INTEGRAL KERNELS ON COMPLEX SYMMETRIC SPACES AND FOR THE DYSON BROWNIAN MOTION

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Abstract. In this article, we consider flat and curved Riemannian symmetric spaces in the complex case and we study their basic integral kernels, in potential and spherical analysis: heat, Newton, Poisson kernels and spherical functions, *i.e.* the kernel of the spherical Fourier transform.

We introduce and exploit a simple new method of construction of these W-invariant kernels by alternating sums. We then use the alternating sum representation of these kernels to obtain their asymptotic behavior. We apply our results to the Dyson Brownian Motion on $\mathbf{R}^d$.

1. Introduction and notations

Analysis on Riemannian symmetric spaces of Euclidean type, also called flat symmetric spaces, continues to develop in recent years ([12, 14, 26, 27]). Its importance is due to its relationship with Dunkl analysis ([6, 23]), in which symmetric spaces of Euclidean type constitute the "geometric case", frequently used as a model case in most challenging open problems of Dunkl theory. A much less known application of symmetric spaces of Euclidean type is stochastic analysis of Dyson Brownian Motion, one of the most important models of non-colliding particles (see e.g. [19]).

The objective of this paper is to study basic integral kernels, in potential theory and spherical analysis: heat, Newton, Poisson kernels, Green function and spherical functions (*i.e.* the kernel of the spherical Fourier transform), in the set-up of flat and curved symmetric spaces of complex type.

Our main results on the exact form and asymptotics of the heat, Poisson and Newton kernels (Theorems 2.1, 4.9, 4.11 and Corollaries 6.5 and 6.6) are crucial for the future development of the potential theory on flat and curved symmetric spaces of complex type, and for the potential theory of the Dyson Brownian Motion. These results are a starting point of research and a source of conjectures for the corresponding kernels in the Weyl-invariant Dunkl setting (for the rank one case, refer to [11]).

The main result on asymptotics of the spherical functions contained in Theorem 5.5 is important from the point of view of spherical analysis on symmetric spaces, because it generalizes significantly the results of Helgason in [14], of Narayanana, Pasquale and Pusti in [21] and of Schapira in [25], for the flat and curved symmetric spaces in the complex case, cf. Remark 5.9.

We recall now some basic terminology and facts about symmetric spaces associated to Cartan motion groups.

Let G be a semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of G. We recall the definition of the Cartan motion group and the flat symmetric space associated with the semisimple Lie group G with maximal compact subgroup K. The Cartan motion group is the semi-direct product $G_0 = K \rtimes \mathfrak{p}$ where the multiplication is defined by $(k_1, X_1) \cdot (k_2, X_2) = (k_1\, k_2,\ \mathrm{Ad}(k_1)(X_2) + X_1)$. The associated flat symmetric space is then $M = \mathfrak{p} \simeq G_0/K$ (the action of G_0 on $\mathfrak{p}$ is given by $(k, X) \cdot Y = \mathrm{Ad}(k)(Y) + X$).

The spherical functions for the symmetric space M are then given by

$$\psi_\lambda(X) = \int_K e^{\lambda(\,\mathrm{Ad}(k)(X))}\, dk \tag{1.1}$$

where λ is a complex linear functional on $\mathfrak{a} \subset \mathfrak{p}$, a Cartan subalgebra of the Lie algebra of G. To extend λ to $X \in \mathrm{Ad}(K)\mathfrak{a} = \mathfrak{p}$, one uses $\lambda(X) = \lambda(\pi_{\mathfrak{a}}(X))$ where $\pi_{\mathfrak{a}}$ is the orthogonal projection with respect to the Killing form (denoted throughout this paper by $\langle \cdot, \cdot \rangle$). Note also that the spherical function for the symmetric space G/K is given by

$$\phi_\lambda(g) = \int_K e^{(\lambda-\rho)(H(g\,k))}\, dg \tag{1.2}$$

where λ is a complex linear functional on $\mathfrak{a}$ and the map H is defined via the Iwasawa decomposition of G, namely $g = k\,e^{H(g)}\,n \in K\,A\,N$ and $\rho = (1/2)\,\sum_{\alpha>0}\,m_\alpha\,\alpha$. Note that in [14, 15, 16], λ is replaced by $i\,\lambda$.

Throughout this paper, we suppose that G is a semisimple complex Lie group. The complex root systems are respectively A_n for $n \geq 1$ (where $\mathfrak{p}$ consists of the $n \times n$ hermitian matrices with trace 0), B_n for $n \geq 2$ (where $\mathfrak{p} = i\,\mathfrak{so}(2\,n+1)$), C_n for $n \geq 3$ (where $\mathfrak{p} = i\,\mathfrak{sp}(n)$) and D_n for $n \geq 4$ (where $\mathfrak{p} = i\,\mathfrak{so}(2\,n)$) for the classical cases and the exceptional root systems E_6, E_7, E_8, F_4 and G_2.

Let Δ be the Laplace-Beltrami operator on M and Δ^W its restriction to W-invariant functions on $\mathfrak{a}$ where W is the corresponding Weyl group. Recall the formula

$$\Delta^W f = \pi^{-1}\,\Delta^{\mathbf{R}^d}(\pi\,f), \tag{1.3}$$

where $\pi(X) = \prod_{\alpha>0}\,\alpha(X)$ (see [16, Chap. II, Theorem 5.37]) in the Euclidean case.

In Section 2, we introduce and exploit a simple new method of construction of important W-invariant kernels on the space M. This approach will allow us to provide asymptotic for these kernels. This method uses the simple form of the operator Δ^W given in (1.3) and our knowledge of the counterparts of the kernels on $\mathbf{R}^d$ as given in Table 1.

We also apply our method to provide a new derivation of the formula for the spherical functions (1.1) in the complex case. We conclude Section 2 by discussing other formulas for the heat kernel in the flat case.

In Section 3, we discuss the corresponding formulas in the curved case still based on the method of alternating sums. This approach lends itself well in the case of the heat kernel and for the spherical functions. For the Newton kernel, the formulas are more complicated and in the case of the Poisson kernels, the answer does not seem to be known.

In Section 4, we discuss the asymptotic behaviour of the Poisson kernel especially when one or both arguments are singular. These results translate well to the Newton kernel.

In Section 5, we compute asymptotics for the spherical functions $\psi_\lambda(Y)$ which can prove challenging when either λ or Y are singular. Our results depend on a property we call "Killing-max" namely the property that for X, $Y \in \overline{\mathfrak{a}^+}$, $\langle X, w\cdot Y\rangle = \langle X, Y\rangle$ if and only if $w \in W_X\,W_Y$ where $W_X = \{w \in W\colon w\cdot X = X\}$. It is known that this property is verified when either X or Y is non singular [14]. We prove in Appendix A, using the classification of Lie algebras, that the Killing-max holds in almost all cases (only in the cases related to the root systems E_6, E_7 and E_8 is the question left unanswered).

We conclude with Section 6 where we apply the previous results to the heat kernel and Poisson and Newton kernels for the Dyson Brownian Motion.

2. Kernels on flat symmetric spaces in the complex case

2.1. **Definitions.** We first recall the classical integral kernels on $\mathbf{R}^d$ in Table 1.

The integral kernels on the flat symmetric space M are considered with respect to the invariant measure $\mu(dY) = \pi^2(Y)\,dY$ on M. Their definition is much the same as in the first column of Table 1 except that the operator Δ^W in (1.3) is used in place of $\Delta^{\mathbf{R}^d}$ and that the measure dY in column 3 of that table is replaced by the measure $\mu(dY)$. We will use the superscript W to distinguish these operators from their Euclidean counterparts given in Table 1. Recall also the formula

$$N^W(X,Y) = \int_0^\infty p_t^W(X,Y)\,dt \tag{2.1}$$

which has a counterpart in the Euclidean case.

2.2. **The method of alternating sums for constructing kernels on M.** This method will be introduced and used in the proof of Theorem 2.1 below.

Theorem 2.1. *Let M be a symmetric space of Euclidean type with G a complex simple Lie group of rank d. Then the following formulas hold.*

(1) *The heat kernel on M is given by*

$$p_t^W(X,Y) = \frac{1}{|W|\,(4\,\pi\,t)^{d/2}\,\pi(X)\,\pi(Y)}\,\sum_{w\in W}\,\epsilon(w)e^{-\frac{|X-w\cdot Y|^2}{4\,t}} \tag{2.2}$$

PDE	Kernel	Solution
$\begin{cases} \Delta^{\mathbf{R}^d} u(X,t) = \frac{\partial}{\partial t} u(X,t) \\ \lim_{t\to 0^+} u(X,t) = f(X) \end{cases}$ $X \in \mathbf{R}^d,\ t > 0$	$p_t(X,Y) = \dfrac{e^{-\frac{\lvert X-Y\rvert^2}{4\,t}}}{(4\,\pi\,t)^{d/2}}$	$u(X,t) = \displaystyle\int_{\mathbf{R}^d} f(Y)\, p_t(X,Y)\, dY$
$\begin{cases} \Delta^{\mathbf{R}^d} u(X) = f(X) \quad \text{on } \mathbf{R}^d \\ \lvert u(X)\rvert \to 0 \text{ as } X \to \infty \end{cases}$	$N(X,Y) = \Phi(X-Y),$	$u(X) = \displaystyle\int_{\mathbf{R}^d} f(Y)\, N(X,Y)\, dY$
$\begin{cases} \Delta^{\mathbf{R}^d} u(X) = 0 \quad \text{on } B(x_0,r) \\ u(X) = f(X) \quad \text{on } \partial B(x_0,r) \end{cases}$	$P(X,Y) = \dfrac{r^2 - \lvert X-X_0\rvert^2}{w_d\, r\, \lvert X-Y\rvert^d}$	$u(X) = \displaystyle\int_{\partial B(x_0,r)} f(Y)\, P(X,Y)\, dY$
$\begin{cases} \Delta^{\mathbf{R}^d} u(X) = f(X) \quad \text{on } B = B(0,1) \\ u(X) = 0 \quad\quad\quad\ \text{on } \partial B \end{cases}$	$G_B(X,Y) = \Phi(X-Y)$ $-\Phi(\lvert X\rvert\,(Y - X/\lvert X\rvert^2))$	$u(X) = \displaystyle\int_{\partial B} f(Y)\, G_B(X,Y)\, dY$

where $w_d = 2\,\pi^{d/2}/\Gamma(d/2)$ (the surface area of a sphere of radius 1 in $\mathbf{R}^d$) and

$$\Phi(X) = \begin{cases} \frac{1}{2\,\pi}\, \ln\lvert X\rvert & \text{if } d = 2 \\ \frac{1}{(2-d)\, w_d}\, \lvert X\rvert^{2-d} & \text{if } d \geq 3 \end{cases}.$$

TABLE 1. The heat kernel p_t, the Newton kernel N, the Poisson kernel P and the Green kernel G_B for $\mathbf{R}^d$

(2) *The Newton kernel on M is given by*

$$N^W(X,Y) = \frac{1}{4\,\pi\,\pi(X)\,\pi(Y)} \sum_{w\in W} \epsilon(w)\, \ln\lvert X - w\cdot Y\rvert \text{ when } d = 2, \tag{2.3}$$

$$N^W(X,Y) = \frac{1}{\lvert W\rvert\,(2-d)\, w_d\, \pi(X)\,\pi(Y)} \sum_{w\in W} \frac{\epsilon(w)}{\lvert X - w\cdot Y\rvert^{d-2}} \text{ when } d \geq 3.$$

(3) *The Poisson kernel of the open unit ball B is given for $X \in B$ and $Y \in \partial B$ by*

$$P^W(X,Y) = \frac{1-\lvert X\rvert^2}{\lvert W\rvert\, w_d\, \pi(X)\,\pi(Y)} \sum_{w\in W} \frac{\epsilon(w)}{\lvert X - w\cdot Y\rvert^d} \tag{2.4}$$

(4) *The Green function of the unit ball is given by*

$$G_B^W(X,Y) = \frac{1}{\lvert W\rvert\,\pi(X)\,\pi(Y)} \sum_{w\in W} \epsilon(w) G_B(X,Y), \tag{2.5}$$

where $G_B(X,Y)$ is the classical Green function of the unit ball B in $\mathbf{R}^d$ (refer to Table 1).

