

CYLINDERS IN MORI FIBER SPACES: FORMS OF THE QUINTIC DEL PEZZO THREEFOLD

ADRIEN DUBOULOZ AND TAKASHI KISHIMOTO

ABSTRACT. Motivated by the general question of existence of open \mathbb{A}^1 -cylinders in higher dimensional projective varieties, we consider the case of Mori Fiber Spaces of relative dimension three, whose general closed fibers are isomorphic to the quintic del Pezzo threefold V_5 , the smooth Fano threefold of index two and degree five. We show that the total spaces of these Mori Fiber Spaces always contain relative \mathbb{A}^2 -cylinders, and we characterize those admitting relative \mathbb{A}^3 -cylinders in terms of the existence of certain special lines in their generic fibers.

INTRODUCTION

An \mathbb{A}_k^r -cylinder in a normal algebraic variety defined over a field k is a Zariski open subset U isomorphic to $Z \times \mathbb{A}_k^r$ for some algebraic variety Z defined over k . In the case where $k = \bar{k}$ is algebraically closed, normal projective varieties V containing \mathbb{A}_k^1 -cylinders have received a lot of attention recently due to the connection between unipotent group actions on their affine cones and polarized \mathbb{A}_k^1 -cylinders in them, that is \mathbb{A}_k^1 -cylinders whose complements are the supports of effective \mathbb{Q} -divisors linearly equivalent to an ample divisor on V (cf. [12, 13]). Certainly, the canonical divisor K_V of a normal projective variety containing an \mathbb{A}_k^r -cylinder for some $r \geq 1$ is not pseudo-effective. Replacing V if necessary by a birational model with at most \mathbb{Q} -factorial terminal singularities, [1] guarantees the existence of a suitable K_V -MMP $V \dashrightarrow X$ whose output X is equipped with a structure of Mori Fiber Space $f : X \rightarrow Y$ over some lower dimensional normal projective variety Y . Since an \mathbb{A}_k^1 -cylinder in X can always be transported back in the initial variety V [4, Lemma 9], total spaces of Mori Fiber Spaces form a natural restricted class in which to search for varieties containing \mathbb{A}_k^1 -cylinders.

In the case where $\dim Y = 0$, X is a Fano variety of Picard number one. The only smooth Fano surface of Picard number is the projective plane \mathbb{P}_k^2 which obviously contains \mathbb{A}_k^1 -cylinders. Several families of examples of smooth Fano varieties of dimension 3 and 4 and Picard number one containing \mathbb{A}_k^1 -cylinders have been constructed [14, 21, 22]. The question of existence of \mathbb{A}_k^1 -cylinders in other possible outputs of MMPs was first considered in [2, 3], in which del Pezzo fibrations $f : X \rightarrow Y$, which correspond to the case where $\dim Y = \dim X - 2 > 0$, were extensively studied. In such a relative context, it is natural to shift the focus to cylinders which are compatible with the fibration structure:

Definition. Let $f : X \rightarrow Y$ be a morphism between normal algebraic varieties defined over a field k and let $U \simeq Z \times \mathbb{A}_k^r$ be an \mathbb{A}_k^r -cylinder inside X . We say that U is *vertical with respect to f* if the restriction $f|_U$ factors as

$$f|_U = h \circ \text{pr}_Z : U \simeq Z \times \mathbb{A}_k^r \xrightarrow{\text{pr}_Z} Z \xrightarrow{h} Y$$

for a suitable morphism $h : Z \rightarrow Y$.

In the present article, we initiate the study of existence of vertical \mathbb{A}^1 -cylinders in Mori Fiber Spaces $f : X \rightarrow Y$ of relative dimension three, whose general fibers are smooth Fano threefolds. Each smooth Fano threefold of Picard number one can appear as a closed fiber of a Mori Fiber Space, and among these,

2000 *Mathematics Subject Classification.* 14E30; 14J30; 14J45; 14R10; 14R25;

Key words and phrases. Fano threefold, Mori Fiber Space, Sarkisov link; Cremona involutions; Cylinders.

The second author was partially funded by Grant-in-Aid for Scientific Research of JSPS No. 15K04805. The research was initiated during a visit of the first author at Saitama University and continued during a stay of the second author at University of Burgundy as a CNRS Research Fellow. The authors thank these institutions for their generous support and the excellent working conditions offered.

it is natural to first restrict to classes which contain a cylinder of the maximal possible dimension, namely the affine space \mathbb{A}_k^3 , and expect that some suitable sub-cylinders of these fiber wise maximal cylinders could arrange into a vertical cylinder with respect to f , possibly of smaller relative dimension. The only four classes of Fano threefold of Picard number one containing \mathbb{A}_k^3 are \mathbb{P}_k^3 , the quadric \mathbb{Q}^3 , the del Pezzo quintic threefold V_5 of index two and degree five [7, 8], and a four dimensional family of prime Fano threefolds V_{22} of genus twelve [6, 20].

The existence inside X of a vertical \mathbb{A}^r -cylinder with respect to $f : X \rightarrow Y$ translates equivalently into that of an \mathbb{A}_K^r -cylinder inside the fiber X_η of f over the generic point η of Y , considered as a variety defined over the function field K of Y [3]. We are therefore led to study the existence of \mathbb{A}_K^r -cylinders inside K -forms of the aforementioned Fano threefolds over non-closed fields K of characteristic zero, that is, smooth projective varieties defined over K whose base extensions to an algebraic closure \overline{K} are isomorphic over \overline{K} to one of these Fano threefolds. The case of \mathbb{P}^3 is easily dispensed: a K -form V of \mathbb{P}^3 contains an \mathbb{A}_K^3 -cylinder if and only if it has a K -rational point hence if and only if it is the trivial K -form \mathbb{P}_K^3 . So equivalently, a Mori Fiber Space $f : X \rightarrow Y$ whose general closed fibers are isomorphic to \mathbb{P}_k^3 contains a vertical \mathbb{A}_k^3 -cylinder if and only if it has a rational section. The case of the quadric \mathbb{Q}^3 is already more intricate: one can deduce from [6] that a K -form V of \mathbb{Q}^3 contains \mathbb{A}_K^3 if and only if it has a hyperplane section defined over K which is a K -rational quadric cone. In this article, we establish a complete characterization of the existence of \mathbb{A}_K^r -cylinders in forms of V_5 which can be summarized as follows:

Theorem. *Let K be a field of characteristic zero and let Y be a K -form of V_5 . Then Y always contains an \mathbb{A}_K^2 -cylinder, and it contains an \mathbb{A}_K^3 -cylinder if and only if it contains a curve $\ell \simeq \mathbb{P}_K^1$ of anticanonical degree $-K_Y \cdot \ell = 2$ and with normal bundle $N_{\ell/Y} \simeq \mathcal{O}_{\mathbb{P}_K^1}(-1) \oplus \mathcal{O}_{\mathbb{P}_K^1}(1)$.*

