots \alpha_p = \prod_{j=1}^k \alpha_j \alpha_{p-j+1} \quad \text{if } p = 2k, \quad k \geq 1,$$

and

$$\alpha_1 \cdots \alpha_p = \alpha_{k+1} \prod_{j=1}^k \alpha_j \alpha_{p-j+1} \quad \text{if } p = 2k+1, \quad k \geq 1.$$

For $j \in \{1, \dots, k\}$ we have

$$\begin{aligned}
\alpha_j \alpha_{p-j+1} &= \frac{\sin(2j\pi/(2p+1))}{1 - \cos(2j\pi/(2p+1))} \frac{\sin((2j-1)\pi/(2p+1))}{1 + \cos((2j-1)\pi/(2p+1))} \\
&= \frac{\cos(j\pi/(2p+1))}{\sin(j\pi/(2p+1))} \frac{\sin((2j-1)\pi/(2p+1))}{1 + \cos((2j-1)\pi/(2p+1))} \\
&\geq \frac{\cos(j\pi/(2p+1))}{\sin(j\pi/(2p+1))} \frac{\sin(j\pi/(2p+1))}{1 + \cos(j\pi/(2p+1))} \\
&= \frac{\cos(j\pi/(2p+1))}{1 + \cos(j\pi/(2p+1))} \geq \frac{\cos(\pi/3)}{1 + \cos(\pi/3)} = \frac{1}{3}.
\end{aligned}$$

Now when $p = 2k + 1$ we have

$$\alpha_{k+1} = \frac{\cos((k+1)\pi/(4k+3))}{\sin((k+1)\pi/(4k+3))} \geq \frac{\cos(\pi/3)}{\sin(\pi/3)} = \frac{\sqrt{3}}{3}.$$

The lemma follows.

Lemma 5 1) We have

$$\frac{r^2}{24} < \frac{a_{2p}}{2} \alpha_{p-1} \quad \text{if } n = 2p \quad \text{and} \quad \frac{r^2}{24} < \frac{a_{2p+1}}{2} \alpha_p \quad \text{if } n = 2p + 1.$$

2) Let $v \in \mathbb{C}$ be such that

$$\operatorname{Re} v < \frac{a}{2} - b,$$

where $b > 0$ is such that

$$b < \frac{a_{2p}}{2} \alpha_{p-1} \quad \text{if } n = 2p \quad \text{and} \quad b < \frac{a_{2p+1}}{2} \alpha_p \quad \text{if } n = 2p + 1.$$

Then

$$|(v - a)^n - v^n| > B_n(b),$$

where

$$B_{2p}(b) = 2p(ba_{2p})^p \quad \text{and} \quad B_{2p+1}(b) = (2p+1)a_{2p+1}^{p+1} \left(b \frac{\sqrt{3}}{3}\right)^p.$$

Proof. 1) is easily verified since $r \leq 2/n$.

2) Let

$$C_n(y) = \prod_{j=1}^{n-1} (b^2 + (y - \frac{a}{2}\alpha_j)^2) \quad , \quad y \in \mathbb{R} .$$

We have

$$C_{2p}(y) = (b^2 + y^2) \prod_{j=1}^{p-1} (b^2 + (y - \frac{a}{2}\alpha_j)^2)(b^2 + (y + \frac{a}{2}\alpha_j)^2) \quad ,$$

and

$$C_{2p+1}(y) = \prod_{j=1}^p (b^2 + (y - \frac{a}{2}\alpha_j)^2)(b^2 + (y + \frac{a}{2}\alpha_j)^2) .$$

Since $a > a_n$ we get

$$(b^2 + (y - \frac{a}{2}\alpha_j)^2)(b^2 + (y + \frac{a}{2}\alpha_j)^2) \geq b^2 a^2 \alpha_j^2 \quad \text{for } 1 \leq j \leq p \quad \text{and } y \in \mathbb{R} .$$

Then Lemma 4 implies that

$$C_{2p}(y) > b^{2p} a_{2p}^{2(p-1)} \quad \text{and} \quad C_{2p+1}(y) > b^{2p} a_{2p+1}^{2p} 3^{-p} \quad \text{for } y \in \mathbb{R} .$$

Now the solutions v_j ($1 \leq j \leq n-1$) of

$$(v - a)^n - v^n = 0 \quad ,$$

are given by

$$v_j = \frac{a}{2}(1 + i\alpha_j) \quad , \quad 1 \leq j \leq n-1 . \quad (1)$$

Therefore

$$\begin{aligned} |(v - a)^n - v^n|^2 &= n^2 a^2 \prod_{j=1}^{n-1} ((\operatorname{Re} v - \frac{a}{2})^2 + (\operatorname{Im} v - \frac{a}{2}\alpha_j)^2) \\ &\geq n^2 a^2 C_n(\operatorname{Im} v) \\ &> B_n(b)^2 . \end{aligned}$$

3. Proof of Theorem 1

Define

$$k(z, u_1, \dots, u_{n-1}) = z^n + \sum_{k=1}^{n-1} (-1)^k \frac{n}{n-k} \\ \times \left(\sum_{1 \leq i_1 < \dots < i_k \leq n-1} u_{i_1} \dots u_{i_k} \right) z^{n-k},$$

for $z, u_1, \dots, u_{n-1} \in \mathbb{C}$. We have

$$k(z, w_1, \dots, w_{n-1}) = p(z) - p(0).$$

Let

$$C = \left\{ z \in \mathbb{C}; \operatorname{Re} z < \frac{a}{2} - \frac{r^2}{24} \right\}.$$

Suppose that $|a - w_k| > 1$ for $k = 1, \dots, n-1$. Then $\operatorname{Re} w_k < a/2$ for $k = 1, \dots, n-1$. Using Lemma 2 with $s = r/2$, Lemma 3 and Remark 1 we obtain $\operatorname{Re} w_k < \frac{a}{2} - \frac{r^2}{24}$ for $k = 1, \dots, n-1$. Theorem 2 implies that there exists $v \in C$ such that

$$-p(0) = k(a, w_1, \dots, w_{n-1}) = k(a, v, \dots, v) = (a - v)^n + (-1)^{n-1} v^n.$$

Lemma 5 with $b = r^2/24$ implies that $v \notin C$ and we reach a contradiction.

Remark 2. Suppose that the zeros of p are not necessarily simple. Using Walsh's Coincidence Theorem we can give a new proof of Sendov's conjecture when $p(0) = 0$. By a rotation, if necessary, we may suppose that p has the form

$$p(z) = (z - a) \prod_{j=1}^{n-1} (z - z_j),$$

where $a \in (0, 1]$ is a simple root. Suppose that $|a - w_k| > 1$ for $k = 1, \dots, n-1$. Since $\operatorname{Re} w_k < a/2$, there exists $t > 0$ such that $\operatorname{Re} w_k < \frac{a}{2} - t$ for $k = 1, \dots, n-1$. Let $E = \{z \in \mathbb{C}; \operatorname{Re} z < \frac{a}{2} - t\}$. Theorem 2 implies that there exists $v \in E$ such that

$$0 = k(a, w_1, \dots, w_{n-1}) = k(a, v, \dots, v) = (a - v)^n + (-1)^{n-1} v^n.$$

Since $v = v_j$ for some $j \in \{1, \dots, n-1\}$, (1) implies that $v \notin E$ and we reach a contradiction.

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