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# CYLINDERS IN MORI FIBER SPACES: FORMS OF THE QUINTIC DEL PEZZO THREEFOLD

ADRIEN DUBOULOZ AND TAKASHI KISHIMOTO

**ABSTRACT.** Motivated by the general question of existence of open  $\mathbb{A}^1$ -cylinders in higher dimensional projective varieties, we consider the case of Mori Fiber Spaces of relative dimension three, whose general closed fibers are isomorphic to the quintic del Pezzo threefold  $V_5$ , the smooth Fano threefold of index two and degree five. We show that the total spaces of these Mori Fiber Spaces always contain relative  $\mathbb{A}^2$ -cylinders, and we characterize those admitting relative  $\mathbb{A}^3$ -cylinders in terms of the existence of certain special lines in their generic fibers.

## INTRODUCTION

An  $\mathbb{A}_k^r$ -cylinder in a normal algebraic variety defined over a field  $k$  is a Zariski open subset  $U$  isomorphic to  $Z \times \mathbb{A}_k^r$  for some algebraic variety  $Z$  defined over  $k$ . In the case where  $k = \bar{k}$  is algebraically closed, normal projective varieties  $V$  containing  $\mathbb{A}_k^1$ -cylinders have received a lot of attention recently due to the connection between unipotent group actions on their affine cones and polarized  $\mathbb{A}_k^1$ -cylinders in them, that is  $\mathbb{A}_k^1$ -cylinders whose complements are the supports of effective  $\mathbb{Q}$ -divisors linearly equivalent to an ample divisor on  $V$  (cf. [12, 13]). Certainly, the canonical divisor  $K_V$  of a normal projective variety containing an  $\mathbb{A}_k^r$ -cylinder for some  $r \geq 1$  is not pseudo-effective. Replacing  $V$  if necessary by a birational model with at most  $\mathbb{Q}$ -factorial terminal singularities, [1] guarantees the existence of a suitable  $K_V$ -MMP  $V \dashrightarrow X$  whose output  $X$  is equipped with a structure of Mori Fiber Space  $f : X \rightarrow Y$  over some lower dimensional normal projective variety  $Y$ . Since an  $\mathbb{A}_k^1$ -cylinder in  $X$  can always be transported back in the initial variety  $V$  [4, Lemma 9], total spaces of Mori Fiber Spaces form a natural restricted class in which to search for varieties containing  $\mathbb{A}_k^1$ -cylinders.

In the case where  $\dim Y = 0$ ,  $X$  is a Fano variety of Picard number one. The only smooth Fano surface of Picard number is the projective plane  $\mathbb{P}_k^2$  which obviously contains  $\mathbb{A}_k^1$ -cylinders. Several families of examples of smooth Fano varieties of dimension 3 and 4 and Picard number one containing  $\mathbb{A}_k^1$ -cylinders have been constructed [14, 21, 22]. The question of existence of  $\mathbb{A}_k^1$ -cylinders in other possible outputs of MMPs was first considered in [2, 3], in which del Pezzo fibrations  $f : X \rightarrow Y$ , which correspond to the case where  $\dim Y = \dim X - 2 > 0$ , were extensively studied. In such a relative context, it is natural to shift the focus to cylinders which are compatible with the fibration structure:

**Definition.** Let  $f : X \rightarrow Y$  be a morphism between normal algebraic varieties defined over a field  $k$  and let  $U \simeq Z \times \mathbb{A}_k^r$  be an  $\mathbb{A}_k^r$ -cylinder inside  $X$ . We say that  $U$  is *vertical with respect to  $f$*  if the restriction  $f|_U$  factors as

$$f|_U = h \circ \text{pr}_Z : U \simeq Z \times \mathbb{A}_k^r \xrightarrow{\text{pr}_Z} Z \xrightarrow{h} Y$$

for a suitable morphism  $h : Z \rightarrow Y$ .

In the present article, we initiate the study of existence of vertical  $\mathbb{A}^1$ -cylinders in Mori Fiber Spaces  $f : X \rightarrow Y$  of relative dimension three, whose general fibers are smooth Fano threefolds. Each smooth Fano threefold of Picard number one can appear as a closed fiber of a Mori Fiber Space, and among these,

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it is natural to first restrict to classes which contain a cylinder of the maximal possible dimension, namely the affine space  $\mathbb{A}_k^3$ , and expect that some suitable sub-cylinders of these fiber wise maximal cylinders could arrange into a vertical cylinder with respect to  $f$ , possibly of smaller relative dimension. The only four classes of Fano threefold of Picard number one containing  $\mathbb{A}_k^3$  are  $\mathbb{P}_k^3$ , the quadric  $\mathbb{Q}^3$ , the del Pezzo quintic threefold  $V_5$  of index two and degree five [7, 8], and a four dimensional family of prime Fano threefolds  $V_{22}$  of genus twelve [6, 20].

The existence inside  $X$  of a vertical  $\mathbb{A}^r$ -cylinder with respect to  $f : X \rightarrow Y$  translates equivalently into that of an  $\mathbb{A}_K^r$ -cylinder inside the fiber  $X_\eta$  of  $f$  over the generic point  $\eta$  of  $Y$ , considered as a variety defined over the function field  $K$  of  $Y$  [3]. We are therefore led to study the existence of  $\mathbb{A}_K^r$ -cylinders inside  $K$ -forms of the aforementioned Fano threefolds over non-closed fields  $K$  of characteristic zero, that is, smooth projective varieties defined over  $K$  whose base extensions to an algebraic closure  $\overline{K}$  are isomorphic over  $\overline{K}$  to one of these Fano threefolds. The case of  $\mathbb{P}^3$  is easily dispensed: a  $K$ -form  $V$  of  $\mathbb{P}^3$  contains an  $\mathbb{A}_K^3$ -cylinder if and only if it has a  $K$ -rational point hence if and only if it is the trivial  $K$ -form  $\mathbb{P}_K^3$ . So equivalently, a Mori Fiber Space  $f : X \rightarrow Y$  whose general closed fibers are isomorphic to  $\mathbb{P}_k^3$  contains a vertical  $\mathbb{A}_k^3$ -cylinder if and only if it has a rational section. The case of the quadric  $\mathbb{Q}^3$  is already more intricate: one can deduce from [6] that a  $K$ -form  $V$  of  $\mathbb{Q}^3$  contains  $\mathbb{A}_K^3$  if and only if it has a hyperplane section defined over  $K$  which is a  $K$ -rational quadric cone. In this article, we establish a complete characterization of the existence of  $\mathbb{A}_K^r$ -cylinders in forms of  $V_5$  which can be summarized as follows:

**Theorem.** *Let  $K$  be a field of characteristic zero and let  $Y$  be a  $K$ -form of  $V_5$ . Then  $Y$  always contains an  $\mathbb{A}_K^2$ -cylinder, and it contains an  $\mathbb{A}_K^3$ -cylinder if and only if it contains a curve  $\ell \simeq \mathbb{P}_K^1$  of anticanonical degree  $-K_Y \cdot \ell = 2$  and with normal bundle  $\mathcal{N}_{\ell/Y} \simeq \mathcal{O}_{\mathbb{P}_K^1}(-1) \oplus \mathcal{O}_{\mathbb{P}_K^1}(1)$ .*