An irreducible curve ℓ of anticanonical degree $-K_Y \cdot \ell = 2$ on a K -form Y of V_5 becomes after base extension to an algebraic closure \overline{K} of K a usual line on $Y_{\overline{K}} \simeq V_5$ embedded into $\mathbb{P}_{\overline{K}}^6$ via its half-anticanonical complete linear system. By combining the previous characterization with a closer study of the Hilbert scheme of such curves ℓ on Y (see §2.1), we derive the following result (see Corollary 13):

Corollary. *Let \overline{k} be an algebraically closed field of characteristic zero and let $f : X \rightarrow C$ be a Mori Fiber Space over a curve C defined over \overline{k} , whose general closed fibers are quintic del Pezzo threefolds V_5 . Then X contains a vertical $\mathbb{A}_{\overline{k}}^3$ -cylinder with respect to f .*

Section 1 contains a brief recollection on the quintic del Pezzo threefold V_5 and its Hilbert scheme of lines. In Section 2, we establish basic geometric properties of forms of V_5 and describe their Hilbert schemes of lines. We also describe an adaptation to non-closed fields of a standard construction of V_5 as the variety of trisecant lines to a Veronese surface in \mathbb{P}^4 , from which we derive for suitable fields the existence of forms of V_5 which do not contain any line with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. The technical core of the article is then Section 3: we give a new construction of a classical rational map, called the double projection from a rational point of a form of V_5 in the form of a Sarkisov link from a \mathbb{P}^1 -bundle over \mathbb{P}^2 explicitly determined by the base locus of a quadratic birational involution of \mathbb{P}^2 . The main results concerning \mathbb{A}^r -cylinders in forms of V_5 are then derived from this construction in Section 4.

1. GEOMETRY OF THE SMOOTH QUINTIC DEL PEZZO THREEFOLD

In the rest of this article, unless otherwise stated, the notations k and \overline{k} refer respectively to a field of characteristic zero and a fixed algebraic closure of k . In this section, we recall without proof classical descriptions and properties of the quintic del Pezzo threefold V_5 over \overline{k} and of its Hilbert scheme of lines.

1.1. Two classical descriptions of V_5 . A quintic del Pezzo threefold V_5 over \overline{k} is a smooth projective threefold whose Picard group is isomorphic to \mathbb{Z} , generated by an ample class H such that $-K_{V_5} = 2H$ and $H^3 = 5$. In other words, V_5 is a smooth Fano threefold of index two and degree five.

1.1.1. Sarkisov links to a smooth quadric in \mathbb{P}^4 . Let us first recall the classical description due to Iskovskikh [11, Chapter II, §1.6]. Letting H be an ample class such that $-K_{V_5} = 2H$, the complete linear system $|H|$ defines a closed embedding $\Phi_{|H|} : V_5 \hookrightarrow \mathbb{P}_{\overline{k}}^6$. A general hyperplane section of V_5 contains a line ℓ whose

normal bundle in V_5 is trivial, and the projection from ℓ induces a birational map $V_5 \dashrightarrow Q$ onto a smooth quadric $Q \subset \mathbb{P}_k^4$. The inverse map $Q \dashrightarrow V_5$ can be described as the blow-up of Q along a rational normal cubic $C \subset Q$ contained in a smooth hyperplane section $Q_0 \simeq \mathbb{P}_k^1 \times \mathbb{P}_k^1$ of Q , followed by the contraction of the proper transform Q'_0 of Q_0 onto the line ℓ . In sum, the projection from the line ℓ induces a Sarkisov link

$$\begin{array}{ccccc}
 & Q'_0 & \xrightarrow{\quad} & \tilde{V}_5 & \xleftarrow{\quad} & Z' \\
 & \swarrow & & \swarrow q & & \searrow q' \\
 \ell & \xrightarrow{\quad} & V_5 & \xrightarrow{\quad} & Q & \xleftarrow{\quad} & C \\
 & & & \text{---} & & & \\
 & & & |H-\ell| & & &
 \end{array}$$

where $q : \tilde{V}_5 \rightarrow V_5$ is the blow-up of V_5 along ℓ with exceptional divisor Q'_0 and $q' : \tilde{V}_5 \rightarrow Q$ is the contraction onto the curve C of the proper transform Z' of the surface $Z \subset V_5$ swept out by lines in V_5 intersecting ℓ .

Since the automorphism group $\mathrm{PGL}_5(\bar{k})$ of \mathbb{P}_k^4 acts transitively on the set of flags $C \subset Q_0 \subset Q$, it follows that over an algebraically closed field \bar{k} , all smooth Fano threefold of Picard number one, index two and degree five embedded into \mathbb{P}_k^6 by their half-anticanonical complete linear system are projectively equivalent.

1.1.2. Quasi-homogeneous space of $\mathrm{PGL}_2(\bar{k})$. We now recall an alternative description of V_5 due to Mukai-Umemura [19] (see also [16, §5.1]). Let $M_d = \mathrm{Sym}^d(\bar{k}^2)^\vee \simeq \bar{k}[x, y]_{(d)}$ be the space of homogeneous polynomials of degree d with coefficients in \bar{k} . The natural action of $\mathrm{GL}_2(\bar{k})$ on M_1 induces a linear action on M_6 , hence an action of $\mathrm{PGL}_2(\bar{k})$ on $\mathbb{P}(M_6) \simeq \mathbb{P}_k^6$. We then have the following description:

Proposition 1. *The Fano threefold V_5 is isomorphic to the closure $\overline{\mathrm{PGL}_2(\bar{k}) \cdot [\phi]}$ of the class of the polynomial $\phi = xy(x^4 + y^4) \in M_6$. Furthermore, the $\mathrm{PGL}_2(\bar{k})$ -orbits on V_5 are described as follows:*

- (i) *The open orbit $O = \mathrm{PGL}_2(\bar{k}) \cdot [\phi]$ with stabilizer equal to the binary octahedral group,*
- (ii) *The 2-dimensional orbit $S_2 = \mathrm{PGL}_2(\bar{k}) \cdot [xy^5]$, which is neither open nor closed, with stabilizer equal to the diagonal torus \mathbb{T} .*
- (iii) *The 1-dimensional closed orbit $C_6 = \mathrm{PGL}_2(\bar{k}) \cdot [x^6]$ with stabilizer equal to the Borel subgroup B of upper triangular matrices.*

It follows from this description that the automorphism $\mathrm{Aut}(V_5)$ is isomorphic to $\mathrm{PGL}_2(\bar{k})$. We also observe that C_6 is a normal rational sextic curve and that the closure $\overline{S_2} = S_2 \cup C_6$ of S_2 is a quadric section of V_5 , hence an anti-canonical divisor on V_5 , which coincides with the *tangential scroll* of C_6 , swept out by the tangent lines to C_6 contained in V_5 . It is singular along C_6 , and its normalization morphism coincides with the map $\nu : \mathbb{P}(M_1) \times \mathbb{P}(M_1) \rightarrow \mathbb{P}(M_6)$, $(f_1, f_2) \mapsto f_1^5 f_2$.

1.2. Lines on V_5 . The family of lines on V_5 is very well-studied [11, 7, 10]. We list below some of its properties which will be useful later on for the study of cylinders on forms of V_5 .

