# On boundary stability of inhomogeneous 2 x 2 1-D hyperbolic systems for the $C^1$ norm.

Amaury Hayat

**Abstract**

We study the exponential stability for the $C^1$ norm of general $2 \times 2$ 1-D quasilinear hyperbolic systems with source terms and boundary controls. When the propagation speeds of the system have the same sign, any nonuniform steady-state can be stabilized using boundary feedbacks that only depend on measurements at the boundaries and we give explicit conditions on the gain of the feedback. In other cases, we exhibit a simple numerical criterion for the existence of basic $C^1$ Lyapunov function, a natural candidate for a Lyapunov function to ensure exponential stability for the $C^1$ norm. We show that, under a simple condition on the source term, the existence of a basic $C^1$ (or $C^p$ , for any $p \geq 1$) Lyapunov function is equivalent to the existence of a basic $H^2$ (or $H^q$ , for any $q \geq 2$) Lyapunov function, its analogue for the $H^2$ norm. Finally, we apply these results to the nonlinear Saint-Venant equations. We show in particular that in the subcritical regime, when the slope is larger than the friction, the system can always be stabilized in the $C^1$ norm using static boundary feedbacks depending only on measurements at the boundaries, which has a large practical interest in hydraulic and engineering applications.

# On boundary stability of inhomogeneous $2 \times 2$ 1-D hyperbolic systems for the $C^1$ norm.

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## 1 Introduction

Hyperbolic systems are widely studied, as their ability to model physical phenomena gives rise to numerous applications. The $2 \times 2$ hyperbolic systems, in particular, are very interesting at two extends: on the one hand they are the simplest systems that present a coupling, and on the other hand, by modeling the systems of two balance laws, they represent a huge number of physical systems from fluid dynamics in rivers and shallow waters [6], to road traffic [1], signal transmission, laser amplification [11], etc. In order to use these models in industrial or practical applications, the question of their stability or their possible stabilization is fundamental. While for linear 1-D systems, or nonlinear 1-D systems without source term, many results exist (see in particular [3, Section 4.5] [7, 16]) the question of the stabilization in general for $2 \times 2$ 1-D nonlinear systems has often been treated for the $H^p$ norm and only few results exist for the more natural $C^1$ (or $C^p$) norm when a source term occurs. In [13], however, were presented some results for the $C^p$ stability ($p \geq 1$) of general $n \times n$ quasilinear hyperbolic system using basic Lyapunov functions for the $C^p$ norm.
In this article we consider the stability for the $C^1$ norm of $2 \times 2$ general quasilinear 1-d hyperbolic systems. We show several results and we use them to study the exponential stability of the general nonlinear Saint-Venant equations for the $C^1$ norm. Firstly introduced in 1871 by Barré de Saint-Venant and used to model flows under shallow water approximation, the Saint-Venant equations can be derived from the Navier-Stokes equations and have been widely used in the last centuries in many areas such as agriculture, river regulation, and hydraulic electricity production. For instance they are used in Belgium for the control of the Meuse and Sambre river (see [8], [10]). Their indisputable usefulness in the field of fluid mechanics or in engineering applications makes them a well-studied example in stability theory ([3], [14], [10]) although their stability for the $C^1$ norm by means of boundary controls seems to be only known so far in the particular case when when both the slope and the friction are sufficiently small (or equivalently the size of the river is sufficiently small) [18].
We first show that the results presented in [13] can be simplified for $2 \times 2$ systems in conditions that are easier to check in practice. In particular, any $2 \times 2$ quasilinear hyperbolic system with propagation speeds of the same sign can be stabilized by means of static boundary feedback and we give here explicit conditions on the gain of the feedbacks to achieve such result. In the general case we also give a simple linear numerical criterion to design good boundary controls and estimate the limit length above which stability is not guaranteed anymore.
Then we deduce a link between the $H^p$ stability and $C^q$ stability under appropriate boundary control for any $p \geq 2$ and $q \geq 1$. In particular we give a practical way to construct a basic Lyapunov function for the $C^1$ norm from a basic quadratic Lyapunov function for the $H^2$ norm and reciprocally.
Finally, we use these results to study the $C^1$ stability of the general nonlinear Saint-Venant equations taking into account the slope and the friction. We show that when the friction is stronger than the slope the system can always be made stable for the $C^1$ norm by applying appropriate boundary controls that are given explicitly. When the slope is higher than the friction, however, there always exists a length above which the system do not admit a basic $C^1$ Lyapunov function that would ensure the stability, whatever the

boundary controls are. This results is all the more interesting that it has been shown that there always exists a basic quadratic $H^2$ Lyapunov function ensuring the stability for the $H^2$ norm under suitable boundary controls (see [14]).
Nevertheless in that last case the results given in this article allow to find good Lyapunov function numerically and estimate the limit length under which the stability can be guaranteed. We provide at the end of this paper numerical computations of this limit for the Saint-Venant equations that illustrate that for most applications the stability can be guaranteed by means of explicit static boundary feedback. This article is organised as follows: in Section 2 we present several properties of $2 \times 2$ quasilinear hyperbolic system, as well as some useful definitions and we review some existing results. Section 3 present the main results for the general case and for the particular case of the Saint-Venant equations. Section 4 is devoted to the proof of the results in the general case and to the link between the $H^p$ and $C^q$ stability, while the proofs of the results about the Saint-Venant equations are given in Section 5. Finally, we provide some numerical computations in Section 6 and some comments in Section 7.

# 2 General considerations and previous results

## 2.1 General considerations

A $2 \times 2$ quasilinear hyperbolic system can be written in the form:

$$\mathbf{Y}_t + F(\mathbf{Y})\mathbf{Y}_x + D(\mathbf{Y}) = 0, \tag{2.1}$$

$$\mathcal{B}(\mathbf{Y}(t,0), \mathbf{Y}(t,L)) = 0. \tag{2.2}$$

As the goal of this study is to deal with the exponential stability of the system around a steady-state we assume that there exists $\mathbf{Y}^*$ a steady-state that we aim at stabilizing. Note that this steady-state is not necessarily uniform and can potentially have large variations of amplitude. As we are looking at the local stability around this steady-state, we study $F$ and $D$ on $\mathcal{U} = \mathcal{B}_{\mathbf{Y}^*,\eta_0}$, the ball of radius $\eta_0$ centered in $\mathbf{Y}^*$ in the space of the continuous functions endowed with the $L^\infty$ norm, for some $\eta_0$ small enough to be precised. We assume that the system is strictly hyperbolic around $\mathbf{Y}^*$ with non vanishing propagation speeds, i.e. non vanishing eigenvalues of $F(\mathbf{Y})$, then $F(\mathbf{Y}^*)$ is diagonalisable and denoting by $N$ a matrix of eigenvector we introduce the following change of variables:

$$\mathbf{u} = N(x)(\mathbf{Y} - \mathbf{Y}^*) \tag{2.3}$$

and the system (2.1)-(2.2) is equivalent to

$$\mathbf{u}_t + A(\mathbf{u}, x)\mathbf{u}_x + B(\mathbf{u}, x) = 0, \tag{2.4}$$

$$\mathcal{B}(N^{-1}(0)\mathbf{u}(t,0) + Y^*(0), N^{-1}(L)\mathbf{u}(t,L) + Y^*(L)) = 0,$$

where

$$A(\mathbf{u}, x) = N(x)F(\mathbf{Y}^* + N^{-1}\mathbf{u})N^{-1}(x), \tag{2.5}$$

$$A(0,x) = \begin{pmatrix} \Lambda_1(x) & 0 \\ 0 & \Lambda_2(x) \end{pmatrix}, \tag{2.6}$$

and $B$ is given in Appendix 8. Let us assume that $F$ and $D$ are $C^1$ on $\mathcal{B}_{\mathbf{Y}^*,\eta_0}$, then $A$ and $B$ are $C^1$ on $\mathcal{B}_{\mathbf{0},\eta_0} \times [0, L]$ (see Appendix 8). As $\mathbf{Y}^*$ is a stationary state, one has $B(\mathbf{0}, \cdot) \equiv 0$ and $B$ can be written:

$$B(\mathbf{u}, x) = M(\mathbf{u}, x).\mathbf{u}. \tag{2.7}$$

Therefore the system (2.1)-(2.2) is now equivalent to

$$u_t + A(\mathbf{u}, x)u_x + M(\mathbf{u}, x).\mathbf{u} = 0, \tag{2.8}$$

$$\mathcal{B}(N^{-1}(0)\mathbf{u}(t,0)+Y^*(0),N^{-1}(L)\mathbf{u}(t,L)+Y^*(L))=0.$$

We can suppose without loss of generality that $\Lambda_1 \geq \Lambda_2$. As the system is strictly hyperbolic with non-vanishing eigenvalues we can denote by $\mathbf{u}_+$ the components associated with positive eigenvalues, i.e. $\Lambda_i > 0$, and $\mathbf{u}_-$ the component associated with negative eigenvalues. We focus now on boundary conditions of the form:

$$\begin{pmatrix}\mathbf{u}_+(0)\\ \mathbf{u}_-(L)\end{pmatrix} = G\begin{pmatrix}\mathbf{u}_+(L)\\ \mathbf{u}_-(0)\end{pmatrix}. \tag{2.9}$$

For the rest of the article, unless otherwise stated, we will assume that $F$, $D$ and $G$ are $C^1$ when dealing with the $C^1$ norm and that $F$,$D$ and $G$ are $C^2$ when dealing with the $H^2$ norm. We also introduce the associated first order compatibility condition on an initial condition $\mathbf{u}^0$:

$$\begin{aligned}
&\begin{pmatrix}\mathbf{u}^0_+(0)\\ \mathbf{u}^0_-(L)\end{pmatrix} = G\begin{pmatrix}\mathbf{u}^0_+(L)\\ \mathbf{u}^0_-(0)\end{pmatrix},\\
&\begin{pmatrix}\big(A(\mathbf{u}^0(0),0)\partial_x\mathbf{u}^0(0)+B(\mathbf{u}^0(0),0)\big)_+\\ \big(A(\mathbf{u}^0(L),L)\partial_x\mathbf{u}^0(L)+B(\mathbf{u}^0(L),L)\big)_-\end{pmatrix} = G'\begin{pmatrix}\mathbf{u}^0_+(L)\\ \mathbf{u}^0_-(0)\end{pmatrix}\\
&\times\begin{pmatrix}\big(A(\mathbf{u}^0(L),L)\partial_x\mathbf{u}^0(L)+B(\mathbf{u}^0(L),L)\big)_+\\ \big(A(\mathbf{u}^0(0),0)\partial_x\mathbf{u}^0(0)+B(\mathbf{u}^0(0),0)\big)_-\end{pmatrix}.
\end{aligned} \tag{2.10}$$

With these boundary conditions the incoming information is a function of the outgoing information which enables the system to be well-posed (see [17], [19] or [3] in particular Theorem 6.4).

**Theorem 2.1.** *Let $T > 0$, there exists $\delta(T) > 0$ and $C(T) > 0$ such that for any $\mathbf{u}_0 \in C^1([0,L])$ satisfying the compatibility conditions (2.10) and*

$$|\mathbf{u}_0|_1 \leq \delta, \tag{2.11}$$

*the system (2.8)–(2.9) with initial condition $\mathbf{u}_0$ has a unique maximal solution $\mathbf{u} \in C^1([0,T]\times[0,L])$ and we have the estimate:*

$$|\mathbf{u}(t,\cdot)|_1 \leq C_1(T)|\mathbf{u}(0,\cdot)|_1,\ \forall\ t\in[0,T]. \tag{2.12}$$

*where $|\cdot|_1$ is the $C^1$ norm that is recalled later on in Definition 2.1. Moreover if $\mathbf{u}_0 \in H^2([0,L])$ and*

$$\|\mathbf{u}_0\|_{H^2((0,L))} \leq \delta, \tag{2.13}$$

*then the solution $\mathbf{u}$ belongs to $C^0([0,T],H^2(0,L))$.*

## 2.2 Context and previous results

**Exponential stability of $2\times 2$ hyperbolic systems.**

- In [3] (see Theorem 4.3) and [7] respectively it has been shown that when there is no source term, i.e. $M \equiv 0$, it is always possible to guarantee the exponential stability of the system (2.8) with boundary controls of the form (2.9), both for the $H^p$ and the $C^q$ norm (with $p \geq 2$ and $q \geq 1$). Moreover, when the system is linear, this is also true for the $L^2$ and $C^0$ norm.

- In [2] the authors study a linear $2\times 2$ system and found a necessary and sufficient interior condition to have existence of quadratic Lyapunov function for the $L^2$ norm with a boundary control of the form (2.9) when the system (2.8) is linear (Theorem 4.1). However it is straightforward to extend this results to the existence of a basic quadratic Lyapunov function for the $H^p$ norm with $p \geq 2$ when the system (2.8) is nonlinear (see Theorem 4.2). At it is mentioned in [2] the existence of a basic quadratic Lyapunov function for the $H^p$ norm implies the exponential stability of the system in the $H^p$ norm.