An irreducible curve  $\ell$  of anticanonical degree  $-K_Y \cdot \ell = 2$  on a  $K$ -form  $Y$  of  $V_5$  becomes after base extension to an algebraic closure  $\overline{K}$  of  $K$  a usual line on  $Y_{\overline{K}} \simeq V_5$  embedded into  $\mathbb{P}_{\overline{K}}^6$  via its half-anticanonical complete linear system. By combining the previous characterization with a closer study of the Hilbert scheme of such curves  $\ell$  on  $Y$  (see §2.1), we derive the following result (see Corollary 13):

**Corollary.** *Let  $\overline{k}$  be an algebraically closed field of characteristic zero and let  $f : X \rightarrow C$  be a Mori Fiber Space over a curve  $C$  defined over  $\overline{k}$ , whose general closed fibers are quintic del Pezzo threefolds  $V_5$ . Then  $X$  contains a vertical  $\mathbb{A}_{\overline{k}}^3$ -cylinder with respect to  $f$ .*

Section 1 contains a brief recollection on the quintic del Pezzo threefold  $V_5$  and its Hilbert scheme of lines. In Section 2, we establish basic geometric properties of forms of  $V_5$  and describe their Hilbert schemes of lines. We also describe an adaptation to non-closed fields of a standard construction of  $V_5$  as the variety of trisecant lines to a Veronese surface in  $\mathbb{P}^4$ , from which we derive for suitable fields the existence of forms of  $V_5$  which do not contain any line with normal bundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ . The technical core of the article is then Section 3: we give a new construction of a classical rational map, called the double projection from a rational point of a form of  $V_5$  in the form of a Sarkisov link from a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^2$  explicitly determined by the base locus of a quadratic birational involution of  $\mathbb{P}^2$ . The main results concerning  $\mathbb{A}^r$ -cylinders in forms of  $V_5$  are then derived from this construction in Section 4.

## 1. GEOMETRY OF THE SMOOTH QUINTIC DEL PEZZO THREEFOLD

In the rest of this article, unless otherwise stated, the notations  $k$  and  $\overline{k}$  refer respectively to a field of characteristic zero and a fixed algebraic closure of  $k$ . In this section, we recall without proof classical descriptions and properties of the quintic del Pezzo threefold  $V_5$  over  $\overline{k}$  and of its Hilbert scheme of lines.

**1.1. Two classical descriptions of  $V_5$ .** A quintic del Pezzo threefold  $V_5$  over  $\overline{k}$  is a smooth projective threefold whose Picard group is isomorphic to  $\mathbb{Z}$ , generated by an ample class  $H$  such that  $-K_{V_5} = 2H$  and  $H^3 = 5$ . In other words,  $V_5$  is a smooth Fano threefold of index two and degree five.

**1.1.1. Sarkisov links to a smooth quadric in  $\mathbb{P}^4$ .** Let us first recall the classical description due to Iskovskikh [11, Chapter II, §1.6]. Letting  $H$  be an ample class such that  $-K_{V_5} = 2H$ , the complete linear system  $|H|$  defines a closed embedding  $\Phi_{|H|} : V_5 \hookrightarrow \mathbb{P}_{\overline{k}}^6$ . A general hyperplane section of  $V_5$  contains a line  $\ell$  whose

normal bundle in  $V_5$  is trivial, and the projection from  $\ell$  induces a birational map  $V_5 \dashrightarrow Q$  onto a smooth quadric  $Q \subset \mathbb{P}_k^4$ . The inverse map  $Q \dashrightarrow V_5$  can be described as the blow-up of  $Q$  along a rational normal cubic  $C \subset Q$  contained in a smooth hyperplane section  $Q_0 \simeq \mathbb{P}_k^1 \times \mathbb{P}_k^1$  of  $Q$ , followed by the contraction of the proper transform  $Q'_0$  of  $Q_0$  onto the line  $\ell$ . In sum, the projection from the line  $\ell$  induces a Sarkisov link

$$\begin{array}{ccccc}
 Q'_0 & \hookrightarrow & \tilde{V}_5 & \xleftarrow{\quad} & Z' \\
 \swarrow & & \downarrow q & & \searrow q' \\
 \ell & \xrightarrow{\quad} & V_5 & \xrightarrow{\quad} & Q \\
 & & \text{---} & & \\
 & & |H-\ell| & & 
 \end{array}$$

where  $q : \tilde{V}_5 \rightarrow V_5$  is the blow-up of  $V_5$  along  $\ell$  with exceptional divisor  $Q'_0$  and  $q' : \tilde{V}_5 \rightarrow Q$  is the contraction onto the curve  $C$  of the proper transform  $Z'$  of the surface  $Z \subset V_5$  swept out by lines in  $V_5$  intersecting  $\ell$ .

Since the automorphism group  $\mathrm{PGL}_5(\bar{k})$  of  $\mathbb{P}_k^4$  acts transitively on the set of flags  $C \subset Q_0 \subset Q$ , it follows that over an algebraically closed field  $\bar{k}$ , all smooth Fano threefold of Picard number one, index two and degree five embedded into  $\mathbb{P}_k^6$  by their half-anticanonical complete linear system are projectively equivalent.

**1.1.2. Quasi-homogeneous space of  $\mathrm{PGL}_2(\bar{k})$ .** We now recall an alternative description of  $V_5$  due to Mukai-Umemura [19] (see also [16, §5.1]). Let  $M_d = \mathrm{Sym}^d(\bar{k}^2)^\vee \simeq \bar{k}[x, y]_{(d)}$  be the space of homogeneous polynomials of degree  $d$  with coefficients in  $\bar{k}$ . The natural action of  $\mathrm{GL}_2(\bar{k})$  on  $M_1$  induces a linear action on  $M_6$ , hence an action of  $\mathrm{PGL}_2(\bar{k})$  on  $\mathbb{P}(M_6) \simeq \mathbb{P}_k^6$ . We then have the following description:

**Proposition 1.** *The Fano threefold  $V_5$  is isomorphic to the closure  $\overline{\mathrm{PGL}_2(\bar{k}) \cdot [\phi]}$  of the class of the polynomial  $\phi = xy(x^4 + y^4) \in M_6$ . Furthermore, the  $\mathrm{PGL}_2(\bar{k})$ -orbits on  $V_5$  are described as follows:*

- (i) *The open orbit  $O = \mathrm{PGL}_2(\bar{k}) \cdot [\phi]$  with stabilizer equal to the binary octahedral group,*
- (ii) *The 2-dimensional orbit  $S_2 = \mathrm{PGL}_2(\bar{k}) \cdot [xy^5]$ , which is neither open nor closed, with stabilizer equal to the diagonal torus  $\mathbb{T}$ .*
- (iii) *The 1-dimensional closed orbit  $C_6 = \mathrm{PGL}_2(\bar{k}) \cdot [x^6]$  with stabilizer equal to the Borel subgroup  $B$  of upper triangular matrices.*

It follows from this description that the automorphism  $\mathrm{Aut}(V_5)$  is isomorphic to  $\mathrm{PGL}_2(\bar{k})$ . We also observe that  $C_6$  is a normal rational sextic curve and that the closure  $\overline{S_2} = S_2 \cup C_6$  of  $S_2$  is a quadric section of  $V_5$ , hence an anti-canonical divisor on  $V_5$ , which coincides with the *tangential scroll* of  $C_6$ , swept out by the tangent lines to  $C_6$  contained in  $V_5$ . It is singular along  $C_6$ , and its normalization morphism coincides with the map  $\nu : \mathbb{P}(M_1) \times \mathbb{P}(M_1) \rightarrow \mathbb{P}(M_6)$ ,  $(f_1, f_2) \mapsto f_1^5 f_2$ .

