



Cylinders in Mori Fiber Spaces: forms of the quintic del Pezzo threefold

Adrien Dubouloz, Takashi Kishimoto

► To cite this version:

Adrien Dubouloz, Takashi Kishimoto. Cylinders in Mori Fiber Spaces: forms of the quintic del Pezzo threefold . 2018. hal-01790674

HAL Id: hal-01790674

<https://hal.science/hal-01790674v1>

Preprint submitted on 13 May 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

CYLINDERS IN MORI FIBER SPACES: FORMS OF THE QUINTIC DEL PEZZO THREEFOLD

ADRIEN DUBOULOZ AND TAKASHI KISHIMOTO

ABSTRACT. Motivated by the general question of existence of open \mathbb{A}^1 -cylinders in higher dimensional projective varieties, we consider the case of Mori Fiber Spaces of relative dimension three, whose general closed fibers are isomorphic to the quintic del Pezzo threefold V_5 , the smooth Fano threefold of index two and degree five. We show that the total spaces of these Mori Fiber Spaces always contain relative \mathbb{A}^2 -cylinders, and we characterize those admitting relative \mathbb{A}^3 -cylinders in terms of the existence of certain special lines in their generic fibers.

INTRODUCTION

An \mathbb{A}_k^r -cylinder in a normal algebraic variety defined over a field k is a Zariski open subset U isomorphic to $Z \times \mathbb{A}_k^r$ for some algebraic variety Z defined over k . In the case where $k = \bar{k}$ is algebraically closed, normal projective varieties V containing \mathbb{A}_k^1 -cylinders have received a lot of attention recently due to the connection between unipotent group actions on their affine cones and polarized \mathbb{A}_k^1 -cylinders in them, that is \mathbb{A}_k^1 -cylinders whose complements are the supports of effective \mathbb{Q} -divisors linearly equivalent to an ample divisor on V (cf. [12, 13]). Certainly, the canonical divisor K_V of a normal projective variety containing an \mathbb{A}_k^r -cylinder for some $r \geq 1$ is not pseudo-effective. Replacing V if necessary by a birational model with at most \mathbb{Q} -factorial terminal singularities, [1] guarantees the existence of a suitable K_V -MMP $V \dashrightarrow X$ whose output X is equipped with a structure of Mori Fiber Space $f : X \rightarrow Y$ over some lower dimensional normal projective variety Y . Since an \mathbb{A}_k^1 -cylinder in X can always be transported back in the initial variety V [4, Lemma 9], total spaces of Mori Fiber Spaces form a natural restricted class in which to search for varieties containing \mathbb{A}_k^1 -cylinders.

In the case where $\dim Y = 0$, X is a Fano variety of Picard number one. The only smooth Fano surface of Picard number is the projective plane \mathbb{P}_k^2 which obviously contains \mathbb{A}_k^1 -cylinders. Several families of examples of smooth Fano varieties of dimension 3 and 4 and Picard number one containing \mathbb{A}_k^1 -cylinders have been constructed [14, 21, 22]. The question of existence of \mathbb{A}_k^1 -cylinders in other possible outputs of MMPs was first considered in [2, 3], in which del Pezzo fibrations $f : X \rightarrow Y$, which correspond to the case where $\dim Y = \dim X - 2 > 0$, were extensively studied. In such a relative context, it is natural to shift the focus to cylinders which are compatible with the fibration structure:

Definition. Let $f : X \rightarrow Y$ be a morphism between normal algebraic varieties defined over a field k and let $U \simeq Z \times \mathbb{A}_k^r$ be an \mathbb{A}_k^r -cylinder inside X . We say that U is *vertical with respect to f* if the restriction $f|_U$ factors as

$$f|_U = h \circ \text{pr}_Z : U \simeq Z \times \mathbb{A}_k^r \xrightarrow{\text{pr}_Z} Z \xrightarrow{h} Y$$

for a suitable morphism $h : Z \rightarrow Y$.

In the present article, we initiate the study of existence of vertical \mathbb{A}^1 -cylinders in Mori Fiber Spaces $f : X \rightarrow Y$ of relative dimension three, whose general fibers are smooth Fano threefolds. Each smooth Fano threefold of Picard number one can appear as a closed fiber of a Mori Fiber Space, and among these,

2000 *Mathematics Subject Classification.* 14E30; 14J30; 14J45; 14R10; 14R25;

Key words and phrases. Fano threefold, Mori Fiber Space, Sarkisov link; Cremona involutions; Cylinders.

The second author was partially funded by Grant-in-Aid for Scientific Research of JSPS No. 15K04805. The research was initiated during a visit of the first author at Saitama University and continued during a stay of the second author at University of Burgundy as a CNRS Research Fellow. The authors thank these institutions for their generous support and the excellent working conditions offered.

it is natural to first restrict to classes which contain a cylinder of the maximal possible dimension, namely the affine space \mathbb{A}_k^3 , and expect that some suitable sub-cylinders of these fiber wise maximal cylinders could arrange into a vertical cylinder with respect to f , possibly of smaller relative dimension. The only four classes of Fano threefold of Picard number one containing \mathbb{A}_k^3 are \mathbb{P}_k^3 , the quadric \mathbb{Q}^3 , the del Pezzo quintic threefold V_5 of index two and degree five [7, 8], and a four dimensional family of prime Fano threefolds V_{22} of genus twelve [6, 20].

The existence inside X of a vertical \mathbb{A}^r -cylinder with respect to $f : X \rightarrow Y$ translates equivalently into that of an \mathbb{A}_K^r -cylinder inside the fiber X_η of f over the generic point η of Y , considered as a variety defined over the function field K of Y [3]. We are therefore led to study the existence of \mathbb{A}_K^r -cylinders inside K -forms of the aforementioned Fano threefolds over non-closed fields K of characteristic zero, that is, smooth projective varieties defined over K whose base extensions to an algebraic closure \overline{K} are isomorphic over \overline{K} to one of these Fano threefolds. The case of \mathbb{P}^3 is easily dispensed: a K -form V of \mathbb{P}^3 contains an \mathbb{A}_K^3 -cylinder if and only if it has a K -rational point hence if and only if it is the trivial K -form \mathbb{P}_K^3 . So equivalently, a Mori Fiber Space $f : X \rightarrow Y$ whose general closed fibers are isomorphic to \mathbb{P}_k^3 contains a vertical \mathbb{A}_k^3 -cylinder if and only if it has a rational section. The case of the quadric \mathbb{Q}^3 is already more intricate: one can deduce from [6] that a K -form V of \mathbb{Q}^3 contains \mathbb{A}_K^3 if and only if it has a hyperplane section defined over K which is a K -rational quadric cone. In this article, we establish a complete characterization of the existence of \mathbb{A}_K^r -cylinders in forms of V_5 which can be summarized as follows:

Theorem. *Let K be a field of characteristic zero and let Y be a K -form of V_5 . Then Y always contains an \mathbb{A}_K^2 -cylinder, and it contains an \mathbb{A}_K^3 -cylinder if and only if it contains a curve $\ell \simeq \mathbb{P}_K^1$ of anticanonical degree $-K_Y \cdot \ell = 2$ and with normal bundle $\mathcal{N}_{\ell/Y} \simeq \mathcal{O}_{\mathbb{P}_K^1}(-1) \oplus \mathcal{O}_{\mathbb{P}_K^1}(1)$.*