- In [13] the author gives a necessary and sufficient condition on (2.8) such that there exists a basic $C^1$ Lyapunov function with a boundary control of the form (2.9), guaranteeing therefore the stability for the $C^1$ norm of the system (2.8)-(2.9). The results can be extended with the same condition to the $C^p$ norm, with $p \geq 1$.

**Exponential stability of the Saint-Venant Equations.** The Saint Venant equations correspond to a system of the form (2.8) where the eigenvalues of $A$ satisfy $\Lambda_1\Lambda_2 < 0$ when the flow is in the fluvial regime and $\Lambda_1\Lambda_2 > 0$ when in the torrential regime. The stability of the Saint-Venant equations has been well-studied in the past twenty years and, to our knowledge, the most advanced contribution in the area would refer, but not exclusively, to the following:

- In [4] the authors show that when there is no slope, i.e. $C \equiv 0$, there always exists a Lyapunov function in the fluvial regime (i.e. the eigenvalues satisfy $\Lambda_1\Lambda_2 < 0$) for the $H^p$ norm for the nonlinear system under boundary controls of the form (2.9) and they give an explicit example. In [14] the authors show that this is true even when the slope is arbitrary.

- In [5] it is found, through a time delay approach, a necessary and sufficient condition for the stability of the linearized system under proportional integral control.

- In [21] and [20, 9] the authors use a backstepping method to stabilize respectively a linear $2 \times 2$ 1-d hyperbolic systems and a nonlinear $2 \times 2$ 1-d hyperbolic systems. These results cover in particular the linearized Saint-Venant equations and the nonlinear Saint-Venant equations. However, in both cases this method gives rise to full-state feedback laws that are harder to implement in practice than static feedback laws depending only on the measurements at the boundaries.

In this article we intend to show that there always exists a Lyapunov function ensuring exponential stability in the $C^1$ (and actually $C^p$) norm under boundary controls of the form (2.9) when the system is in the fluvial regime and the slope is smaller than the friction. However, in the fluvial regime when the slope is larger than the friction, there exists a maximal length $L_{max}$ beyond which there never exists a basic $C^1$ Lyapunov function whatever the boundary controls are. Nevertheless, this maximal length $L_{max}$ can be estimated numerically and can be shown to be large enough to ensure the feasibility of nearly all hydraulic applications.

**Notations and definitions.** We recall the definition of the $C^1$ norm:

**Definition 2.1.** Let $\mathbf{U} \in C^0([0,L],\mathbb{R}^2)$, its $C^0$ norm $|\mathbf{U}|_0$ is defined by:

$$|\mathbf{U}|_0 = \max(\|U_1\|_\infty, \|U_2\|_\infty), \tag{2.14}$$

and if $\mathbf{U} \in C^1([0,L],\mathbb{R}^2)$, its $C^1$ norm $|\mathbf{U}|_1$ is defined by

$$|\mathbf{U}|_1 = |\mathbf{U}|_0 + |\partial_x \mathbf{U}|_0. \tag{2.15}$$

We recall the definition of exponential stability for the $C^1$ (resp. $H^2$) norm:

**Definition 2.2.** The null steady-state $\mathbf{u}^* \equiv 0$ of the system (2.8)–(2.9) is said exponentially stable for the $C^1$ (resp. $H^2$) norm if there exists $\gamma > 0$, $\delta > 0$ and $C > 0$ such that for any $\mathbf{u}_0 \in C^1([0,L])$ (resp. $H^2([0,L])$) satisfying the compatibility conditions (2.10) and such that $\|u_0\|_{C^1([0,L])} \leq \delta$ (resp.$\|u_0\|_{H^2((0,L))} \leq \delta$), the system (2.8)–(2.9) has a unique solution $\mathbf{u} \in C^1([0,+\infty) \times [0,L])$ (resp. $\mathbf{u} \in C^1([0,+\infty) \times [0,L]) \cap C^0([0,+\infty), H^2((0,L)))$) and

$$\begin{aligned} \|\mathbf{u}(t,\cdot)\|_{C^1([0,L])} &\leq Ce^{-\gamma t}\|\mathbf{u}_0\|_{C^1([0,L])},\ \forall\ t \in [0,+\infty) \\ (\text{resp. } \|\mathbf{u}(t,\cdot)\|_{H^2((0,L))} &\leq Ce^{-\gamma t}\|\mathbf{u}_0\|_{H^2((0,L))},\ \forall\ t \in [0,+\infty)), \end{aligned} \tag{2.16}$$

**Remark 2.1.** *The exponential stability of the steady state $\mathbf{u}^* \equiv 0$ of the system (2.8)–(2.9) is equivalent to the exponential stability of the steady-states $\mathbf{Y}^*$ ((2.16) could in fact even be seen as a definition of the exponential stability of $\mathbf{Y}^*$). We see here one of the interests of the change of variables given by (2.3): from the stabilization of a potentially nonuniform steady-state the problem is reduced to the stabilization of a null steady-state.*

We now recall the definition of two useful tools. The first one deals with the basic $C^1$ Lyapunov functions described in [13]:

**Definition 2.3.** We call *basic $C^1$ Lyapunov function* for the system (2.8),(2.9) the function $V : C^1([0,L]) \to \mathbb{R}_+$ defined by:

$$\begin{aligned} V(\mathbf{U}) =& |\sqrt{f_1}U_1, \sqrt{f_2}U_2|_0 \\ &+ |(A(\mathbf{U},\cdot).\mathbf{U}_x + B(\mathbf{U},\cdot))_1\sqrt{f_1}, (A(\mathbf{U},\cdot).\mathbf{U}_x + B(\mathbf{U},\cdot))_2\sqrt{f_2}|_0, \end{aligned} \tag{2.17}$$

where $f_1$ and $f_2$ belong to $C^1([0,L],\mathbb{R}_+^*)$, and such that there exists $\gamma > 0$ and $\eta > 0$ such that for any $\mathbf{u} \in C^1([0,L])$ solution of the system (2.8),(2.9) with $|\mathbf{u}^0|_1 \leq \eta$ and for any $T > 0$:

$$\frac{dV(\mathbf{u})}{dt} \leq -\gamma V(\mathbf{u}), \tag{2.18}$$

in a distributional sense on $(0,T)$. In that case $f_1$ and $f_2$ are called coefficients of the basic $C^1$ Lyapunov function.

**Remark 2.2.** *Note that for any $u \in C^1([0,L]\times[0,T])$ solution of (2.8), one has*

$$\begin{aligned} V(\mathbf{u}(t,\cdot)) =& |\sqrt{f_1}u_1(t,\cdot), \sqrt{f_2}u_2(t,\cdot)|_0, \\ &+ |(u_1(t,\cdot))_t\sqrt{f_1}, (u_2(t,\cdot))_t\sqrt{f_2}|_0. \end{aligned} \tag{2.19}$$

*The previous definition (2.17) of $V$ is only stated to show that $V$ is in fact a function on $C^1([0,L])$ and therefore only depends on $t$ through $\mathbf{u}$. Besides, one could wonder why using a weight $\sqrt{f_i}$ instead of $f_i$ in the definition. The goal is to facilitate the comparison with the existing definition of basic quadratic Lyapunov functions for the $L^2$ (resp. $H^2$) norm introduced by Jean-Michel Coron and Georges Bastin in [2] and recalled below.*

**Definition 2.4.** We call *basic quadratic Lyapunov function* for the $L^2$ norm (resp. for the $H^2$ norm) and for the system (2.8), (2.9) the function $V$ defined on $L^2(0,L)$ (resp. $H^2(0,L)$) by:

$$\begin{aligned} V(\mathbf{U}) &= \int_0^L q_1U_1^2 + q_2U_2^2 dx \\ \Big(\text{resp. } V(\mathbf{U}) &= \int_0^L q_1U_1^2 + q_2U_2^2 dx \\ &+ \int_0^L (A(\mathbf{U},x).\mathbf{U}_x + B(\mathbf{U},x))_1^2 q_1 + (A(\mathbf{U},x).\mathbf{U}_x + B(\mathbf{U},x))_1^2 q_2 dx \\ &+ \int_0^L q_1(\partial_{\mathbf{U}}A.[A.\mathbf{U}_x + B].\mathbf{U}_x + A\frac{d}{dx}(A(\mathbf{U},x).\mathbf{U}_x + B(\mathbf{U},x)) + \partial_{\mathbf{U}}B.[A.\mathbf{U}_x + B])_1^2 \\ &+ q_2(\partial_{\mathbf{U}}A.[A.\mathbf{U}_x + B].\mathbf{U}_x + A\frac{d}{dx}(A(\mathbf{U},x).\mathbf{U}_x + B(\mathbf{U},x)) + \partial_{\mathbf{U}}B.[A.\mathbf{U}_x + B])_2^2 dx\Big), \end{aligned} \tag{2.20}$$

where $q_1$ and $q_2$ belong to $C^1([0,L],\mathbb{R}_+^*)$ and such that there exists $\gamma > 0$ and $\eta > 0$ such that for any $\mathbf{u} \in L^2(0,L)$ (resp. $H^2(0,L)$) solution of the system (2.8),(2.9) with $\left|\mathbf{u}^0\right|_{L^2(0,L)} \leq \eta$ (resp. $\left|\mathbf{u}^0\right|_{H^2(0,L)} \leq \eta$) and any $T > 0$

$$\frac{dV(\mathbf{u}(t))}{dt} \leq -\gamma V(\mathbf{u}(t)), \tag{2.21}$$

in a distributional sense on $(0,T)$. The function $q_1$ and $q_2$ are called *coefficients* of the *basic quadratic Lyapunov function*.

**Remark 2.3.** *As for the basic $C^1$ Lyapunov functions, note that for any $\mathbf{u} \in C^0([0,T], H^2(0,L))$ solution to (2.8) the expression (2.20) of a basic quadratic Lyapunov function for the $H^2$ norm becomes*

$$\begin{aligned} V(\mathbf{U}) = &\int_0^L q_1 u_1^2 + q_2 u_2^2 dx \\ &+ \int_0^L (u_1)_t^2 q_1 + (u_2)_t^2 q_2 dx \\ &+ \int_0^L (u_1)_{tt}^2 q_1 + (u_2)_{tt}^2 q_2 dx, \end{aligned} \tag{2.22}$$

*which justifies the expression (2.20).*

**Remark 2.4.** (*Lyapunov functions and stability*)

- *The existence of a basic $C^1$ Lyapunov function for a quasilinear hyperbolic system implies the exponential stability for the $C^1$ norm of this system. A proof for the general case is given in [13].*
- *Similarly the existence a basic quadratic Lyapunov function for the $L^2$ (resp. $H^2$) norm implies the exponential stability of the system for the $L^2$ (resp. $H^2$) norm (see for instance the proof in [3] and in particular (4.50)).*

Finally we introduce the following notations, useful for the rest of the article,

$$\begin{aligned} \varphi_1 &= \exp\left(\int_0^x \frac{M_{11}(0,s)}{\Lambda_1} ds\right), \\ \varphi_2 &= \exp\left(\int_0^x \frac{M_{22}(0,s)}{\Lambda_2} ds\right), \\ \varphi &= \frac{\varphi_1}{\varphi_2}, \end{aligned} \tag{2.23}$$

$$\begin{aligned} a &= \varphi M_{12}(0,\cdot), \\ b &= M_{21}(0,\cdot)/\varphi. \end{aligned} \tag{2.24}$$

While the function $\varphi_1$ and $\varphi_2$ represent the influence of the diagonal terms of $M(0,\cdot)$ that would lead to an exponential variation of the amplitude on $[0,L]$ in the absence of coupling between $u_1$ and $u_2$, the function $a$ and $b$ represent the coupling term of $M(0,\cdot)$ after a change of variables on the system to remove the diagonal coefficients of $M$ (see (4.1) and (4.4)).

We can now state the main results.

# 3 Main results

## 3.1 Stability of a general $2 \times 2$ hyperbolic system for the $C^1$ norm

**Theorem 3.1.** *Let a $2 \times 2$ quasilinear hyperbolic system of the form (2.8) be such that $\Lambda_1 \Lambda_2 > 0$. Assume that*

$$\begin{aligned} G'(0) &= \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \text{ , where} \\ k_1^2 &< \exp\left(\int_0^L 2\frac{M_{11}(0,s)}{|\Lambda_1|} - 2\max\left(\left|\frac{a(s)}{\Lambda_1}\right|, \left|\frac{b(s)}{\Lambda_2}\right|\right) ds\right), \\ k_2^2 &< \exp\left(\int_0^L 2\frac{M_{22}(0,s)}{|\Lambda_2|} - 2\max\left(\left|\frac{a(s)}{\Lambda_1}\right|, \left|\frac{b(s)}{\Lambda_2}\right|\right) ds\right). \end{aligned} \tag{3.1}$$

*Then there exists a basic $C^1$ Lyapunov function and a basic quadratic $H^2$ Lyapunov function. In particular, the null steady-state $\mathbf{u}^* \equiv 0$ of the system (2.8)–(2.9) is exponentially stable for the $C^1$ and the $H^2$ norms.*

This theorem is a direct consequence of Theorem 3.1 in [13] and will be proven in Appendix 10. From this theorem, when the eigenvalues of the hyperbolic system have the same sign, the coupling between the two equations does not raise any obstruction to the stability in the $H^2$ and in $C^1$ norm, so this case poses no challenge. We will therefore focus on the case where the eigenvalues have opposite signs, and without loss of generality we can assume that $\Lambda_1 > 0$ and $\Lambda_2 < 0$.