**1.2. Lines on  $V_5$ .** The family of lines on  $V_5$  is very well-studied [11, 7, 10]. We list below some of its properties which will be useful later on for the study of cylinders on forms of  $V_5$ .

First, by a *line* on  $V_5$ , we mean an integral curve  $\ell \subset V_5$  of anticanonical degree  $-K_{V_5} \cdot \ell = 2$ . It thus corresponds through the half-anticanonical embedding  $\Phi_{|H|} : V_5 \hookrightarrow \mathbb{P}_k^6$  to a usual line in  $\mathbb{P}_k^6$  which is contained in the image of  $V_5$ . A general line  $\ell$  in  $V_5$  has trivial normal bundle, whereas there is a one-dimensional subfamily of lines with normal bundle  $\mathcal{N}_{\ell/V_5} \simeq \mathcal{O}_{\mathbb{P}_k^1}(-1) \oplus \mathcal{O}_{\mathbb{P}_k^1}(1)$ , which we call *special lines*.

The Hilbert scheme  $\mathcal{H}(V_5)$  of lines in  $V_5$  is isomorphic to  $\mathbb{P}_k^2$ , and the evaluation map  $v : \mathcal{U} \rightarrow V_5$  from the universal family  $\mathcal{U} \rightarrow \mathcal{H}(V_5)$  is a finite morphism of degree 3. There are thus precisely three lines counted with multiplicities passing through a given closed point of  $V_5$ .

The curve in  $\mathcal{H}(V_5) \simeq \mathbb{P}_k^2$  that parametrizes special lines in  $V_5$  is a smooth conic  $C$ . The restriction of  $v$  to  $\mathcal{U}|_C$  is injective, and in the description of  $V_5$  as a quasi-homogeneous space of  $\mathrm{PGL}_2(\bar{k})$  given in § 1.1.2 above,  $v(\mathcal{U}|_C)$  is the tangential scroll  $\overline{S_2}$  to the rational normal sextic  $C_6$ , while  $C_6$  itself coincides with the image of the intersection of  $\mathcal{U}|_C$  with the ramification locus of  $v$ . In particular, special lines on  $V_5$  never intersect each others. Furthermore, there are three lines with trivial normal bundle through any point in  $V_5 \setminus \overline{S_2}$ , a line with trivial normal bundle and a special line through any point of  $S_2$ , and a unique special line through every point of  $C_6$ .

2. FORMS OF  $V_5$  OVER NON-CLOSED FIELDS

A  $k$ -form of  $V_5$  is a smooth projective variety  $Y$  defined over  $k$  such that  $Y_{\bar{k}}$  is isomorphic to  $V_5$ . In this subsection, we establish basic properties of these forms and their Hilbert schemes of lines.

**2.1. Hilbert scheme of lines on a  $k$ -form of  $V_5$ .** Let  $Y$  be a  $k$ -form of  $V_5$ , let  $\mathcal{H}(Y)$  be the Hilbert scheme of irreducible curves of  $-K_Y$ -degree equal to 2 on  $Y$  and let  $v : \mathcal{U} \rightarrow Y$  be the evaluation map from the universal family  $\mathcal{U} \rightarrow \mathcal{H}(Y)$ . By § 1.2,  $\mathcal{H}(Y)_{\bar{k}} = \mathcal{H}(Y_{\bar{k}})$  is isomorphic to  $\mathbb{P}_{\bar{k}}^2$ , i.e.  $\mathcal{H}(Y)$  is a  $k$ -form of  $\mathbb{P}_{\bar{k}}^2$ . Furthermore, since the smooth conic parametrizing special lines on  $Y_{\bar{k}}$  is invariant under the action of the Galois group  $\text{Gal}(\bar{k}/k)$ , it corresponds to a smooth curve  $C \subset \mathcal{H}(Y)$  defined over  $k$ , and such that  $C_{\bar{k}} \simeq \mathbb{P}_{\bar{k}}^1$ .

**Lemma 2.** *Let  $Y$  be a  $k$ -form of  $V_5$ .*

a) *The Hilbert scheme  $\mathcal{H}(Y)$  of lines on  $Y$  is isomorphic to  $\mathbb{P}_k^2$ , in particular  $Y$  always contains lines defined over  $k$  with trivial normal bundles.*

b) *The following assertions are equivalent:*

- (i)  *$Y$  contains a special line defined over  $k$ ,*
- (ii) *The conic  $C$  has a  $k$ -rational point,*
- (iii) *The surface  $v(\mathcal{U}|_C)$  has a  $k$ -rational point,*
- (iv) *The image of the intersection of  $\mathcal{U}|_C$  with the ramification locus of  $v$  has a  $k$ -rational point.*

*Proof.* Since  $\mathcal{H}(Y)$  is a  $k$ -form of  $\mathbb{P}_{\bar{k}}^2$ , to prove the first assertion it is enough to show that  $\mathcal{H}(Y)$  has a  $k$ -rational point. Since  $C_{\bar{k}} \simeq \mathbb{P}_{\bar{k}}^1$ , there exists a quadratic extension  $k \subset k'$  such that  $C(k')$  is nonempty, so that in particular  $\mathcal{H}(Y)_{k'} \simeq \mathbb{P}_{k'}^2$ . Let  $p$  be a  $k'$ -rational point of  $C_{k'}$ . If  $p$  is invariant under the action of the Galois group  $\text{Gal}(k'/k)$ , then it corresponds to a  $k$ -rational point of  $C$ , hence of  $\mathcal{H}(Y)$ , and we are done. Otherwise, its Galois conjugate  $\bar{p}$  is a  $k'$ -rational point of  $C_{k'}$  distinct from  $p$ , and then the tangent lines  $T_p C_{k'}$  and  $T_{\bar{p}} C_{k'}$  to  $C_{k'}$  at  $p$  and  $\bar{p}$  respectively intersect each other at unique point. The latter is thus  $\text{Gal}(k'/k)$ -invariant, hence corresponds to a  $k$ -rational point of  $\mathcal{H}(Y) \setminus C$ . The existence of lines with trivial normal bundles defined over  $k$  then follows from the description of  $\mathcal{H}(V_5)$  given in § 1.2.