An irreducible curve ℓ of anticanonical degree $-K_Y \cdot \ell = 2$ on a K -form Y of V_5 becomes after base extension to an algebraic closure \overline{K} of K a usual line on $Y_{\overline{K}} \simeq V_5$ embedded into $\mathbb{P}_{\overline{K}}^6$ via its half-anticanonical complete linear system. By combining the previous characterization with a closer study of the Hilbert scheme of such curves ℓ on Y (see §2.1), we derive the following result (see Corollary 13):

Corollary. *Let \overline{k} be an algebraically closed field of characteristic zero and let $f : X \rightarrow C$ be a Mori Fiber Space over a curve C defined over \overline{k} , whose general closed fibers are quintic del Pezzo threefolds V_5 . Then X contains a vertical $\mathbb{A}_{\overline{k}}^3$ -cylinder with respect to f .*

Section 1 contains a brief recollection on the quintic del Pezzo threefold V_5 and its Hilbert scheme of lines. In Section 2, we establish basic geometric properties of forms of V_5 and describe their Hilbert schemes of lines. We also describe an adaptation to non-closed fields of a standard construction of V_5 as the variety of trisecant lines to a Veronese surface in \mathbb{P}^4 , from which we derive for suitable fields the existence of forms of V_5 which do not contain any line with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. The technical core of the article is then Section 3: we give a new construction of a classical rational map, called the double projection from a rational point of a form of V_5 in the form of a Sarkisov link from a \mathbb{P}^1 -bundle over \mathbb{P}^2 explicitly determined by the base locus of a quadratic birational involution of \mathbb{P}^2 . The main results concerning \mathbb{A}^r -cylinders in forms of V_5 are then derived from this construction in Section 4.

1. GEOMETRY OF THE SMOOTH QUINTIC DEL PEZZO THREEFOLD

In the rest of this article, unless otherwise stated, the notations k and \overline{k} refer respectively to a field of characteristic zero and a fixed algebraic closure of k . In this section, we recall without proof classical descriptions and properties of the quintic del Pezzo threefold V_5 over \overline{k} and of its Hilbert scheme of lines.

1.1. Two classical descriptions of V_5 . A quintic del Pezzo threefold V_5 over \overline{k} is a smooth projective threefold whose Picard group is isomorphic to \mathbb{Z} , generated by an ample class H such that $-K_{V_5} = 2H$ and $H^3 = 5$. In other words, V_5 is a smooth Fano threefold of index two and degree five.

1.1.1. Sarkisov links to a smooth quadric in \mathbb{P}^4 . Let us first recall the classical description due to Iskovskikh [11, Chapter II, §1.6]. Letting H be an ample class such that $-K_{V_5} = 2H$, the complete linear system $|H|$ defines a closed embedding $\Phi_{|H|} : V_5 \hookrightarrow \mathbb{P}_{\overline{k}}^6$. A general hyperplane section of V_5 contains a line ℓ whose

normal bundle in V_5 is trivial, and the projection from ℓ induces a birational map $V_5 \dashrightarrow Q$ onto a smooth quadric $Q \subset \mathbb{P}_k^4$. The inverse map $Q \dashrightarrow V_5$ can be described as the blow-up of Q along a rational normal cubic $C \subset Q$ contained in a smooth hyperplane section $Q_0 \simeq \mathbb{P}_k^1 \times \mathbb{P}_k^1$ of Q , followed by the contraction of the proper transform Q'_0 of Q_0 onto the line ℓ . In sum, the projection from the line ℓ induces a Sarkisov link

$$\begin{array}{ccccc}
 Q'_0 & \hookrightarrow & \tilde{V}_5 & \xleftarrow{\quad} & Z' \\
 \swarrow & & \downarrow q & & \searrow q' \\
 \ell & \xrightarrow{\quad} & V_5 & \xrightarrow{\quad} & Q \\
 & & \text{---} & & \\
 & & |H-\ell| & &
 \end{array}$$

where $q : \tilde{V}_5 \rightarrow V_5$ is the blow-up of V_5 along ℓ with exceptional divisor Q'_0 and $q' : \tilde{V}_5 \rightarrow Q$ is the contraction onto the curve C of the proper transform Z' of the surface $Z \subset V_5$ swept out by lines in V_5 intersecting ℓ .

Since the automorphism group $\mathrm{PGL}_5(\bar{k})$ of \mathbb{P}_k^4 acts transitively on the set of flags $C \subset Q_0 \subset Q$, it follows that over an algebraically closed field \bar{k} , all smooth Fano threefold of Picard number one, index two and degree five embedded into \mathbb{P}_k^6 by their half-anticanonical complete linear system are projectively equivalent.

1.1.2. Quasi-homogeneous space of $\mathrm{PGL}_2(\bar{k})$. We now recall an alternative description of V_5 due to Mukai-Umemura [19] (see also [16, §5.1]). Let $M_d = \mathrm{Sym}^d(\bar{k}^2)^\vee \simeq \bar{k}[x, y]_{(d)}$ be the space of homogeneous polynomials of degree d with coefficients in \bar{k} . The natural action of $\mathrm{GL}_2(\bar{k})$ on M_1 induces a linear action on M_6 , hence an action of $\mathrm{PGL}_2(\bar{k})$ on $\mathbb{P}(M_6) \simeq \mathbb{P}_k^6$. We then have the following description:

Proposition 1. *The Fano threefold V_5 is isomorphic to the closure $\overline{\mathrm{PGL}_2(\bar{k}) \cdot [\phi]}$ of the class of the polynomial $\phi = xy(x^4 + y^4) \in M_6$. Furthermore, the $\mathrm{PGL}_2(\bar{k})$ -orbits on V_5 are described as follows:*

- (i) *The open orbit $O = \mathrm{PGL}_2(\bar{k}) \cdot [\phi]$ with stabilizer equal to the binary octahedral group,*
- (ii) *The 2-dimensional orbit $S_2 = \mathrm{PGL}_2(\bar{k}) \cdot [xy^5]$, which is neither open nor closed, with stabilizer equal to the diagonal torus \mathbb{T} .*
- (iii) *The 1-dimensional closed orbit $C_6 = \mathrm{PGL}_2(\bar{k}) \cdot [x^6]$ with stabilizer equal to the Borel subgroup B of upper triangular matrices.*

It follows from this description that the automorphism $\mathrm{Aut}(V_5)$ is isomorphic to $\mathrm{PGL}_2(\bar{k})$. We also observe that C_6 is a normal rational sextic curve and that the closure $\overline{S_2} = S_2 \cup C_6$ of S_2 is a quadric section of V_5 , hence an anti-canonical divisor on V_5 , which coincides with the *tangential scroll* of C_6 , swept out by the tangent lines to C_6 contained in V_5 . It is singular along C_6 , and its normalization morphism coincides with the map $\nu : \mathbb{P}(M_1) \times \mathbb{P}(M_1) \rightarrow \mathbb{P}(M_6)$, $(f_1, f_2) \mapsto f_1^5 f_2$.