**Theorem 3.2.** *Let a $2 \times 2$ quasilinear hyperbolic system be of the form (2.8), where $A$ and $B$ are $C^3$ functions with $\Lambda_1 > 0$ and $\Lambda_2 < 0$. There exists a control of the form (2.9) such that there exists a basic $C^1$ Lyapunov function, if and only if*

$$d_1' = \frac{|a(x)|}{\Lambda_1} d_2, \tag{3.2}$$

$$d_2' = -\frac{|b(x)|}{|\Lambda_2|} d_1, \tag{3.3}$$

*admit a positive solution $d_1$, $d_2$ on $[0, L]$ or equivalently*

$$\begin{aligned} \eta' &= \left|\frac{a}{\Lambda_1}\right| + \left|\frac{b}{\Lambda_2}\right| \eta^2, \\ \eta(0) &= 0, \end{aligned} \tag{3.4}$$

*admits a solution on $[0, L]$, where $a$ and $b$ are defined in (2.24).*
*Moreover if one of the previous condition is verified and*

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_2^2 < \varphi(L)^2 \left(\frac{d_2(L)}{d_1(L)}\right)^2 \text{ and } k_1^2 < \left(\frac{d_1(0)}{d_2(0)}\right)^2, \tag{3.5}$$

*where $d_1$ and $d_2$ are any positive solution of (3.2)–(3.3), then the system (2.8)–(2.9) is exponentially stable for the $C^1$ norm.*

**Remark 3.1.** *This result can be used in general to find good Lyapunov functions numerically and to estimate the limit length under which the stability is guaranteed by solving linear ODEs which are quite simple to handle.*

The third equivalence together with the criterion given in [2] (recalled in Section 4) can be used to show a link between the $H^2$ and $C^1$ stability. This link is given in the following corollary

**Corollary 1.** *Let a $2 \times 2$ quasilinear hyperbolic system be of the form (2.8),(2.9), where $A$ and $B$ are $C^3$ functions and such that $\Lambda_1 > 0$ and $\Lambda_2 < 0$.*

1. *If there exists a basic $C^1$ Lyapunov function then there exists a boundary control of the form (2.9) such that there exists a basic quadratic Lyapunov function for the $H^2$ norm.*
   *Moreover, if $M_{12}(0, \cdot) M_{21}(0, \cdot) \geq 0$, then the converse is true.*

2. *In particular if the system (2.8),(2.9) admits a basic $C^1$ Lyapunov function and*

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_2^2 < \varphi(L)^2 \left(\frac{d_2(L)}{d_1(L)}\right)^2 \text{ and } k_1^2 < \left(\frac{d_1(0)}{d_2(0)}\right)^2, \tag{3.6}$$

   *where $d_1$ and $d_2$ are positive solutions of (3.2)–(3.3), then under the same boundary control there exists a basic quadratic Lyapunov function for the $H^2$ norm.*
   *Conversly if the system admits a basic quadratic $H^2$ Lyapunov function and $M_{12}(0, \cdot) M_{21}(0, \cdot) \geq 0$ and*

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_2^2 < \left(\frac{\varphi(L)}{\eta(L)}\right)^2 \text{ and } k_1^2 < \eta(0)^2, \tag{3.7}$$

   *where $\eta$ is a positive solution of*

$$\eta' = \left|\frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\eta^2\right|, \tag{3.8}$$

   *then there exists a basic $C^1$ Lyapunov function.*

**Remark 3.2.**

- *The existence of a positive solution to (3.8) is guaranteed by [2] when there exists a basic quadratic Lyapunov function for the $H^2$ norm. This result is recalled in Theorem 4.2.*

- *The converse of 1. is wrong in general. An example where the system admits a basic quadratic $H^2$ Lyapunov function but no basic $C^1$ Lyapunov function, whatever are the boundary controls, is provided in Appendix 9.*

- *To our knowledge the only such link that existed so far consists in the trivial case where $B \equiv 0$ and where there consequently always exists both a basic quadratic $H^2$ Lyapunov function and a basic $C^1$ Lyapunov function. This link can be in fact extended to the $H^p$ and $C^q$ stability with $p \geq 2$ and $q \geq 1$ with the same condition (see Section 6).*

This theoretical link can be complemented by the following practical theorem that enables to construct basic quadratic $H^2$ Lyapunov functions from basic $C^1$ Lyapunov functions and conversely when possible.

**Theorem 3.3.** *If there exists a boundary control of the form (2.9) such that there exists a basic $C^1$ Lyapunov function with coefficients $g_1$ and $g_2$, then for any $0 < \varepsilon < \min_{[0,L]}((\varphi_2/\varphi_1)\sqrt{g_1/g_2})/L$ there exists a boundary control of the form (2.9) such that*

$$\frac{1}{\Lambda_1}\left(\sqrt{\frac{g_1}{g_2}}\varphi_1\varphi_2 - \varphi_1^2\varepsilon Id\right) \quad \text{and} \quad \frac{1}{|\Lambda_2|}\sqrt{\frac{g_2}{g_1}}\varphi_1\varphi_2 \tag{3.9}$$

*are coefficients of a basic quadratic Lyapunov functions for the $H^2$ norm, where $Id$ refers to the identity function.*

*If there exists a basic quadratic $H^2$ Lyapunov function with coefficients $(q_1, q_2)$ and if $M_{12}(0,\cdot)M_{21}(0,\cdot) \geq 0$, then for all $A \geq 0$ and $\varepsilon > 0$ there exists a boundary control of the form (2.9) such that $g_1$ and $g_2$ defined by:*

$$g_1(x) = A\exp\left(2\int_0^x \frac{M_{11}(0,\cdot)}{\Lambda_1} - \frac{|M_{12}(0,\cdot)|}{\Lambda_1}\sqrt{\frac{|\Lambda_1|q_1}{|\Lambda_2|q_2}}ds - \varepsilon x\right), \tag{3.10}$$

$$g_2 = \frac{|\Lambda_2|q_2}{\Lambda_1 q_1}g_1, \tag{3.11}$$

*induce a basic $C^1$ Lyapunov function.*

## 3.2 Stability of the general Saint-Venant equations for the $C^1$ norm

We introduce the nonlinear Saint-Venant equations with a slope and a dissipative source term resulting from the friction:

$$\begin{aligned} &\partial_t H + \partial_x(HV) = 0, \\ &\partial_t V + \partial_x\left(\frac{V^2}{2} + gH\right) + \left(\frac{kV^2}{H} - C\right) = 0, \end{aligned} \tag{3.12}$$

where $k > 0$ is the constant friction coefficient, $g$ is the acceleration of gravity, and $C$ is the constant slope coefficient. We denote by $(H^*, V^*)$ the steady-state around which we want to stabilize the system, and we assume $gH^* - V^{*2} > 0$ such that the propagation speeds have opposite signs, i.e. the system is in fluvial regime (see [2] in particular (63)). The case where the propagation speeds have same sign raises no difficulty and is treated by Theorem 3.1. We show two results depending on whether the slope or the friction is the most influent.

**Theorem 3.4.** *Consider the nonlinear Saint-Venant equations (3.12) with the boundary control:*

$$\begin{aligned} h(t,0) &= b_1 v(t,0), \\ h(t,L) &= b_2 v(t,L), \end{aligned} \tag{3.13}$$

*such that*

$$b_1 \in \left( -\frac{H^*(0)}{V^*(0)}, -\frac{V^*(0)}{g} \right) \text{ and } b_2 \in \mathbb{R} \setminus \left[ -\frac{H^*(L)}{V^*(L)}, -\frac{V^*(0)}{g} \right]. \tag{3.14}$$

*If $kV^{*2}(0)/H^*(0) > C$, this system admits a basic $C^1$ Lyapunov function and the steady-state $(H^*, V^*)$ is exponentially stable for the $C^1$ norm.*

**Remark 3.3.** *It could seem surprising at first that the condition (3.14) that appears is the same as the condition that appears for the existence of a basic quadratic $H^2$ Lyapunov function (see [14]). This is an illustration of the second part of Corollary 1.*

**Theorem 3.5.** *Consider the nonlinear Saint-Venant equations (3.12) on a domain $[0, L]$. If $kV^{*2}(0)/H^*(0) < C$ then:*

1. *There exists $L_1 > 0$ such that if $L < L_1$, there exists boundary controls of the form (2.9) such that the system admits a basic $C^1$ Lyapunov function and $(H^*, V^*)$ is exponentially stable for the $C^1$ norm.*

2. *There exists $L_2 > 0$ independent from the boundary control such that, if $L > L_2$, the system does not admit a basic $C^1$ Lyapunov function.*

**Remark 3.4.** *This last result is all the more interesting since it has been shown that for any $L > 0$ the system always admits a basic quadratic $H^2$ Lyapunov function (see [14]).*

# 4 $C^1$ stability of a 2×2 quasilinear hyperbolic system and link with basic quadratic $H^2$ Lyapunov functions

In this section we prove Theorem 3.2, Corollary 1 and Theorem 3.3. For convenience in the computations, let us first introduce the following change of variables to remove the diagonal coefficients of the source term:

$$\begin{aligned} z_1(t,x) &= \varphi_1(x) u_1(t,x), \\ z_2(t,x) &= \varphi_2(x) u_2(t,x), \end{aligned} \tag{4.1}$$

where $\varphi_1$ and $\varphi_2$ are given by (2.23). This change of variables can be found in [2] and is inspired from [15, Chapter 9]. Then the system (2.8) becomes

$$\mathbf{z}_t + A_2(\mathbf{z}, x)\mathbf{z}_x + M_2(\mathbf{z}, x)\mathbf{z} = 0, \tag{4.2}$$

where

$$A_2(0,x) = A(0,x), \tag{4.3}$$

$$M_2(0,x) = \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix}, \tag{4.4}$$

with $a$ and $b$ given by (2.24), and (2.9) becomes:

$$\begin{pmatrix} \mathbf{z}_+(0) \\ \mathbf{z}_-(L) \end{pmatrix} = G_1 \begin{pmatrix} \mathbf{z}_+(L) \\ \mathbf{z}_-(0) \end{pmatrix}. \tag{4.5}$$

where $G_1$ as the same regularity than $G$. Showing the existence of a basic $C^1$ Lyapunov function (resp. a basic quadratic $H^2$ Lyapunov function) for the system (2.8)–(2.9) is obviously equivalent to showing the existence of a basic $C^1$ Lyapunov function (resp. a basic quadratic $H^2$ Lyapunov function) for the system (4.2), (4.5), and the stability of the steady-state $\mathbf{u}^* \equiv 0$ in (2.8)–(2.9) is equivalent to the stability of the steady-state $\mathbf{z}^* \equiv 0$ in (4.2), (4.5). We now state two useful Lemma that can be found for instance in [12]:

**Lemma 4.1.** *Let $n \in \mathbb{N}^*$. Consider the ODE problem*

$$\begin{aligned} y' &= f(x, y, s), \\ y(0) &= y_0, \end{aligned} \tag{4.6}$$

*where $y_0 \in \mathbb{R}^n$. If $f \in C^0(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and is locally Lipschitz in $y$ for any $s \in \mathbb{R}$, then for all $s \in \mathbb{R}$ (4.6) has a maximum solution $y_s$ defined on an interval $I_s$, and the function $(x, s) \to y_s(x)$ is continuous on $\{(x, s) \in \mathbb{R}^2 : s \in \mathbb{R}, x \in I_s\}$.*

**Lemma 4.2.** *Let $L > 0$ and let $g$ and $f$ be continuous functions on $[0, L] \times \mathbb{R}_+$ and locally Lipshitz with respect to their second variable such that*

$$g(x, y) \geq f(x, y) \geq 0, \ \forall \ (x, y) \in [0, L] \times \mathbb{R}_+. \tag{4.7}$$

*If there exists a solution $y_1$ on $[0, L]$ to*

$$\begin{aligned} y_1' &= g(x, y_1), \\ y_1(0) &= y_0, \end{aligned} \tag{4.8}$$

*with $y_0 \in \mathbb{R}_+$, then there exists a solution $y$ on $[0, L]$ to*

$$\begin{aligned} y' &= f(x, y), \\ y(0) &= y_0, \end{aligned} \tag{4.9}$$

*and in addition $0 \leq y \leq y_1$ on $[0, L]$.*

Let us now prove Theorem 3.2, which is mainly based on the results in [13].