First, by a *line* on V_5 , we mean an integral curve $\ell \subset V_5$ of anticanonical degree $-K_{V_5} \cdot \ell = 2$. It thus corresponds through the half-anticanonical embedding $\Phi_{|H|} : V_5 \hookrightarrow \mathbb{P}_k^6$ to a usual line in \mathbb{P}_k^6 which is contained in the image of V_5 . A general line ℓ in V_5 has trivial normal bundle, whereas there is a one-dimensional subfamily of lines with normal bundle $\mathcal{N}_{\ell/V_5} \simeq \mathcal{O}_{\mathbb{P}_k^1}(-1) \oplus \mathcal{O}_{\mathbb{P}_k^1}(1)$, which we call *special lines*.

The Hilbert scheme $\mathcal{H}(V_5)$ of lines in V_5 is isomorphic to \mathbb{P}_k^2 , and the evaluation map $v : \mathcal{U} \rightarrow V_5$ from the universal family $\mathcal{U} \rightarrow \mathcal{H}(V_5)$ is a finite morphism of degree 3. There are thus precisely three lines counted with multiplicities passing through a given closed point of V_5 .

The curve in $\mathcal{H}(V_5) \simeq \mathbb{P}_k^2$ that parametrizes special lines in V_5 is a smooth conic C . The restriction of v to $\mathcal{U}|_C$ is injective, and in the description of V_5 as a quasi-homogeneous space of $\mathrm{PGL}_2(\bar{k})$ given in § 1.1.2 above, $v(\mathcal{U}|_C)$ is the tangential scroll $\overline{S_2}$ to the rational normal sextic C_6 , while C_6 itself coincides with the image of the intersection of $\mathcal{U}|_C$ with the ramification locus of v . In particular, special lines on V_5 never intersect each others. Furthermore, there are three lines with trivial normal bundle through any point in $V_5 \setminus \overline{S_2}$, a line with trivial normal bundle and a special line through any point of S_2 , and a unique special line through every point of C_6 .

2. FORMS OF V_5 OVER NON-CLOSED FIELDS

A k -form of V_5 is a smooth projective variety Y defined over k such that $Y_{\bar{k}}$ is isomorphic to V_5 . In this subsection, we establish basic properties of these forms and their Hilbert schemes of lines.

2.1. Hilbert scheme of lines on a k -form of V_5 . Let Y be a k -form of V_5 , let $\mathcal{H}(Y)$ be the Hilbert scheme of irreducible curves of $-K_Y$ -degree equal to 2 on Y and let $v : \mathcal{U} \rightarrow Y$ be the evaluation map from the universal family $\mathcal{U} \rightarrow \mathcal{H}(Y)$. By § 1.2, $\mathcal{H}(Y)_{\bar{k}} = \mathcal{H}(Y_{\bar{k}})$ is isomorphic to $\mathbb{P}_{\bar{k}}^2$, i.e. $\mathcal{H}(Y)$ is a k -form of $\mathbb{P}_{\bar{k}}^2$. Furthermore, since the smooth conic parametrizing special lines on $Y_{\bar{k}}$ is invariant under the action of the Galois group $\text{Gal}(\bar{k}/k)$, it corresponds to a smooth curve $C \subset \mathcal{H}(Y)$ defined over k , and such that $C_{\bar{k}} \simeq \mathbb{P}_{\bar{k}}^1$.

Lemma 2. *Let Y be a k -form of V_5 .*

a) *The Hilbert scheme $\mathcal{H}(Y)$ of lines on Y is isomorphic to $\mathbb{P}_{\bar{k}}^2$, in particular Y always contains lines defined over k with trivial normal bundles.*

b) *The following assertions are equivalent:*

- (i) *Y contains a special line defined over k ,*
- (ii) *The conic C has a k -rational point,*
- (iii) *The surface $v(\mathcal{U}|_C)$ has a k -rational point,*
- (iv) *The image of the intersection of $\mathcal{U}|_C$ with the ramification locus of v has a k -rational point.*

Proof. Since $\mathcal{H}(Y)$ is a k -form of $\mathbb{P}_{\bar{k}}^2$, to prove the first assertion it is enough to show that $\mathcal{H}(Y)$ has a k -rational point. Since $C_{\bar{k}} \simeq \mathbb{P}_{\bar{k}}^1$, there exists a quadratic extension $k \subset k'$ such that $C(k')$ is nonempty, so that in particular $\mathcal{H}(Y)_{k'} \simeq \mathbb{P}_{k'}^2$. Let p be a k' -rational point of $C_{k'}$. If p is invariant under the action of the Galois group $\text{Gal}(k'/k)$, then it corresponds to a k -rational point of C , hence of $\mathcal{H}(Y)$, and we are done. Otherwise, its Galois conjugate \bar{p} is a k' -rational point of $C_{k'}$ distinct from p , and then the tangent lines $T_p C_{k'}$ and $T_{\bar{p}} C_{k'}$ to $C_{k'}$ at p and \bar{p} respectively intersect each other at unique point. The latter is thus $\text{Gal}(k'/k)$ -invariant, hence corresponds to a k -rational point of $\mathcal{H}(Y) \setminus C$. The existence of lines with trivial normal bundles defined over k then follows from the description of $\mathcal{H}(V_5)$ given in § 1.2.

The second assertion is an immediate consequence of the facts that special lines in $Y_{\bar{k}}$ are in one-to-one correspondence with closed points of the image D of the intersection of $(\mathcal{U}|_C)_{\bar{k}}$ with the ramification locus of v , and that $v(\mathcal{U}|_C)_{\bar{k}}$ coincides with the surface swept out by the tangent lines to D . \square

Corollary 3. *A k -form Y of V_5 is k -rational and the natural map $\text{Pic}(Y) \rightarrow \text{Pic}(Y_{\bar{k}})$ is an isomorphism.*

Proof. By Lemma 2 a), Y contains a line $\ell \simeq \mathbb{P}_k^1$ with trivial normal bundle. The surface Z in $Y_{\bar{k}}$ swept out by lines intersecting $\ell_{\bar{k}}$ is defined over k . As in § 1.1.1, the composition of the blow-up $q : \tilde{Y} \rightarrow Y$ of ℓ with exceptional divisor $Q'_0 \simeq \mathbb{P}_k^1 \times \mathbb{P}_k^1$ followed by the contraction $q' : \tilde{Y} \rightarrow Q$ of the proper transform of Z' of Z yields a birational map $Y \dashrightarrow Q$ defined over k onto a smooth quadric $Q \subset \mathbb{P}_k^4$, which maps Q'_0 onto a hyperplane section Q_0 of Q . Since Q contains k -rational points, it is k -rational. The natural map $\text{Pic}(Y) \rightarrow \text{Pic}(Y_{\bar{k}})$ is an isomorphism if and only $-K_Y$ is divisible in $\text{Pic}(Y)$. But since $-K_Q \sim 3Q_0$ and $q'^* Q_0 = Q'_0 + Z'$, we deduce from the ramification formula for q and q' that

$$-K_Y \sim q_*(-K_{\tilde{Y}}) \sim q_*(-q'^* K_Q - Z') \sim q_*(3Q'_0 + 2Z') \sim 2Z.$$

\square

2.2. Varieties of trisecant lines to Veronese surfaces in \mathbb{P}^4 . In this subsection, we review a third classical construction of V_5 as the variety of trisecant line to the Veronese surface in $\mathbb{P}_{\bar{k}}^4$ which, when performed over k gives rise, depending on the choices made, to nontrivial k -forms of V_5 .