Proof. It is based on the following steps:

(1) Write a kernel on $\mathbf{R}^d$ where d is the rank of M;
(2) Exploit the formula (1.3);
(3) Apply the W-invariance (the kernels on M must be W-invariant).

We give the proof in the Poisson kernel case; the other proofs are similar. The Poisson kernel of $B(0,1)$ in the Euclidean case is

$$P(X,Y) = \frac{1-\lvert X\rvert^2}{w_d\, \lvert X-Y\rvert^d}.$$

If f is harmonic with respect to Δ^W then $\pi\, f$ is harmonic with respect to $\Delta^{\mathbf{R}^d}$. Hence

$$\pi(X)\, f(X) = \int_{\partial B} \frac{1-\lvert X\rvert^2}{w_d\, \lvert X-Y\rvert^d}\, \pi(Y)\, f(Y)\, dY.$$

This is equivalent to

$$f(X) = \int_{\partial B} \frac{1-\lvert X\rvert^2}{w_d\, \pi(X)\,\pi(Y)}\, \frac{1}{\lvert X-Y\rvert^d}\, f(Y)\, \pi(Y)^2\, dY.$$

The reproducing kernel $\frac{1-|X|^2}{w_d\,\pi(X)\,\pi(Y)}\,\frac{1}{|X-Y|^d}$ is not W-invariant. We write the last equation $|W|$ times, replacing X by $w\cdot X$

$$f(X)=f(w\cdot X)=\int_{\partial B}\frac{1-|X|^2}{w_d\,\pi(w\cdot X)\,\pi(Y)}\,\frac{1}{|w\cdot X-Y|^d}\,f(Y)\,\pi(Y)^2\,dY$$
$$=\int_{\partial B}\frac{1-|X|^2}{w_d\,\pi(X)\,\pi(Y)}\,\frac{\epsilon(w)}{|X-w\cdot Y|^d}\,f(Y)\,\pi(Y)^2\,dY$$

and we sum up the $|W|$ equations. We obtain

$$f(X)=\frac{1}{|W|\,w_d}\int_{\partial B}\frac{1-|X|^2}{\pi(X)\,\pi(Y)}\sum_{w\in W}\frac{\epsilon(w)}{|X-w\cdot Y|^d}\,f(Y)\,\pi(Y)^2\,dY.$$

□

For the root systems of type A, we obtain the following determinantal formula for the heat kernel on M.

Corollary 2.2. *Consider the complex flat symmetric space M with the root system $\Sigma=A_d$. Let $g_t(u,v)=\frac{1}{\sqrt{4\,\pi\,t}}\,e^{-|u-v|^2/4t}$ be the 1-dimensional classical heat kernel. The heat kernel on M is given by*

$$p_t^W(X,Y)=\frac{1}{|W|\,\pi(X)\,\pi(Y)}\,\det\left(g_t(x_i,y_j)\right). \tag{2.6}$$

Proof. Formula (2.6) follows from Theorem 2.1(1) and the definition of determinant. □

Let us resume the method of alternating sums, applied in the proof of Theorem 2.1. An Euclidean kernel $\mathcal{K}^{\mathbf{R}^d}(X,Y)$ (heat kernel, potential kernel, Poisson kernel, ...) for the Laplacian $\Delta^{\mathbf{R}^d}$ is transformed in the following way into the kernel $\mathcal{K}^W$ acting on W-invariant functions on M:

$$\mathcal{K}^W(X,Y)=\frac{1}{|W|\,\pi(X)\,\pi(Y)}\sum_{w\in W}\epsilon(w)\mathcal{K}_{\mathbf{R}^d}(X,w\cdot Y). \tag{2.7}$$

The formulas (2.2)-(2.6) are new. However, in the harmonic analysis of flat symmetric spaces of complex type the alternating sum formula (2.8) for a spherical function on M given below is well known (see [16, Chap. IV, Proposition 4.10]). Our method leading to formula (2.7) will allow us to give a new simple proof of formula (2.8) given below.

We preface the result by a technical lemma which gives the values of alternating sums of the powers of the scalar product $\langle\lambda,X\rangle$.

Lemma 2.3. *Let γ be the number of positive roots and $k\in\mathbf{N}$.*

(i) *When $k=0,\,1,\,\dots\gamma-1$, we have $\sum_{w\in W}\epsilon(w)\,\langle\lambda,w\cdot X\rangle^k=0$ for all λ and x.*
(ii) $\sum_{w\in W}\epsilon(w)\,\langle\lambda,w\cdot X\rangle^\gamma=\frac{2^\gamma\gamma!}{\pi(\rho)}\,\pi(\lambda)\,\pi(X)$ *for all λ and x.*
(iii) *If $k\geq\gamma+1$ and $\lambda=0$ or $X=0$ then*

$$\frac{\sum_{w\in W}\epsilon(w)\,\langle\lambda,w\cdot X\rangle^k}{\pi(\lambda)\,\pi(X)}=0.$$

Proof. The polynomial $S_k(\lambda,X)=\sum_{w\in W}\epsilon(w)\langle\lambda,w\cdot X\rangle^k$ is skew-symmetric in λ and X and therefore it can be factorized as $S_k(\lambda,X)=\pi(\lambda)\,\pi(X)\,R_k(\lambda,X)$ where $R_k(\lambda,X)$ is a polynomial.

When $k\leq\gamma-1$, the degree of $S_k(\lambda,X)$ is at most k in λ and k in X, whereas the degree of $\pi(\lambda)\,\pi(X)$ equals γ in λ and γ in X. Thus $S_k(\lambda,X)=0$.

When $k\geq\gamma+1$ and $s,\,t\geq0$, the numerator $S_k(\lambda,X)$ satisfies $S_k(s\lambda,t\,X)=s^kt^kS_k(\lambda,X)$, so $R_k(s\lambda,t\,X)=s^{k-\gamma}t^{k-\gamma}R_k(\lambda,X)$ and thus $R_k(0,X)=0$, $R_k(\lambda,0)=0$ and (iii) follows.

When $k=\gamma$ we have $S_\gamma(\lambda,X)=C\,\pi(\lambda)\,\pi(X)$. We use the differential operator $\partial(\pi)$ verifying

$$\partial(\pi)|_\lambda\,e^{\langle\lambda,X\rangle}=\pi(X)e^{\langle\lambda,X\rangle}.$$

By comparing the terms which are constant in λ on both sides of this equality and using an argument by homogeneity, we obtain

$$\partial(\pi)|_\lambda \ \langle \lambda, X\rangle^\gamma = \gamma!\,\pi(X) \ \text{ and } \ \partial(\pi)|_\lambda \ S_\gamma(\lambda, X) = \gamma! \sum_{w\in W} \epsilon(w)\,\pi(w\cdot X) = \gamma!|W|\,\pi(X) \neq 0.$$

In order to determine the constant C, we apply the operator $\partial(\pi)$ to both sides of the last equality and use the formula

$$\partial(\pi)\,\pi = \frac{|W|\,\pi(\rho)}{2^\gamma},$$

see [16, Chap. II, Cor. 5.36] (note that a different normalization of ρ is used in Helgason's book). □

Proposition 2.4. *A spherical function $\psi_\lambda(X)$ on M is given by the formula*

$$\psi_\lambda(X) = \frac{\pi(\rho)}{2^\gamma \pi(\lambda)\,\pi(X)} \sum_{w\in W} \epsilon(w) e^{\langle \lambda, w\cdot X\rangle}, \tag{2.8}$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha \alpha = \sum_{\alpha\in\Sigma^+} \alpha$ and $\gamma = |\Sigma^+|$ is the number of positive roots.

Proof. We give a new simple proof of formula (2.8), based on the method of alternating sums. We start with the property

$$\Delta_X^{\mathbf{R}^d} e^{\langle \lambda, X\rangle} = |\lambda|^2\, e^{\langle \lambda, w\cdot X\rangle}$$

Similarly as in the proof of Theorem 2.1, we obtain that

$$\Delta_X^W \left(\frac{\sum_{w\in W} \epsilon(w)\, e^{\langle \lambda, X\rangle}}{\pi(X)} \right) = |\lambda|^2 \, \frac{\sum_{w\in W} \epsilon(w)\, e^{\langle \lambda, X\rangle}}{\pi(X)}.$$

By Lemma 2.3,

$$\sum_{w\in W} \frac{\epsilon(w)\, e^{\langle \lambda, X\rangle}}{\pi(X)}\bigg|_{X=0} = \frac{2^\gamma\,\pi(\lambda)}{\pi(\rho)}.$$

Thus $\psi(X) = \frac{\pi(\rho)}{2^\gamma\,\pi(\lambda)\,\pi(X)} \sum_{w\in W} \epsilon(w) e^{\langle \lambda, w\cdot X\rangle}$ is a W-invariant function such that $\Delta^W \psi = |\lambda|^2\psi$ and $\psi(0) = 1$. This means that $\psi = \psi_\lambda$. □

2.3. **Relations between the heat kernel and spherical functions.** In this section we give formulas relating the heat kernel $p_t^W(X, Y)$ with the spherical functions on M.

Proposition 2.5. *Let M be a flat symmetric space of complex type. The following formula holds*

$$p_t(X, Y) = \frac{1}{|W|\,2^d\,\pi^{d/2}\,\pi(\rho)}\, t^{-\frac{d}{2}-\gamma}\, e^{\frac{-|X|^2-|Y|^2}{4t}}\, \psi_X\left(\frac{Y}{2t}\right). \tag{2.9}$$

Proof. We start with formula (2.2), in which we write

$$e^{-\frac{|X-w\cdot Y|^2}{4\,t}} = e^{\frac{-|X|^2-|Y|^2}{4t}} e^{\frac{\langle X, w\cdot Y\rangle}{2t}}.$$

By formula (2.8) we obtain

$$\frac{1}{\pi(X)\,\pi(Y)} \sum_{w\in W} \epsilon(w)\, e^{\frac{\langle X, w\cdot Y\rangle}{2t}} = \frac{1}{(2\,t)^\gamma\,\pi(X)\,\pi\left(\frac{Y}{2t}\right)} \sum_{w\in W} \epsilon(w)\, e^{\frac{\langle X, w\cdot Y\rangle}{2t}} = t^{-\gamma}\,\frac{1}{\pi(\rho)}\,\psi_X\left(\frac{Y}{2t}\right).$$

and formula (2.9) follows. □

Let us finish by pointing out other formulas relating the heat kernel $p_t(X, Y)$ with the centered heat kernel $h_t(X) = p_t(X, 0)$, with the product formula kernel $k(H, x, y)$ and with the spherical functions ψ_λ.