*Proof of Theorem 3.2.* Let a $2 \times 2$ quasilinear hyperbolic system be of the form (4.2). Using Theorem 3.2 in [13] on (4.2) we know that there exists a boundary control of the form of (2.9) such that there exists a basic $C^1$ Lyapunov function if and only if:

$$\begin{aligned} f_1' &\leq -\frac{2|a(x)|}{\Lambda_1} \frac{f_1^{3/2}}{\sqrt{f_2}}, \\ f_2' &\geq \frac{2|b(x)|}{|\Lambda_2|} \frac{f_2^{3/2}}{\sqrt{f_1}}, \end{aligned} \tag{4.10}$$

admit a solution on [0,L] with $f_1 > 0$ and $f_2 > 0$ on $[0, L]$. But as $f_1$ and $f_2$ are positive this is equivalent to say that:

$$\left(\frac{1}{\sqrt{f_1}}\right)' \geq \frac{|a(x)|}{\Lambda_1} \frac{1}{\sqrt{f_2}}, \tag{4.11}$$

$$\left(\frac{1}{\sqrt{f_2}}\right)' \leq -\frac{|b(x)|}{|\Lambda_2|} \frac{1}{\sqrt{f_1}}. \tag{4.12}$$

Denoting $d_1 = 1/\sqrt{f_1}$ and $d_2 = 1/\sqrt{f_2}$ and checking that $(f_1, f_2) \in \mathbb{R}_+^*$ is equivalent to $(d_1, d_2) \in \mathbb{R}_+^*$, the existence of a solution with positive components to (4.10) is equivalent to having a solution with positive components on $[0, L]$ to the system:

$$\begin{aligned} d_1' &\geq \frac{|a(x)|}{\Lambda_1} d_2, \\ d_2' &\leq -\frac{|b(x)|}{|\Lambda_2|} d_1. \end{aligned} \tag{4.13}$$

Let us show that this is equivalent to the existence of a solution with positive components on $[0, L]$ to the system (3.2)-(3.3). One way is obvious: if there exists a solution with positive components to (3.2)-(3.3)

then it is also a solution with positive components to (4.13). Let us show the other way: suppose that there exists a solution $(d_1, d_2)$ to (4.13) with positive components on $[0, L]$. Then:

$$\left(\frac{d_1}{d_2}\right)' \geq \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|}\left(\frac{d_1}{d_2}\right)^2. \tag{4.14}$$

Hence from Lemma 4.2 the system:

$$\eta' = \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|}\eta^2, \tag{4.15}$$

$$\eta(0) = \frac{d_1(0)}{d_2(0)}, \tag{4.16}$$

admits a solution on $[0, L]$. We can now define $g_2$ as the unique solution of:

$$\begin{aligned} g_2' &= -\eta \left|\frac{b(x)}{\Lambda_2}\right| g_2, \\ g_2(0) &= d_2(0) > 0, \end{aligned} \tag{4.17}$$

and $g_1 = \eta g_2$. Thus $g_1$ and $g_2$ exist on $[0, L]$, and take only positive values and

$$g_1' = \frac{|a(x)|}{\Lambda_1} g_2, \tag{4.18}$$

$$g_2' = -\left|\frac{b(x)}{\Lambda_2}\right| g_1. \tag{4.19}$$

Therefore this system admits a solution $(g_1, g_2)$ with positive components on $[0, L]$. This ends the proof of the first equivalence.

To prove the second equivalence, note from the previous that if there exists a solution to (4.13) with positive components on $[0, L]$ then there exists a function $\eta$ on $[0, L]$ such that:

$$\begin{aligned} \eta' &= \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|}\eta^2, \\ \eta(0) &> 0. \end{aligned} \tag{4.20}$$

Therefore by comparison the system :

$$\begin{aligned} \eta' &= \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|}\eta^2, \\ \eta(0) &= 0, \end{aligned} \tag{4.21}$$

admits a solution on $[0, L]$.

Conversely, if (4.21) admits a solution on $[0, L]$ then there exists $\varepsilon > 0$ such that:

$$\begin{aligned} \eta' &= \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|}\eta^2, \\ \eta(0) &= \varepsilon, \end{aligned} \tag{4.22}$$

admits a solution $\eta_\varepsilon$ on $[0, L]$. Defining as previously $g_2$ the unique solution of:

$$\begin{aligned} g_2' &= -\eta_\varepsilon \left|\frac{b(x)}{\Lambda_2}\right| g_2, \\ g_2(0) &= \eta_\varepsilon(0) > 0, \end{aligned} \tag{4.23}$$

and $g_1 = \eta_\varepsilon g_2$, then $g_1$ and $g_2$ and $(g_1, g_2)$ is solution on $[0, L]$ of the system (3.2)-(3.3). This ends the proof of the second equivalence.

It remains now only to prove that if one of the previous conditions is verified, and if the boundary conditions (2.9) satisfy (3.5), then the system (2.8)–(2.9) is exponentially stable for the $C^1$ norm. Suppose that the system (3.2)-(3.3) admits a solution $(d_1, d_2)$ on $[0, L]$ where $d_1$ and $d_2$ are positive, then from the previous, (4.10) admits a solution $(f_1, f_2)$ on $[0, L]$ where $f_1 = d_1^{-2}$ and $f_2 = d_2^{-2}$ are positive. Therefore, as $y \to y_1^{3/2}/\sqrt{y_2}$ is $C^1$ and hence locally Lipshitz on $\mathbb{R}_+^*$, and from Lemma 4.1, there exists $\sigma_1 > 0$ such that for all $0 \leq \sigma < \sigma_1$ there exists on $[0, L]$ a solution $(f_{1,\sigma}, f_{2,\sigma})$ of the system

$$\begin{aligned} f'_{1,\sigma} &= -\frac{2|a(x)|}{\Lambda_1} \frac{f_{1,\sigma}^{3/2}}{\sqrt{f_{2,\sigma}}} - \sigma, \\ f'_{2,\sigma} &= \frac{2|b(x)|}{|\Lambda_2|} \frac{f_{2,\sigma}^{3/2}}{\sqrt{f_{1,\sigma}}} + \sigma, \end{aligned} \tag{4.24}$$

with $f_{1,\sigma} > 0$ and $f_{2,\sigma} > 0$. Note that from the proof of Theorem 3.1 in [13] (see in particular (4.29),(4.37),(4.45) and note that $K := G'(0)$), when these $f_{1,\sigma}$ and $f_{2,\sigma}$ exist, one only needs to show the following condition to have a basic $C^1$ Lyapunov function:

$$\begin{aligned} &\exists \alpha > 0,\ \exists \mu > 0,\ \exists p_1 \geq 0 : \forall p \geq p_1, \\ &\Lambda_1(L) f_{1,\sigma}(L)^p (z_1(t,L))^{2p} e^{-2p\mu L} - |\Lambda_2(L)| f_{2,\sigma}(L)^p \left( G_1'(0). \begin{pmatrix} z_1(t,L) \\ z_2(t,0) \end{pmatrix} \right)_2^{2p} e^{2p\mu L} \\ &+ |\Lambda_2(0)| f_{2,\sigma}(0)^p z_2^{2p}(t,0) - \Lambda_1(0) f_{1,\sigma}(0)^p \left( G_1'(0). \begin{pmatrix} z_1(t,L) \\ z_2(t,0) \end{pmatrix} \right)_1^{2p} > \alpha(z_1^{2p} + z_2^{2p}) \end{aligned} \tag{4.25}$$

where $G_1$ is given by (4.5). As (4.25) only needs to be true for one particular $\sigma > 0$ and using that $f_{1,\sigma}$ and $f_{2,\sigma}$ are continuous with $\sigma$ and $f_{i,0} = f_i$ for $i \in \{1, 2\}$, by continuity one only needs to show that:

$$\begin{aligned} &\exists \alpha > 0,\ \exists \mu > 0,\ \exists p_1 \geq 0 : \forall p \geq p_1, \\ &\Lambda_1(L) f_1(L)^p (z_1(t,L))^{2p} e^{-2p\mu L} - |\Lambda_2(L)| f_2(L)^p \left( G_1'(0). \begin{pmatrix} z_1(t,L) \\ z_2(t,0) \end{pmatrix} \right)_2^{2p} e^{2p\mu L} \\ &+ |\Lambda_2(0)| f_2(0)^p z_2^{2p}(t,0) - \Lambda_1(0) f_1(0)^p \left( G_1'(0). \begin{pmatrix} z_1(t,L) \\ z_2(t,0) \end{pmatrix} \right)_1^{2p} > \alpha(z_1^{2p} + z_2^{2p}). \end{aligned} \tag{4.26}$$

Now under hypothesis (3.5) and with the change of variables (4.1) we have

$$G_1'(0) = \begin{pmatrix} 0 & k_1 \\ \frac{k_2}{\varphi(L)} & 0 \end{pmatrix}. \tag{4.27}$$

Therefore the condition (4.26) becomes:

$$\begin{aligned} &\exists \alpha > 0,\ \exists \mu > 0,\ \exists p_1 \geq 0 : \forall p \geq p_1, \\ &(\Lambda_1(L) f_1(L)^p e^{-2p\mu L} - k_2^{2p} \varphi^{-2p}(L) |\Lambda_2|(L) f_2(L)^p e^{2p\mu L})(z_1(t,L))^{2p} \\ &+ (|\Lambda_2(0)| f_2(0)^p - k_1^{2p} \Lambda_1(0) f_1(0)^p) z_2^{2p}(t,0) > \alpha(z_1^{2p} + z_2^{2p}). \end{aligned} \tag{4.28}$$

But as $f_1 = d_1^{-2}$ and $f_2 = d_2^{-2}$, from (3.5)

$$\begin{aligned} \varphi^{-2}(L) f_2(L) k_2^2 &< f_1(L), \\ f_1(0) k_1^2 &< f_2(0). \end{aligned} \tag{4.29}$$

Therefore by continuity there exists $\alpha > 0$, $\mu > 0$ and $p_1 \geq 0$ such that

$$\begin{aligned} \forall p \geq p_1,\ e^{2\mu L} (|\Lambda_2|(L))^{1/p} \varphi^{-2}(L) f_2(L) k_2^2 &< f_1(L) (\Lambda_1(L))^{1/p} e^{-2\mu L}, \\ \text{and} \quad (\Lambda_1(0))^{1/p} f_1(0) k_1^2 &< f_2(0) (|\Lambda_2|(0))^{1/p}, \end{aligned} \tag{4.30}$$

and therefore (4.28) is verified, hence the system admits a basic $C^1$ Lyapunov function and is exponentially stable for the $C^1$ norm. This ends the proof of Theorem 3.2. □

**Remark 4.1.** *This theorem has a theoretical interest as it gives a simple criterion to ensure the stability of the system, but it has also a numerical interest. By computing numerically $d_2$ and seeking the first point where it vanishes, one can find the limit length $L_{max}$ above which there cannot exist a basic $C^1$ Lyapunov function and under which the stability is guaranteed. Then the coefficients of the boundary feedback control can also be designed numerically using $d_1$ and $d_2$ thus computed. Moreover, finding $d_1$ and $d_2$ only consists in solving two linear ODEs and is therefore computationally very easy to achieve. An example is given with the Saint-Venant equations in Section 5 to illustrate this statement. Finally the second equivalence is useful to show Corollary 1.*

Before proving Corollary 1, let us first state the following theorem dealing with the stability in the $L^2$ norm of linear hyperbolic systems:

**Theorem 4.1** (Bastin and Coron [2])**.** *Let a linear hyperbolic system be of the form:*

$$z_t + \begin{pmatrix} \Lambda_1(x) & 0 \\ 0 & \Lambda_2(x) \end{pmatrix} z_x + \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix} z = 0, \tag{4.31}$$

*with $\Lambda_1 > 0$ and $\Lambda_2 < 0$. There exists a boundary control of the form (2.9) such that there exists a basic quadratic Lyapunov function for the $L^2$ norm and for this sytem if and only if there exists a function $\eta$ on $[0, L]$ solution of:*

$$\begin{aligned} &\eta' = \left| \frac{b}{|\Lambda_2|}\eta^2 + \frac{a}{\Lambda_1} \right|, \\ &\eta(0) = 0. \end{aligned} \tag{4.32}$$

*Besides for any $\sigma > 0$ such that*

$$\begin{aligned} &\eta'_\sigma = \left| \frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\eta_\sigma^2 \right|, \\ &\eta_\sigma(0) = \sigma, \end{aligned} \tag{4.33}$$

*has a solution $\eta_\sigma$ on $[0, L]$ then*

$$G_1'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix} \text{ with } l_1^2 < \eta_\sigma^2(0) \text{ and } l_2^2 < \frac{1}{\eta_\sigma^2(L)}, \tag{4.34}$$

*are suitable boundary conditions such that there exists a quadratic $L^2$ Lyapunov function for the system (4.31), (4.34).*

Such Lyapunov function guarantees the global exponential stability in the $L^2$ norm for a linear system under suitable boundary controls of the form (2.9). This result can be extended to the stability in the $H^2$ norm when the system is nonlinear, namely we have:

**Theorem 4.2.** *Let a quasilinear hyperbolic system be of the form (4.2) with $\Lambda_1 > 0$ and $\Lambda_2 < 0$, where the $\Lambda_i$ are defined in (2.6). There exists a boundary control of the form (4.5) such that there exists a basic quadratic Lyapunov function for the $H^2$ norm for this system if and only if there exists a function $\eta$ on $[0, L]$ solution of:*

$$\begin{aligned} &\eta' = \left| \frac{b}{|\Lambda_2|}\eta^2 + \frac{a}{\Lambda_1} \right|, \\ &\eta(0) = 0. \end{aligned} \tag{4.35}$$

*Besides for any $\sigma > 0$ such that*

$$\begin{aligned} &\eta'_\sigma = \left| \frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\eta_\sigma^2 \right|, \\ &\eta_\sigma(0) = \sigma, \end{aligned} \tag{4.36}$$

*has a solution* $\eta_\sigma$ *on* $[0,L]$ *then*

$$G_1'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix} \text{ with } l_1^2 < \eta_\sigma^2(0) \text{ and } l_2^2 < \frac{1}{\eta_\sigma^2(L)}, \tag{4.37}$$

*are suitable boundary conditions such that there exists a quadratic* $H^2$ *Lyapunov function for the system (4.2), (4.37).*

The proof of this theorem is straightforward and is given in Appendix 10. Knowing Theorem 4.2, we can prove Corollary 1.