Let V be a k -vector space of dimension 3, let $f \in \text{Sym}^2 V^*$ be a homogeneous form defining a smooth conic $Q \subset \mathbb{P}(V)$, and let $W = \text{Sym}^2(V^*/\langle f \rangle)$. Recall that a closed subscheme $Z \subset \mathbb{P}(V^*)$ defined over k is called *apolar* to Q if the class of $[f] \in \mathbb{P}(\text{Sym}^2 V^*)$ lies in the linear span of the image of Z by the second Veronese embedding $v_2 : \mathbb{P}(V^*) \hookrightarrow \mathbb{P}(\text{Sym}^2 V^*)$. When $Z = \{[\ell_1], [\ell_2], [\ell_3]\} \subset \mathbb{P}(V^*)$ consists of three distinct k -rational points, this says equivalently that there are scalars $\lambda_i \in k$, $i = 1, 2, 3$, such that $f = \lambda_1 \ell_1^2 + \lambda_2 \ell_2^2 + \lambda_3 \ell_3^2$. We denote by $\text{VSP}(f)$ the variety of closed subschemes of length 3 of $\mathbb{P}(V^*)$ which are apolar to Q .

Let $\pi_{[f]} : \mathbb{P}(\text{Sym}^2 V^*) \dashrightarrow \mathbb{P}(W)$ be the projection from $[f]$. Since Q is smooth, the composition $\pi_{[f]} \circ v_2$ is a closed embedding of $\mathbb{P}(V^*)$, whose image is a Veronese surface X in $\mathbb{P}(W)$. Since X is defined over k ,

the closed subscheme Y of the Grassmannian $G(1, \mathbb{P}(W_{\bar{k}}))$ of lines in $\mathbb{P}(W_{\bar{k}})$ consisting of trisecant lines to $X_{\bar{k}}$, that is, lines ℓ in $\mathbb{P}(W_{\bar{k}})$ such that $\ell \cdot X_{\bar{k}}$ is a closed subscheme of length 3 of ℓ , is defined over k , and we obtain an identification between $\text{VSP}(f)$ and Y .

Proposition 4. (see [9, §2.3]) *With the notation above, $Y = \text{VSP}(f)$ is a k -form of V_5 . Furthermore, the stratification of $Y_{\bar{k}}$ by the types of the corresponding closed subschemes of length 3 of $\mathbb{P}(V^*)$ is related to that of V_5 as a quasi-homogenous space of $\text{PGL}_2(\bar{k})$ given in § 1.1.2 as follows:*

- (i) *Points of the open orbit O correspond to reduced subschemes $\{[\ell_1], [\ell_2], [\ell_3]\}$ apolar to $Q_{\bar{k}}$ such that none of the $[\ell_i]$ belongs to the dual conic $Q_{\bar{k}}^* \subset \mathbb{P}(V_{\bar{k}}^*)$ of $Q_{\bar{k}}$.*
- (ii) *Points in the 2-dimensional orbit S_2 correspond to non-reduced subschemes $\{2[\ell_1], [\ell_2]\}$ where $[\ell_1]$ belongs to $Q_{\bar{k}}^*$ and $[\ell_2]$ is a point of the tangent line $T_{[\ell_1]}Q_{\bar{k}}^*$ to $Q_{\bar{k}}^*$ at $[\ell_1]$ distinct from $[\ell_1]$.*
- (iii) *Points in the 1-dimensional orbit C_6 correspond to non-reduced subschemes $\{3[\ell]\}$ where $[\ell]$ is a point of $Q_{\bar{k}}^*$.*

The Hilbert scheme $\mathcal{H}(Y)$ of lines on a k -form $Y = \text{VSP}(f) \subset G(1, \mathbb{P}(W))$ can be explicitly described as follows. Viewing $G(1, \mathbb{P}(W))$ as a closed subscheme of $\mathbb{P}(\Lambda^2 W)$ via the Plücker embedding, $\mathcal{H}(Y)$ is a closed subscheme of the Hilbert scheme $\mathcal{H}(G(1, \mathbb{P}(W)))$ of lines on $G(1, \mathbb{P}(W))$. The latter is isomorphic to the flag variety $\mathcal{F}(0, 2, \mathbb{P}(W)) \subset \mathbb{P}(W) \times G(2, \mathbb{P}(W))$ of pairs (x, P) consisting of a point $x \in \mathbb{P}(W)$ and a 2-dimensional linear space $P \subset \mathbb{P}(W)$ containing it.

Proposition 5. (see e.g. [10, §1.2]) *Let $f \in \text{Sym}^2 V^*$ be a homogeneous form defining a smooth conic $Q \subset \mathbb{P}(V)$, let $W = \text{Sym}^2(V^*/\langle f \rangle)$, and let $\text{VSP}(f) \subset G(1, \mathbb{P}(W))$ be the variety of trisecant lines to the Veronese surface $X = \pi_{[f]} \circ v_2(\mathbb{P}(V^*))$.*

- (i) *The projection $\text{pr}_1 : \mathcal{F}(0, 2, \mathbb{P}(W)) \rightarrow \mathbb{P}(W)$ restricts to a closed embedding $\mathcal{H}(\text{VSP}(f)) \hookrightarrow \mathbb{P}(W)$ whose image is equal to $X \simeq \mathbb{P}(V^*)$.*
- (ii) *The image of the smooth conic $C \subset \mathcal{H}(\text{VSP}(f))$ parametrizing special lines in $\text{VSP}(f)$ coincides with the conic $Q^* \subset \mathbb{P}(V^*)$ dual to $Q \subset \mathbb{P}(V)$.*
- (iii) *For every line ℓ in $\text{VSP}(f)_{\bar{k}}$ defined by a point $x \in \mathbb{P}(V_{\bar{k}}^*)$, the set of lines in $\text{VSP}(f)_{\bar{k}}$ which intersect ℓ is parametrized by the conjugate line of x with respect to $Q_{\bar{k}}^*$.*

As a consequence, we obtain the following characterization of which k -forms $Y = \text{VSP}(f)$ of V_5 contain special lines defined over k .