The density $h_t(X) = p_t(X, 0)$ is well known when G is a semisimple complex Lie group; we can simply deduce it from formula (2.9)

$$h_t(X) = p_t(X, 0) = \frac{1}{|W|\,2^d\,\pi^{d/2}\,\pi(\rho)}\, t^{-\frac{d}{2}-\gamma}\, e^{-|X|^2/(4\,t)}. \tag{2.10}$$

Remark 2.6. *The passage from $h_t(X)$ to $p_t(X,Y)$ is well understood at the group level:*

$$p_t(g,h) = h_t(h^{-1}g),$$

which is equivalent to

$$p_t(X,Y) = \int_K h_t(e^{-Y}\,k^{-1}\,e^X)\,dk$$

and to

$$p_t(X,Y) = \int_{\mathfrak{a}} h_t(H)\,k(H,-Y,X)\,\pi(H)\,dH, \tag{2.11}$$

where the last formula contains the product formula kernel k which is defined by

$$\int_{\mathfrak{a}} \psi_\lambda(e^H)\,k(H,X,Y)\,\pi(H)\,dH = \psi_\lambda(e^X)\,\psi_\lambda(e^Y) = \int_K \psi_\lambda(e^X\,k\,e^Y)\,dk.$$

Although we provide a more direct proof, the preceding remark was instrumental in obtaining the next result.

Proposition 2.7. *The following formula holds:*

$$p_t(X,Y) = C\int_{\mathfrak{a}} e^{-|\lambda|^2 t}\,\psi_{i\,\lambda}(X)\,\psi_{-i\,\lambda}(Y)\,\pi(\lambda)^2\,d\lambda \tag{2.12}$$

where $C > 0$.

Proof. We show that the right hand side of equation (2.12) satisfies the definition of the heat kernel. For a test function f, consider

$$u(X,t) = C\int_{\mathfrak{a}}\int_{\mathfrak{a}} e^{-|\lambda|^2 t}\,\psi_{i\,\lambda}(X)\,\psi_{-i\,\lambda}(Y)\,K\,\pi(\lambda)^2\,d\lambda\,f(Y)\,\pi(Y)^2\,dY$$

where $K\,\pi(\lambda)^2\,d\lambda$ is Plancherel measure.

The fact that $\Delta^W\,u(X,t) = \frac{\partial}{\partial t}\,u(X,t)$ follows easily from the fact that $\Delta^W\,\psi_{i\,\lambda}(X) = -|\lambda|^2\,\psi_{i\,\lambda}(X)$ and $\frac{\partial}{\partial t}\,e^{-|\lambda|^2 t} = e^{-|\lambda|^2 t}$. Now, using Fubini's theorem,

$$\begin{aligned} u(X,t) &= C\,K\int_{\mathfrak{a}} e^{-|\lambda|^2 t}\left[\int_{\mathfrak{a}} \psi_{-i\,\lambda}(Y)\,f(Y)\,\pi(Y)^2\,dY\right]\psi_{i\,\lambda}(X)\,\pi(\lambda)^2\,d\lambda \\ &= C\,K\int_{\mathfrak{a}} e^{-|\lambda|^2 t}\,\tilde{f}(\lambda)\,\psi_{i\,\lambda}(X)\,\pi(\lambda)^2\,d\lambda \end{aligned}$$

which tends to $f(X)$ as $t\to\infty$ by the dominated convergence theorem. □

Remark 2.8. *In the above reasoning, the constant C depends on the normalization of the various measures (refer to* [16, Theorem 7.5]*). In order to be consistent with our previous formulas for p_t, we need*

$$C = \frac{(-1)^\gamma\,4^\gamma}{|W|\,2^d\,\pi^d\,\pi(\rho)^2}.$$

To see this, it suffices to let $Y = 0$ in (2.12)*, integrate with respect to λ and compare the result with* (2.10)*.*

Formula (2.12), after necessary adaptations, is true in the whole generality of Riemannian symmetric spaces (the proof is essentially the same as for Proposition 2.7):

Proposition 2.9. *The following formula holds*

$$p_t(X,Y) = C\int_{\mathfrak{a}^*} e^{-(|\lambda|^2+|\rho|^2)\,t}\,\phi_{i\,\lambda}(X)\,\phi_{-i\,\lambda}(Y)\,\frac{d\lambda}{|c(\lambda)|^2}. \tag{2.13}$$

Remark 2.10. *The heat kernel estimates on symmetric spaces (*[1] *and references therein) are based on the inverse spherical Fourier transform formula*

$$h_t(X) = C\int_{\mathfrak{a}} e^{-(|\lambda|^2+|\rho|^2)\,t}\,\phi_{i\,\lambda}(X)\,\frac{d\lambda}{|c(\lambda)|^2},$$

which is a special case of (2.13) *when $Y = 0$. Thus one may hope that estimates of $p_t(X,Y)$ can be deduced from* (2.13)*.*

3. Curved symmetric spaces $X = G/K$ with G complex

The alternating sum formulas given in Section 2.2 have analogs in the curved complex case, considered in this section. To underline the difference with the flat case, we denote the spherical and potential analysis objects on X with a tilde ($\tilde{}$). The kernels in this section are with respect to the invariant measure $\delta(Y)\,dY$ where

$$\delta(Y) = \prod_{\alpha>0} \sinh^2 \alpha(Y).$$

The following method of construction of kernels is similar to the one presented in Section 2.2.

(1) Exploit the formula for the Laplace-Beltrami operator on X ([16, Chap. II, Theorem 5.37]):

$$\tilde{\Delta}^W f = \delta^{-1/2}\,(\Delta^{\mathbf{R}^d} - |\rho|^2)(\delta^{1/2}\,f)$$

(2) Apply the W-invariance.

In this way, the Euclidean kernel $\mathcal{K}_{\Delta^{\mathbf{R}^d}-|\rho|^2}(X,Y)$ (heat, potential, Poisson, ...) for the operator $\Delta^{\mathbf{R}^d} - |\rho|^2$ is transformed into the kernels $\tilde{\mathcal{K}}$ for G/K:

$$\tilde{\mathcal{K}}(X,Y) = \frac{1}{\delta^{1/2}(X)\,\delta^{1/2}(Y)} \sum_{w\in W} \epsilon(w)\mathcal{K}_{\Delta^{\mathbf{R}^d}-|\rho|^2}(X, w\cdot Y). \tag{3.1}$$

3.1. The heat kernel.

Proposition 3.1. *The following formula holds for the heat kernel of symmetric spaces of non-Euclidean type with G complex*

$$\tilde{p}_t(X,Y) = \frac{e^{-|\rho|^2 t}}{|W|\,(4\,\pi\,t)^{d/2}\,\delta^{1/2}(X)\,\delta^{1/2}(Y)} \sum_{w\in W} \epsilon(w)\,e^{-\frac{|X-w\cdot Y|^2}{4\,t}}. \tag{3.2}$$

Proof. Let $p_t(X,Y)$ be the Euclidean heat kernel. Then the heat kernel for $\Delta^{\mathbf{R}^d} - |\rho|^2$ is

$$e^{-|\rho|^2 t} p_t(X,Y).$$

We apply the construction (3.1) to the kernel $e^{-|\rho|^2 t}\,p_t(X,Y)$. □

For the root systems of type A, we obtain the following determinantal formula for the heat kernel on X.

Corollary 3.2. *Consider the complex curved symmetric space X with the root system $\Sigma = A_d$. Let $g_t(u,v) = \frac{1}{\sqrt{4\,\pi\,t}}\,e^{-|u-v|^2/4t}$ be the 1-dimensional classical heat kernel. The heat kernel on M is given by*

$$\tilde{p}_t(X,Y) = \frac{e^{-|\rho|^2 t}}{|W|\,\delta^{1/2}(X)\,\delta^{1/2}(Y)}\,\det\left(g_t(x_i,y_j)\right). \tag{3.3}$$

Proof. Formula (3.3) follows from Proposition 3.1 and the definition of determinant. □

Proposition 3.3. Relation between heat kernels in the flat and curved cases. *We have*

$$\tilde{p}_t(X,Y) = e^{-|\rho|^2 t}\,\frac{\pi(X)\pi(Y)}{\delta^{1/2}(X)\,\delta^{1/2}(Y)}\,p_t^W(X,Y). \tag{3.4}$$

Proof. We compare the formulas (3.2) and (2.2). □

Proposition 3.4. Relation between heat kernel and spherical functions in the curved case. *Let X be a curved symmetric space of complex type. Let ϕ_λ denote a spherical function on X. The following formula holds*

$$\tilde{p}_t(X,Y) = \frac{2^{\gamma-d}}{|W|\,\pi^{d/2}\,\pi(\rho)}\,t^{-\frac{d}{2}-\gamma}\,e^{\frac{-|X|^2-|Y|^2}{4t}}\,\phi_X\left(\frac{Y}{2t}\right)\,f(t,x,y) \tag{3.5}$$

where the correction factor $f(t,x,y)$ equals

$$f(t,x,y) = \frac{\pi(X)\,t^\gamma\,\delta^{1/2}\left(\frac{Y}{2t}\right)}{\delta^{1/2}(X)\,\delta^{1/2}(Y)}.$$

Proof. We use formulas (2.9), (3.4) and the formula relating spherical functions in the flat and curved cases (refer to [12, 16]). □

The density $\tilde{h}_t(X) = \tilde{p}_t(X, 0)$ is well known (refer to [9]). We can deduce it from formula (3.5), taking the limit as $Y \to 0$

$$\tilde{h}_t(X) = \tilde{p}_t(X, 0) = \frac{2^{-d}}{|W|\,\pi(\rho)\,\pi^{d/2}}\, t^{-\frac{d}{2}-\gamma} e^{\frac{-|X|^2}{4t}}\, \frac{\pi(X)}{\delta^{1/2}(X)}.$$

The passage from $\tilde{h}_t(X)$ to $\tilde{p}_t(X, Y)$ is well known at the group level:

$$\tilde{p}_t(g, h) = \tilde{h}_t(h^{-1}g),$$

which is equivalent to

$$\tilde{p}_t(X, Y) = \int_K \tilde{h}_t(e^{-Y}\, k^{-1}\, e^X)\, dk$$

and to

$$\tilde{p}_t(X, Y) = \int_{\mathfrak{a}} \tilde{h}_t(H)\, k(H, -Y, X)\, \delta(H)\, dH, \tag{3.6}$$

where the last formula contains the product formula kernel k which is defined by

$$\int_{\mathfrak{a}} \phi_\lambda(e^H)\, k(H, X, Y)\, \delta(H)\, dH = \phi_\lambda(e^X)\, \phi_\lambda(e^Y) = \int_K \phi_\lambda(e^X\, k\, e^Y)\, dk.$$

3.2. **Other kernels.** In order to express the potential kernel $\tilde{N}(X, Y)$ by the formula (3.1), we need to know the Newton kernel of the Schrödinger operator $\Delta^{\mathbf{R}^d} - |\rho|^2$, *i.e.* the $|\rho|^2$-potential ($|\rho|^2$-resolvent) of $\Delta^{\mathbf{R}^d}$ This can be computed, and the answer contains the Bessel function of third type $K_{n/2}$ (refer to [1])

For the Poisson kernel, we need to know the Poisson kernel for the Schrödinger operator $\Delta^{\mathbf{R}^d} - |\rho|^2$. This result does not seem to be known, despite the fact that the analytic and stochastic theory of Schrödinger operator is very well developed [3].