1.2. Lines on V_5 . The family of lines on V_5 is very well-studied [11, 7, 10]. We list below some of its properties which will be useful later on for the study of cylinders on forms of V_5 .

First, by a *line* on V_5 , we mean an integral curve $\ell \subset V_5$ of anticanonical degree $-K_{V_5} \cdot \ell = 2$. It thus corresponds through the half-anticanonical embedding $\Phi_{|H|} : V_5 \hookrightarrow \mathbb{P}_k^6$ to a usual line in \mathbb{P}_k^6 which is contained in the image of V_5 . A general line ℓ in V_5 has trivial normal bundle, whereas there is a one-dimensional subfamily of lines with normal bundle $\mathcal{N}_{\ell/V_5} \simeq \mathcal{O}_{\mathbb{P}_k^1}(-1) \oplus \mathcal{O}_{\mathbb{P}_k^1}(1)$, which we call *special lines*.

The Hilbert scheme $\mathcal{H}(V_5)$ of lines in V_5 is isomorphic to \mathbb{P}_k^2 , and the evaluation map $v : \mathcal{U} \rightarrow V_5$ from the universal family $\mathcal{U} \rightarrow \mathcal{H}(V_5)$ is a finite morphism of degree 3. There are thus precisely three lines counted with multiplicities passing through a given closed point of V_5 .

The curve in $\mathcal{H}(V_5) \simeq \mathbb{P}_k^2$ that parametrizes special lines in V_5 is a smooth conic C . The restriction of v to $\mathcal{U}|_C$ is injective, and in the description of V_5 as a quasi-homogeneous space of $\mathrm{PGL}_2(\bar{k})$ given in § 1.1.2 above, $v(\mathcal{U}|_C)$ is the tangential scroll $\overline{S_2}$ to the rational normal sextic C_6 , while C_6 itself coincides with the image of the intersection of $\mathcal{U}|_C$ with the ramification locus of v . In particular, special lines on V_5 never intersect each others. Furthermore, there are three lines with trivial normal bundle through any point in $V_5 \setminus \overline{S_2}$, a line with trivial normal bundle and a special line through any point of S_2 , and a unique special line through every point of C_6 .

2. FORMS OF V_5 OVER NON-CLOSED FIELDS

A k -form of V_5 is a smooth projective variety Y defined over k such that $Y_{\bar{k}}$ is isomorphic to V_5 . In this subsection, we establish basic properties of these forms and their Hilbert schemes of lines.

2.1. Hilbert scheme of lines on a k -form of V_5 . Let Y be a k -form of V_5 , let $\mathcal{H}(Y)$ be the Hilbert scheme of irreducible curves of $-K_Y$ -degree equal to 2 on Y and let $v : \mathcal{U} \rightarrow Y$ be the evaluation map from the universal family $\mathcal{U} \rightarrow \mathcal{H}(Y)$. By § 1.2, $\mathcal{H}(Y)_{\bar{k}} = \mathcal{H}(Y_{\bar{k}})$ is isomorphic to $\mathbb{P}_{\bar{k}}^2$, i.e. $\mathcal{H}(Y)$ is a k -form of $\mathbb{P}_{\bar{k}}^2$. Furthermore, since the smooth conic parametrizing special lines on $Y_{\bar{k}}$ is invariant under the action of the Galois group $\text{Gal}(\bar{k}/k)$, it corresponds to a smooth curve $C \subset \mathcal{H}(Y)$ defined over k , and such that $C_{\bar{k}} \simeq \mathbb{P}_{\bar{k}}^1$.

Lemma 2. *Let Y be a k -form of V_5 .*

a) *The Hilbert scheme $\mathcal{H}(Y)$ of lines on Y is isomorphic to \mathbb{P}_k^2 , in particular Y always contains lines defined over k with trivial normal bundles.*

b) *The following assertions are equivalent:*

- (i) *Y contains a special line defined over k ,*
- (ii) *The conic C has a k -rational point,*
- (iii) *The surface $v(\mathcal{U}|_C)$ has a k -rational point,*
- (iv) *The image of the intersection of $\mathcal{U}|_C$ with the ramification locus of v has a k -rational point.*

Proof. Since $\mathcal{H}(Y)$ is a k -form of $\mathbb{P}_{\bar{k}}^2$, to prove the first assertion it is enough to show that $\mathcal{H}(Y)$ has a k -rational point. Since $C_{\bar{k}} \simeq \mathbb{P}_{\bar{k}}^1$, there exists a quadratic extension $k \subset k'$ such that $C(k')$ is nonempty, so that in particular $\mathcal{H}(Y)_{k'} \simeq \mathbb{P}_{k'}^2$. Let p be a k' -rational point of $C_{k'}$. If p is invariant under the action of the Galois group $\text{Gal}(k'/k)$, then it corresponds to a k -rational point of C , hence of $\mathcal{H}(Y)$, and we are done. Otherwise, its Galois conjugate \bar{p} is a k' -rational point of $C_{k'}$ distinct from p , and then the tangent lines $T_p C_{k'}$ and $T_{\bar{p}} C_{k'}$ to $C_{k'}$ at p and \bar{p} respectively intersect each other at unique point. The latter is thus $\text{Gal}(k'/k)$ -invariant, hence corresponds to a k -rational point of $\mathcal{H}(Y) \setminus C$. The existence of lines with trivial normal bundles defined over k then follows from the description of $\mathcal{H}(V_5)$ given in § 1.2.

The second assertion is an immediate consequence of the facts that special lines in $Y_{\bar{k}}$ are in one-to-one correspondence with closed points of the image D of the intersection of $(\mathcal{U}|_C)_{\bar{k}}$ with the ramification locus of v , and that $v(\mathcal{U}|_C)_{\bar{k}}$ coincides with the surface swept out by the tangent lines to D . \square

Corollary 3. *A k -form Y of V_5 is k -rational and the natural map $\text{Pic}(Y) \rightarrow \text{Pic}(Y_{\bar{k}})$ is an isomorphism.*

Proof. By Lemma 2 a), Y contains a line $\ell \simeq \mathbb{P}_k^1$ with trivial normal bundle. The surface Z in $Y_{\bar{k}}$ swept out by lines intersecting $\ell_{\bar{k}}$ is defined over k . As in § 1.1.1, the composition of the blow-up $q : \tilde{Y} \rightarrow Y$ of ℓ with exceptional divisor $Q'_0 \simeq \mathbb{P}_k^1 \times \mathbb{P}_k^1$ followed by the contraction $q' : \tilde{Y} \rightarrow Q$ of the proper transform of Z' of Z yields a birational map $Y \dashrightarrow Q$ defined over k onto a smooth quadric $Q \subset \mathbb{P}_k^4$, which maps Q'_0 onto a hyperplane section Q_0 of Q . Since Q contains k -rational points, it is k -rational. The natural map $\text{Pic}(Y) \rightarrow \text{Pic}(Y_{\bar{k}})$ is an isomorphism if and only $-K_Y$ is divisible in $\text{Pic}(Y)$. But since $-K_Q \sim 3Q_0$ and $q'^* Q_0 = Q'_0 + Z'$, we deduce from the ramification formula for q and q' that

$$-K_Y \sim q_*(-K_{\tilde{Y}}) \sim q_*(-q'^* K_Q - Z') \sim q_*(3Q'_0 + 2Z') \sim 2Z.$$

\square

2.2. Varieties of trisecant lines to Veronese surfaces in \mathbb{P}^4 . In this subsection, we review a third classical construction of V_5 as the variety of trisecant line to the Veronese surface in $\mathbb{P}_{\bar{k}}^4$ which, when performed over k gives rise, depending on the choices made, to nontrivial k -forms of V_5 .