Corollary 6. *The k -form $Y = \text{VSP}(f)$ of V_5 contains a special line defined over k if and only if the conic $Q = V(f) \subset \mathbb{P}(V)$ has a k -rational point.*

Proof. By combining Lemma 2 b) and Proposition 5 (ii), we deduce that Y contains a special line defined over k if and only if the conic $Q^* \subset \mathbb{P}(V^*)$ dual to Q has a k -rational point, hence if and only if Q has a k -rational point. \square

Example 7. Let $k = \mathbb{C}(s, t)$ with s, t algebraically independent over \mathbb{C} , and let $f = x^2 + sy^2 + tz^2 \in k[x, y, z]$. Since the conic $Q = V(f) \subset \mathbb{P}_k^2 = \text{Proj}_k(k[x, y, z])$ has no k -rational point, the variety $Y = \text{VSP}(f)$ is a k -form of V_5 which does not contain any special line defined over k .

3. DOUBLE PROJECTION FROM A RATIONAL POINT

Let $V_5 \hookrightarrow \mathbb{P}_k^6$ be embedded by its half-anticanonical complete linear system. For every closed point $y \in V_5$, the linear system $|\mathcal{O}_{V_5}(1) \otimes \mathfrak{m}_y^2|$ of hyperplane sections of V_5 which are singular at y defines a rational map $\psi_y : V_5 \dashrightarrow \mathbb{P}_k^2$ called the *double projection from y* , whose description is classical knowledge in the birational geometry of threefolds [8, 15]. In this section, inspired by [24], we give an explicit “reverse construction” of these maps, valid over any field k of characteristic zero, formulated in terms of Sarkisov links performed from certain locally trivial \mathbb{P}^1 -bundles over \mathbb{P}^2 associated to standard birational quadratic involutions of \mathbb{P}^2 .

3.1. Recollection on standard quadratic involutions of \mathbb{P}^2 . Let $\sigma : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$ be a birational quadratic involution of \mathbb{P}_k^2 and let $\Gamma_\sigma \subset \mathbb{P}_k^2 \times \mathbb{P}_k^2$ be its graph. Via the two projections $\tau = \text{pr}_1 : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$ and $\tau' = \text{pr}_2 : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$, Γ_σ is canonically identified with the blow-up of the scheme-theoretic base loci $Z = \text{Bs}(\sigma)$ and $Z' = \text{Bs}(\sigma^{-1})$ respectively. We denote by $e = \tau^* Z$ and $e' = \tau'^* Z'$ the scheme-theoretic inverse images

of Z and Z' respectively in Γ_σ . It is well-known that Z and Z' are local complete intersection 0-dimensional closed sub-scheme of length 3 of \mathbb{P}_k^2 with one of the following possible structure:

(Type I) Z (resp. Z') is smooth and its base extension to \bar{k} consists of three non-collinear points p_1, p_2, p_3 (resp. p'_1, p'_2, p'_3) whose union is defined over k . The surface Γ_σ is smooth, $e_{\bar{k}} = e_{p_1} + e_{p_2} + e_{p_3}$, $e'_k = e'_{p'_1} + e'_{p'_2} + e'_{p'_3}$ where the e_{p_i} and $e'_{p'_i}$ are (-1) -curves.

(Type II) Z (resp. Z') consists of the disjoint union (p_1, \mathbf{p}_2) (resp. (p'_1, \mathbf{p}'_2)) of a smooth k -rational point p_1 (resp. p'_1) and a 0-dimensional sub-scheme \mathbf{p}_2 (resp. \mathbf{p}'_2) of length 2 locally isomorphic to $V(x, y^2)$ and supported at a k -rational point p_2 (resp. p'_2). We have $e = e_{p_1} + 2e_{p_2}$, $e' = e'_{p'_1} + 2e'_{p'_2}$ where e_{p_1} and e_{p_2} (resp. $e'_{p'_1}$ and $e'_{p'_2}$) are smooth k -rational curves with self-intersections -1 and $-\frac{1}{2}$ respectively, and Γ_σ has a unique A_1 -singularity at the k -rational point $e_{p_2} \cap e'_{p'_2}$.

(Type III) Z (resp. Z') is supported on a unique k -rational point p (resp. p') and is locally isomorphic to $V(y^3, x - y^2)$. We have $e = 3e_p$ and $e' = 3e'_{p'}$ where e_p and $e'_{p'}$ are smooth k -rational curves with self-intersection $-\frac{1}{3}$, and Γ_σ has a unique A_2 -singularity at the k -rational point $e_p \cap e'_{p'}$.

The following lemma whose proof is left to the reader records for later use some additional basic properties of the surfaces Γ_σ .

Lemma 8. *With the notation above, the following hold:*

- a) *The canonical divisor K_{Γ_σ} is linearly equivalent to $-e - e'$.*
- b) *The pull-back by $\tau : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$ (resp. $\tau' : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$) of a general line $\ell \simeq \mathbb{P}_k^1$ in \mathbb{P}_k^2 is \mathbb{Q} -linearly equivalent to $\frac{1}{3}(2e + e')$ (resp. $\frac{1}{3}(e + 2e')$).*

3.2. Locally trivial \mathbb{P}^1 -bundles over \mathbb{P}^2 associated to quadratic involutions. Let $\sigma : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$ be a birational quadratic involution of \mathbb{P}_k^2 with graph $\Gamma_\sigma \subset \mathbb{P}_k^2 \times \mathbb{P}_k^2$, and let $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}_k^2}$ be the ideal sheaf of its scheme-theoretic base locus Z .

Lemma 9. *There exists a locally free sheaf \mathcal{E} of rank 2 on \mathbb{P}_k^2 and an exact sequence*

$$(3.1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}_k^2}(-1) \xrightarrow{s} \mathcal{E} \rightarrow \mathcal{I}_Z \rightarrow 0.$$

Proof. Since Z is a local complete intersection of codimension 2 in \mathbb{P}_k^2 , the existence of \mathcal{E} with the required properties follows from Serre correspondence [23]. More precisely, the local-to-global spectral sequence

$$E_2^{p,q} = H^p(\mathbb{P}_k^2, \mathcal{E}xt^q(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}_k^2}(-1))) \Rightarrow \text{Ext}^{p+q}(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}_k^2}(-1))$$

degenerates to a long exact sequence

$$0 \rightarrow H^1(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(-1)) \rightarrow \text{Ext}^1(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}_k^2}(-1)) \rightarrow H^0(Z, \det \mathcal{N}_{Z/\mathbb{P}_k^2} \otimes \mathcal{O}_Z(-1)) \rightarrow H^2(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(-1)).$$

Since $H^i(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(-1)) = 0$ for $i = 1, 2$ and Z is 0-dimensional, this sequence provides an isomorphism

$$\text{Ext}^1(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}_k^2}(-1)) \simeq H^0(Z, \det \mathcal{N}_{Z/\mathbb{P}_k^2} \otimes \mathcal{O}_Z(-1)) \simeq H^0(Z, \mathcal{O}_Z).$$

The extension corresponding via this isomorphism to the constant section $1 \in H^0(Z, \mathcal{O}_Z)$ has the desired property. \square