4. Asymptotic behavior of the kernels

To simplify the notation, we will write $f \overset{Y_0}{\sim} g$ if $\lim_{X \to Y_0} \frac{f(X)}{g(X)} = 1$.

The main results of this Section are Theorems 4.9 and 4.11 which give asymptotics of the Poisson and Newton kernels of the flat symmetric complex space M. In their proofs, we need some knowledge of Dunkl analysis on $\mathbf{R}^d$.

Consider $\mathbf{R}^d$ with a root system Σ. The basic information on the Dunkl analysis in this context can be found in [23]. Denote the Dunkl Laplacian by Δ_k and the intertwining operator by V_k.

Recall now the formula of Dunkl ([5, 6]) for the Dunkl-Poisson kernel of the unit centered open ball $B = B(0, 1)$.

$$P_k(X, Y) = \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|\,w_d}\, V_k\left[\frac{1 - |X|^2}{(1 - 2\langle X, \cdot\rangle + |X|^2)^{\gamma + d/2}}\right](Y), \quad X \in B,\ Y \in \partial B,\ \gamma = \frac{1}{2}\sum_{\alpha \in R} k_\alpha. \tag{4.1}$$

The complex flat symmetric spaces M correspond to the formula (4.1) in the W-invariant case and with $k_\alpha = 1$. Then $\gamma = |\Sigma_+|$ expresses the number of positive roots.

A formula for the Dunkl-Newton kernel $N_k(X, Y)$, analogous to (4.1), was proven in [8].

4.1. **Poisson kernel of the flat symmetric complex space.** The following technical results will prove useful further on.

Lemma 4.1. *$\partial(\pi)\,|X|^{-d} = 2^\gamma \prod_{k=0}^{\gamma-1}(-d/2 - k)\,\pi(X)\,|X|^{-d-2\,\gamma}$ and $\partial(\pi)\,\log|X| = (-2)^{\gamma-1}\,(\gamma - 1)!\,\pi(X)\,|X|^{-2\,\gamma}$.*

Proof. We see easily that $|X|^{d+2\,\gamma}\,\partial(\pi)\,|X|^{-d}$ is a skew polynomial of degree at most γ. It must therefore be a constant multiple of $\pi(X)$. The rest follows easily. A similar reasoning applies for the computation of $\partial(\pi)\,\log|X|$. □

Proposition 4.2. *Let $A(X, Y) = \frac{1}{\pi(X)\,\pi(Y)} \sum_{w \in W} \frac{\epsilon(w)}{|X - w\cdot Y|^d}$. Then $A(0, Y) = \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)}\,|Y|^{-d-2\,\gamma}$.*

Proof. Note first that $\partial(\pi)_X\,|X-Y|^{-d} = 2^\gamma\,\prod_{k=0}^{\gamma-1}(-d/2-k)\,\pi(X-Y)\,|X-Y|^{-d-2\,\gamma}$. Consider $B(X,Y) = \pi(X)\,A(X,Y) = \frac{1}{\pi(Y)}\,\sum_{w\in W}\,\epsilon(w)\,|X-w\cdot Y|^{-d}$. We apply the differential operator $\pi(\partial)|_{X=0}$ to B. We find

$$\begin{aligned}\partial(\pi)(\pi)\,A(0,Y) &= 2^\gamma\,\prod_{k=0}^{\gamma-1}(-d/2-k)\,\frac{1}{\pi(Y)}\,\sum_{w\in W}\,\epsilon(w)\,\pi(X-w\cdot Y)\,|X-w\cdot Y|^{-d-2\,\gamma}\Big|_{X=0}\\ &= (-1)^\gamma\,2^\gamma\,\prod_{k=0}^{\gamma-1}(-d/2-k)\,|W|\,|Y|^{-d-2\,\gamma}\end{aligned}$$

Finally,

$$\begin{aligned}A(0,Y) &= \frac{(-1)^\gamma\,2^\gamma\,\prod_{k=0}^{\gamma-1}(-d/2-k)\,|W|}{\partial(\pi)(\pi)}\,|Y|^{-d-2\,\gamma} = \frac{(-1)^\gamma\,2^\gamma\,\prod_{k=0}^{\gamma-1}(-d/2-k)\,|W|}{\pi(\rho)\,|W|/2^\gamma}\,|Y|^{-d-2\,\gamma}\\ &= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)}\,|Y|^{-d-2\,\gamma}.\end{aligned}$$

□

Corollary 4.3. *For $X\notin W\cdot Y$,*

$$\begin{aligned}P^W(0,Y) &= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)}\\ N^W(0,Y) &= \frac{-2^{2\,\gamma-1}\,(\gamma-1)!}{4\,\pi\,\pi(\rho)}\,|Y|^{-2\,\gamma}\ \textit{if } d=2\\ N^W(0,Y) &= \frac{2^{2\,\gamma}\,((d-2)/2)_\gamma}{|W|\,(d-2)\,w_d\,\pi(\rho)}\,|Y|^{2-d-2\,\gamma}\ \textit{if } d\geq 3\end{aligned}$$

Proposition 4.4. *The Poisson kernel of the unit ball on the flat symmetric complex space M is given by*

$$P^W(X,Y) = \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|\,w_d}\,\mathcal{A}^*\left(\frac{1-|X|^2}{(1-2\,\langle X,\cdot\rangle+|X|^2)^{\gamma+d/2}}\right)(Y), \tag{4.2}$$

where $\mathcal{A}^*$ denotes the dual Abel transform on M.

Recall that the dual of the Abel transform can be defined by the equation

$$\mathcal{A}^*(f)(X) = \int_K f(\pi_{\mathfrak{a}}(\ \mathrm{Ad}(k)\,X))\,dk$$

where, as before, $\pi_{\mathfrak{a}}$ is the orthogonal projection from $\mathfrak{p}$ to $\mathfrak{a}$ with respect to the Killing form. Note in particular that $\mathcal{A}^*(e^{\lambda(\cdot)})(X) = \psi_\lambda(X)$. Note also (see [16, Ch. IV, Theorem 10.11]) that unless $C(X)$ reduces to $\{X\}$, there exists a density $K(H,X)$ such that

$$\mathcal{A}^*(f)(X) = \int_{C(X)} f(H)\,K(H,X)\,dH.$$

Proof of Proposition 4.4. It should be noted that for f Weyl-invariant, $\mathcal{A}(f) = V_k(f)$ (refer to [23]). Since the argument of $\mathcal{A}^*$ in (4.2) is not Weyl-invariant, some proof is needed. Let $K(Z,Y)$ be the kernel of the Abel

transform. Using (4.1), we have

$$\begin{aligned}
P^W(X,Y) &= \frac{\sum_{w,w_0\in W} P(w\cdot X, w_0\cdot Y)}{|W|^2}\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|^3\,w_d}\sum_{w,w_0\in W}\int_{C(w_0\cdot Y)}\frac{1-|w\cdot X|^2}{(1-2\,\langle w\cdot X,Z\rangle+|w\cdot X|^2)^{\gamma+d/2}}\,d\mu_{w_0\cdot Y}(Z)\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|^3\,w_d}\,(1-|X|^2)\int_{C(Y)}\sum_{w,w_0\in W}\frac{1}{(1-2\,\langle w\cdot X,Z\rangle+|X|^2)^{\gamma+d/2}}\,d\mu_Y(w_0^{-1}\cdot Z)\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|^3\,w_d}\,(1-|X|^2)\int_{C(Y)}\overbrace{\sum_{w,w_0\in W}\frac{1}{(1-2\,\langle w\cdot X,w_0\cdot Z\rangle+|X|^2)^{\gamma+d/2}}}^{\text{Weyl-invariant}}\,d\mu_Y(Z)\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|^3\,w_d}\,(1-|X|^2)\int_{C(Y)}\sum_{w,w_0\in W}\frac{1}{(1-2\,\langle w\cdot X,w_0\cdot Z\rangle+|X|^2)^{\gamma+d/2}}\,K(Z,Y)\,dZ\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|^3\,w_d},(1-|X|^2)\int_{C(Y)}\sum_{w,w_0\in W}\frac{1}{(1-2\,\langle X,w^{-1}w_0\cdot Z\rangle+|X|^2)^{\gamma+d/2}}\,K(Z,Y)\,dZ\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|^3\,w_d}\,(1-|X|^2)\int_{C(Y)}\sum_{w,w_0\in W}\frac{1}{(1-2\,\langle X,Z\rangle+|X|^2)^{\gamma+d/2}}\,K(w_0^{-1}\,w\,Z,Y)\,dZ\\
&= \frac{2^{2\,\gamma}\,(d/2)_\gamma}{\pi(\rho)\,|W|\,w_d}\,(1-|X|^2)\int_{C(Y)}\frac{1}{(1-2\,\langle X,Z\rangle+|X|^2)^{\gamma+d/2}}\,K(Z,Y)\,dZ.
\end{aligned}$$

Note that Corollary 4.3 explains the origin of our normalizing constant which is different from what is found in [5, 6] (note that $V_k(1)=1$). □

Corollary 4.5. *The Newton kernel of the flat symmetric complex space M is given by*

$$N^W(X,Y)=\frac{2^{2\,\gamma}\,((d-2)/2)_\gamma}{|W|\,(d-2)\,w_d\,\pi(\rho)}\,\mathcal{A}^*\left((|Y|^2-2\,\langle X,\cdot\rangle+|X|^2)^{(2-d-2\,\gamma)/2}\right)(Y).$$

Proof. We apply the same computations as for the Poisson kernel to formula [8, (6.1)] (the constant has been adjusted to follow our conventions as per Corollary 4.3). □

We now start to study the asymptotic behavior of the Poisson kernel $P^W(X,Y)$. Let us introduce some notations. We define

$$R(X,Y)=\sum_{w\in W}\frac{\epsilon(w)}{|X-w\cdot Y|^d}\quad\text{and}\quad T(X,Y)=\frac{R(X,Y)}{\pi(X)\pi(Y)}$$

and therefore,

$$P^W(X,Y)=\frac{1-|X|^2}{|W|\,w_d}\,T(X,Y).$$

The function $R(X,Y)$ is defined for $X,Y\in\mathfrak{a}$ such that $X\notin W\cdot Y=\{w\cdot Y|\ w\in W\}$. We will denote this domain by

$$D:=\{(X,Y)\in\mathfrak{a}^2|\ X\notin W\cdot Y\}$$

The function $T(X,Y)$ is, for now, defined for non-singular $X,Y\in\mathfrak{a}$ such that $X\notin WY$. We will see in Proposition 4.8 that the function $T(X,Y)$ extends by continuity to an analytic function on the domain D.

Studying the properties of $P^W(X,Y)$ is equivalent to studying the properties of $T(X,Y)$ and $R(X,Y)$. We will give some of them in Proposition 4.8. We start by introducing two auxiliary results.