The second assertion is an immediate consequence of the facts that special lines in  $Y_{\bar{k}}$  are in one-to-one correspondence with closed points of the image  $D$  of the intersection of  $(\mathcal{U}|_C)_{\bar{k}}$  with the ramification locus of  $v$ , and that  $v(\mathcal{U}|_C)_{\bar{k}}$  coincides with the surface swept out by the tangent lines to  $D$ .  $\square$

**Corollary 3.** *A  $k$ -form  $Y$  of  $V_5$  is  $k$ -rational and the natural map  $\text{Pic}(Y) \rightarrow \text{Pic}(Y_{\bar{k}})$  is an isomorphism.*

*Proof.* By Lemma 2 a),  $Y$  contains a line  $\ell \simeq \mathbb{P}_k^1$  with trivial normal bundle. The surface  $Z$  in  $Y_{\bar{k}}$  swept out by lines intersecting  $\ell_{\bar{k}}$  is defined over  $k$ . As in § 1.1.1, the composition of the blow-up  $q : \tilde{Y} \rightarrow Y$  of  $\ell$  with exceptional divisor  $Q'_0 \simeq \mathbb{P}_k^1 \times \mathbb{P}_k^1$  followed by the contraction  $q' : \tilde{Y} \rightarrow Q$  of the proper transform of  $Z'$  of  $Z$  yields a birational map  $Y \dashrightarrow Q$  defined over  $k$  onto a smooth quadric  $Q \subset \mathbb{P}_k^4$ , which maps  $Q'_0$  onto a hyperplane section  $Q_0$  of  $Q$ . Since  $Q$  contains  $k$ -rational points, it is  $k$ -rational. The natural map  $\text{Pic}(Y) \rightarrow \text{Pic}(Y_{\bar{k}})$  is an isomorphism if and only  $-K_Y$  is divisible in  $\text{Pic}(Y)$ . But since  $-K_Q \sim 3Q_0$  and  $q'^* Q_0 = Q'_0 + Z'$ , we deduce from the ramification formula for  $q$  and  $q'$  that

$$-K_Y \sim q_*(-K_{\tilde{Y}}) \sim q_*(-q'^* K_Q - Z') \sim q_*(3Q'_0 + 2Z') \sim 2Z.$$

$\square$

**2.2. Varieties of trisecant lines to Veronese surfaces in  $\mathbb{P}^4$ .** In this subsection, we review a third classical construction of  $V_5$  as the variety of trisecant line to the Veronese surface in  $\mathbb{P}_{\bar{k}}^4$  which, when performed over  $k$  gives rise, depending on the choices made, to nontrivial  $k$ -forms of  $V_5$ .

Let  $V$  be a  $k$ -vector space of dimension 3, let  $f \in \text{Sym}^2 V^*$  be a homogeneous form defining a smooth conic  $Q \subset \mathbb{P}(V)$ , and let  $W = \text{Sym}^2(V^*/\langle f \rangle)$ . Recall that a closed subscheme  $Z \subset \mathbb{P}(V^*)$  defined over  $k$  is called *apolar* to  $Q$  if the class of  $[f] \in \mathbb{P}(\text{Sym}^2 V^*)$  lies in the linear span of the image of  $Z$  by the second Veronese embedding  $v_2 : \mathbb{P}(V^*) \hookrightarrow \mathbb{P}(\text{Sym}^2 V^*)$ . When  $Z = \{[\ell_1], [\ell_2], [\ell_3]\} \subset \mathbb{P}(V^*)$  consists of three distinct  $k$ -rational points, this says equivalently that there are scalars  $\lambda_i \in k$ ,  $i = 1, 2, 3$ , such that  $f = \lambda_1 \ell_1^2 + \lambda_2 \ell_2^2 + \lambda_3 \ell_3^2$ . We denote by  $\text{VSP}(f)$  the variety of closed subschemes of length 3 of  $\mathbb{P}(V^*)$  which are apolar to  $Q$ .

Let  $\pi_{[f]} : \mathbb{P}(\text{Sym}^2 V^*) \dashrightarrow \mathbb{P}(W)$  be the projection from  $[f]$ . Since  $Q$  is smooth, the composition  $\pi_{[f]} \circ v_2$  is a closed embedding of  $\mathbb{P}(V^*)$ , whose image is a Veronese surface  $X$  in  $\mathbb{P}(W)$ . Since  $X$  is defined over  $k$ ,

the closed subscheme  $Y$  of the Grassmannian  $G(1, \mathbb{P}(W_{\bar{k}}))$  of lines in  $\mathbb{P}(W_{\bar{k}})$  consisting of trisecant lines to  $X_{\bar{k}}$ , that is, lines  $\ell$  in  $\mathbb{P}(W_{\bar{k}})$  such that  $\ell \cdot X_{\bar{k}}$  is a closed subscheme of length 3 of  $\ell$ , is defined over  $k$ , and we obtain an identification between  $\text{VSP}(f)$  and  $Y$ .

**Proposition 4.** (see [9, §2.3]) *With the notation above,  $Y = \text{VSP}(f)$  is a  $k$ -form of  $V_5$ . Furthermore, the stratification of  $Y_{\bar{k}}$  by the types of the corresponding closed subschemes of length 3 of  $\mathbb{P}(V^*)$  is related to that of  $V_5$  as a quasi-homogenous space of  $\text{PGL}_2(\bar{k})$  given in § 1.1.2 as follows:*

- (i) *Points of the open orbit  $O$  correspond to reduced subschemes  $\{[\ell_1], [\ell_2], [\ell_3]\}$  apolar to  $Q_{\bar{k}}$  such that none of the  $[\ell_i]$  belongs to the dual conic  $Q_{\bar{k}}^* \subset \mathbb{P}(V_{\bar{k}}^*)$  of  $Q_{\bar{k}}$ .*
- (ii) *Points in the 2-dimensional orbit  $S_2$  correspond to non-reduced subschemes  $\{2[\ell_1], [\ell_2]\}$  where  $[\ell_1]$  belongs to  $Q_{\bar{k}}^*$  and  $[\ell_2]$  is a point of the tangent line  $T_{[\ell_1]}Q_{\bar{k}}^*$  to  $Q_{\bar{k}}^*$  at  $[\ell_1]$  distinct from  $[\ell_1]$ .*
- (iii) *Points in the 1-dimensional orbit  $C_6$  correspond to non-reduced subschemes  $\{3[\ell]\}$  where  $[\ell]$  is a point of  $Q_{\bar{k}}^*$ .*

The Hilbert scheme  $\mathcal{H}(Y)$  of lines on a  $k$ -form  $Y = \text{VSP}(f) \subset G(1, \mathbb{P}(W))$  can be explicitly described as follows. Viewing  $G(1, \mathbb{P}(W))$  as a closed subscheme of  $\mathbb{P}(\Lambda^2 W)$  via the Plücker embedding,  $\mathcal{H}(Y)$  is a closed subscheme of the Hilbert scheme  $\mathcal{H}(G(1, \mathbb{P}(W)))$  of lines on  $G(1, \mathbb{P}(W))$ . The latter is isomorphic to the flag variety  $\mathcal{F}(0, 2, \mathbb{P}(W)) \subset \mathbb{P}(W) \times G(2, \mathbb{P}(W))$  of pairs  $(x, P)$  consisting of a point  $x \in \mathbb{P}(W)$  and a 2-dimensional linear space  $P \subset \mathbb{P}(W)$  containing it.