Let V be a k -vector space of dimension 3, let $f \in \text{Sym}^2 V^*$ be a homogeneous form defining a smooth conic $Q \subset \mathbb{P}(V)$, and let $W = \text{Sym}^2(V^*/\langle f \rangle)$. Recall that a closed subscheme $Z \subset \mathbb{P}(V^*)$ defined over k is called *apolar* to Q if the class of $[f] \in \mathbb{P}(\text{Sym}^2 V^*)$ lies in the linear span of the image of Z by the second Veronese embedding $v_2 : \mathbb{P}(V^*) \hookrightarrow \mathbb{P}(\text{Sym}^2 V^*)$. When $Z = \{[\ell_1], [\ell_2], [\ell_3]\} \subset \mathbb{P}(V^*)$ consists of three distinct k -rational points, this says equivalently that there are scalars $\lambda_i \in k$, $i = 1, 2, 3$, such that $f = \lambda_1 \ell_1^2 + \lambda_2 \ell_2^2 + \lambda_3 \ell_3^2$. We denote by $\text{VSP}(f)$ the variety of closed subschemes of length 3 of $\mathbb{P}(V^*)$ which are apolar to Q .

Let $\pi_{[f]} : \mathbb{P}(\text{Sym}^2 V^*) \dashrightarrow \mathbb{P}(W)$ be the projection from $[f]$. Since Q is smooth, the composition $\pi_{[f]} \circ v_2$ is a closed embedding of $\mathbb{P}(V^*)$, whose image is a Veronese surface X in $\mathbb{P}(W)$. Since X is defined over k ,

the closed subscheme Y of the Grassmannian $G(1, \mathbb{P}(W_{\bar{k}}))$ of lines in $\mathbb{P}(W_{\bar{k}})$ consisting of trisecant lines to $X_{\bar{k}}$, that is, lines ℓ in $\mathbb{P}(W_{\bar{k}})$ such that $\ell \cdot X_{\bar{k}}$ is a closed subscheme of length 3 of ℓ , is defined over k , and we obtain an identification between $\text{VSP}(f)$ and Y .

Proposition 4. (see [9, §2.3]) *With the notation above, $Y = \text{VSP}(f)$ is a k -form of V_5 . Furthermore, the stratification of $Y_{\bar{k}}$ by the types of the corresponding closed subschemes of length 3 of $\mathbb{P}(V^*)$ is related to that of V_5 as a quasi-homogenous space of $\text{PGL}_2(\bar{k})$ given in § 1.1.2 as follows:*

- (i) *Points of the open orbit O correspond to reduced subschemes $\{[\ell_1], [\ell_2], [\ell_3]\}$ apolar to $Q_{\bar{k}}$ such that none of the $[\ell_i]$ belongs to the dual conic $Q_{\bar{k}}^* \subset \mathbb{P}(V_{\bar{k}}^*)$ of $Q_{\bar{k}}$.*
- (ii) *Points in the 2-dimensional orbit S_2 correspond to non-reduced subschemes $\{2[\ell_1], [\ell_2]\}$ where $[\ell_1]$ belongs to $Q_{\bar{k}}^*$ and $[\ell_2]$ is a point of the tangent line $T_{[\ell_1]}Q_{\bar{k}}^*$ to $Q_{\bar{k}}^*$ at $[\ell_1]$ distinct from $[\ell_1]$.*
- (iii) *Points in the 1-dimensional orbit C_6 correspond to non-reduced subschemes $\{3[\ell]\}$ where $[\ell]$ is a point of $Q_{\bar{k}}^*$.*

The Hilbert scheme $\mathcal{H}(Y)$ of lines on a k -form $Y = \text{VSP}(f) \subset G(1, \mathbb{P}(W))$ can be explicitly described as follows. Viewing $G(1, \mathbb{P}(W))$ as a closed subscheme of $\mathbb{P}(\Lambda^2 W)$ via the Plücker embedding, $\mathcal{H}(Y)$ is a closed subscheme of the Hilbert scheme $\mathcal{H}(G(1, \mathbb{P}(W)))$ of lines on $G(1, \mathbb{P}(W))$. The latter is isomorphic to the flag variety $\mathcal{F}(0, 2, \mathbb{P}(W)) \subset \mathbb{P}(W) \times G(2, \mathbb{P}(W))$ of pairs (x, P) consisting of a point $x \in \mathbb{P}(W)$ and a 2-dimensional linear space $P \subset \mathbb{P}(W)$ containing it.

Proposition 5. (see e.g. [10, §1.2]) *Let $f \in \text{Sym}^2 V^*$ be a homogeneous form defining a smooth conic $Q \subset \mathbb{P}(V)$, let $W = \text{Sym}^2(V^*/\langle f \rangle)$, and let $\text{VSP}(f) \subset G(1, \mathbb{P}(W))$ be the variety of trisecant lines to the Veronese surface $X = \pi_{[f]} \circ v_2(\mathbb{P}(V^*))$.*

- (i) *The projection $\text{pr}_1 : \mathcal{F}(0, 2, \mathbb{P}(W)) \rightarrow \mathbb{P}(W)$ restricts to a closed embedding $\mathcal{H}(\text{VSP}(f)) \hookrightarrow \mathbb{P}(W)$ whose image is equal to $X \simeq \mathbb{P}(V^*)$.*
- (ii) *The image of the smooth conic $C \subset \mathcal{H}(\text{VSP}(f))$ parametrizing special lines in $\text{VSP}(f)$ coincides with the conic $Q^* \subset \mathbb{P}(V^*)$ dual to $Q \subset \mathbb{P}(V)$.*
- (iii) *For every line ℓ in $\text{VSP}(f)_{\bar{k}}$ defined by a point $x \in \mathbb{P}(V_{\bar{k}}^*)$, the set of lines in $\text{VSP}(f)_{\bar{k}}$ which intersect ℓ is parametrized by the conjugate line of x with respect to $Q_{\bar{k}}^*$.*

As a consequence, we obtain the following characterization of which k -forms $Y = \text{VSP}(f)$ of V_5 contain special lines defined over k .

Corollary 6. *The k -form $Y = \text{VSP}(f)$ of V_5 contains a special line defined over k if and only if the conic $Q = V(f) \subset \mathbb{P}(V)$ has a k -rational point.*

Proof. By combining Lemma 2 b) and Proposition 5 (ii), we deduce that Y contains a special line defined over k if and only if the conic $Q^* \subset \mathbb{P}(V^*)$ dual to Q has a k -rational point, hence if and only if Q has a k -rational point. \square

Example 7. Let $k = \mathbb{C}(s, t)$ with s, t algebraically independent over \mathbb{C} , and let $f = x^2 + sy^2 + tz^2 \in k[x, y, z]$. Since the conic $Q = V(f) \subset \mathbb{P}_k^2 = \text{Proj}_k(k[x, y, z])$ has no k -rational point, the variety $Y = \text{VSP}(f)$ is a k -form of V_5 which does not contain any special line defined over k .