*Proof of Corollary 1.* Let a $2 \times 2$ quasilinear hyperbolic system be of the form (4.2).
Let us suppose that there exists a boundary control such that there exists a basic $C^1$ Lyapunov function. Then from Theorem 3.2 there exists $\eta_1$ solution on $[0,L]$ of:

$$\begin{aligned} &\eta_1' = \left|\frac{a}{\Lambda_1}\right| + \left|\frac{b}{\Lambda_2}\right| \eta_1^2, \\ &\eta_1(0) = 0, \end{aligned} \tag{4.38}$$

and from Lemma 4.2 there also exists $\eta$ solution on $[0,L]$ of

$$\begin{aligned} &\eta' = \left|\frac{a}{|\Lambda_1|} + \frac{b}{|\Lambda_2|}\eta^2\right|, \\ &\eta(0) = 0. \end{aligned} \tag{4.39}$$

Therefore, from Theorem 4.2 there exists a boundary control of the form (2.9) such that there exists a basic quadratic Lyapunov function for the $H^2$ norm for this system.
Let us suppose now that $M_{12}(0,\cdot).M_{22}(0,\cdot) \geq 0$, then from (2.24) $ab \geq 0$. Thus (4.39) and (4.38) are the same equations and therefore, from Theorems 3.2 and Theorem 4.2 if there exists a boundary control of the form (4.5) such that there exists a basic quadratic Lyapunov function for the $H^2$ norm, then there also exists a boundary control of the form (4.5) such that the system admits a basic $C^1$ Lyapunov function. This ends the proof of the first part of Corollary 1.

Let us now show the second part of Corollary 1. As previously, from (4.1) we only need to show the result for the equivalent system (4.2), (4.5). Observe first that from Theorem 4.2, for any $\sigma > 0$ such that

$$\begin{aligned} &\eta_2' = \left|\frac{a}{|\Lambda_1|} + \frac{b}{|\Lambda_2|}\eta_2^2\right|, \\ &\eta_2(0) = \sigma, \end{aligned} \tag{4.40}$$

has a solution $\eta_2$ on $[0,L]$, then

$$G_1'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix} \quad \text{with} \quad l_1^2 < \eta_2^2(0) \quad \text{and} \quad l_2^2 < \frac{1}{\eta_2^2(L)}, \tag{4.41}$$

are suitable boundary conditions such that there exists a basic quadratic $H^2$ Lyapunov function for the system (4.2), (4.5), where $G_1$ is given by (4.5). Now, let us suppose that there exists a basic $C^1$ Lyapunov function for the system (4.2), (4.5). From the first part of Corollary 1, there exists a boundary control of the form (4.5) such that there exists a basic quadratic Lyapunov function for the $H^2$ norm. Let us suppose that

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_2^2 < \varphi(L)^2 \left(\frac{d_2(L)}{d_1(L)}\right)^2 \text{ and } k_1^2 < \left(\frac{d_1(0)}{d_2(0)}\right)^2, \tag{4.42}$$

where $d_1$ and $d_2$ are positive solutions of (3.2)–(3.3). Note that defining $\eta_3 = d_1/d_2$ and $\sigma = \eta_3(0) = d_1(0)/d_2(0) > 0$ the condition (4.42) is equivalent to

$$G_1'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix} \quad \text{with} \quad l_1^2 < \eta_3^2(0) \quad \text{and} \quad l_2^2 < \frac{1}{\eta_3^2(L)}. \tag{4.43}$$

As from (3.2)-(3.3),

$$\begin{aligned} \eta_3' = \frac{d_1'}{d_2} - \frac{d_1 d_2'}{d_2^2} = \left|\frac{a}{\Lambda_1}\right| + \left|\frac{b}{\Lambda_2}\right| \eta_3^2, \\ \text{and } \eta_3(0) = \sigma, \end{aligned} \tag{4.44}$$

then from Lemma 4.2 the problem (4.40) has a solution $\eta_2$ on $[0, L]$ and $\eta_3(L) \geq \eta_2(L)$. Therefore $G'(0)$ also satisfies (4.41). Hence there exists a basic quadratic Lyapunov function for the $H^2$ norm and the system is exponentially stable for the $H^2$ norm.

Let us now show the other way. Suppose that $M_{12}(0,\cdot)M_{22}(0,\cdot) \geq 0$ and that the system admits a basic quadratic Lyapunov function for the $H^2$ norm. Then from Theorem 4.2 and by continuity of the solutions with respect to the initial conditions there exists $\sigma > 0$ such that:

$$\begin{aligned} \eta_2' = \left|\frac{a}{|\Lambda_1|} + \frac{b}{|\Lambda_2|}\eta_2^2\right|, \\ \eta_2(0) = \sigma. \end{aligned} \tag{4.45}$$

has a solution on $[0, L]$, that we denote $\eta_2$. From hypothesis (3.7) there exists such $\sigma > 0$ such that the condition (3.7) is still satisfied with $\eta_2$. We can define

$$d_2 = \exp\left(-\int_0^x \eta_2(s)\left|\frac{b(s)}{\Lambda_2(s)}\right| ds\right), \tag{4.46}$$

$$d_1 = \eta_2 d_2. \tag{4.47}$$

As $|ab| = ab$, then $(d_1, d_2)$ is a solution of (3.2)–(3.3) and $d_1 > 0$ and $d_2 > 0$, and from (3.7) and (4.46)–(4.47).

$$k_2^2 < \varphi(L)^2 \left(\frac{d_2(L)}{d_1(L)}\right)^2 \text{ and } k_1^2 < \left(\frac{d_1(0)}{d_2(0)}\right)^2. \tag{4.48}$$

Hence from Theorem 3.2 there exists a basic $C^1$ Lyapunov function. This ends the proof of Corollary 1. □

*Proof of Theorem 3.3.* Let us first note from (4.1) that $(g_1, g_2)$ are the coefficients of a basic $C^1$ Lyapunov function for the system (2.8) if and only if $(f_1, f_2)$ are the coefficients of a basic $C^1$ Lyapunov for the system (4.2) with

$$f_i = \frac{g_i}{\varphi_i^2},\ i \in \{1, 2\}, \tag{4.49}$$

Therefore, we will first prove the result for (4.2) and then use the change of coordinates (4.1) and the transformation (4.49) to come back to the system (2.8). Let a $2 \times 2$ quasilinear hyperbolic system be of the form (4.2). Suppose that there exist boundary controls of the form (2.9) such that there exists a basic $C^1$ Lyapunov function with coefficients $f_1$ and $f_2$. Then from [13] (see in particular Theorem 3.2), one has:

$$\begin{aligned} f_1' \leq -\frac{2|a(x)|}{\Lambda_1}\frac{f_1^{3/2}}{\sqrt{f_2}}, \\ f_2' \geq \frac{2|b(x)|}{|\Lambda_2|}\frac{f_2^{3/2}}{\sqrt{f_1}}, \end{aligned} \tag{4.50}$$

Now let us denote $d_1 = f_1^{-1/2}$ and $d_2 = f_2^{-1/2}$, then

$$d_1' \geq \frac{|a(x)|}{\Lambda_1} d_2, \tag{4.51}$$

$$d_2' \leq -\frac{|b(x)|}{|\Lambda_2|} d_1. \tag{4.52}$$

Therefore:

$$-\left(\frac{d_1}{d_2}\right)'\left(\frac{d_2}{d_1}\right)' \geq \left(\left|\frac{b}{\Lambda_2}\right|\left(\frac{d_1}{d_2}\right)^2+\left|\frac{a}{\Lambda_1}\right|\right)\left(\left|\frac{b}{\Lambda_2}\right|+\left|\frac{a}{\Lambda_1}\right|\left(\frac{d_2}{d_1}\right)^2\right). \tag{4.53}$$

Hence:

$$-\left(\frac{d_1}{d_2}\right)'\left(\frac{d_2}{d_1}\right)' \geq \left(\left|\frac{b}{\Lambda_2}\right|\left(\frac{d_1}{d_2}\right)+\left|\frac{a}{\Lambda_1}\right|\left(\frac{d_2}{d_1}\right)\right)^2. \tag{4.54}$$

Now let us take $\varepsilon > 0$ such that $\varepsilon L < \min_{[0,L]}(d_2/d_1)$. Note that from (4.49) this is equivalent to $\varepsilon L < \min_{[0,L]}\left(\varphi_2/\varphi_1\sqrt{g_1/g_2}\right)$, where $g_1$ and $g_2$ are the coefficients of the basic $C^1$ Lyapunov function for the original system (2.8). We have:

$$\left(\frac{d_1(x)}{d_2(x)}\right)' \geq 0 \tag{4.55}$$

$$\text{and } -\left(\frac{d_1(x)}{d_2(x)}\right)'\left(\frac{d_2(x)}{d_1(x)}-\varepsilon x\right)' > \left(\left|\frac{b(x)}{\Lambda_2(x)}\right|\left(\frac{d_1(x)}{d_2(x)}\right)+\left|\frac{a(x)}{\Lambda_1(x)}\right|\left(\frac{d_2(x)}{d_1(x)}-\varepsilon x\right)\right)^2. \tag{4.56}$$

It can be shown (see [2] or [3] in particular Theorem 6.10 for more details) that this condition implies that

$$\frac{1}{\Lambda_1}\left(\frac{d_2}{d_1}-\varepsilon Id\right) \quad \text{and} \quad \frac{d_1}{|\Lambda_2|d_2}$$

are the coefficients of a basic quadratic Lyapunov function for the $H^2$ norm for the system (4.2) for some boundary controls of the form (4.5). Equivalently this means that, for some boundary controls of the form (4.5),

$$\begin{aligned} V(t) = &\int_0^L \frac{1}{\Lambda_1}\left(\sqrt{\frac{f_1(x)}{f_2(x)}}-\varepsilon x\right) z_1^2(t,x) + \frac{1}{|\Lambda_2|}\sqrt{\frac{f_2}{f_1}} z_2^2(t,x)dx \\ &+ \int_0^L \frac{1}{\Lambda_1}\left(\sqrt{\frac{f_1(x)}{f_2(x)}}-\varepsilon x\right)(\partial_t z_1)^2(t,x) + \frac{1}{|\Lambda_2|}\sqrt{\frac{f_2}{f_1}}(\partial_t z_2)^2(t,x)dx \\ &+ \int_0^L \frac{1}{\Lambda_1}\left(\sqrt{\frac{f_1(x)}{f_2(x)}}-\varepsilon x\right)(\partial_{tt}^2 z_1)^2(t,x) + \frac{1}{|\Lambda_2|}\sqrt{\frac{f_2}{f_1}}(\partial_{tt}^2 z_2)^2(t,x)dx \end{aligned} \tag{4.57}$$

is a Lyapunov function for the $H^2$ norm. Therefore using (4.49) and performing the inverse change of coordinates to go from (4.2) to (2.8), $V$ can also be written as

$$\begin{aligned} V(t) = &\int_0^L \frac{1}{\Lambda_1}\left(\sqrt{\frac{g_1}{g_2}}-\frac{\varphi_1}{\varphi_2}\varepsilon x\right)\varphi_1\varphi_2 u_1^2(t,x) + \frac{1}{|\Lambda_2|}\sqrt{\frac{g_2}{g_1}}\varphi_1\varphi_2 u_2^2(t,x)dx \\ &+ \int_0^L \frac{1}{\Lambda_1}\left(\sqrt{\frac{g_1}{g_2}}-\frac{\varphi_1}{\varphi_2}\varepsilon x\right)\varphi_1\varphi_2(\partial_t u_1(t,x))^2 + \frac{1}{|\Lambda_2|}\sqrt{\frac{g_2}{g_1}}\varphi_1\varphi_2(\partial_t u_2(t,x))^2 dx \\ &+ \int_0^L \frac{1}{\Lambda_1}\left(\sqrt{\frac{g_1}{g_2}}-\frac{\varphi_1}{\varphi_2}\varepsilon x\right)\varphi_1\varphi_2(\partial_{tt}^2 u_1(t,x))^2 + \frac{1}{|\Lambda_2|}\sqrt{\frac{g_2}{g_1}}\varphi_1\varphi_2(\partial_{tt}^2 u_2(t,x))^2 dx, \end{aligned} \tag{4.58}$$

where $g_1$ and $g_2$ are the coefficients of the basic $C^1$ Lyapunov function of the system (2.8), and this concludes the proof of the first part of Theorem 3.3.