**Proposition 5.** (see e.g. [10, §1.2]) *Let  $f \in \text{Sym}^2 V^*$  be a homogeneous form defining a smooth conic  $Q \subset \mathbb{P}(V)$ , let  $W = \text{Sym}^2(V^*/\langle f \rangle)$ , and let  $\text{VSP}(f) \subset G(1, \mathbb{P}(W))$  be the variety of trisecant lines to the Veronese surface  $X = \pi_{[f]} \circ v_2(\mathbb{P}(V^*))$ .*

- (i) *The projection  $\text{pr}_1 : \mathcal{F}(0, 2, \mathbb{P}(W)) \rightarrow \mathbb{P}(W)$  restricts to a closed embedding  $\mathcal{H}(\text{VSP}(f)) \hookrightarrow \mathbb{P}(W)$  whose image is equal to  $X \simeq \mathbb{P}(V^*)$ .*
- (ii) *The image of the smooth conic  $C \subset \mathcal{H}(\text{VSP}(f))$  parametrizing special lines in  $\text{VSP}(f)$  coincides with the conic  $Q^* \subset \mathbb{P}(V^*)$  dual to  $Q \subset \mathbb{P}(V)$ .*
- (iii) *For every line  $\ell$  in  $\text{VSP}(f)_{\bar{k}}$  defined by a point  $x \in \mathbb{P}(V_{\bar{k}}^*)$ , the set of lines in  $\text{VSP}(f)_{\bar{k}}$  which intersect  $\ell$  is parametrized by the conjugate line of  $x$  with respect to  $Q_{\bar{k}}^*$ .*

As a consequence, we obtain the following characterization of which  $k$ -forms  $Y = \text{VSP}(f)$  of  $V_5$  contain special lines defined over  $k$ .

**Corollary 6.** *The  $k$ -form  $Y = \text{VSP}(f)$  of  $V_5$  contains a special line defined over  $k$  if and only if the conic  $Q = V(f) \subset \mathbb{P}(V)$  has a  $k$ -rational point.*

*Proof.* By combining Lemma 2 b) and Proposition 5 (ii), we deduce that  $Y$  contains a special line defined over  $k$  if and only if the conic  $Q^* \subset \mathbb{P}(V^*)$  dual to  $Q$  has a  $k$ -rational point, hence if and only if  $Q$  has a  $k$ -rational point.  $\square$

**Example 7.** Let  $k = \mathbb{C}(s, t)$  with  $s, t$  algebraically independent over  $\mathbb{C}$ , and let  $f = x^2 + sy^2 + tz^2 \in k[x, y, z]$ . Since the conic  $Q = V(f) \subset \mathbb{P}_k^2 = \text{Proj}_k(k[x, y, z])$  has no  $k$ -rational point, the variety  $Y = \text{VSP}(f)$  is a  $k$ -form of  $V_5$  which does not contain any special line defined over  $k$ .

### 3. DOUBLE PROJECTION FROM A RATIONAL POINT

Let  $V_5 \hookrightarrow \mathbb{P}_k^6$  be embedded by its half-anticanonical complete linear system. For every closed point  $y \in V_5$ , the linear system  $|\mathcal{O}_{V_5}(1) \otimes \mathfrak{m}_y^2|$  of hyperplane sections of  $V_5$  which are singular at  $y$  defines a rational map  $\psi_y : V_5 \dashrightarrow \mathbb{P}_k^2$  called the *double projection from  $y$* , whose description is classical knowledge in the birational geometry of threefolds [8, 15]. In this section, inspired by [24], we give an explicit “reverse construction” of these maps, valid over any field  $k$  of characteristic zero, formulated in terms of Sarkisov links performed from certain locally trivial  $\mathbb{P}^1$ -bundles over  $\mathbb{P}^2$  associated to standard birational quadratic involutions of  $\mathbb{P}^2$ .

**3.1. Recollection on standard quadratic involutions of  $\mathbb{P}^2$ .** Let  $\sigma : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$  be a birational quadratic involution of  $\mathbb{P}_k^2$  and let  $\Gamma_\sigma \subset \mathbb{P}_k^2 \times \mathbb{P}_k^2$  be its graph. Via the two projections  $\tau = \text{pr}_1 : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$  and  $\tau' = \text{pr}_2 : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$ ,  $\Gamma_\sigma$  is canonically identified with the blow-up of the scheme-theoretic base loci  $Z = \text{Bs}(\sigma)$  and  $Z' = \text{Bs}(\sigma^{-1})$  respectively. We denote by  $e = \tau^*Z$  and  $e' = \tau'^*Z'$  the scheme-theoretic inverse images



of  $Z$  and  $Z'$  respectively in  $\Gamma_\sigma$ . It is well-known that  $Z$  and  $Z'$  are local complete intersection 0-dimensional closed sub-scheme of length 3 of  $\mathbb{P}_k^2$  with one of the following possible structure:

(Type I)  $Z$  (resp.  $Z'$ ) is smooth and its base extension to  $\bar{k}$  consists of three non-collinear points  $p_1, p_2, p_3$  (resp.  $p'_1, p'_2, p'_3$ ) whose union is defined over  $k$ . The surface  $\Gamma_\sigma$  is smooth,  $e_{\bar{k}} = e_{p_1} + e_{p_2} + e_{p_3}$ ,  $e'_{\bar{k}} = e'_{p'_1} + e'_{p'_2} + e'_{p'_3}$  where the  $e_{p_i}$  and  $e'_{p'_i}$  are  $(-1)$ -curves.

(Type II)  $Z$  (resp.  $Z'$ ) consists of the disjoint union  $(p_1, \mathbf{p}_2)$  (resp.  $(p'_1, \mathbf{p}'_2)$ ) of a smooth  $k$ -rational point  $p_1$  (resp.  $p'_1$ ) and a 0-dimensional sub-scheme  $\mathbf{p}_2$  (resp.  $\mathbf{p}'_2$ ) of length 2 locally isomorphic to  $V(x, y^2)$  and supported at a  $k$ -rational point  $p_2$  (resp.  $p'_2$ ). We have  $e = e_{p_1} + 2e_{p_2}$ ,  $e' = e'_{p'_1} + 2e'_{p'_2}$  where  $e_{p_1}$  and  $e_{p_2}$  (resp.  $e'_{p'_1}$  and  $e'_{p'_2}$ ) are smooth  $k$ -rational curves with self-intersections  $-1$  and  $-\frac{1}{2}$  respectively, and  $\Gamma_\sigma$  has a unique  $A_1$ -singularity at the  $k$ -rational point  $e_{p_2} \cap e'_{p'_2}$ .

(Type III)  $Z$  (resp.  $Z'$ ) is supported on a unique  $k$ -rational point  $p$  (resp.  $p'$ ) and is locally isomorphic to  $V(y^3, x - y^2)$ . We have  $e = 3e_p$  and  $e' = 3e'_{p'}$  where  $e_p$  and  $e'_{p'}$  are smooth  $k$ -rational curves with self-intersection  $-\frac{1}{3}$ , and  $\Gamma_\sigma$  has a unique  $A_2$ -singularity at the  $k$ -rational point  $e_p \cap e'_{p'}$ .

The following lemma whose proof is left to the reader records for later use some additional basic properties of the surfaces  $\Gamma_\sigma$ .