Let $\pi : \mathbb{P}(\mathcal{E}) = \text{Proj}(\text{Sym} \cdot \mathcal{E}) \rightarrow \mathbb{P}_k^2$ be the locally trivial \mathbb{P}^1 -bundle associated with the locally free sheaf \mathcal{E} of rank 2 as in Lemma 9. Since $\det \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}_k^2}(-1)$, the canonical sheaf $\omega_{\mathbb{P}(\mathcal{E})}$ of $\mathbb{P}(\mathcal{E})$ is isomorphic to

$$(3.2) \quad \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2) \otimes \pi^* \det \mathcal{E} \otimes \pi^* \omega_{\mathbb{P}_k^2} \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^2}(-4).$$

The surjection $\mathcal{E} \rightarrow \mathcal{I}_Z \rightarrow 0$ defines a closed embedding $\mathbb{P}(\mathcal{I}_Z) = \text{Proj}(\text{Sym} \cdot \mathcal{I}_Z) \hookrightarrow \mathbb{P}(\mathcal{E})$. Since Z is a local complete intersection, the canonical homomorphism of graded $\mathcal{O}_{\mathbb{P}_k^2}$ -algebras $\text{Sym} \cdot \mathcal{I}_Z \rightarrow \mathcal{R}(\mathcal{I}_Z) = \bigoplus_{n \geq 0} \mathcal{I}_Z \cdot t^n$ is an isomorphism [17, Théorème 1], and it follows that the restriction $\pi : \mathbb{P}(\mathcal{I}_Z) \rightarrow \mathbb{P}_k^2$ is isomorphic to the blow-up $\tau : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$ of Z . We can thus identify from now on Γ_σ with the closed sub-scheme $\mathbb{P}(\mathcal{I}_Z)$ of $\mathbb{P}(\mathcal{E})$. The composition of $\pi^* s : \pi^* \mathcal{O}_{\mathbb{P}_k^2}(-1) \rightarrow \pi^* \mathcal{E}$ with the canonical surjection $\pi^* \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ defines a global section

$$\bar{s} \in \text{Hom}_{\mathbb{P}(\mathcal{E})}(\pi^* \mathcal{O}_{\mathbb{P}_k^2}(-1), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \simeq H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^2}(1))$$

whose zero locus $V(\bar{s})$ coincides with Γ_σ .

Letting ξ and A be the classes in the divisor class group of $\mathbb{P}(\mathcal{E})$ of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and of the inverse image of a general line in \mathbb{P}_k^2 by π respectively, we have $\Gamma_\sigma = V(\bar{s}) \sim \xi + A$. On the other hand, it follows from (3.2) that the canonical divisor $K_{\mathbb{P}(\mathcal{E})}$ of $\mathbb{P}(\mathcal{E})$ is linearly equivalent to $-2(\Gamma_\sigma + A) \sim -2(\xi + 2A)$. Since $c_1(\mathcal{E}) = -1$ and $c_2(\mathcal{E}) = 3$ by construction, we derive the following numerical information:

$$(3.3) \quad K_{\mathbb{P}(\mathcal{E})}^3 = -32, \quad K_{\mathbb{P}(\mathcal{E})}^2 \cdot \Gamma_\sigma = 4, \quad K_{\mathbb{P}(\mathcal{E})} \cdot \Gamma_\sigma^2 = 2, \quad \Gamma_\sigma^3 = -2.$$

3.3. Construction of Sarkisov links.

Proposition 10. *Let $\pi : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}_k^2$ be the \mathbb{P}^1 -bundle associated to a quadratic involution $\sigma : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$ with graph $\Gamma_\sigma \subset \mathbb{P}(\mathcal{E})$ as in §3.2. Then there exists a k -form Y of V_5 and a Sarkisov link*

$$\begin{array}{ccccc} \Gamma_\sigma & \xrightarrow{\quad} & \mathbb{P}(\mathcal{E}) & \xrightarrow{\quad \varphi \quad} & X^+ & \xleftarrow{\quad} & \Gamma_\sigma^+ \\ & \searrow \tau & \swarrow \pi & \searrow & \swarrow \theta & & \searrow \theta|_{\Gamma^+} \\ & & \mathbb{P}_k^2 & & X_0 & & Y & \xleftarrow{\quad} & \{y\} \end{array}$$

where:

- (i) $\varphi : \mathbb{P}(\mathcal{E}) \dashrightarrow X^+$ is a flop whose flopping locus coincides with the support of $e' \subset \Gamma_\sigma$,
- (ii) $\Gamma_\sigma^+ \simeq \mathbb{P}_k^2$ is the proper transform of Γ_σ and $\theta|_{\Gamma_\sigma^+} : \Gamma_\sigma \dashrightarrow \Gamma_\sigma^+$ is the contraction of e' ,
- (iii) $\theta : X^+ \rightarrow Y$ is the divisorial contraction of Γ_σ^+ to a smooth k -rational point $y \in Y$,
- (iv) The support of the image in Y of the flopped locus $e^+ \subset X^+$ of φ coincides with the union of the lines in $Y_{\bar{k}}$ passing through y .

Proof. We denote $\mathbb{P}(\mathcal{E})$ and $\Gamma_\sigma \subset \mathbb{P}(\mathcal{E})$ simply by X and Γ . Since $-K_X \sim 2\xi + 4A \sim 2\Gamma + 2A$ we see that $-K_X$ is nef and that any irreducible curve $C \subset X_{\bar{k}}$ such that $-K_{X_{\bar{k}}} \cdot C \leq 0$ is contained in $\Gamma_{\bar{k}}$. By the adjunction formula and Lemma 8 a), we have

$$\Gamma^2 = -K_\Gamma - 2A \cdot \Gamma \sim_{\mathbb{Q}} \frac{1}{3}(-e + e') \quad \text{and} \quad \Gamma^2 - K_\Gamma \sim_{\mathbb{Q}} \frac{2}{3}(e + 2e'),$$

which implies by adjunction again that

$$-K_{X_{\bar{k}}} \cdot C = (\Gamma_{\bar{k}}^2 - K_{\Gamma_{\bar{k}}}) \cdot C = \frac{2}{3}(e_{\bar{k}} + 2e'_{\bar{k}}) \cdot C.$$

Since $e + 2e'$ is τ -ample and τ' -numerically trivial by virtue of Lemma 8 b), we conclude that the irreducible curves $C \subset X_{\bar{k}}$ such that $-K_{X_{\bar{k}}} \cdot C = 0$ are precisely the irreducible components of the exceptional locus $e'_{\bar{k}}$ of $\tau'_{\bar{k}} : \Gamma_{\bar{k}} \rightarrow \mathbb{P}_k^2$ (see § 3.1 for the notation).

Let $\varphi : X \dashrightarrow X^+$ be the flop of the union of the irreducible components of $e'_{\bar{k}}$. Since the union of these components is defined over k , so is the union e^+ of the flopped curves of φ , and so X^+ is a smooth threefold defined over k and φ is a birational map defined over k restricting to an isomorphism between $X \setminus e'$ and $X^+ \setminus e^+$. Let Γ^+ and A^+ be the proper transforms in X^+ of Γ and A respectively. By construction, the restriction $\varphi|_\Gamma : \Gamma \dashrightarrow \Gamma^+$ coincide with $\tau' : \Gamma \rightarrow \mathbb{P}_k^2$. Since $K_{X^+} \sim -2(\Gamma^+ + A^+)$, we deduce from the adjunction formula that

$$-(\Gamma^+)^2 = K_{\Gamma^+} + 2A^+ \cdot \Gamma^+ = \tau'_*(K_\Gamma + 2A \cdot \Gamma) = \frac{1}{3}\tau'_*e,$$

which is linearly equivalent to a line $\ell \simeq \mathbb{P}_k^1$ in $\Gamma^+ \simeq \mathbb{P}_k^2$. The normal bundle $\mathcal{N}_{\Gamma^+/X^+}$ of Γ^+ in X^+ is thus isomorphic to $\mathcal{O}_{\mathbb{P}_k^2}(-1)$.