3. DOUBLE PROJECTION FROM A RATIONAL POINT

Let $V_5 \hookrightarrow \mathbb{P}_k^6$ be embedded by its half-anticanonical complete linear system. For every closed point $y \in V_5$, the linear system $|\mathcal{O}_{V_5}(1) \otimes \mathfrak{m}_y^2|$ of hyperplane sections of V_5 which are singular at y defines a rational map $\psi_y : V_5 \dashrightarrow \mathbb{P}_k^2$ called the *double projection from y* , whose description is classical knowledge in the birational geometry of threefolds [8, 15]. In this section, inspired by [24], we give an explicit “reverse construction” of these maps, valid over any field k of characteristic zero, formulated in terms of Sarkisov links performed from certain locally trivial \mathbb{P}^1 -bundles over \mathbb{P}^2 associated to standard birational quadratic involutions of \mathbb{P}^2 .

3.1. Recollection on standard quadratic involutions of \mathbb{P}^2 . Let $\sigma : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$ be a birational quadratic involution of \mathbb{P}_k^2 and let $\Gamma_\sigma \subset \mathbb{P}_k^2 \times \mathbb{P}_k^2$ be its graph. Via the two projections $\tau = \text{pr}_1 : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$ and $\tau' = \text{pr}_2 : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$, Γ_σ is canonically identified with the blow-up of the scheme-theoretic base loci $Z = \text{Bs}(\sigma)$ and $Z' = \text{Bs}(\sigma^{-1})$ respectively. We denote by $e = \tau^*Z$ and $e' = \tau'^*Z'$ the scheme-theoretic inverse images

of Z and Z' respectively in Γ_σ . It is well-known that Z and Z' are local complete intersection 0-dimensional closed sub-scheme of length 3 of \mathbb{P}_k^2 with one of the following possible structure:

(Type I) Z (resp. Z') is smooth and its base extension to \bar{k} consists of three non-collinear points p_1, p_2, p_3 (resp. p'_1, p'_2, p'_3) whose union is defined over k . The surface Γ_σ is smooth, $e_{\bar{k}} = e_{p_1} + e_{p_2} + e_{p_3}$, $e'_{\bar{k}} = e'_{p'_1} + e'_{p'_2} + e'_{p'_3}$ where the e_{p_i} and $e'_{p'_i}$ are (-1) -curves.

(Type II) Z (resp. Z') consists of the disjoint union (p_1, \mathbf{p}_2) (resp. (p'_1, \mathbf{p}'_2)) of a smooth k -rational point p_1 (resp. p'_1) and a 0-dimensional sub-scheme \mathbf{p}_2 (resp. \mathbf{p}'_2) of length 2 locally isomorphic to $V(x, y^2)$ and supported at a k -rational point p_2 (resp. p'_2). We have $e = e_{p_1} + 2e_{p_2}$, $e' = e'_{p'_1} + 2e'_{p'_2}$ where e_{p_1} and e_{p_2} (resp. $e'_{p'_1}$ and $e'_{p'_2}$) are smooth k -rational curves with self-intersections -1 and $-\frac{1}{2}$ respectively, and Γ_σ has a unique A_1 -singularity at the k -rational point $e_{p_2} \cap e'_{p'_2}$.

(Type III) Z (resp. Z') is supported on a unique k -rational point p (resp. p') and is locally isomorphic to $V(y^3, x - y^2)$. We have $e = 3e_p$ and $e' = 3e'_{p'}$ where e_p and $e'_{p'}$ are smooth k -rational curves with self-intersection $-\frac{1}{3}$, and Γ_σ has a unique A_2 -singularity at the k -rational point $e_p \cap e'_{p'}$.

The following lemma whose proof is left to the reader records for later use some additional basic properties of the surfaces Γ_σ .

Lemma 8. *With the notation above, the following hold:*

- a) *The canonical divisor K_{Γ_σ} is linearly equivalent to $-e - e'$.*
- b) *The pull-back by $\tau : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$ (resp. $\tau' : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$) of a general line $\ell \simeq \mathbb{P}_k^1$ in \mathbb{P}_k^2 is \mathbb{Q} -linearly equivalent to $\frac{1}{3}(2e + e')$ (resp. $\frac{1}{3}(e + 2e')$).*

3.2. Locally trivial \mathbb{P}^1 -bundles over \mathbb{P}^2 associated to quadratic involutions. Let $\sigma : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$ be a birational quadratic involution of \mathbb{P}_k^2 with graph $\Gamma_\sigma \subset \mathbb{P}_k^2 \times \mathbb{P}_k^2$, and let $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}_k^2}$ be the ideal sheaf of its scheme-theoretic base locus Z .

Lemma 9. *There exists a locally free sheaf \mathcal{E} of rank 2 on \mathbb{P}_k^2 and an exact sequence*

$$(3.1) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}_k^2}(-1) \xrightarrow{s} \mathcal{E} \rightarrow \mathcal{I}_Z \rightarrow 0.$$

Proof. Since Z is a local complete intersection of codimension 2 in \mathbb{P}_k^2 , the existence of \mathcal{E} with the required properties follows from Serre correspondence [23]. More precisely, the local-to-global spectral sequence

$$E_2^{p,q} = H^p(\mathbb{P}_k^2, \mathcal{E} \otimes \mathcal{I}_Z^q) \Rightarrow \text{Ext}^{p+q}(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}_k^2}(-1))$$

degenerates to a long exact sequence

$$0 \rightarrow H^1(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(-1)) \rightarrow \text{Ext}^1(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}_k^2}(-1)) \rightarrow H^0(Z, \det \mathcal{N}_{Z/\mathbb{P}_k^2} \otimes \mathcal{O}_Z(-1)) \rightarrow H^2(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(-1)).$$

Since $H^i(\mathbb{P}_k^2, \mathcal{O}_{\mathbb{P}_k^2}(-1)) = 0$ for $i = 1, 2$ and Z is 0-dimensional, this sequence provides an isomorphism

$$\text{Ext}^1(\mathcal{I}_Z, \mathcal{O}_{\mathbb{P}_k^2}(-1)) \simeq H^0(Z, \det \mathcal{N}_{Z/\mathbb{P}_k^2} \otimes \mathcal{O}_Z(-1)) \simeq H^0(Z, \mathcal{O}_Z).$$

The extension corresponding via this isomorphism to the constant section $1 \in H^0(Z, \mathcal{O}_Z)$ has the desired property. \square