To show the second part of Theorem 3.3 suppose that $M_{12}M_{21} \geq 0$. Therefore from (2.24), $ab \geq 0$. Suppose also that $(l_1, l_2)$ are the coefficients of a basic quadratic Lyapunov functions for the $H^2$ norm for the system (4.2), (4.5). Define $h_i = |\Lambda_i| l_i$ and

$$f_1(x) = A\exp\left(-\int_0^x 2\frac{|a|}{\Lambda_1}\sqrt{\frac{h_1}{h_2}}ds - \varepsilon x\right), \tag{4.59}$$

$$f_2 = \frac{h_2}{h_1}f_1, \tag{4.60}$$

where $A > 0$ and $\varepsilon > 0$ are taken arbitrarly. We have:

$$f_1' = -2\frac{|a|}{\Lambda_1}\sqrt{\frac{h_1}{h_2}}f_1 - \varepsilon f_1 < -2\frac{|a|}{\Lambda_1}\sqrt{\frac{h_1}{h_2}}f_1 = -2\frac{|a|}{\Lambda_1}\frac{f_1^{3/2}}{\sqrt{f_2}}, \tag{4.61}$$

and

$$\left(\left(\frac{h_2}{h_1}\right)'\right)^2 = \left(h_2'\frac{1}{h_1} + \left(\frac{1}{h_1}\right)'h_2\right)^2 \geq 4h_2'\left(\frac{1}{h_1}\right)'\frac{h_2}{h_1}. \tag{4.62}$$

Besides $l_i$ are the coefficients of a basic quadratic Lyapunov function for the $H^2$ norm for (4.2) therefore (see [2], in particular (41)-(43)) $h_1' < 0$, $h_2' > 0$ and

$$h_2'\left(\frac{1}{h_1}\right)' > \left(\frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\frac{h_2}{h_1}\right)^2. \tag{4.63}$$

Let us denote

$$I_1 := \left(h_2'\left(\frac{1}{h_1}\right)'\right)^{1/2} - \left|\frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\frac{h_2}{h_1}\right| > 0. \tag{4.64}$$

Thus from (4.61), (4.62) and (4.63):

$$\begin{aligned} f_2' &\geq 2\sqrt{\frac{h_2}{h_1}}\left|\frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\frac{h_2}{h_1}\right|f_1 + 2\sqrt{\frac{h_2}{h_1}}I_1f_1 + f_1'\frac{h_2}{h_1} \\ &= 2\sqrt{\frac{h_2}{h_1}}\left|\frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|}\frac{h_2}{h_1}\right|f_1 + 2\sqrt{\frac{h_2}{h_1}}I_1f_1 - 2\frac{|a|}{\Lambda_1}\sqrt{\frac{h_2}{h_1}}f_1 - \varepsilon f_1\frac{h_2}{h_1}. \end{aligned} \tag{4.65}$$

Assuming now that $\varepsilon < 2\ \min_{[0,L]}(I_1\sqrt{h_1/h_2})$, we have, as $|ab| = ab$ and $h_1$ and $h_2$ are positive,

$$\begin{aligned} f_2' &> 2\sqrt{\frac{h_2}{h_1}}\frac{|b|}{|\Lambda_2|}\frac{h_2}{h_1}f_1 \\ &= 2\frac{|b|}{|\Lambda_2|}\frac{f_2^{3/2}}{\sqrt{f_1}}. \end{aligned} \tag{4.66}$$

Therefore from (4.65) and (4.66) and Theorem 3.1 in [13], there exists a boundary control of the form (4.5) such that $(f_1, f_2)$ induce a basic $C^1$ Lyapunov function for the system (4.2). Performing the inverse change of coordinates and using (4.1) there exists a boundary control of the form (2.9) such that $(f_1, f_2)$ induce a $C^1$ basic Lyapunov function, where

$$g_1(x) = A\exp\left(2\int_0^x \frac{M_{11}(0,\cdot)}{\Lambda_1} - \frac{|M_{12}(0,\cdot)|}{\Lambda_1}\sqrt{\frac{|\Lambda_1|q_1}{|\Lambda_2|q_2}}ds - \varepsilon x\right), \tag{4.67}$$

$$g_2 = \frac{|\Lambda_2|q_2}{|\Lambda_1|q_1}g_1, \tag{4.68}$$

and $(q_1, q_2)$ are the coefficients inducing a basic quadratic Lyapunov function for the $H^2$ norm for the system (2.8)–(2.9). This ends the proof of Theorem 3.3.

□

# 5 An application to the Saint-Venant equations

In this section we will show Theorem 3.4 and Theorem 3.5. Before proving these results, we recall some properties of the Saint-Venant equations. The steady-states $(H^*, V^*)$ of (3.12) are the solutions of:

$$\begin{aligned} \partial_x(H^*V^*) &= 0, \\ \partial_x\left(\frac{V^{*2}}{2} + gH^*\right) &= \left(C - \frac{kV^{*2}}{H^*}\right). \end{aligned} \tag{5.1}$$

Under the assumption of physical fluvial (also called subcritical) regime, i.e. $0 < V^* < \sqrt{gH^*}$, these equations reduce to:

$$\begin{aligned} H_x^* &= -H^* \frac{V_x^*}{V^*}, \\ V_x^* &= V^* \frac{\frac{kV^{*2}}{H^*} - C}{gH^* - V^{*2}}, \end{aligned} \tag{5.2}$$

and have a unique maximal solution for a given $H^*(0)$ and $V^*(0)$ verifying $V^*(0) < \sqrt{gH^*(0)}$. Observe now that there are three different cases:

- $\frac{kV^{*2}(0)}{H^*(0)} > C$, i.e. the friction is larger than the slope. In this case $V^*$ is an increasing function, $H^*$ is a decreasing function and therefore the friction stays larger than the slope on the whole domain, i.e. $kV^{*2}/H^* > C$.
- $\frac{kV^{*2}(0)}{H^*(0)} = C$, in this case the steady-states are uniform and thus defined on $[0, +\infty)$ and in particular they are defined on $[0, L]$ for any $L > 0$.
- $\frac{kV^{*2}(0)}{H^*(0)} < C$, in this case $V^*$ is an decreasing function, $H^*$ is an increasing function, therefore the slope stays larger than the friction on the whole domain. Note that the system moves away from the critical regime, therefore the solution $(H^*, V^*)$ is defined on $[0, +\infty)$ and in particular it is defined on $[0, L]$ for any $L > 0$. A more rigorous proof will be given later on (see the the beginning of the proof of Theorem 3.5).

Therefore, it is enough to look at the difference between the friction and the slope at the initial point $x = 0$ to know whether the slope or the friction is larger on the whole domain.
We will consider a steady-state $(H^*, V^*)$ with $H(0)^* = H_0^*$ and $V(0)^* = V_0^*$ the associated initial conditions, and we define now the perturbations:

$$h = H - H^* \text{ and } v = V - V^*. \tag{5.3}$$

Assuming subcritical regime, i.e. $V^* < \sqrt{gH^*}$, the Saint-Venant equations (3.12) can be transformed using the transformation described by (2.1)–(2.2)→(2.8)–(2.9) in

$$\partial_t \mathbf{u} + A(\mathbf{u}, x)\partial_x \mathbf{u} + M(\mathbf{u}, x)\mathbf{u} = 0, \tag{5.4}$$

where

$$\mathbf{u} = \begin{pmatrix} v + h\sqrt{\frac{g}{H^*}} \\ v - h\sqrt{\frac{g}{H^*}} \end{pmatrix}, \tag{5.5}$$

$$\Lambda_1 = V^* + \sqrt{gH^*}, \tag{5.6}$$

$$\Lambda_2 = V^* - \sqrt{gH^*}, \tag{5.7}$$

$$\begin{aligned} M(0,\cdot) =& \frac{kV^{*2}}{H^*} \begin{pmatrix} -\frac{3}{4(\sqrt{gH^*}+V^*)} + \frac{1}{V^*} - \frac{1}{2\sqrt{gH^*}} & -\frac{1}{4(\sqrt{gH^*}+V^*)} + \frac{1}{V^*} + \frac{1}{2\sqrt{gH^*}} \\ \frac{1}{4(\sqrt{gH^*}-V^*)} + \frac{1}{V^*} - \frac{1}{2\sqrt{gH^*}} & \frac{3}{4(\sqrt{gH^*}-V^*)} + \frac{1}{V^*} + \frac{1}{2\sqrt{gH^*}} \end{pmatrix} \\ & - C \begin{pmatrix} -\frac{3}{4(\sqrt{gH^*}+V^*)} & -\frac{1}{4(\sqrt{gH^*}+V^*)} \\ \frac{1}{4(\sqrt{gH^*}-V^*)} & \frac{3}{4(\sqrt{gH^*}-V^*)} \end{pmatrix}. \end{aligned} \tag{5.8}$$

Observe that the system is indeed strictly hyperbolic under small perturbations as $\Lambda_1 > 0$ and $\Lambda_2 < 0$. The derivation of $\Lambda_1$, $\Lambda_2$ and $M(0,\cdot)$ will not be detailed here but is quite straightforward and the expression (5.6)–(5.8) can be found for instance in [3, Section 1.4.2].

*Proof of Theorem 3.4.* Let us suppose that the flow is in the fluvial regime on $[0, L]$, therefore

$$V^*(x) < \sqrt{gH^*(x)}, \ \forall\ x \in [0, L], \tag{5.9}$$

and suppose that $kV_0^{*2}/H_0^* > C$. From the previous we have $kV^{*2}/H^* > C$ on $[0, L]$. Thus from (5.8) $M_{12}(0, \cdot) \geq 0$ and $M_{21}(0, \cdot) \geq 0$. Before going any further let us note that we know from [14] that there exists a basic quadratic $H^2$ function for the system (5.4) for some boundary controls of the form (4.5), therefore Corollary 1 applies and there exists a boundary control of the form (4.5) such that there exists a basic $C^1$ Lyapunov function for this system.

Moreover from [14] (see Lemma 3.1), we know that

$$\eta = \frac{\varphi_1 |\Lambda_2|}{\varphi_2 \Lambda_1}, \tag{5.10}$$

is a positive solution of:

$$\eta' = \frac{b}{|\Lambda_2|}\eta^2 + \frac{a}{\Lambda_1}. \tag{5.11}$$

Therefore from the second part of Corollary 1 one has that if

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \tag{5.12}$$

$$\text{with } k_2^2 < \frac{\varphi^2(L)}{\eta^2(L)} \text{ and } k_1^2 < \eta^2(0), \tag{5.13}$$

where $G$ is given by (2.9) and $\eta$ by (5.10), then the system (5.4), (2.9) admits a basic $C^1$ Lyapunov function and is exponentially stable for the $C^1$ norm.