The divisor class group of X^+ is freely generated by Γ^+ and A^+ . The Mori cone $\text{NE}(X^+)$ is spanned by two extremal rays: one R_1 corresponding to the flopped curves of φ and a second one R_2 which is K_{X^+} -negative. Since $\Gamma^+ \cdot C \geq 0$ for any irreducible curve not contained in Γ^+ , including thus the irreducible components of e^+ , whereas $\Gamma^+ \cdot \ell = -1$ for any line $\ell \simeq \mathbb{P}_k^1$ in $\Gamma^+ \simeq \mathbb{P}_k^2$, it follows that R_2 is generated by the class of ℓ . The extremal contraction associated to R_2 is thus the divisorial contraction $\theta : X^+ \rightarrow Y$ of $\Gamma^+ \simeq \mathbb{P}_k^2$ to a smooth k -rational point p of a smooth projective threefold Y .

Since $-K_{X^+} = 2(\Gamma^+ + A^+)$, we conclude that the image H of A^+ by θ is an ample divisor on Y generating the divisor class group of Y and such that $-K_Y = 2H$. Furthermore, we have

$$\begin{aligned} K_Y^3 &= \theta^*(K_Y)^3 = (K_{X^+} - 2\Gamma^+)^3 \\ &= K_{X^+}^3 - 6K_{X^+}^2 \cdot \Gamma^+ + 12K_{X^+} \cdot (\Gamma^+)^2 - 8(\Gamma^+)^3 \\ &= K_X^3 - 6K_X^2 \cdot \Gamma + 12K_X \cdot \Gamma - 8 \end{aligned}$$

so that $K_Y^3 = -40$ by (3.3). Altogether, this shows $Y_{\bar{k}}$ is a smooth Fano threefold of Picard number 1, index 2 and degree $d = H^3 = 5$, hence is isomorphic to V_5 .

The fact that the support of the image in Y of e^+ coincides with the union of the lines in $Y_{\bar{k}}$ passing through p is clear by construction. \square

By construction, the proper transform by the reverse composition $\psi_y = \pi \circ \varphi^{-1} \circ \theta^{-1} : Y \dashrightarrow \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^2$ of the complete linear system of lines in \mathbb{P}_k^2 consists of divisors H on Y singular at y and such that $-K_Y = 2H$. This shows that $\psi_{y,\bar{k}} : Y_{\bar{k}} \simeq V_5 \dashrightarrow \mathbb{P}_k^2$ coincides with the double projection from the point y . Conversely, given any k -form Y of V_5 , Corollary 3 ensures that $-K_Y$ is divisible, equal to $2H$ for some ample divisor H on Y . So given any k -rational point $y \in Y$, the double projection $\psi_y : Y \dashrightarrow \mathbb{P}_k^2$ from y is defined over k , given by the linear system $|\mathcal{O}_Y(H) \otimes \mathfrak{m}_y^2|$, and it coincides with the composition $\pi \circ \varphi^{-1} \circ \theta^{-1} : Y \dashrightarrow \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^2$ for a suitable quadratic birational involution σ of \mathbb{P}_k^2 .

4. APPLICATION : CYLINDERS IN FORMS OF THE QUINTIC DEL PEZZO THREEFOLD

Theorem 11. *Let Y be a k -form of V_5 , let $y \in Y$ be a k -rational point and let C be the union of the lines in $Y_{\bar{k}}$ passing through y . Then $Y \setminus C$ has the structure of a Zariski locally trivial \mathbb{A}^1 -bundle $\rho : Y \setminus C \rightarrow \mathbb{P}_k^2 \setminus Z$ over the complement of a closed sub-scheme $Z \subset \mathbb{P}_k^2$ of length 3 with as many irreducible geometric components as C .*

Proof. Indeed, the birational map $\xi = \theta \circ \varphi : \mathbb{P}(\mathcal{E}) \dashrightarrow Y$ constructed in Proposition 10 restricts to an isomorphism between $Y \setminus C$ and the complement of the proper transform Γ in $\mathbb{P}(\mathcal{E})$ of the exceptional divisor of the blow-up of Y at y . On the other hand, it follows from the construction of \mathcal{E} in § 3.2 that Γ is the graph of a quadratic birational involution σ of \mathbb{P}_k^2 and that the restriction of $\pi : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}_k^2$ to the complement of Γ is a locally trivial \mathbb{A}^1 -bundle over the complement $\mathbb{P}_k^2 \setminus Z$ of the base locus Z of σ . \square

Theorem 12. *Every k -form Y of V_5 contains a Zariski open \mathbb{A}_k^2 -cylinder. Furthermore, Y contains \mathbb{A}_k^3 as a Zariski open subset if and only if it contains a special line defined over k .*

Proof. By Lemma 2, Y contains a line $\ell \simeq \mathbb{P}_k^1$ with trivial normal bundle. Projecting from ℓ as in the proof of Corollary 3, we obtain a birational map $Y \dashrightarrow Q$ onto a smooth quadric $Q \subset \mathbb{P}_k^4$ defined over k , which restricts to an isomorphism between the complement of the surface $Z \subset Y$ defined over k swept out by the lines in $Y_{\bar{k}}$ intersecting $\ell_{\bar{k}}$ and the complement of a hyperplane section $Q_0 \simeq \mathbb{P}_k^1 \times \mathbb{P}_k^1$ of Q . The complement $Q \setminus Q_0$ is isomorphic to the smooth affine quadric $\tilde{Q}_0 = \{xv - yu = 1\} \subset \mathbb{A}_k^4$, which contains for instance the principal open \mathbb{A}_k^2 -cylinder $\tilde{Q}_{0,x} \simeq \text{Spec}(k[x^{\pm 1}][y, u])$.

By Lemma 2 b), Y contains a special line if and only if it has a k -rational point p through which there exists at most two lines in $Y_{\bar{k}}$. Letting C be the union of these lines, the previous theorem implies that $\rho : Y \setminus C \rightarrow \mathbb{P}_k^2 \setminus Z$ is a Zariski locally trivial \mathbb{A}^1 -bundle over the complement of a 0-dimensional closed subset Z defined over k whose support consists of at most two points. Letting $L \simeq \mathbb{P}_k^1$ be any line in \mathbb{P}_k^2 containing the support of Z , the restriction of ρ over $\mathbb{P}_k^2 \setminus L \simeq \mathbb{A}_k^2$ is then a trivial \mathbb{A}^1 -bundle, so that $Y \setminus \rho^{-1}(L)$ is a Zariski open subset of Y isomorphic to \mathbb{A}_k^3 .