**Lemma 8.** *With the notation above, the following hold:*

- a) *The canonical divisor  $K_{\Gamma_\sigma}$  is linearly equivalent to  $-e - e'$ .*
- b) *The pull-back by  $\tau : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$  (resp.  $\tau' : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$ ) of a general line  $\ell \simeq \mathbb{P}_k^1$  in  $\mathbb{P}_k^2$  is  $\mathbb{Q}$ -linearly equivalent to  $\frac{1}{3}(2e + e')$  (resp.  $\frac{1}{3}(e + 2e')$ ).*

**3.2. Locally trivial  $\mathbb{P}^1$ -bundles over  $\mathbb{P}^2$  associated to quadratic involutions.** Let  $\sigma : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$  be a birational quadratic involution of  $\mathbb{P}_k^2$  with graph  $\Gamma_\sigma \subset \mathbb{P}_k^2 \times \mathbb{P}_k^2$ , and let  $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}_k^2}$  be the ideal sheaf of its scheme-theoretic base locus  $Z$ .

**Lemma 9.** *There exists a locally free sheaf  $\mathcal{E}$  of rank 2 on  $\mathbb{P}_k^2$  and an exact sequence*

$$(3.1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}_k^2}(-1) \xrightarrow{s} \mathcal{E} \rightarrow \mathcal{I}_Z \rightarrow 0.$$

*Proof.* Since  $Z$  is a local complete intersection of codimension 2 in  $\mathbb{P}_k^2$ , the existence of  $\mathcal{E}$  with the required properties follows from Serre correspondence [23]. More precisely, the local-to-global spectral sequence

$$E_2^{p,q} = H^p(\mathbb{P}_k^2, \mathcal{E} \otimes \mathcal{I}_Z^q) \Rightarrow \text{Ext}^{p+q}(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}_k^2}(-1))$$

degenerates to a long exact sequence

$$0 \rightarrow H^1(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(-1)) \rightarrow \text{Ext}^1(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}_k^2}(-1)) \rightarrow H^0(Z, \det \mathcal{N}_{Z/\mathbb{P}_k^2} \otimes \mathcal{O}_Z(-1)) \rightarrow H^2(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(-1)).$$

Since  $H^i(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(-1)) = 0$  for  $i = 1, 2$  and  $Z$  is 0-dimensional, this sequence provides an isomorphism

$$\text{Ext}^1(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}_k^2}(-1)) \simeq H^0(Z, \det \mathcal{N}_{Z/\mathbb{P}_k^2} \otimes \mathcal{O}_Z(-1)) \simeq H^0(Z, \mathcal{O}_Z).$$

The extension corresponding via this isomorphism to the constant section  $1 \in H^0(Z, \mathcal{O}_Z)$  has the desired property.  $\square$

Let  $\pi : \mathbb{P}(\mathcal{E}) = \text{Proj}(\text{Sym} \mathcal{E}) \rightarrow \mathbb{P}_k^2$  be the locally trivial  $\mathbb{P}^1$ -bundle associated with the locally free sheaf  $\mathcal{E}$  of rank 2 as in Lemma 9. Since  $\det \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}_k^2}(-1)$ , the canonical sheaf  $\omega_{\mathbb{P}(\mathcal{E})}$  of  $\mathbb{P}(\mathcal{E})$  is isomorphic to

$$(3.2) \quad \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2) \otimes \pi^* \det \mathcal{E} \otimes \pi^* \omega_{\mathbb{P}_k^2} \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^2}(-4).$$

The surjection  $\mathcal{E} \rightarrow \mathcal{I}_Z \rightarrow 0$  defines a closed embedding  $\mathbb{P}(\mathcal{I}_Z) = \text{Proj}(\text{Sym} \mathcal{I}_Z) \hookrightarrow \mathbb{P}(\mathcal{E})$ . Since  $Z$  is a local complete intersection, the canonical homomorphism of graded  $\mathcal{O}_{\mathbb{P}_k^2}$ -algebras  $\text{Sym} \mathcal{I}_Z \rightarrow \mathcal{R}(\mathcal{I}_Z) = \bigoplus_{n \geq 0} \mathcal{I}_Z \cdot t^n$  is an isomorphism [17, Théorème 1], and it follows that the restriction  $\pi : \mathbb{P}(\mathcal{I}_Z) \rightarrow \mathbb{P}_k^2$  is isomorphic to the blow-up  $\tau : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$  of  $Z$ . We can thus identify from now on  $\Gamma_\sigma$  with the closed sub-scheme  $\mathbb{P}(\mathcal{I}_Z)$  of  $\mathbb{P}(\mathcal{E})$ . The composition of  $\pi^* s : \pi^* \mathcal{O}_{\mathbb{P}_k^2}(-1) \rightarrow \pi^* \mathcal{E}$  with the canonical surjection  $\pi^* \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$  defines a global section

$$\bar{s} \in \text{Hom}_{\mathbb{P}(\mathcal{E})}(\pi^* \mathcal{O}_{\mathbb{P}_k^2}(-1), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \simeq H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^2}(1))$$

whose zero locus  $V(\bar{s})$  coincides with  $\Gamma_\sigma$ .

Letting  $\xi$  and  $A$  be the classes in the divisor class group of  $\mathbb{P}(\mathcal{E})$  of  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$  and of the inverse image of a general line in  $\mathbb{P}_k^2$  by  $\pi$  respectively, we have  $\Gamma_\sigma = V(\bar{s}) \sim \xi + A$ . On the other hand, it follows from (3.2) that the canonical divisor  $K_{\mathbb{P}(\mathcal{E})}$  of  $\mathbb{P}(\mathcal{E})$  is linearly equivalent to  $-2(\Gamma_\sigma + A) \sim -2(\xi + 2A)$ . Since  $c_1(\mathcal{E}) = -1$  and  $c_2(\mathcal{E}) = 3$  by construction, we derive the following numerical information:

$$(3.3) \quad K_{\mathbb{P}(\mathcal{E})}^3 = -32, \quad K_{\mathbb{P}(\mathcal{E})}^2 \cdot \Gamma_\sigma = 4, \quad K_{\mathbb{P}(\mathcal{E})} \cdot \Gamma_\sigma^2 = 2, \quad \Gamma_\sigma^3 = -2.$$

### 3.3. Construction of Sarkisov links.

**Proposition 10.** *Let  $\pi : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}_k^2$  be the  $\mathbb{P}^1$ -bundle associated to a quadratic involution  $\sigma : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$  with graph  $\Gamma_\sigma \subset \mathbb{P}(\mathcal{E})$  as in §3.2. Then there exists a  $k$ -form  $Y$  of  $V_5$  and a Sarkisov link*

$$\begin{array}{ccccccc} \Gamma_\sigma & \xrightarrow{\quad} & \mathbb{P}(\mathcal{E}) & \xrightarrow{\quad \varphi \quad} & X^+ & \xleftarrow{\quad} & \Gamma_\sigma^+ \\ & \searrow \tau & \swarrow \pi & \searrow & \swarrow \theta & & \searrow \theta|_{\Gamma^+} \\ & & \mathbb{P}_k^2 & & X_0 & & \\ & & & & \searrow \theta & & \\ & & & & Y & \xleftarrow{\quad} & \{y\} \end{array}$$

where:

- (i)  $\varphi : \mathbb{P}(\mathcal{E}) \dashrightarrow X^+$  is a flop whose flopping locus coincides with the support of  $e' \subset \Gamma_\sigma$ ,
- (ii)  $\Gamma_\sigma^+ \simeq \mathbb{P}_k^2$  is the proper transform of  $\Gamma_\sigma$  and  $\theta|_{\Gamma_\sigma^+} : \Gamma_\sigma \dashrightarrow \Gamma_\sigma^+$  is the contraction of  $e'$ ,
- (iii)  $\theta : X^+ \rightarrow Y$  is the divisorial contraction of  $\Gamma_\sigma^+$  to a smooth  $k$ -rational point  $y \in Y$ ,
- (iv) The support of the image in  $Y$  of the flopped locus  $e^+ \subset X^+$  of  $\varphi$  coincides with the union of the lines in  $Y_{\bar{k}}$  passing through  $y$ .