It is therefore enough to show that the boundary conditions (3.13) under hypothesis (3.14) are equivalent to boundary conditions of the form (2.9) satisfying the previous condition (5.12)–(5.13) in the new system (5.4), obtained by the change of variables (5.5). Observe that from (3.13) we have

$$\begin{aligned} v(t,0) + h(t,0)\sqrt{\frac{g}{H^*(0)}} &= k_1 \left( v(t,0) - h(t,0)\sqrt{\frac{g}{H^*(0)}} \right), \\ v(t,L) - h(t,L)\sqrt{\frac{g}{H^*(L)}} &= k_2 \left( v(t,L) + h(t,L)\sqrt{\frac{g}{H^*(0)}} \right), \end{aligned} \tag{5.14}$$

where

$$k_1 := \left( \frac{1 + \sqrt{\frac{g}{H^*(0)}} b_1}{1 - \sqrt{\frac{g}{H^*(0)}} b_1} \right), \tag{5.15}$$

$$\frac{1}{k_2} := \left( \frac{1 + \sqrt{\frac{g}{H^*(L)}} b_2}{1 - \sqrt{\frac{g}{H^*(L)}} b_2} \right), \tag{5.16}$$

and therefore:

$$\begin{aligned} u_1(t,0) &= k_1 u_2(t,0), \\ u_2(t,L) &= k_2 u_1(t,L). \end{aligned} \tag{5.17}$$

Therefore after the change of variables given by (5.5), the boundary conditions (3.13) are equivalent to boundary conditions of the form (2.9) satisfying (5.12). All it remains to do is to prove that under the hypothesis (3.14), the boundary conditions (5.17) also satisfy the condition (5.13). Now observe that from

(5.10), (5.15) and (5.16), the condition (5.13) becomes:

$$\left(\frac{1+\sqrt{\frac{g}{H^*(0)}}b_1}{1-\sqrt{\frac{g}{H^*(0)}}b_1}\right)^2 < \left(\frac{\Lambda_2(0)}{\Lambda_1(0)}\right)^2, \tag{5.18}$$

$$\left(\frac{1+\sqrt{\frac{g}{H^*(L)}}b_2}{1-\sqrt{\frac{g}{H^*(L)}}b_2}\right)^2 > \left(\frac{\Lambda_2(L)}{\Lambda_1(L)}\right)^2, \tag{5.19}$$

which is equivalent to

$$\left(1-\left(\frac{\Lambda_2(0)}{\Lambda_1(0)}\right)^2\right)+\left(1-\left(\frac{\Lambda_2(0)}{\Lambda_1(0)}\right)^2\right)\left(\sqrt{\frac{g}{H^*(0)}}b_1\right)^2+2\left(1+\left(\frac{\Lambda_2(0)}{\Lambda_1(0)}\right)^2\right)\sqrt{\frac{g}{H^*(0)}}b_1<0, \tag{5.20}$$

$$\left(1-\left(\frac{\Lambda_2(L)}{\Lambda_1(L)}\right)^2\right)+\left(1-\left(\frac{\Lambda_2(L)}{\Lambda_1(L)}\right)^2\right)\left(\sqrt{\frac{g}{H^*(L)}}b_2\right)^2+2\left(1+\left(\frac{\Lambda_2(L)}{\Lambda_1(L)}\right)^2\right)\sqrt{\frac{g}{H^*(L)}}b_2>0, \tag{5.21}$$

and from (5.6) and (5.7) this is equivalent to having

$$b_1 \in \left(-\frac{H^*(0)}{V^*(0)}, -\frac{V^*(0)}{g}\right) \text{ and } b_2 \in \mathbb{R}\setminus\left[-\frac{H^*(L)}{V^*(L)}, -\frac{V^*(0)}{g}\right]. \tag{5.22}$$

which is exactly (3.14). Therefore under boundary conditions (3.13) and hypothesis (3.14) the system admits a basic $C^1$ Lyapunov function and is therefore stable for the $C^1$ norm, this ends the proof. □

*Proof of Theorem 3.5.* Let us suppose that $C - kV_0^{*2}/H_0^* > 0$ and $gH_0^* > V_0^{*2}$. Then there exists a unique maximal solution $(H^*, V^*)$ to the equations (5.2). Let us prove that this solution is defined on $[0, +\infty)$. Denoting $L_0 \in (0, \infty]$ the limit such that the maximal solution is defined on $[0, L_0)$, we have from the beginning of this section, in particular (5.9), that for all $x \in [0, L_0)$, $H^*$ and $V^*$ are continuous, positive, and:

$$\frac{kV^{*2}}{H^*} > 0, \tag{5.23}$$

$$gH^* > V^{*2}. \tag{5.24}$$

Therefore from (5.2), $H^*$ is an increasing function and $V^*$ is a decreasing function. Besides, as $H^*V^*$ remains constant, both $H^*$ and $V^*$ remain positive. From (5.2) we can get an estimate on the growth of $H^*$:

$$H_x^*(1-\frac{V^{*2}}{gH^*}) = \frac{C}{g} - \frac{kV^{*2}}{gH^*}, \tag{5.25}$$

therefore

$$\frac{C}{g} - \frac{kV_0^{*2}}{gH_0^*} \le H_x^*(1-\frac{V^{*2}}{gH^*}) \le \frac{C}{g}, \tag{5.26}$$

hence

$$0 < \frac{C}{g} - \frac{kV_0^{*2}}{gH_0^*} \le H_x^* \le \frac{C}{g(1-\frac{V_0^{*2}}{gH_0^*})}. \tag{5.27}$$

Thus $H_x^*$ is bounded, Hence $L_0 = +\infty$, as $H^*$ is an increasing function and cannot explode in finite length and as $V^* = H_0^*V_0^*/H^*$ from (5.2).

Consider now the Saint-Venant equations transformed into the system (5.4) with (5.5)–(5.8). The first part of the theorem is straightforward from Theorem 3.2 as there exists $L_1 > 0$ such that (3.4) admits a solution

on $[0, L_1]$. Let us now suppose by contradiction that for any $L > 0$ there exists a basic $C^1$ Lyapunov function for the system. Then from Theorem 3.2, for any $L > 0$ there exists a solution $\eta$ on $[0, L]$ of

$$\begin{aligned} \eta' &= \left|\frac{a}{\Lambda_1}\right| + \left|\frac{b}{\Lambda_2}\right| \eta^2, \\ \eta(0) &= 0, \end{aligned} \tag{5.28}$$

where $a$, $b$, $\Lambda_1$ and $\Lambda_2$ are given by (2.24), (5.6) and (5.7). From (5.27), $H$ goes to $+\infty$ when $x$ goes to $+\infty$, and from (5.8) we have

$$\begin{aligned} \frac{M_{11}(0,\cdot)}{\Lambda_1} + \frac{M_{22}(0,\cdot)}{|\Lambda_2|} =& \frac{3C}{4}\left(\frac{1}{\Lambda_1^2} - \frac{1}{\Lambda_2^2}\right) + \frac{kV^*}{H^*}\left(\frac{1}{\Lambda_1} + \frac{1}{|\Lambda_2|}\right) \\ &+ \frac{kV^{*2}}{H^*}\left[\frac{3}{4}\left(\frac{1}{|\Lambda_2|^2} - \frac{1}{\Lambda_1^2}\right) + (\frac{1}{|\Lambda_2|} - \frac{1}{\Lambda_1})\frac{1}{2\sqrt{gH^*}}\right], \\ =& \frac{-3C\sqrt{gH^*}V^*}{(gH^* - V^{*2})^2} + \sqrt{\frac{g}{H^*}}\frac{2kV^*}{gH^* - V^{*2}} + \frac{kV^{*2}}{H^*}\left[\frac{3\sqrt{gH^*}V^*}{(gH^* - V^{*2})^2} + \frac{V^*}{(gH^* - V^{*2})\sqrt{gH^*}}\right], \\ =& \sqrt{\frac{g}{H^*}}\frac{1}{(gH^* - V^{*2})}\left[\frac{-3CQ}{gH^* - V^{*2}} + \frac{2kQ}{H^*}\right] \\ &+ \frac{kQ^2}{H^{*3}}\left[\sqrt{\frac{g}{H^*}}\frac{3Q}{(gH^* - V^{*2})^2} + \frac{2Q}{H^*(gH^* - V^{*2})\sqrt{gH^*}}\right], \\ =& \sqrt{\frac{g}{H^*}}\frac{1}{(gH^* - V^{*2})}\left[\frac{-3CQ + 2kQg - 2kQ\frac{Q^2}{H^{*3}}}{gH^* - V^{*2}}\right] + O\left(\frac{1}{H^{*5}}\right), \\ =& O\left(\frac{1}{H^{*5/2}}\right). \end{aligned} \tag{5.29}$$

We used here that $H^*V^*$ is constant and therefore $1/(gH^* - V^{*2}) = O\left(1/H^*\right)$ when $H^*$ (or equivalently $x$) goes to infinity. But from (5.27) we know that $H^* \geq \left(C/g - kV_0^{*2}/gH_0^*\right)x + H_0^*$. Therefore

$$\frac{M_{11}(0,x)}{\Lambda_1} + \frac{M_{22}(0,x)}{|\Lambda_2|} = O\left(\frac{1}{x^{5/2}}\right) \text{ for } x \to +\infty, \tag{5.30}$$

and thus is integrable. Hence

$$\lim_{x\to+\infty} \frac{\varphi_1(x)}{\varphi_2(x)} = C_2 > 0. \tag{5.31}$$

Let us look at $a/\Lambda_1$ and $b/|\Lambda_2|$. We have

$$\begin{aligned} \frac{a}{\Lambda_1} =& \varphi\frac{M_{12}(0,\cdot)}{\Lambda_1} \\ =& \varphi\left(\frac{C - \frac{kV^{*2}}{H^*}}{4(\sqrt{gH^*} + V^*)} + \frac{kV^{*2}}{H^*\Lambda_1}\left[\frac{1}{V^*} + \frac{1}{2\sqrt{gH^*}}\right]\right) \\ =& \frac{\varphi}{gH^*}\left(\frac{C - \frac{kV^{*2}}{H^*}}{4(1 + \frac{V^*}{\sqrt{gH^*}})^2} + \frac{kQ^2g}{H^{*2}(\sqrt{gH^*} + V^*)}\left[\frac{H^*}{Q} + \frac{1}{2\sqrt{gH^*}}\right]\right). \end{aligned} \tag{5.32}$$

Therefore

$$\lim_{x\to+\infty} \frac{agH^*}{\Lambda_1} = \frac{C_2C}{4}. \tag{5.33}$$

Similarly we can obtain:

$$\lim_{x\to+\infty} \frac{bgH^*}{|\Lambda_2|} = -\frac{C}{4C_2}. \tag{5.34}$$

Therefore there exists $x_1 \in (0,\infty)$ such that for all $x > x_1$

$$\left|\frac{a}{\Lambda_1}\right| \geq \frac{C_2 C}{5gH^*}, \tag{5.35}$$

$$\left|\frac{b}{\Lambda_2}\right| \geq \frac{C}{5C_2 gH^*}. \tag{5.36}$$

Let $L > x_1$, by assumption equation (5.28) has a solution $\eta$ defined on $[0, L]$ and from (5.35) and (5.36), for all $x > x_1$

$$\eta' \geq \frac{C}{5gH^*}(C_2 + \frac{\eta^2}{C_2}) \geq \frac{C_3 C}{5gH^*}(1+\eta^2), \tag{5.37}$$

where $C_3 = \min\left(C_2, \frac{1}{C_2}\right)$. From (5.27) right-hand side and (5.37) we have

$$\frac{\eta'}{(1+\eta^2)} \geq \frac{C_3 C}{5gH^*} \geq \frac{C_3\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{5(x + \frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C} gH_0^*)}, \tag{5.38}$$

hence

$$\int_{x_1}^{x} \frac{\eta'}{(1+\eta^2)} dx \geq \int_{x_1}^{x} \frac{C_3\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{5\left(x + \frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C} gH_0^*\right)} dx. \tag{5.39}$$

Thus

$$\arctan(\eta(x)) - \arctan(\eta(x_1)) \geq \frac{C_3}{5}\left(1-\frac{V_0^{*2}}{gH_0^*}\right) \ln\left(\frac{x + \frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C} gH_0^*}{x_1 + \frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C} gH_0^*}\right). \tag{5.40}$$

Note that the right-hand side does not depend on $\eta$ and $L$ and that

$$\lim_{x\to+\infty} \frac{C_3}{5}\left(1-\frac{V_0^{*2}}{gH_0^*}\right) \ln\left(\frac{x + \frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C} gH_0^*}{x_1 + \frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C} gH_0^*}\right) = +\infty. \tag{5.41}$$

Therefore, as this is true for any $L > 0$ we can choose $L$ such that

$$\frac{C_3}{5}\left(1-\frac{V_0^{*2}}{gH_0^*}\right) \ln\left(\frac{L + \frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C} gH_0^*}{x_1 + \frac{\left(1-\frac{V_0^{*2}}{gH_0^*}\right)}{C} gH_0^*}\right) \geq \frac{\pi}{2}. \tag{5.42}$$

By hypothesis, there still exist a function $\eta$ that verifies (5.40), is positive and defined on $[0, L]$ with this choice of $L$. Hence, $\arctan(\eta(L)) < \pi/2$. But, as $\eta$ is positive, $\arctan(\eta(x_1)) > 0$ so we have from (5.40) and (5.42)

$$\arctan(\eta(L)) > \frac{\pi}{2} \tag{5.43}$$

Hence we have a contradiction. Therefore there exists $L_2 > 0$ such that for any $L > L_2$ there do not exist a basic $C^1$ Lyapunov function whatever the boundary conditions are. This ends the proof. □

# 6 Numerical estimation

From Theorem 3.4 when $kV_0^{*2}/H_0^* \geq C$ there exists explicit static boundary controls under which the general Saint-Venant equations are exponentially stable for the $C^1$ norm, whatever the length of the channel. When $kV_0^{*2}/H_0^* < C$ no such explicit result exists but in practice we can however use Theorem 3.2 to find the limit length under which stability can be guaranteed. We provide here some numerical estimations under reasonable conditions ($Q^* = 1\ m^2.s^{-1}$, $V_0^* = 0.5\ m.s^{-1}$, $k = 0.002$). On Figure 1, one can see that for a 50km channel with a constant slope such that $C = 2kV_0^{*2}/H_0^*$, $\eta$ exists and there is no problem. In Figure 2, we extended the channel until the limit length $L_{max}$ for this system and it appears that $L_{max} > 10^4 km$ and that $H^*(L_{max}) > 100m$ which is quite unrealistic in current hydraulic applications. This suggest that for nearly all hydraulic applications it will be possible to design boundary conditions such that there exists a basic $C^1$ Lyapunov function that ensures the stability of the system for the $C^1$ norm.