Conversely, suppose that Y contains a Zariski open subset $U \simeq \mathbb{A}_k^3$. Since U is affine, $D = Y \setminus U$ has pure codimension 1 in Y . Since $Y_{\bar{k}} \simeq V_5$ and $Y_{\bar{k}} \setminus D_{\bar{k}} \simeq U_{\bar{k}} \simeq \mathbb{A}_k^3$, according to [7, Corollary 1.2, b] and [8], we have the following alternative for the divisor $D_{\bar{k}}$ in $Y_{\bar{k}}$:

- 1) $D_{\bar{k}}$ is a normal del Pezzo surface of degree 5 with a unique singular point q of type A_4 ,
- 2) $D_{\bar{k}}$ is a non-normal del Pezzo surface of degree 5 whose singular locus is a special line ℓ in $Y_{\bar{k}}$, and $D_{\bar{k}}$ is swept out by lines in V_5 which intersect ℓ .

In the first case, there exists a unique special line ℓ in $Y_{\bar{k}}$ passing through q , which is then automatically contained in $D_{\bar{k}}$. Since D is defined over k and q is the unique singular point of $D_{\bar{k}}$, q corresponds to a

k -rational point of D , so that ℓ is actually a special line in Y defined over k . In the second case, since ℓ is the singular locus of $D_{\bar{k}}$, it is invariant under the action of the Galois group $\text{Gal}(\bar{k}/k)$, hence is defined over k . So in both cases, Y contains a special line defined over k . \square

Corollary 13. *Every form of V_5 defined over a C_1 -field k contains \mathbb{A}_k^3 as a Zariski open subset.*

Proof. This follows from Lemma 2 b) and the previous theorem since every conic over a C_1 -field has a rational point. \square

Example 14. The $\mathbb{C}(s, t)$ -form $Y = \text{VSP}(f)$ of V_5 associated to the quadratic form $f = x^2 + sy^2 + tz^2 \in \mathbb{C}(s, t)[x, y, z]$ considered in Example 7 does not contain any special line defined over $\mathbb{C}(s, t)$, hence does not contain $\mathbb{A}_{\mathbb{C}(s, t)}^3$ as a Zariski open subset.

REFERENCES

1. C. Birkar, P. Cascini, C.D. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, Journal of AMS, 23 (2010), 405-468.
2. A. Dubouloz and T. Kishimoto, *Explicit biregular/birational geometry of affine threefolds: completions of \mathbb{A}^3 into del Pezzo fibrations and Mori conic bundles*, Adv. Stud. Pure Math. 75, Mathematical Society of Japan, Tokyo, 2017, 49-71.
3. A. Dubouloz and T. Kishimoto, *Cylinders in del Pezzo fibrations*, arXiv:1607.00840v1 [math.AG], 2016, Israel Journal of Mathematics DOI 10.1007/s11856-018-1679-z (to appear).
4. A. Dubouloz and T. Kishimoto, *Deformations of \mathbb{A}^1 -cylindrical varieties*, arXiv:1710.09108v1 [math.AG] 25 Oct 2017.
5. M. Furushima, *Singular del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space \mathbb{C}^3* , Nagoya Math. J., 104 (1986), 1-28.
6. M. Furushima, *The complete classification of compactifications of \mathbb{C}^3 which are projective manifolds with the second Betti number one*, Math. Ann., 297 (1993), 627-662.
7. M. Furushima and N. Nakayama, *The family of lines on the Fano threefolds V_5* , Nagoya Math. J., 116 (1989), 111-122.
8. M. Furushima and N. Nakayama, *A new construction of a compactification of \mathbb{C}^3* , Tôhoku Math. J., 41 (1989), 543-560.
9. A. Iliev, *Lines on the Gushel' threefold*, Indag. Mathem., N.S., 5 (3) (1994), 307-320.
10. A. Iliev, *The Fano surface of the Gushel threefold*, Compositio Mathematica 94, n^o1 (1994), 81-107.
11. V.A. Iskovskikh, *Anticanonical models of three-dimensional algebraic varieties*, J Math Sci (1980) 13 (6), 745-814.
12. T. Kishimoto, Y. Prokhorov and M. Zaidenberg, *Group actions on affine cones*. Affine algebraic geometry, 123-163. Peter Russell's Festschrift, CRM Proc. Lecture Notes, 54, Amer. Math. Soc., Providence, RI, 2011.
13. T. Kishimoto, Y. Prokhorov and M. Zaidenberg, *\mathbb{G}_a -actions on affine cones*, Transformation Groups, 18 (2013), 1137-1153.
14. T. Kishimoto, Y. Prokhorov and M. Zaidenberg, *Affine cones over Fano threefolds and additive group actions*, Osaka J. Math., 51 (2014), 1093-1112.
15. I.Y. Kolpakov-Miroshnichenko and Y.G. Prokhorov, *Rationality construction of fields of invariants of some finite four-dimensional linear groups associated with Fano threefolds*, Math Notes 51 (1) (1992), 74-76.
16. A. G. Kuznetsov, Y. G. Prokhorov and C. A. Shramov, *Hilbert schemes of lines and conics and automorphism groups of Fano threefolds*, Jpn. J. Math. (2018) 13 (1), 109-185.
17. A. Micali, *Sur les algèbres universelles*, Annales de l'Institut Fourier, Volume 14 (1964) no. 2, p. 33-87.
18. S. Mori and S. Mukai, *On Fano 3-folds with $B_2 \geq 2$* , Advanced Studies in Pure Mathematics, 1 (1983), 101-129.
19. S. Mukai and H. Umemura, *Minimal rational threefolds*, Lect. Notes Math., 1016 (1983), 490-518.
20. Yu. Prokhorov, *Fano threefolds of genus 12 and compactifications of \mathbb{C}^3* , St. Petersburg Math. J., 3 (1992), 855-864.
21. Y. Prokhorov, M. Zaidenberg, *Examples of cylindrical Fano fourfolds*, Eur. J. Math., 2 (2016), 262-282.
22. Y. Prokhorov, M. Zaidenberg, *New examples of cylindrical Fano fourfolds*, New examples of cylindrical Fano fourfolds, Adv. Stud. Pure Math. 75, Mathematical Society of Japan, Tokyo, 2017, 443-463.
23. J.P. Serre, *Sur les modules projectifs*, Séminaire Dubreil-Pisot (1960/61), Secr. Math. Paris, exposé 2, (1961).
24. K. Takeuchi, *Some birational maps of Fano 3-folds*, Compositio Math., 71 (1989), 265-283.

IMB UMR5584, CNRS, UNIV. BOURGOGNE FRANCHE-COMTÉ, F-21000 DIJON, FRANCE
E-mail address: adrien.dubouloz@u-bourgogne.fr

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, SAITAMA UNIVERSITY, SAITAMA 338-8570, JAPAN
E-mail address: tkishimo@rimath.saitama-u.ac.jp