Lemma 4.6. *Assume $a_1,\ldots,a_n$ are not all 0 and let U be an open set. Let q be an analytic function on U which is 0 whenever $\sum_{k=1}^d a\,x_k=0$. Then $q(X)=\left(\sum_{k=1}^d a\,x_k\right)r(X)$ where r is a analytic function on U.*

Proof. Using a change of variable, we can assume that $a_1 = 1$ and $a_i = 0$ for $i > 1$. It is also enough to show that for every $X_0 = (b_1, \dots, b_n) \in U$, there exists $\epsilon > 0$ such that the result holds in the ball $B(X_0, \epsilon)$. If $X_0 \neq 0$, then pick $\epsilon > 0$ small enough so that $(x_1, \dots, x_d) \in U$ implies $x_1 \neq 0$. Then we can pick $r(X) = q(X)/x_1$.

Suppose now that $b_1 = 0$. We then have $q(x_1, \dots, x_d) = x_1 \overbrace{\int_0^1 \frac{\partial}{\partial x_1}(t\, x_1, x_2, \dots, x_n)\, dt}^{r(X)}$ for $(x_1, \dots, x_n) \in B(X_0, r) \subset U$. □

Proposition 4.7. *Let $p(X) = \prod_{i=1}^d \langle \alpha_i, X\rangle$ where no α_i's is a multiple of another α_j and let U be an open set. If q is an analytic function on U which is 0 whenever $\alpha_i(X) = 0$ for some i then $q(X) = p(X)\, r(X)$ where r is an analytic function on U.*

Proof. We use induction on n. Lemma 4.6 shows that the result is true for $n = 1$. Assume it is true for $n-1$, $n \geq 2$ and write $q(X) = \prod_{i=1}^{n-1} \langle \alpha_i, X\rangle\, r(X)$. Since $q(X) = 0$ when $\langle \alpha_n, X\rangle = 0$, we conclude that $r(X) = 0$ on the set $\{X | \langle \alpha_n, X\rangle = 0 \text{ and } \langle \alpha_i, X\rangle \neq 0,\ i < n\}$. By continuity, we deduce that $r(X) = 0$ when $\langle \alpha_n, X\rangle = 0$ and, using Lemma 4.6 once more, we can conclude. □

Proposition 4.8.

1. *(Symmetry in X and Y) $R(X, Y) = R(Y, X)$ and $T(X, Y) = T(Y, X)$.*
2. *(Skew-symmetry) $R(w_0 X, Y) = \epsilon(w_0)\, R(X, Y)$ and $R(X, w_0 Y) = \epsilon(w_0)\, R(X, Y)$.*
3. *(Nullity of R on singular arguments) $R(X, Y)$ is zero whenever at least one of X or Y is singular.*
4. *(analytic factorization of R, analytic extension of T to D.) There exists a function f analytic on D such that $R(X, Y) = \pi(X)\pi(Y) f(X, Y)$ on D. Equivalently, the function T extends to an analytic function on D.*
5. *(Non-nullity of T and P^W) When $X \in B$ and $Y \in \partial B$ then $T(X, Y) > 0$ and $P^W(X, Y) > 0$.*

Proof. The proof of (1) and (2) is straightforward.

3. Suppose $\alpha(Y) = Y$. We use Property 2 and $\epsilon(\sigma_\alpha) = -1$ where σ_α is the reflection with respect to the hyperplane $\{\alpha = 0\}$. Since $R(X, Y)/\pi(Y)$ is analytic, the statement follows.
4. This follows from Proposition 4.7.
5. This follows from Proposition 4.4. The dual Abel integral transform of a strictly positive function is strictly positive.

□

Theorem 4.9. *Let $Y_0 \in \partial B$, $\Sigma' = \{\alpha \in \Sigma |\ \alpha(Y_0) = 0\}$ and $\Sigma'_+ = \Sigma' \cap \Sigma^+$. Then*

$$P^W(X, Y_0) \overset{Y_0}{\sim} \frac{2^{2\gamma'}\, (d/2)_{\gamma'}}{|W'|\, w_d\, \pi'(\rho')\, (\pi''(Y_0))^2}\, \frac{1 - |X|^2}{|X - Y_0|^{2\gamma' + d}} \tag{4.3}$$

where $\gamma' = |\Sigma'_+|$ is the number of positive roots annihilating Y_0, $W' = \{w \in W | w \cdot Y = Y\}$, $\pi'(Y) = \prod_{\alpha \in \Sigma'_+} \langle \alpha, Y\rangle$ and $\pi''(Y) = \prod_{\alpha \in \Sigma_+ \setminus \Sigma'_+} \langle \alpha, Y\rangle$.

Proof. In this proof, we consider $X \in V = B(Y_0, \epsilon)$ with $\epsilon > 0$ fixed and chosen in such a way that

$$\alpha(\bar{V}) \subset (0, \infty) \text{ for } \alpha \in \Sigma_+ \setminus \Sigma'_+ \quad \text{and} \quad w\, V \cap V = \emptyset \text{ for every } w \in W \setminus W'.$$

Using Theorem 2.1, we have

$$P^W(X, Y) = \frac{1}{|W|\, w_d}\, \frac{1 - |X|^2}{\pi(X)\, \pi(Y)} \sum_{w \in W} \frac{\epsilon(w)}{|X - w \cdot Y|^d}.$$

We consider $X \in V \setminus \{Y_0\}$ and we deal with

$$T(X, Y_0) = \frac{|W|\, w_d}{1 - |X|^2}\, P^W(X, Y_0) = \frac{1}{\pi(X)\, \pi(Y_0)} \sum_{w \in W} \frac{\epsilon(w)}{|X - w \cdot Y_0|^d} = \frac{R(X, Y_0)}{\pi(X)\, \pi(Y_0)}. \tag{4.4}$$

By Proposition 4.8 applied to the root systems Σ and Σ', all the expressions in (4.4) are well defined for $X \in V \setminus \{Y_0\}$, if needed in the limit sense.

We decompose the sum $\sum_{w\in W}$ into two terms, the first being the sum over the subgroup $W'=\{w\in W|w\cdot Y_0=Y_0\}$ which is the Weyl group of the root subsystem Σ'. We obtain

$$T(X,Y_0)=\frac{\sum_{w\in W}\epsilon(w)|X-w\cdot Y_0|^{-d}}{\pi(X)\,\pi(Y_0)}=\frac{\sum_{w\in W'}\epsilon(w)|X-w\cdot Y_0|^{-d}}{\pi(X)\,\pi(Y_0)}+\frac{\sum_{w\in W\setminus W'}\epsilon(w)|X-w\cdot Y_0|^{-d}}{\pi(X)\,\pi(Y_0)}.$$

By Proposition 4.8, all the expressions in the last formula are well defined for $X\in V\setminus\{Y_0\}$, if needed in the limit sense. Denote

$$T_1(X,Y_0)=\frac{\sum_{w\in W'}\epsilon(w)|X-w\cdot Y_0|^{-d}}{\pi(X)\,\pi(Y_0)}\quad\text{and}\quad T_2(X,Y_0)=\frac{\sum_{w\in W\setminus W'}\epsilon(w)|X-w\cdot Y_0|^{-d}}{\pi(X)\,\pi(Y_0)}.$$

Let $\pi'(X)=\prod_{\alpha\in\Sigma'_+}\alpha(X)$ and $\pi''(X)=\prod_{\alpha\in\Sigma_+\setminus\Sigma'_+}\alpha(X)$. Observe that by Theorem 2.1,

$$\pi''(X)\,\pi''(Y_0)\,T_1(X,Y_0)=\frac{\sum_{w\in W'}\epsilon(w)|X-w\cdot Y_0|^{-d}}{\pi'(X)\,\pi'(Y_0)}=\frac{|W'|\,w_d}{1-|X|^2}\,(P^W)^{\Sigma'}(X,Y_0)$$

where $P^{W'}(X,Y)$ is the Poisson kernel for the flat symmetric space $(\mathbf{R}^d,\Sigma')$ corresponding to the complex root system Σ'. The convex envelope $C'(Y_0)=\mathrm{conv}(W'\,Y_0)=\{Y_0\}$, so by Proposition 4.4 and the properties of $\mathcal{A}^*$,

$$\begin{aligned}\frac{1}{1-|X|^2}\,P^{W'}(X,Y_0)&=\frac{2^{2\,\gamma'}\,(d/2)_{\gamma'}}{\pi(\rho')\,|W'|\,w_d}\int_{C(Y_0)}\frac{1}{(1-2\,\langle X,Z\rangle+|X|^2)^{\gamma'+d/2}}\delta_{\{Y_0\}}(dZ)\\&=\frac{2^{2\,\gamma'}\,(d/2)_{\gamma'}}{\pi(\rho')\,|W'|\,w_d}\,\frac{1}{|X-Y_0|^{2\gamma'+d}}\end{aligned}\tag{4.5}$$

where $X\in B\cap V$.

We now prove that the function $X\mapsto T_2(X,Y_0)$ is bounded on V, which, together with (4.5), will conclude the proof. We denote by

$$N(X,Y)=\sum_{w\in W\setminus W'}\epsilon(w)|X-w\cdot Y|^{-d}$$

the numerator of T_2. Observe that $N(X,Y)$ is an analytic function on $V\times V$. The function

$$T_2(X,Y)=\frac{\sum_{w\in W\setminus W'}\epsilon(w)|X-w\cdot Y|^{-d}}{\pi(X)\,\pi(Y)}$$

is well defined and analytic for $(X,Y)\in V\times V\setminus D$ with $D=\{(X,Y)\in\mathfrak{a}\times\mathfrak{a}\colon X=Y\}$, since $T(X,Y)$ and $T_1(X,Y)$ have these properties by Proposition 4.8 and $T_2=T-T_1$.

This implies that if $X'\in V$ or $Y'\in V$ are singular (*i.e.* $\alpha(X')=0$ or $\alpha(Y')=0$ for some $\alpha\in\Sigma'_+$) and $X'\neq Y'$ then the numerator $N(X',Y')=0$ since otherwise the limit $N(X,Y)/\pi(X)\pi(Y)$ could not exist when $(X,Y)\to(X',Y')$.

We deduce that if $X'\in V$ or $Y'\in V$ and $\alpha(X')=0$ or $\alpha(Y')=0$ for some $\alpha\in\Sigma'_+$) then $N(X',Y')=0$. This is also true for $X'=Y'$ since such points are limits when t tends to 1 of $(t\,X',Y')$ with singular $t\,X'\neq Y'$ and $N(t\,X',Y')$ converges to $N(X',Y')$.