Let $\pi : \mathbb{P}(\mathcal{E}) = \text{Proj}(\text{Sym} \mathcal{E}) \rightarrow \mathbb{P}_k^2$ be the locally trivial \mathbb{P}^1 -bundle associated with the locally free sheaf \mathcal{E} of rank 2 as in Lemma 9. Since $\det \mathcal{E} \simeq \mathcal{O}_{\mathbb{P}_k^2}(-1)$, the canonical sheaf $\omega_{\mathbb{P}(\mathcal{E})}$ of $\mathbb{P}(\mathcal{E})$ is isomorphic to

$$(3.2) \quad \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2) \otimes \pi^* \det \mathcal{E} \otimes \pi^* \omega_{\mathbb{P}_k^2} \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-2) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^2}(-4).$$

The surjection $\mathcal{E} \rightarrow \mathcal{I}_Z \rightarrow 0$ defines a closed embedding $\mathbb{P}(\mathcal{I}_Z) = \text{Proj}(\text{Sym} \mathcal{I}_Z) \hookrightarrow \mathbb{P}(\mathcal{E})$. Since Z is a local complete intersection, the canonical homomorphism of graded $\mathcal{O}_{\mathbb{P}_k^2}$ -algebras $\text{Sym} \mathcal{I}_Z \rightarrow \mathcal{R}(\mathcal{I}_Z) = \bigoplus_{n \geq 0} \mathcal{I}_Z \cdot t^n$ is an isomorphism [17, Théorème 1], and it follows that the restriction $\pi : \mathbb{P}(\mathcal{I}_Z) \rightarrow \mathbb{P}_k^2$ is isomorphic to the blow-up $\tau : \Gamma_\sigma \rightarrow \mathbb{P}_k^2$ of Z . We can thus identify from now on Γ_σ with the closed sub-scheme $\mathbb{P}(\mathcal{I}_Z)$ of $\mathbb{P}(\mathcal{E})$. The composition of $\pi^* s : \pi^* \mathcal{O}_{\mathbb{P}_k^2}(-1) \rightarrow \pi^* \mathcal{E}$ with the canonical surjection $\pi^* \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ defines a global section

$$\bar{s} \in \text{Hom}_{\mathbb{P}(\mathcal{E})}(\pi^* \mathcal{O}_{\mathbb{P}_k^2}(-1), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \simeq H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^* \mathcal{O}_{\mathbb{P}_k^2}(1))$$

whose zero locus $V(\bar{s})$ coincides with Γ_σ .

Letting ξ and A be the classes in the divisor class group of $\mathbb{P}(\mathcal{E})$ of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and of the inverse image of a general line in \mathbb{P}_k^2 by π respectively, we have $\Gamma_\sigma = V(\bar{s}) \sim \xi + A$. On the other hand, it follows from (3.2) that the canonical divisor $K_{\mathbb{P}(\mathcal{E})}$ of $\mathbb{P}(\mathcal{E})$ is linearly equivalent to $-2(\Gamma_\sigma + A) \sim -2(\xi + 2A)$. Since $c_1(\mathcal{E}) = -1$ and $c_2(\mathcal{E}) = 3$ by construction, we derive the following numerical information:

$$(3.3) \quad K_{\mathbb{P}(\mathcal{E})}^3 = -32, \quad K_{\mathbb{P}(\mathcal{E})}^2 \cdot \Gamma_\sigma = 4, \quad K_{\mathbb{P}(\mathcal{E})} \cdot \Gamma_\sigma^2 = 2, \quad \Gamma_\sigma^3 = -2.$$

3.3. Construction of Sarkisov links.

Proposition 10. *Let $\pi : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}_k^2$ be the \mathbb{P}^1 -bundle associated to a quadratic involution $\sigma : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$ with graph $\Gamma_\sigma \subset \mathbb{P}(\mathcal{E})$ as in §3.2. Then there exists a k -form Y of V_5 and a Sarkisov link*

$$\begin{array}{ccccccc} \Gamma_\sigma & \xrightarrow{\quad} & \mathbb{P}(\mathcal{E}) & \xrightarrow{\quad \varphi \quad} & X^+ & \xleftarrow{\quad} & \Gamma_\sigma^+ \\ & \searrow \tau & \swarrow \pi & \searrow & \swarrow \theta & \searrow \theta|_{\Gamma^+} & \\ & & \mathbb{P}_k^2 & & X_0 & & \\ & & & & \searrow \theta & & \\ & & & & Y & \xleftarrow{\quad} & \{y\} \end{array}$$

where:

- (i) $\varphi : \mathbb{P}(\mathcal{E}) \dashrightarrow X^+$ is a flop whose flopping locus coincides with the support of $e' \subset \Gamma_\sigma$,
- (ii) $\Gamma_\sigma^+ \simeq \mathbb{P}_k^2$ is the proper transform of Γ_σ and $\theta|_{\Gamma_\sigma^+} : \Gamma_\sigma \dashrightarrow \Gamma_\sigma^+$ is the contraction of e' ,
- (iii) $\theta : X^+ \rightarrow Y$ is the divisorial contraction of Γ_σ^+ to a smooth k -rational point $y \in Y$,
- (iv) The support of the image in Y of the flopped locus $e^+ \subset X^+$ of φ coincides with the union of the lines in $Y_{\bar{k}}$ passing through y .

Proof. We denote $\mathbb{P}(\mathcal{E})$ and $\Gamma_\sigma \subset \mathbb{P}(\mathcal{E})$ simply by X and Γ . Since $-K_X \sim 2\xi + 4A \sim 2\Gamma + 2A$ we see that $-K_X$ is nef and that any irreducible curve $C \subset X_{\bar{k}}$ such that $-K_{X_{\bar{k}}} \cdot C \leq 0$ is contained in $\Gamma_{\bar{k}}$. By the adjunction formula and Lemma 8 a), we have

$$\Gamma^2 = -K_\Gamma - 2A \cdot \Gamma \sim_{\mathbb{Q}} \frac{1}{3}(-e + e') \quad \text{and} \quad \Gamma^2 - K_\Gamma \sim_{\mathbb{Q}} \frac{2}{3}(e + 2e'),$$

which implies by adjunction again that

$$-K_{X_{\bar{k}}} \cdot C = (\Gamma_{\bar{k}}^2 - K_{\Gamma_{\bar{k}}}) \cdot C = \frac{2}{3}(e_{\bar{k}} + 2e'_{\bar{k}}) \cdot C.$$

Since $e + 2e'$ is τ -ample and τ' -numerically trivial by virtue of Lemma 8 b), we conclude that the irreducible curves $C \subset X_{\bar{k}}$ such that $-K_{X_{\bar{k}}} \cdot C = 0$ are precisely the irreducible components of the exceptional locus $e'_{\bar{k}}$ of $\tau'_{\bar{k}} : \Gamma_{\bar{k}} \rightarrow \mathbb{P}_{\bar{k}}^2$ (see § 3.1 for the notation).