*Proof.* We denote  $\mathbb{P}(\mathcal{E})$  and  $\Gamma_\sigma \subset \mathbb{P}(\mathcal{E})$  simply by  $X$  and  $\Gamma$ . Since  $-K_X \sim 2\xi + 4A \sim 2\Gamma + 2A$  we see that  $-K_X$  is nef and that any irreducible curve  $C \subset X_{\bar{k}}$  such that  $-K_{X_{\bar{k}}} \cdot C \leq 0$  is contained in  $\Gamma_{\bar{k}}$ . By the adjunction formula and Lemma 8 a), we have

$$\Gamma^2 = -K_\Gamma - 2A \cdot \Gamma \sim_{\mathbb{Q}} \frac{1}{3}(-e + e') \quad \text{and} \quad \Gamma^2 - K_\Gamma \sim_{\mathbb{Q}} \frac{2}{3}(e + 2e'),$$

which implies by adjunction again that

$$-K_{X_{\bar{k}}} \cdot C = (\Gamma_{\bar{k}}^2 - K_{\Gamma_{\bar{k}}}) \cdot C = \frac{2}{3}(e_{\bar{k}} + 2e'_{\bar{k}}) \cdot C.$$

Since  $e + 2e'$  is  $\tau$ -ample and  $\tau'$ -numerically trivial by virtue of Lemma 8 b), we conclude that the irreducible curves  $C \subset X_{\bar{k}}$  such that  $-K_{X_{\bar{k}}} \cdot C = 0$  are precisely the irreducible components of the exceptional locus  $e'_{\bar{k}}$  of  $\tau'_{\bar{k}} : \Gamma_{\bar{k}} \rightarrow \mathbb{P}_{\bar{k}}^2$  (see § 3.1 for the notation).

Let  $\varphi : X \dashrightarrow X^+$  be the flop of the union of the irreducible components of  $e'_{\bar{k}}$ . Since the union of these components is defined over  $k$ , so is the union  $e^+$  of the flopped curves of  $\varphi$ , and so  $X^+$  is a smooth threefold defined over  $k$  and  $\varphi$  is a birational map defined over  $k$  restricting to an isomorphism between  $X \setminus e'$  and  $X^+ \setminus e^+$ . Let  $\Gamma^+$  and  $A^+$  be the proper transforms in  $X^+$  of  $\Gamma$  and  $A$  respectively. By construction, the restriction  $\varphi|_\Gamma : \Gamma \dashrightarrow \Gamma^+$  coincide with  $\tau' : \Gamma \rightarrow \mathbb{P}_k^2$ . Since  $K_{X^+} \sim -2(\Gamma^+ + A^+)$ , we deduce from the adjunction formula that

$$-(\Gamma^+)^2 = K_{\Gamma^+} + 2A^+ \cdot \Gamma^+ = \tau'_*(K_\Gamma + 2A \cdot \Gamma) = \frac{1}{3}\tau'_*e,$$

which is linearly equivalent to a line  $\ell \simeq \mathbb{P}_k^1$  in  $\Gamma^+ \simeq \mathbb{P}_k^2$ . The normal bundle  $\mathcal{N}_{\Gamma^+/X^+}$  of  $\Gamma^+$  in  $X^+$  is thus isomorphic to  $\mathcal{O}_{\mathbb{P}_k^2}(-1)$ .

The divisor class group of  $X^+$  is freely generated by  $\Gamma^+$  and  $A^+$ . The Mori cone  $\text{NE}(X^+)$  is spanned by two extremal rays: one  $R_1$  corresponding to the flopped curves of  $\varphi$  and a second one  $R_2$  which is  $K_{X^+}$ -negative. Since  $\Gamma^+ \cdot C \geq 0$  for any irreducible curve not contained in  $\Gamma^+$ , including thus the irreducible components of  $e^+$ , whereas  $\Gamma^+ \cdot \ell = -1$  for any line  $\ell \simeq \mathbb{P}_k^1$  in  $\Gamma^+ \simeq \mathbb{P}_k^2$ , it follows that  $R_2$  is generated by the class of  $\ell$ . The extremal contraction associated to  $R_2$  is thus the divisorial contraction  $\theta : X^+ \rightarrow Y$  of  $\Gamma^+ \simeq \mathbb{P}_k^2$  to a smooth  $k$ -rational point  $p$  of a smooth projective threefold  $Y$ .



Since  $-K_{X^+} = 2(\Gamma^+ + A^+)$ , we conclude that the image  $H$  of  $A^+$  by  $\theta$  is an ample divisor on  $Y$  generating the divisor class group of  $Y$  and such that  $-K_Y = 2H$ . Furthermore, we have

$$\begin{aligned} K_Y^3 &= \theta^*(K_Y)^3 = (K_{X^+} - 2\Gamma^+)^3 \\ &= K_{X^+}^3 - 6K_{X^+}^2 \cdot \Gamma^+ + 12K_{X^+} \cdot (\Gamma^+)^2 - 8(\Gamma^+)^3 \\ &= K_X^3 - 6K_X^2 \cdot \Gamma + 12K_X \cdot \Gamma - 8 \end{aligned}$$

so that  $K_Y^3 = -40$  by (3.3). Altogether, this shows  $Y_{\bar{k}}$  is a smooth Fano threefold of Picard number 1, index 2 and degree  $d = H^3 = 5$ , hence is isomorphic to  $V_5$ .

The fact that the support of the image in  $Y$  of  $e^+$  coincides with the union of the lines in  $Y_{\bar{k}}$  passing through  $p$  is clear by construction.  $\square$

By construction, the proper transform by the reverse composition  $\psi_y = \pi \circ \varphi^{-1} \circ \theta^{-1} : Y \dashrightarrow \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^2$  of the complete linear system of lines in  $\mathbb{P}_k^2$  consists of divisors  $H$  on  $Y$  singular at  $y$  and such that  $-K_Y = 2H$ . This shows that  $\psi_{y,\bar{k}} : Y_{\bar{k}} \simeq V_5 \dashrightarrow \mathbb{P}_k^2$  coincides with the double projection from the point  $y$ . Conversely, given any  $k$ -form  $Y$  of  $V_5$ , Corollary 3 ensures that  $-K_Y$  is divisible, equal to  $2H$  for some ample divisor  $H$  on  $Y$ . So given any  $k$ -rational point  $y \in Y$ , the double projection  $\psi_y : Y \dashrightarrow \mathbb{P}_k^2$  from  $y$  is defined over  $k$ , given by the linear system  $|\mathcal{O}_Y(H) \otimes \mathfrak{m}_y^2|$ , and it coincides with the composition  $\pi \circ \varphi^{-1} \circ \theta^{-1} : Y \dashrightarrow \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^2$  for a suitable quadratic birational involution  $\sigma$  of  $\mathbb{P}_k^2$ .