By Proposition 4.7, there exists a function $F(X,Y)$ analytic on $V\times V$ such that

$$N(X,Y)=\pi'(X)\pi'(Y)F(X,Y),\qquad X,Y\in V$$

and, finally,

$$T_2(X,Y)=\frac{F(X,Y)}{\pi''(X)\pi''(Y)},\qquad X,Y\in V$$

(we have $\min_{X\in\bar V}\pi''(X)>0$ since $\pi''(\bar V)\subset(0,\infty)$). In particular, the function $X\mapsto T_2(X,Y_0)$ is bounded on V. □

Remark 4.10. *For the asymptotic properties of P^W, besides the alternating sum formula, the approach via the Dunkl formula* (4.1) *and dual Abel transform, i.e. the Proposition 4.2 is needed. We use it to compute the leading term $T_1(X,Y)$ in $T(X,Y)$.*

4.2. **Asymptotic behavior of the Newton kernel on flat complex symmetric spaces.** Using the same approach as in the proof of Theorem 4.9 together with Corollary 4.5, we conclude that

Theorem 4.11. *Let $Y_0 \in \overline{\mathfrak{a}^+}$. If $d = 2$ and α, β are the simple roots then*

$$N^W(X,0) = \frac{-2^{2\gamma-1}\,(\gamma-1)!}{4\,\pi\,\pi(\rho)}\,|X|^{-2\gamma} \text{ (case } Y_0 = 0\text{)},$$
$$N^W(X,Y_0) \overset{Y_0}{\sim} \frac{-2^{\gamma-1}\,(\gamma-1)!\,|W|}{4\,\pi\,\pi''(Y_0)^2\,\langle\alpha,\alpha\rangle}\,|X-Y_0|^{-2} \text{ where } Y_0 \neq 0,\ \alpha(Y_0) = 0 \text{ and } \beta(Y_0) \neq 0.$$

If $d \geq 3$

$$N^W(X,Y_0) \overset{Y_0}{\sim} \frac{2^{2\gamma'}\,((d-2)/2)_{\gamma'}}{|W'|\,(2-d)\,w_d\,\pi(\rho')\,(\pi''(Y_0))^2}\,\frac{1}{|X-Y_0|^{2\gamma'+d-2}} \tag{4.6}$$

where $\gamma' = |\Sigma'_+|$ is the number of positive roots annihilating Y_0, $W' = \{w \in W | w\cdot Y = Y\}$ and $\pi''(Y) = \prod_{\alpha\in\Sigma_+\setminus\Sigma'_+}\langle\alpha, Y\rangle$.

5. Asymptotic behavior of spherical functions on flat complex symmetric spaces

In this section we consider spherical functions on M, satisfying the formula

$$\psi_\lambda(Y) = \frac{\pi(\rho)}{2^\gamma\pi(\lambda)\,\pi(Y)}\sum_{w\in W}\epsilon(w)e^{\langle\lambda,w\cdot Y\rangle},\quad \lambda\in\mathfrak{a}^{\mathbf{C}},\ Y\in\mathfrak{a}^{\mathbf{R}}. \tag{5.1}$$

Note that our notation is different from that of Helgason (in his notation the function given by (5.1) is denoted $\psi_{-i\,\lambda}$).

The following technical lemma will prove useful later in this section.

Lemma 5.1. *Suppose G_1 and G_2 are subgroups of the finite group G. Then $|G_1\,G_2|\,|G_1\cap G_2| = |G_1|\,|G_2|$.*

Proof. The group $G_1\times G_2$ acts on the set $G_1\,G_2\subset G$ via $g_1,g_2)(g) = g_1\,g\,g_2^{-1}$. Clearly the action is transitive. The stabilizer of $e\in G_1\,G_2$ (e being the identity) is easily seen to be isomorphic to $G_1\cap G_2$. The orbit-stabilizer theorem ([24, Theorem 5.8]) implies then that $|G_1\,G_2|\,|G_1\cap G_2| = |G_1|\,|G_2|$. □

We introduce here some notation. If $X\in\mathfrak{a}$, we denote by Σ_X^+ the positive root system $\Sigma_X^+ = \{\alpha\in\Sigma^+ \colon \alpha(X) = 0\}$ and by W_X the Weyl group generated by the symmetries s_α with $s_\alpha\in\Sigma_X^+$ (consequently, $W_X = \{w\in W\colon w\cdot X = X\}$). We also write $\pi_X(Y) = \prod_{\alpha\in\Sigma_X^+}\alpha(Y)$ and $c_X = \partial(\pi_X)(\pi_X)$ (this derivative is constant on $\mathfrak{a}$).

For $X\in\mathfrak{a}$ we define the polynomial $\pi'_X(Y)$ by $\pi(Y) = \pi_X(Y)\pi'_X(Y)$. Denote

$$W(\lambda_0,Y_0) = \{w\in W\colon \langle\lambda_0, w\cdot Y_0\rangle = \langle\lambda_0,Y_0\rangle\}.$$

Remark 5.2. *We conjecture that the property $W(\lambda_0,Y_0) = W_{\lambda_0}\,W_{Y_0}$ is valid for all root systems. In Appendix A, we provide a series of proofs that cover all cases except for the exceptional root systems of type E. We also point out that if one of λ_0 or Y_0 is regular then this property is also verified, see* [14].

Denote the Weyl subgroup $W_{\lambda_0,Y_0} = W_{\lambda_0}\cap W_{Y_0} = \{w\in W\colon w\cdot\lambda_0 = \lambda_0 \text{ and } w\cdot Y = Y\}$. The group W_{λ_0,Y_0} corresponds to the root system $\Sigma^+_{\lambda_0,Y_0} = \Sigma^+_{\lambda_0}\cap\Sigma^+_{Y_0}$. We write $\pi_0(Y) = \pi_{\lambda_0,Y_0}(Y) = \prod_{\alpha\in\Sigma^+_{\lambda_0,Y_0}}\alpha(Y)$ and $c_{\lambda_0,Y_0} = \partial(\pi_{\lambda_0,Y_0})(\pi_{\lambda_0,Y_0})$. Denote by $\mathcal{M}$ the set of positive roots that are neither in $\Sigma^+_{\lambda_0}$ nor in $\Sigma^+_{Y_0}$, *i.e.* $\mathcal{M} = \Sigma^+\setminus(\Sigma^+_{\lambda_0}\cup\Sigma^+_{Y_0})$. We also write $\pi_{\mathcal{M}}(X) = \prod_{\alpha\in\mathcal{M}}\alpha(X)$.

Proposition 5.3.

(i) *If $w\in W_Y$ then $\pi_Y(w\cdot X) = \epsilon(w)\,\pi_Y(X)$.*
(ii) *If $w\in W_Y$ then $\pi_Y(\partial)[f(w\cdot Y)] = \epsilon(w)(\pi_Y(\partial)f)(w\cdot Y)$.*

Proof. The property (i) is well known [15]. The property (ii) is straightforward for $f(X) = e^{\langle Z,X\rangle}$ and extends by linear density. □

Proposition 5.4. *Let λ_0, Y_0 be singular. The asymptotics of $\psi_{\lambda_0}(t\,Y_0)$ when $t\to\infty$ are given by the following formula:*

$$\psi_{\lambda_0}(t\,Y_0)\sim C(\lambda_0,Y_0)t^{|\Sigma^+_{Y_0}|-|\Sigma^+|}\sum_{w\in W(\lambda_0,Y_0)}\epsilon(w)\pi_{Y_0}(\partial^Y)\,\left(\pi_{\lambda_0}(w\cdot Y)e^{\langle\lambda_0,w\cdot Y\rangle}\right)\big|_{Y=t\,Y_0} \tag{5.2}$$

where $C(\lambda_0,Y_0)=(c_{\lambda_0}\,c_{Y_0}\,\pi'_{\lambda_0}(\lambda_0)\,\pi'_{Y_0}(Y_0))^{-1}$.

When $W(\lambda_0,Y_0)=W_{\lambda_0}\,W_{Y_0}$, the last formula simplifies to

$$\psi_{\lambda_0}(t\,Y_0)\sim C_1(\lambda_0,Y_0)t^{|\Sigma^+_{Y_0}|-|\Sigma^+|}\pi_{Y_0}(\partial^Y)\left(\pi_{\lambda_0}(Y)e^{\langle\lambda_0,Y\rangle}\right)\big|_{Y=t\,Y_0} \tag{5.3}$$

where $C_1(\lambda_0,Y_0)=C(\lambda_0,Y_0)|W_{\lambda_0}|\,|W_{Y_0}|/|W_{\lambda_0,Y_0}|$.

Proof. We start with the alternating sum formula for the spherical function ψ_λ, written in the following way

$$\pi(\lambda)\pi(Y)\psi_\lambda(Y)=\sum_{w\in W}\epsilon(w)e^{\langle\lambda,w\cdot Y\rangle} \tag{5.4}$$

We write $\pi(\lambda)=\pi_{\lambda_0}(\lambda)\pi'_{\lambda_0}(\lambda)$ and $\pi(Y)=\pi_{Y_0}(Y)\pi'_{Y_0}(Y)$. We apply the operator $L=\pi_{Y_0}(\partial^Y)\pi_{\lambda_0}(\partial^\lambda)$ to both sides of (5.4). Using the fact that $\pi_{\lambda_0}(\partial^\lambda)\,e^{\langle\lambda,w\cdot Y\rangle}=\pi_{\lambda_0}(w\cdot Y)e^{\langle\lambda,w\cdot Y\rangle}$, we obtain

$$c_{\lambda_0}\,c_{Y_0}\,\pi'_{\lambda_0}(\lambda_0)\,\pi'_{Y_0}(t\,Y_0)\,\psi_{\lambda_0}(t\,Y_0)=\sum_{w\in W}\epsilon(w)\,\pi_{Y_0}(\partial^Y)\,\left(\pi_{\lambda_0}(w\cdot Y)\,e^{\langle\lambda_0,w\cdot Y\rangle}\right)\big|_{Y=t\,Y_0}.$$

In order to get the exact asymptotics of $\psi_{\lambda_0}(t\,Y_0)$, we only need to deal with $w\in W$ such that $\langle\lambda_0,w\cdot Y_0\rangle=\langle\lambda_0,Y_0\rangle$. This gives the asymptotics (5.2).

We now assume that $W(\lambda_0,Y_0)=W_{\lambda_0}\,W_{Y_0}$. The asymptotics (5.2) simplify, since by Proposition 5.3, we obtain for $w=w_1\,w_2$ with $w_1\in W_{\lambda_0}$ and $w_2\in W_{Y_0}$

$$\begin{aligned}\pi_{Y_0}(\partial^Y)\left(\pi_{\lambda_0}(w\cdot Y)\,e^{\langle\lambda_0,w\cdot Y\rangle}\right)&=\epsilon(w_1)\pi_{Y_0}(\partial^Y)\left(\pi_{\lambda_0}(w_2Y)\,e^{\langle\lambda_0,w_2Y\rangle}\right)=\epsilon(w_1)\,\epsilon(w_2)\pi_{Y_0}(\partial^Y)\left(\pi_{\lambda_0}(Y)\,e^{\langle\lambda_0,Y\rangle}\right)\\&=\epsilon(w)\,\pi_{Y_0}(\partial^Y)\left(\pi_{\lambda_0}(Y)\,e^{\langle\lambda_0,Y\rangle}\right).\end{aligned}$$

Using Lemma 5.1, we have $|W_{\lambda_0}W_{Y_0}|=|W_{\lambda_0}||W_{Y_0}|/|W_{\lambda_0,Y_0}|$.. We obtain the formula (5.3). □

Theorem 5.5. *Let λ_0, Y_0 be singular. Assume that $W(\lambda_0,Y_0)=W_{\lambda_0}\,W_{Y_0}$. Then the asymptotics of $\psi_{\lambda_0}(t\,Y_0)$ when $t\to\infty$ are given by the following formula:*

$$\psi_{\lambda_0}(t\,Y_0)\sim D(\lambda_0,Y_0)\,t^{-m}\,e^{t\,\langle\lambda_0,Y_0\rangle} \tag{5.5}$$

where m is the number of positive roots that are neither in $\Sigma^+_{\lambda_0}$ nor in $\Sigma^+_{Y_0}$ i.e.