Let $\varphi : X \dashrightarrow X^+$ be the flop of the union of the irreducible components of $e'_{\bar{k}}$. Since the union of these components is defined over k , so is the union e^+ of the flopped curves of φ , and so X^+ is a smooth threefold defined over k and φ is a birational map defined over k restricting to an isomorphism between $X \setminus e'$ and $X^+ \setminus e^+$. Let Γ^+ and A^+ be the proper transforms in X^+ of Γ and A respectively. By construction, the restriction $\varphi|_\Gamma : \Gamma \dashrightarrow \Gamma^+$ coincide with $\tau' : \Gamma \rightarrow \mathbb{P}_k^2$. Since $K_{X^+} \sim -2(\Gamma^+ + A^+)$, we deduce from the adjunction formula that

$$-(\Gamma^+)^2 = K_{\Gamma^+} + 2A^+ \cdot \Gamma^+ = \tau'_*(K_\Gamma + 2A \cdot \Gamma) = \frac{1}{3}\tau'_*e,$$

which is linearly equivalent to a line $\ell \simeq \mathbb{P}_k^1$ in $\Gamma^+ \simeq \mathbb{P}_k^2$. The normal bundle $\mathcal{N}_{\Gamma^+/X^+}$ of Γ^+ in X^+ is thus isomorphic to $\mathcal{O}_{\mathbb{P}_k^2}(-1)$.

The divisor class group of X^+ is freely generated by Γ^+ and A^+ . The Mori cone $\text{NE}(X^+)$ is spanned by two extremal rays: one R_1 corresponding to the flopped curves of φ and a second one R_2 which is K_{X^+} -negative. Since $\Gamma^+ \cdot C \geq 0$ for any irreducible curve not contained in Γ^+ , including thus the irreducible components of e^+ , whereas $\Gamma^+ \cdot \ell = -1$ for any line $\ell \simeq \mathbb{P}_k^1$ in $\Gamma^+ \simeq \mathbb{P}_k^2$, it follows that R_2 is generated by the class of ℓ . The extremal contraction associated to R_2 is thus the divisorial contraction $\theta : X^+ \rightarrow Y$ of $\Gamma^+ \simeq \mathbb{P}_k^2$ to a smooth k -rational point p of a smooth projective threefold Y .

Since $-K_{X^+} = 2(\Gamma^+ + A^+)$, we conclude that the image H of A^+ by θ is an ample divisor on Y generating the divisor class group of Y and such that $-K_Y = 2H$. Furthermore, we have

$$\begin{aligned} K_Y^3 &= \theta^*(K_Y)^3 = (K_{X^+} - 2\Gamma^+)^3 \\ &= K_{X^+}^3 - 6K_{X^+}^2 \cdot \Gamma^+ + 12K_{X^+} \cdot (\Gamma^+)^2 - 8(\Gamma^+)^3 \\ &= K_X^3 - 6K_X^2 \cdot \Gamma + 12K_X \cdot \Gamma - 8 \end{aligned}$$

so that $K_Y^3 = -40$ by (3.3). Altogether, this shows $Y_{\bar{k}}$ is a smooth Fano threefold of Picard number 1, index 2 and degree $d = H^3 = 5$, hence is isomorphic to V_5 .

The fact that the support of the image in Y of e^+ coincides with the union of the lines in $Y_{\bar{k}}$ passing through p is clear by construction. \square

By construction, the proper transform by the reverse composition $\psi_y = \pi \circ \varphi^{-1} \circ \theta^{-1} : Y \dashrightarrow \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^2$ of the complete linear system of lines in \mathbb{P}_k^2 consists of divisors H on Y singular at y and such that $-K_Y = 2H$. This shows that $\psi_{y,\bar{k}} : Y_{\bar{k}} \simeq V_5 \dashrightarrow \mathbb{P}_k^2$ coincides with the double projection from the point y . Conversely, given any k -form Y of V_5 , Corollary 3 ensures that $-K_Y$ is divisible, equal to $2H$ for some ample divisor H on Y . So given any k -rational point $y \in Y$, the double projection $\psi_y : Y \dashrightarrow \mathbb{P}_k^2$ from y is defined over k , given by the linear system $|\mathcal{O}_Y(H) \otimes \mathfrak{m}_y^2|$, and it coincides with the composition $\pi \circ \varphi^{-1} \circ \theta^{-1} : Y \dashrightarrow \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}_k^2$ for a suitable quadratic birational involution σ of \mathbb{P}_k^2 .

4. APPLICATION : CYLINDERS IN FORMS OF THE QUINTIC DEL PEZZO THREEFOLD

Theorem 11. *Let Y be a k -form of V_5 , let $y \in Y$ be a k -rational point and let C be the union of the lines in $Y_{\bar{k}}$ passing through y . Then $Y \setminus C$ has the structure of a Zariski locally trivial \mathbb{A}^1 -bundle $\rho : Y \setminus C \rightarrow \mathbb{P}_k^2 \setminus Z$ over the complement of a closed sub-scheme $Z \subset \mathbb{P}_k^2$ of length 3 with as many irreducible geometric components as C .*

Proof. Indeed, the birational map $\xi = \theta \circ \varphi : \mathbb{P}(\mathcal{E}) \dashrightarrow Y$ constructed in Proposition 10 restricts to an isomorphism between $Y \setminus C$ and the complement of the proper transform Γ in $\mathbb{P}(\mathcal{E})$ of the exceptional divisor of the blow-up of Y at y . On the other hand, it follows from the construction of \mathcal{E} in § 3.2 that Γ is the graph of a quadratic birational involution σ of \mathbb{P}_k^2 and that the restriction of $\pi : \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}_k^2$ to the complement of Γ is a locally trivial \mathbb{A}^1 -bundle over the complement $\mathbb{P}_k^2 \setminus Z$ of the base locus Z of σ . \square

Theorem 12. *Every k -form Y of V_5 contains a Zariski open \mathbb{A}_k^2 -cylinder. Furthermore, Y contains \mathbb{A}_k^3 as a Zariski open subset if and only if it contains a special line defined over k .*

Proof. By Lemma 2, Y contains a line $\ell \simeq \mathbb{P}_k^1$ with trivial normal bundle. Projecting from ℓ as in the proof of Corollary 3, we obtain a birational map $Y \dashrightarrow Q$ onto a smooth quadric $Q \subset \mathbb{P}_k^4$ defined over k , which restricts to an isomorphism between the complement of the surface $Z \subset Y$ defined over k swept out by the lines in $Y_{\bar{k}}$ intersecting $\ell_{\bar{k}}$ and the complement of a hyperplane section $Q_0 \simeq \mathbb{P}_k^1 \times \mathbb{P}_k^1$ of Q . The complement $Q \setminus Q_0$ is isomorphic to the smooth affine quadric $\tilde{Q}_0 = \{xv - yu = 1\} \subset \mathbb{A}_k^4$, which contains for instance the principal open \mathbb{A}_k^2 -cylinder $\tilde{Q}_{0,x} \simeq \text{Spec}(k[x^{\pm 1}][y, u])$.

By Lemma 2 b), Y contains a special line if and only if it has a k -rational point p through which there exists at most two lines in $Y_{\bar{k}}$. Letting C be the union of these lines, the previous theorem implies that $\rho : Y \setminus C \rightarrow \mathbb{P}_k^2 \setminus Z$ is a Zariski locally trivial \mathbb{A}^1 -bundle over the complement of a 0-dimensional closed subset Z defined over k whose support consists of at most two points. Letting $L \simeq \mathbb{P}_k^1$ be any line in \mathbb{P}_k^2 containing the support of Z , the restriction of ρ over $\mathbb{P}_k^2 \setminus L \simeq \mathbb{A}_k^2$ is then a trivial \mathbb{A}^1 -bundle, so that $Y \setminus \rho^{-1}(L)$ is a Zariski open subset of Y isomorphic to \mathbb{A}_k^3 .