#### 4. APPLICATION : CYLINDERS IN FORMS OF THE QUINTIC DEL PEZZO THREEFOLD

**Theorem 11.** *Let  $Y$  be a  $k$ -form of  $V_5$ , let  $y \in Y$  be a  $k$ -rational point and let  $C$  be the union of the lines in  $Y_{\bar{k}}$  passing through  $y$ . Then  $Y \setminus C$  has the structure of a Zariski locally trivial  $\mathbb{A}^1$ -bundle  $\rho : Y \setminus C \rightarrow \mathbb{P}_k^2 \setminus Z$  over the complement of a closed sub-scheme  $Z \subset \mathbb{P}_k^2$  of length 3 with as many irreducible geometric components as  $C$ .*

*Proof.* Indeed, the birational map  $\xi = \theta \circ \varphi : \mathbb{P}(\mathcal{E}) \dashrightarrow Y$  constructed in Proposition 10 restricts to an isomorphism between  $Y \setminus C$  and the complement of the proper transform  $\Gamma$  in  $\mathbb{P}(\mathcal{E})$  of the exceptional divisor of the blow-up of  $Y$  at  $y$ . On the other hand, it follows from the construction of  $\mathcal{E}$  in § 3.2 that  $\Gamma$  is the graph of a quadratic birational involution  $\sigma$  of  $\mathbb{P}_k^2$  and that the restriction of  $\pi : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}_k^2$  to the complement of  $\Gamma$  is a locally trivial  $\mathbb{A}^1$ -bundle over the complement  $\mathbb{P}_k^2 \setminus Z$  of the base locus  $Z$  of  $\sigma$ .  $\square$

**Theorem 12.** *Every  $k$ -form  $Y$  of  $V_5$  contains a Zariski open  $\mathbb{A}_k^2$ -cylinder. Furthermore,  $Y$  contains  $\mathbb{A}_k^3$  as a Zariski open subset if and only if it contains a special line defined over  $k$ .*

*Proof.* By Lemma 2,  $Y$  contains a line  $\ell \simeq \mathbb{P}_k^1$  with trivial normal bundle. Projecting from  $\ell$  as in the proof of Corollary 3, we obtain a birational map  $Y \dashrightarrow Q$  onto a smooth quadric  $Q \subset \mathbb{P}_k^4$  defined over  $k$ , which restricts to an isomorphism between the complement of the surface  $Z \subset Y$  defined over  $k$  swept out by the lines in  $Y_{\bar{k}}$  intersecting  $\ell_{\bar{k}}$  and the complement of a hyperplane section  $Q_0 \simeq \mathbb{P}_k^1 \times \mathbb{P}_k^1$  of  $Q$ . The complement  $Q \setminus Q_0$  is isomorphic to the smooth affine quadric  $\tilde{Q}_0 = \{xv - yu = 1\} \subset \mathbb{A}_k^4$ , which contains for instance the principal open  $\mathbb{A}_k^2$ -cylinder  $\tilde{Q}_{0,x} \simeq \text{Spec}(k[x^{\pm 1}][y, u])$ .

By Lemma 2 b),  $Y$  contains a special line if and only if it has a  $k$ -rational point  $p$  through which there exists at most two lines in  $Y_{\bar{k}}$ . Letting  $C$  be the union of these lines, the previous theorem implies that  $\rho : Y \setminus C \rightarrow \mathbb{P}_k^2 \setminus Z$  is a Zariski locally trivial  $\mathbb{A}^1$ -bundle over the complement of a 0-dimensional closed subset  $Z$  defined over  $k$  whose support consists of at most two points. Letting  $L \simeq \mathbb{P}_k^1$  be any line in  $\mathbb{P}_k^2$  containing the support of  $Z$ , the restriction of  $\rho$  over  $\mathbb{P}_k^2 \setminus L \simeq \mathbb{A}_k^2$  is then a trivial  $\mathbb{A}^1$ -bundle, so that  $Y \setminus \rho^{-1}(L)$  is a Zariski open subset of  $Y$  isomorphic to  $\mathbb{A}_k^3$ .

Conversely, suppose that  $Y$  contains a Zariski open subset  $U \simeq \mathbb{A}_k^3$ . Since  $U$  is affine,  $D = Y \setminus U$  has pure codimension 1 in  $Y$ . Since  $Y_{\bar{k}} \simeq V_5$  and  $Y_{\bar{k}} \setminus D_{\bar{k}} \simeq U_{\bar{k}} \simeq \mathbb{A}_{\bar{k}}^3$ , according to [7, Corollary 1.2, b)] and [8], we have the following alternative for the divisor  $D_{\bar{k}}$  in  $Y_{\bar{k}}$ :

- 1)  $D_{\bar{k}}$  is a normal del Pezzo surface of degree 5 with a unique singular point  $q$  of type  $A_4$ ,
- 2)  $D_{\bar{k}}$  is a non-normal del Pezzo surface of degree 5 whose singular locus is a special line  $\ell$  in  $Y_{\bar{k}}$ , and  $D_{\bar{k}}$  is swept out by lines in  $V_5$  which intersect  $\ell$ .

In the first case, there exists a unique special line  $\ell$  in  $Y_{\bar{k}}$  passing through  $q$ , which is then automatically contained in  $D_{\bar{k}}$ . Since  $D$  is defined over  $k$  and  $q$  is the unique singular point of  $D_{\bar{k}}$ ,  $q$  corresponds to a

$k$ -rational point of  $D$ , so that  $\ell$  is actually a special line in  $Y$  defined over  $k$ . In the second case, since  $\ell$  is the singular locus of  $D_{\bar{k}}$ , it is invariant under the action of the Galois group  $\text{Gal}(\bar{k}/k)$ , hence is defined over  $k$ . So in both cases,  $Y$  contains a special line defined over  $k$ .  $\square$

**Corollary 13.** *Every form of  $V_5$  defined over a  $C_1$ -field  $k$  contains  $\mathbb{A}_k^3$  as a Zariski open subset.*

*Proof.* This follows from Lemma 2 b) and the previous theorem since every conic over a  $C_1$ -field has a rational point.  $\square$

**Example 14.** The  $\mathbb{C}(s, t)$ -form  $Y = \text{VSP}(f)$  of  $V_5$  associated to the quadratic form  $f = x^2 + sy^2 + tz^2 \in \mathbb{C}(s, t)[x, y, z]$  considered in Example 7 does not contain any special line defined over  $\mathbb{C}(s, t)$ , hence does not contain  $\mathbb{A}_{\mathbb{C}(s, t)}^3$  as a Zariski open subset.

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