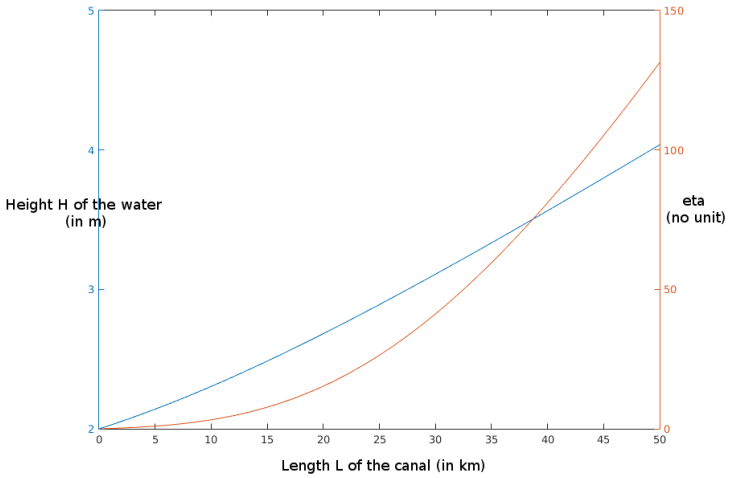


Figure 1: In the $x$-axis is represented the length of the water channel, in blue the height of the water for the stationary state and in red the value of $\eta$.

# 7 Further details

The previous results were derived for the $C^1$ and the $H^2$ norm but they can actually be extended to the $C^q$ and the $H^p$ norm with the same conditions, for any $p \in \mathbb{N}^* \setminus \{1\}$ and $q \in \mathbb{N}^*$. To show that, one only needs to realize that Theorem 3.1 and 3.2 in [13] and Theorem 4.1 (and therefore Theorem 4.2) are true for the $C^q$ and the $H^p$ norm with the same conditions, for any $p \in \mathbb{N}^* \setminus \{1\}$ and $q \in \mathbb{N}^*$.
In conclusion we gave explicit conditions on the gain of the feedbacks to get exponential stability for the $C^1$ norm for $2 \times 2$ quasilinear hyperbolic systems with propagation speeds of the same sign. In the general case we derived a simple criterion for the existence of basic $C^1$ Lyapunov functions and a practical way to derive admissible static feedback gains when this criterion is satisfied, simply by solving an ODE. We showed that under some conditions on the coefficients of the source term the existence of a $H^p$ and $C^q$ basic Lyapunov function for any $p \in \mathbb{N}^* \setminus \{1\}$ and $q \in \mathbb{N}^*$ are equivalent and that, in the general case, the existence of a $C^q$ Lyapunov function for any $q \in \mathbb{N}^*$ for some appropriate boundary controls implies the existence of a basic quadratic $H^p$ Lyapunov function for any $p \in \mathbb{N}^* \setminus \{1\}$ for some appropriate boundary controls. Finally we showed that when the friction is larger than the slope the general nonlinear Saint-Venant equations can be stabilized for the $C^1$ norm by means of simple pointwise feedback, and we gave explicit conditions on the feedbacks. When the the slope is larger than the friction no such general result can be shown. However, we

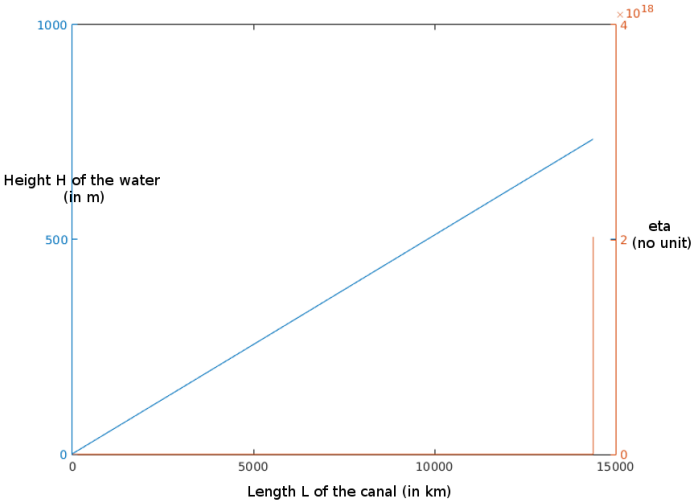


Figure 2: In the $x$-axis is represented the length of the water channel, in blue the height of the water in stationary state and in red the value of $\eta$, one can see the limit length $L_{max}$ at which $\eta$ explodes.

showed that for nearly all applications the Saint-Venant equations can be stabilized by means of such simple feedbacks.

# 8 Explicit form of $B$ and regularity of $A$ and $B$

Applying the transformation (2.3) on the system (2.8), $B$ is given by:

$$B(\mathbf{u}, x) = N(x)(F(\mathbf{Y})(\mathbf{Y}^*_x + (N^{-1}(x))'\mathbf{u}) + D(\mathbf{Y})). \tag{8.1}$$

As $\mathbf{Y}^*$ is a steady-state, it verifies the equation:

$$F(\mathbf{Y}^*)\partial_x \mathbf{Y}^* = -G(\mathbf{Y}^*). \tag{8.2}$$

Thus if we suppose that $F$ and $G$ are $C^p$ on $\mathcal{B}_{\mathbf{Y}^*,\eta_0}$, where $p \in \mathbb{N}^*$, as $F$ is strictly hyperbolic with non-vanishing eigenvalues, $\mathbf{Y}^*$ is $C^{p+1}$ on $[0, L]$. Therefore, using (2.5) and (8.1), $A$ and $B$ are also $C^p$ on $\mathcal{B}_{\mathbf{0},\eta_0} \times [0, L]$.

# 9 Counter exemple of the converse of Corollary 1 in general

As mentionned earlier, from [14] we know that for any $L > 0$ the system (5.4) corresponding to the Saint-Venant equations with boundary conditions (4.5) admits a basic quadratic $H^2$ Lyapunov function. However from Theorem 3.5 we know that there exists $L_{max}$ such that for $L > L_{max}$ the system does not admit a $C^1$ Lyapunov function whatever the boundary control is. This is a counter exemple of the Corollary 1 when one cannot ensure that $M_{12}$ and $M_{21}$ have the same sign.

# 10 Proof of Theorem 3.1

In this Theorem we rely mainly on Theorem 3.1 of [13]. Let a quasilinear $2 \times 2$ hyperbolic system of the form (2.8)–(2.9) be with $\Lambda_1\Lambda_2 > 0$. Without loss of generality we can assume that $\Lambda_1 > 0$ and $\Lambda_2 > 0$. As previously this system is equivalent to the system (4.2), (4.5) (see Section 4) and the existence of a basic $C^1$

(resp. basic quadratic $H^2$) Lyapunov function for this system is equivalent to the existence of a basic $C^1$ (resp. basic quadratic $H^2$) Lyapunov function for the system (2.8)–(2.9). Let us suppose that

$$\begin{aligned} G'(0) &= \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \text{ such that} \\ k_1^2 &< \exp\left( \int_0^L 2\frac{M_{11}(0,s)}{\Lambda_1} - 2\max\left( \frac{|a(s)|}{\Lambda_1}, \frac{|b(s)|}{\Lambda_2} \right) ds \right), \\ k_2^2 &< \exp\left( \int_0^L 2\frac{M_{22}(0,s)}{\Lambda_2} - 2\max\left( \frac{|a(s)|}{\Lambda_1}, \frac{|b(s)|}{\Lambda_2} \right) ds \right), \end{aligned} \tag{10.1}$$

then from (4.1) we have

$$\begin{aligned} G_1'(0) &= \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix} \text{ such that} \\ l_1^2 &< \exp\left( -2\int_0^L \max\left( \frac{|a(s)|}{\Lambda_1}, \frac{|b(s)|}{\Lambda_2} \right) ds \right), \\ l_2^2 &< \exp\left( -2\int_0^L \max\left( \frac{|a(s)|}{\Lambda_1}, \frac{|b(s)|}{\Lambda_2} \right) ds \right), \end{aligned} \tag{10.2}$$

where $G_1$ is defined in (4.5). Let us define $f_1 = f_2 = f$ by

$$f(x) = \exp\left( -2\int_0^x \max\left( \frac{|a(s)|}{\Lambda_1(s)}, \frac{|b(s)|}{\Lambda_2(s)} \right) ds \right), \forall x \in [0, L]. \tag{10.3}$$

Then $f$ is positive and $C^1$ on $[0, L]$ and

$$\begin{aligned} &f' \leq -2\frac{|a(x)|}{\Lambda_1(x)}\frac{f^{3/2}}{f^{1/2}}, \\ &\text{and } f' \leq -2\frac{|b(x)|}{\Lambda_2(x)}\frac{f^{3/2}}{f^{1/2}}. \end{aligned} \tag{10.4}$$

Besides, as $G_1'(0)$ is diagonal, if we define $\rho_\infty : M \to \min(\|\Delta M \Delta^{-1}\|_\infty : \Delta \in D_2^+)$ where $D_2^+$ is the space of diagonal $2 \times 2$ matrix with positive coefficients we have from (10.2):

$$\rho_\infty(G_1'(0)) = \max(l_1, l_2) < \sqrt{\frac{f(L)}{f(0)}}. \tag{10.5}$$

Therefore, from (10.4) and (10.5) and Theorem 3.1 in [13], there exists a basic $C^1$ Lyapunov function and therefore the system (2.8)–(2.9) is stable for the $C^1$ norm.
Let us now show the stability for the $H^2$ norm by showing that there exists a basic quadratic $H^2$ Lyapunov function for the system (4.2), (4.5). From (10.5) and by continuity we know that there exists $\sigma > 0$ such that

$$\max(l_1, l_2) < \sqrt{\frac{g(L)}{g(0)}}, \tag{10.6}$$

where $g$ is defined by

$$g(x) = \exp\left( -2\int_0^x \max\left( \frac{|a(s)|}{\Lambda_1(s)}, \frac{|b(s)|}{\Lambda_2(s)} \right) ds \right) - \sigma, \forall x \in [0, L]. \tag{10.7}$$

We want now to be able to apply Theorem 6.6 in [3] which would give the result. Note that if we now define $q_1 = g/\Lambda_1$ and $q_2 = g/\Lambda_2$ we have

$$\begin{aligned} &-\begin{pmatrix}(\Lambda_1 q_1)' & 0\\ 0 & (\Lambda_2 q_2)'\end{pmatrix} + \begin{pmatrix} q_1 & 0\\ 0 & q_2\end{pmatrix}\begin{pmatrix} 0 & a\\ b & 0\end{pmatrix} + \begin{pmatrix} 0 & a\\ b & 0\end{pmatrix}^T \begin{pmatrix} q_1 & 0\\ 0 & q_2\end{pmatrix}\\ &= \begin{pmatrix} -g' & g\left(\frac{b}{\Lambda_2}+\frac{a}{\Lambda_1}\right)\\ g\left(\frac{b}{\Lambda_2}+\frac{a}{\Lambda_1}\right) & -g'\end{pmatrix}, \end{aligned} \tag{10.8}$$

and this matrix is positive definite as $g' < 0$ and:

$$g'^2 - g^2\left(\frac{b}{\Lambda_2}+\frac{a}{\Lambda_1}\right)^2 > \left(4\max\left(\frac{|a|}{\Lambda_1},\frac{|b|}{\Lambda_2}\right)^2 - \left(\frac{b}{\Lambda_2}+\frac{a}{\Lambda_1}\right)^2\right) g^2 \geq 0. \tag{10.9}$$

Besides

$$\begin{aligned} &\begin{pmatrix} q_1(L)\Lambda_1(L) & 0\\ 0 & q_2(L)\Lambda_2(L)\end{pmatrix} - G_1'(0)^T \begin{pmatrix} q_1(0)\Lambda_1(0) & 0\\ 0 & q_2(0)\Lambda_2(0)\end{pmatrix} G_1'(0)\\ &= \begin{pmatrix} g(L) - (1-\sigma)l_1^2 & 0\\ 0 & g(L) - (1-\sigma)l_2^2\end{pmatrix} \end{aligned} \tag{10.10}$$

is positive semi-definite from (10.6). Therefore from Theorem 6.6 in [3] we know that the system admits a basic quadratic $H^2$ Lyapunov function and is stable for the $H^2$ norm. This ends the proof of Theorem 3.1.

**Remark 10.1.** *Although the existence of a basic quadratic $H^2$ Lyapunov function is not stated directly in Theorem 6.6 in [3] one can easily check that the theorem actually proves the $H^2$ stability by showing that there exists a basic quadratic $H^2$ Lyapunov function as defined in Definition 2.4 (see in particular Lemma 6.8 in [3]).*

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