Conversely, suppose that Y contains a Zariski open subset $U \simeq \mathbb{A}_k^3$. Since U is affine, $D = Y \setminus U$ has pure codimension 1 in Y . Since $Y_{\bar{k}} \simeq V_5$ and $Y_{\bar{k}} \setminus D_{\bar{k}} \simeq U_{\bar{k}} \simeq \mathbb{A}_{\bar{k}}^3$, according to [7, Corollary 1.2, b)] and [8], we have the following alternative for the divisor $D_{\bar{k}}$ in $Y_{\bar{k}}$:

- 1) $D_{\bar{k}}$ is a normal del Pezzo surface of degree 5 with a unique singular point q of type A_4 ,
- 2) $D_{\bar{k}}$ is a non-normal del Pezzo surface of degree 5 whose singular locus is a special line ℓ in $Y_{\bar{k}}$, and $D_{\bar{k}}$ is swept out by lines in V_5 which intersect ℓ .

In the first case, there exists a unique special line ℓ in $Y_{\bar{k}}$ passing through q , which is then automatically contained in $D_{\bar{k}}$. Since D is defined over k and q is the unique singular point of $D_{\bar{k}}$, q corresponds to a

k -rational point of D , so that ℓ is actually a special line in Y defined over k . In the second case, since ℓ is the singular locus of $D_{\bar{k}}$, it is invariant under the action of the Galois group $\text{Gal}(\bar{k}/k)$, hence is defined over k . So in both cases, Y contains a special line defined over k . \square

Corollary 13. *Every form of V_5 defined over a C_1 -field k contains \mathbb{A}_k^3 as a Zariski open subset.*

Proof. This follows from Lemma 2 b) and the previous theorem since every conic over a C_1 -field has a rational point. \square

Example 14. The $\mathbb{C}(s, t)$ -form $Y = \text{VSP}(f)$ of V_5 associated to the quadratic form $f = x^2 + sy^2 + tz^2 \in \mathbb{C}(s, t)[x, y, z]$ considered in Example 7 does not contain any special line defined over $\mathbb{C}(s, t)$, hence does not contain $\mathbb{A}_{\mathbb{C}(s, t)}^3$ as a Zariski open subset.

REFERENCES

1. C. Birkar, P. Cascini, C.D. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, Journal of AMS, 23 (2010), 405-468.
2. A. Dubouloz and T. Kishimoto, *Explicit biregular/birational geometry of affine threefolds: completions of \mathbb{A}^3 into del Pezzo fibrations and Mori conic bundles*, Adv. Stud. Pure Math. 75, Mathematical Society of Japan, Tokyo, 2017, 49-71.
3. A. Dubouloz and T. Kishimoto, *Cylinders in del Pezzo fibrations*, arXiv:1607.00840v1 [math.AG], 2016, Israel Journal of Mathematics DOI 10.1007/s11856-018-1679-z (to appear).
4. A. Dubouloz and T. Kishimoto, *Deformations of \mathbb{A}^1 -cylindrical varieties*, arXiv:1710.09108v1 [math.AG] 25 Oct 2017.
5. M. Furushima, *Singular del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space \mathbb{C}^3* , Nagoya Math. J., 104 (1986), 1-28.
6. M. Furushima, *The complete classification of compactifications of \mathbb{C}^3 which are projective manifolds with the second Betti number one*, Math. Ann., 297 (1993), 627-662.
7. M. Furushima and N. Nakayama, *The family of lines on the Fano threefolds V_5* , Nagoya Math. J., 116 (1989), 111-122.
8. M. Furushima and N. Nakayama, *A new construction of a compactification of \mathbb{C}^3* , Tôhoku Math. J., 41 (1989), 543-560.
9. A. Iliev, *Lines on the Gushel' threefold*, Indag. Mathem., N.S., 5 (3) (1994), 307-320.
10. A. Iliev, *The Fano surface of the Gushel threefold*, Compositio Mathematica 94, n°1 (1994), 81-107.
11. V.A. Iskovskikh, *Anticanonical models of three-dimensional algebraic varieties*, J Math Sci (1980) 13 (6), 745-814.
12. T. Kishimoto, Y. Prokhorov and M. Zaidenberg, *Group actions on affine cones*. Affine algebraic geometry, 123-163. Peter Russell's Festschrift, CRM Proc. Lecture Notes, 54, Amer. Math. Soc., Providence, RI, 2011.
13. T. Kishimoto, Y. Prokhorov and M. Zaidenberg, *\mathbb{G}_a -actions on affine cones*, Transformation Groups, 18 (2013), 1137-1153.
14. T. Kishimoto, Y. Prokhorov and M. Zaidenberg, *Affine cones over Fano threefolds and additive group actions*, Osaka J. Math., 51 (2014), 1093-1112.
15. I.Y. Kolpakov-Miroshnichenko and Y.G. Prokhorov, *Rationality construction of fields of invariants of some finite four-dimensional linear groups associated with Fano threefolds*, Math Notes 51 (1) (1992), 74-76.
16. A. G. Kuznetsov, Y. G. Prokhorov and C. A. Shramov, *Hilbert schemes of lines and conics and automorphism groups of Fano threefolds*, Jpn. J. Math. (2018) 13 (1), 109-185.
17. A. Micali, *Sur les algèbres universelles*, Annales de l'Institut Fourier, Volume 14 (1964) no. 2, p. 33-87.
18. S. Mori and S. Mukai, *On Fano 3-folds with $B_2 \geq 2$* , Advanced Studies in Pure Mathematics, 1 (1983), 101-129.
19. S. Mukai and H. Umemura, *Minimal rational threefolds*, Lect. Notes Math., 1016 (1983), 490-518.
20. Yu. Prokhorov, *Fano threefolds of genus 12 and compactifications of \mathbb{C}^3* , St. Petersburg Math. J., 3 (1992), 855-864.
21. Y. Prokhorov, M. Zaidenberg, *Examples of cylindrical Fano fourfolds*, Eur. J. Math., 2 (2016), 262-282.
22. Y. Prokhorov, M. Zaidenberg, *New examples of cylindrical Fano fourfolds*, New examples of cylindrical Fano fourfolds, Adv. Stud. Pure Math. 75, Mathematical Society of Japan, Tokyo, 2017, 443-463.
23. J.P. Serre, *Sur les modules projectifs*, Séminaire Dubreil-Pisot (1960/61), Secr. Math. Paris, exposé 2, (1961).
24. K. Takeuchi, *Some birational maps of Fano 3-folds*, Compositio Math., 71 (1989), 265-283.

IMB UMR5584, CNRS, UNIV. BOURGOGNE FRANCHE-COMTÉ, F-21000 DIJON, FRANCE
E-mail address: adrien.dubouloz@u-bourgogne.fr

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, SAITAMA UNIVERSITY, SAITAMA 338-8570, JAPAN
E-mail address: tkishimo@rimath.saitama-u.ac.jp