$$m=card\mathcal{M}=|\Sigma|^+-(|\Sigma^+_{\lambda_0}|+|\Sigma^+_{Y_0}|-|\Sigma^+_{\lambda_0}\cap\Sigma^+_{Y_0}|)$$

and

$$D(\lambda_0,Y_0)=\frac{c_{\lambda_0,Y_0}}{c_{\lambda_0}\,c_{Y_0}}\,\frac{|W_{\lambda_0}|\,|W_{Y_0}|}{|W_{Y_0}\cap W_{\lambda_0}|}\frac{1}{\pi_{\mathcal{M}}(\lambda_0)\,\pi_{\mathcal{M}}(Y_0)}.$$

Remark 5.6. *When Y_0 is regular, the method of proof used in Theorem 4.9 for the asymptotics of the Poisson kernel could have been used here. When both λ_0 and Y_0 are singular, that approach fails to apply.*

Proof. Using Leibniz formula, we have

$$\begin{aligned}&\pi_{Y_0}(\partial^Y)\,\left(\pi_{\lambda_0}(Y)\,e^{\langle\lambda_0,Y\rangle}\right)\big|_{Y=t\,Y_0}\\&=\pi_0(\partial^Y)\prod_{\alpha\in\Sigma^+_{Y_0}\setminus\Sigma^+_{\lambda_0}}\partial^Y(A_\alpha)\,\left(\pi_{\lambda_0}(Y)e^{\langle\lambda_0,Y\rangle}\right)\big|_{Y=t\,Y_0}\\&=\pi_0(\partial^Y)\left(\prod_{\alpha\in\Sigma^+_{Y_0}\setminus\Sigma^+_{\lambda_0}}\langle\lambda_0,\alpha\rangle\,\pi_{\lambda_0}(Y)\,e^{\langle\lambda_0,Y\rangle}+P(Y)\,e^{\langle\lambda_0,Y\rangle}\right)\big|_{Y=t\,Y_0}\end{aligned}$$

The number of factors in each term $P(Y)$ of the form $\langle \eta, Y\rangle$ where η is a root, is strictly less than the number of factors in π_{λ_0} *i.e.* less than $|\Sigma^+_{\lambda_0}|$.

In the expression in the last line, all derivatives involving the term $e^{\langle \lambda_0, Y\rangle}$ give 0 since $\beta(\lambda_0) = 0$ for $\beta \in \Sigma^+_{Y_0} \cap \Sigma^+_{\lambda_0}$.

In the derivatives of $\pi_{\lambda_0}(Y)$, any term that contains $\langle \beta, Y\rangle$ with $\beta \in \Sigma^+_{Y_0} \cap \Sigma^+_{\lambda_0}$ will be zero when Y is replaced by $t\, Y_0$. Thus, for a non-zero result, the operator $\pi_0(\partial^Y)$ must be applied to $\pi_0(Y)$, what gives $c_{\lambda_0, Y_0} > 0$. We obtain

$$\begin{aligned}
&\pi_{Y_0}(\partial^Y)\, \left(\pi_{\lambda_0}(Y)\, e^{\langle \lambda_0, Y\rangle}\right)\big|_{Y = t\, Y_0} \\
&= \prod_{\alpha \in \Sigma^+_{Y_0} \setminus \Sigma^+_{\lambda_0}} \langle \lambda_0, \alpha\rangle \prod_{\gamma \in \Sigma^+_{\lambda_0} \setminus \Sigma^+_{Y_0}} \langle \gamma, t\, Y_0\rangle\, c_{\lambda_0, Y_0}\, e^{t\, \langle \lambda_0, Y_0\rangle} + \pi_0(\partial^Y) P(t\, Y_0) e^{t\langle \lambda_0, Y_0\rangle} \\
&= c_{\lambda_0, Y_0}\, t^{|\Sigma^+_{\lambda_0}| - |\Sigma^+_{\lambda_0} \cap \Sigma^+_{Y_0}|} \prod_{\alpha \in \Sigma^+_{Y_0} \setminus \Sigma^+_{\lambda_0}} \langle \lambda_0, \alpha\rangle \prod_{\gamma \in \Sigma^+_{\lambda_0} \setminus \Sigma^+_{Y_0}} \langle \gamma, Y_0\rangle\ e^{t\, \langle \lambda_0, Y_0\rangle} + \text{negligible terms.}
\end{aligned}$$

We labeled as "negligible terms" the terms with the derivatives involving $P(Y)$. They have the number of factors of the form $\langle \eta, t\, Y_0\rangle$ strictly less then $|\Sigma^+_{\lambda_0}| - |\Sigma^+_{\lambda_0, Y_0}|$, so strictly less than the term $\prod_{\gamma \in \Sigma^+_{\lambda_0} \setminus \Sigma^+_{Y_0}} \langle \gamma, t\, Y_0\rangle$. The rest follows from the definition of $C(\lambda_0, Y_0)$. □

Remark 5.7. *We can give a more explicit expression for the constant D, using the formula*

$$\partial(\pi)\, \pi = \frac{|W|\, \pi(\rho)}{2^\gamma},$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha = \sum_{\alpha \in \Sigma^+} \alpha$ and $\gamma = |\Sigma^+|$ is the number of positive roots.

For $X \in \mathfrak{a}$, denote $p_X = \pi_X(\rho_X)$. Analogously, we define p_{X_1, X_2} for the root system annihilating both elements $X_1, X_2 \in \mathfrak{a}$. We have

$$D(\lambda_0, Y_0) = \frac{2^{\gamma_{\lambda_0, Y_0} - \gamma_{\lambda_0} - \gamma_{Y_0}}}{\pi_{\mathcal{M}}(\lambda_0)\, \pi_{\mathcal{M}}(Y_0)}\, \frac{p_{\lambda_0, Y_0}}{p_{\lambda_0}\, p_{Y_0}},$$

and therefore

$$\lim_{t \to \infty} \frac{\psi_{\lambda_0}(t\, Y_0)}{t^{-m}\, e^{t\langle \lambda_0, Y_0\rangle}} = \frac{2^{\gamma_{\lambda_0, Y_0} - \gamma_{\lambda_0} - \gamma_{Y_0}}}{\pi_{\mathcal{M}}(\lambda_0)\, \pi_{\mathcal{M}}(Y_0)}\, \frac{p_{\lambda_0, Y_0}}{p_{\lambda_0}\, p_{Y_0}}$$

The following corollary may be found in [14]. Our method is simpler and proves the stronger result given in Theorem 5.5.

Corollary 5.8. (Helgason-Johnson theorem in the flat complex case) *The spherical function ψ_λ is bounded on $\mathfrak{a}$ if and only if $\lambda \in i\, \mathfrak{a}$.*

Proof. If λ is purely imaginary then ψ_λ is bounded by one by equation (1.1). The rest follows from Theorem 5.5. □

Remark 5.9. *The reader will want to compare Theorem 5.5 with those of* [21, 25] *taking into account the relationship between the spherical functions in the flat case and those in the curved case for the complex Lie groups. In these cases, Theorem 5.5 strengthens the estimates of* [21, 25] *providing the exact asymptotics. We conjecture that asymptotics with appropriate constants and not only estimates hold in the results of Narayanana, Pasquale and Pusti* [21] *and Schapira* [25].

6. Kernels of the Dyson Brownian motion

6.1. **Definition and transition density of the Dyson Brownian motion.** When a probabilist looks at formula (1.3), he or she sees in it the generator of the Doob h-transform (refer to [22]) of the Brownian Motion on $\mathbf{R}^d$ with the excessive function $h(X) = \pi(X)$. For the root system A_d on $\mathbf{R}^d$, the operator Δ^W is the generator of the Dyson Brownian Motion on $\mathbf{R}^d$ ([4]), *i.e.* the d Brownian independent particles $B^{(1)}_t, \ldots, B^{(d)}_t$ conditioned not to collide. More generally, for any root system Σ, on $\mathbf{R}^d$ the construction of a Dyson Brownian Motion as a Brownian Motion conditioned not to touch the walls of the positive Weyl chamber, can be done ([10]).

Let us recall basic facts about the Doob h-transform and the Dyson Brownian Motion. Let Σ be a root system on $\mathbf{R}^d$ and $\pi(X)=\prod_{\alpha>0}\langle\alpha,X\rangle$. It is known that π is $\Delta_{\mathbf{R}^d}$-harmonic on $\mathbf{R}^d$ ([10]), so in particular π is excessive.

Definition 6.1. *Let Σ be a root system on $\mathbf{R}^d$ and $\pi(X)=\prod_{\alpha>0}\langle\alpha,X\rangle$. The Dyson Brownian Motion D_t^Σ on the positive Weyl chamber $\mathfrak{a}^+$ is defined as the h-Doob transform of the Brownian Motion on $\mathbf{R}^d$, with $h=\pi$, i.e. its transition density is equal to*

$$p_t^D(X,Y)=\frac{\pi(Y)}{\pi(X)}p_t^{\text{killed}}(X,Y), \tag{6.1}$$

where $p_t^{\text{killed}}(X,Y)$ is the transition density of the Brownian Motion killed at the first strictly positive time of touching $\partial\mathfrak{a}^+$.

The infinitesimal generator of D_t^Σ is given by the formula([22])

$$\Delta^W f=\pi^{-1}\Delta^{\mathbf{R}^d}(\pi f)$$

which coincides with formula (1.3). The only difference with the symmetric flat complex case is that no invariant measure $\pi^2(Y)\,dY$ appears for the integral kernels in the Dyson Brownian Motion case. Consequently, we obtain

Corollary 6.2. *The transition density and the heat kernel of the Dyson Brownian Motion D_t^Σ on $\mathbf{R}^d$ is given by the formula*

$$p_t^D(X,Y)=\frac{\pi(Y)}{|W|\,\pi(X)}\sum_{w\in W}\epsilon(w)\,h_t(X-w\cdot Y),$$

where $p_t(X,Y)=h_t(X-Y)=\frac{1}{(4\,\pi\,t)^{d/2}}e^{-\frac{|X-Y|^2}{4\,t}}$ is the Euclidean heat kernel on $\mathbf{R}^d$.

In the case $\Sigma=A_p$ we have

$$p_t^D(X,Y)=\frac{\pi(Y)}{|W|\,\pi(X)}\det\left(g_t(x_i,y_j)\right),$$

where $g_t(u,v)=\frac{1}{\sqrt{4\,\pi t}}\,e^{-|u-v|^2/4t}$ is the 1-dimensional classical heat kernel.

Proof. We use Theorem 2.1(1) and Corollary 2.2. □

Comparing the formulas from Corollary 6.2 with formula (6.1), we obtain the following formulas for the heat kernel of the Brownian Motion killed at the first strictly positive time of touching a wall of the positive Weyl chamber.

Corollary 6.3. *The transition density for the Brownian Motion killed when exiting the positive Weyl chamber is given by the formula*

$$p_t^{\text{killed}}(X,Y)=\frac{1}{|W|}\sum_{w\in W}\epsilon(w)\,h_t(X-w\cdot Y). \tag{6.2}$$

In the case $\Sigma=A_p$ we have

$$p_t^{\text{killed}}(X,Y)=\frac{1}{|W|}\det\left(g_t(x_i,y_j)\right), \tag{6.3}$$

Remark 6.4. *Karlin and McGregor* [18] *showed formula* (6.3) *by different methods. In* [10]*, formulas for $p_t^{killed}(X,Y)$ for the root systems B_n and D_n are proven. Our method of alternating sums provides a simple proof of formula* (6.2) *valid for any root system Σ.*

6.2. **Poisson and Newton kernels for the Dyson Brownian Motion.** The alternating sum formulas for the integral Poisson and Newton kernels P^D and N^D of the Dyson Brownian Motion D_t^Σ can be easily deduced from their counterparts (see Theorem 2.1) for the flat symmetric complex spaces M, just by multiplying P^W and N^W by $\pi(Y)^2$.

Similarly, Theorems 4.9 and 4.11 imply asymptotics for the Poisson and Newton kernels for the Dyson Brownian Motion. For completeness and for their applications in the potential theory of the process D_t^Σ, we state these results here.

Corollary 6.5.

$$P^D(X,Y) = \frac{(1-|X|^2)\,\pi(Y)}{|W|\,w_d\,\pi(X)} \sum_{w\in W} \frac{\epsilon(w)}{|X-w\cdot Y|^d}$$

$$N^D(X,Y) = \frac{\pi(Y)}{4\,\pi\,\pi(X)} \sum_{w\in W} \epsilon(w)\,\ln|X-w\cdot Y| \ \textit{when } d=2,$$

$$N^D(X,Y) = \frac{\pi(Y)}{|W|\,(2-d)\,w_d\,\pi(X)} \sum_{w\in W} \frac{\epsilon(w)}{|X-w\cdot Y|^{d-2}} \ \textit{when } d\geq 3.$$

Corollary 6.6. *Let* $Y\in\overline{\mathfrak{a}^+}$ $\Sigma'=\{\alpha\in\Sigma|\ \alpha(Y_0)=0\}$, $\Sigma'_+=\Sigma'\cap\Sigma^+$, $\gamma'=|\Sigma'_+|$, $W'=\{w\in W|w\cdot Y=Y\}$ *and* $\pi'(X)=\prod_{\alpha\in\Sigma'_+}\langle\alpha,X\rangle$.

(i) *Let* $Y_0\in\partial B$. *Then*

$$P^D(X,Y_0)\overset{Y_0}{\sim}\frac{2^{2\,\gamma'}\,(d/2)_{\gamma'}\,\pi'(X)^2}{|W|\,w_d\,\pi'(\rho')}\,\frac{1-|X|^2}{|X-Y_0|^{2\gamma'+d}}.$$

(ii) *If* $d=2$, α, β *are simple roots,* $\alpha(Y_0)=0$, $\beta(Y_0)\neq 0$, *then* $N^D(X,Y_0)\overset{Y_0}{\sim}\frac{-2^{\gamma-1}\,(\gamma-1)!\,|W|\pi'(X)^2}{4\,\pi\,\langle\alpha,\alpha\rangle}\,|X-Y_0|^{-2}$.

(iii) *If* $d\geq 3$, *then* $N^D(X,Y_0)\overset{Y_0}{\sim}\frac{2^{2\,\gamma'}\,((d-2)/2)_{\gamma'}\,\pi'(X)^2}{|W'|\,(2-d)\,w_d\,\pi(\rho')}\,\frac{1}{|X-Y_0|^{2\gamma'+d-2}}$.

Appendix A. The Killing-max property

The aim of this appendix is to find precise conditions on $w\in W$ under which

$$\langle\lambda,w\cdot Y\rangle=\langle\lambda,Y\rangle. \tag{A.1}$$

Definition A.1. *Let* $W_\lambda=\{w\in W\colon w\cdot\lambda=\lambda\}$ *(similarly for* W_Y*). We will say that the property Killing-max is satisfied if* (A.1) *is verified if and only if* $w\in W_\lambda\,W_Y$.

Remark A.2. *It is clear that the condition* $w\in W_\lambda\,W_Y$ *is sufficient. Property Killing-max is also satisfied whenever at least one of* λ *or* Y *is regular (refer to* [14]*). We observe also that this property only depends on the action of the Weyl group on the Cartan subalgebra* $\mathfrak{a}$. *Given that* $\langle\lambda,w\cdot Y\rangle=\langle w^{-1}\lambda,Y\rangle$, *this problem is symmetric in* λ *and* Y.

In Table 2, we describe the action of the Weyl group on the Cartan subalgebra in the case of the noncompact and complex simple Lie algebras. Note that in the case of $(\mathfrak{f}_{4(-26)},\mathfrak{so}(9))$, which is not in the table, the Killing-max property is trivially true since the rank of the space is 1.

A.1. Type A_n $(\mathfrak{sl}(n+1,\mathbf{F}))$.

Lemma A.3 ("max principle" for permutations)**.** *Let* λ, $Y\in\mathbf{R}^n$ *with their entries in decreasing order and let* $w\in S_n$ *be a permutation. Suppose that the block of* λ_1 *in* λ *has length* $j_0\geq 1$ *and that the block of* Y_1 *in* Y *has length* $i_0\geq 1$. *If* $\min w^{-1}(\{1,\dots,i_0\})>j_0$ *then* $\langle\lambda,w\cdot Y\rangle\ <\ \langle\lambda,Y\rangle$.

Remark A.4. *The lemma states that if* $\langle\lambda,w\cdot Y\rangle=\langle\lambda,Y\rangle$ *then the permutation* w *is such that "*$\max Y$ *meets* $\max\lambda$*",* i.e. *there exists* $k\leq j_0$ *such that* $(w\cdot Y)_k=y_1$.

Symmetric space	Description of $X \in \mathfrak{a}^+$	Action of $w \in W$, the Weyl group	Underlying root system
$\mathbf{SL}(n,\mathbf{F})/\mathbf{SU}(n,\mathbf{F})$, $\mathbf{F}=\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, $n\geq 2$, $\mathbf{F}=\mathbf{O}$, $n=3$ (*i.e.* $\mathbf{E}_6/\mathbf{F}_4$)	$X=\mathrm{diag}[x_1,\dots,x_n]$, $\sum_{i=1}^n x_i=0$, $x_1>\cdots>x_n$	$w\in S_n$ permutes the entries x_i	A_{n-1}
$\mathbf{SO}(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$, $1\leq p<q$, $\mathbf{SU}(p,q)/\mathbf{SU}(p)\times\mathbf{SU}(q)$ and $\mathbf{Sp}(p,q)/\mathbf{Sp}(p)\times\mathbf{Sp}(q)$, $1\leq p\leq q$,	$X=\begin{bmatrix}0&\mathcal{D}_X&0\\ \mathcal{D}_X&0&0\\ 0&0&0\end{bmatrix}$, $\mathcal{D}_X=\mathrm{diag}[x_1,\dots,x_p]$, $x_1>\cdots>x_p>0$	w permutes the x_i's and changes any number of signs	B_n
$\mathbf{SO}(p,p)/\mathbf{SO}(p)\times\mathbf{SO}(p)$, $p\geq 2$	$X=\begin{bmatrix}0&\mathcal{D}_X\\ \mathcal{D}_X&0\end{bmatrix}$, $\mathcal{D}_X=\mathrm{diag}[x_1,\dots,x_p]$, $x_1>\cdots>x_{p-1}>\lvert x_p\rvert$	w permutes the x_i's and changes any **even** number of signs	D_n
$\mathbf{SO}^*(2\,n)/\mathbf{U}(n)$, $n\geq 3$	$X=\left[\begin{array}{c\|c}0_{n\times n}&\mathcal{E}_X\\ \hline -\mathcal{E}_X&0_{n\times n}\end{array}\right]$, $\mathcal{E}_X=\sum_{k=1}^{[n/2]} x_k\,F_{2\,k,2\,k+1}$, $x_1>\cdots>x_{n/2}>0$	w permutes the x_i's and changes any number of signs	B_n
$\mathbf{Sp}(n,\mathbf{R})/\mathbf{U}(n)$ and $\mathbf{Sp}(n,\mathbf{C})/\mathbf{Sp}(n)$, $n\geq 1$	$\begin{bmatrix}0&i\,\mathcal{D}_X\\ -i\,\mathcal{D}_X&0\end{bmatrix}$, $\mathcal{D}_X=\mathrm{diag}[x_1,\dots,x_p]$, $x_1>\cdots>x_{p-1}>x_p>0$	w permutes the x_i's and changes any number of signs	C_n
$\mathbf{SO}(2\,n,\mathbf{C})/\mathbf{SO}(2\,n)$, $n\geq 3$	$X=i\sum_{k=1}^n x_k\,F_{2\,k-1,2\,k}$, $x_1>\cdots>x_{p-1}>\lvert x_p\rvert$	w permutes the x_i's and changes any **even** number of signs	D_n
$\mathbf{SO}(2\,n+1,\mathbf{C})/\mathbf{SO}(2\,n+1)$, $n\geq 2$	$X=i\sum_{k=1}^n x_k\,F_{2\,k-1,2\,k}$, $x_1>\cdots>x_{p-1}>x_p>0$	w permutes the x_i's and changes any number of signs	B_n
$\mathbf{F}_4^{\mathbf{C}}/\mathbf{F}_4$, $(\mathfrak{f}_{4(4)},\mathfrak{sp}(3)+\mathfrak{su}(2))$	$X=[x_1,x_2,x_3,x_4]$, $x_2>x_3>x_4>0$, $x_1>x_2+x_3+x_4$	Refer to [2]	F_4
$\mathbf{G}_2^{\mathbf{C}}/\mathbf{G}_2$, $(\mathfrak{g}_{2(2)},\mathfrak{su}(3)+\mathfrak{su}(2))$	$X=\mathrm{diag}[x_1,x_2,x_1-x_2,0,x_2-x_1,-x_2,-x_1]$, $x_1>x_2>x_1/2$	Refer to [20]	G_2

TABLE 2. Action of the Weyl group (except for E_6, E_7 and E_8)

Proof. Without loss of generality, we may assume that $\lambda\neq\lambda_1\,1^n$ and $Y\neq y_1\,1^n$. Let $i=\min w^{-1}(\{1,\dots,i_0\})$. By assumption, the first y_1 appears in $w\cdot Y$ at the i-th position with $i>j_0$. Let $w(1)=k$, *i.e.* $w\cdot Y$ begins with y_k. We have $y_k<y_1$ and $\lambda_i<\lambda_1$. Consider $w_0=(1i\,)\,w$; we then have

$$\langle\lambda,w_0\cdot Y\rangle-\langle\lambda,w\cdot Y\rangle\;=(\lambda_1-\lambda_i)\,(y_1-y_k)>0.$$

By the standard property of the Weyl group, $\langle\lambda,w_0\cdot Y\rangle\leq\langle\lambda,Y\rangle$. Hence, $\langle\lambda,w\cdot Y\rangle<\langle\lambda,Y\rangle$. □

Corollary A.5. *Property Killing-max is verified in the case of the root system A_n.*

Proof. We use the same notation as in Lemma A.3 and in its proof. Suppose $\langle\lambda,Y\rangle=\langle\lambda,w\cdot Y\rangle$. We use induction on n. The result is clear for $n=1$. Then by the lemma, there exists $i\leq j_0$ such that $w(i)\leq i_0$. We apply the induction hypothesis to $\lambda=(\lambda_2,\dots,\lambda_n)$ and to $Y=(y_2,\dots,y_n)$ the result follows.

Let $\lambda'=(\lambda_2,\dots,\lambda_n)$, $Y'=(y_2,\dots,y_n)$ and note that $[y_{w(1)},\dots,\widehat{y_{w(i)}},\dots,y_{w(n)}]$ is a permutation of Y' (say $w'\cdot Y'$). We have $\langle\lambda,w\cdot Y\rangle=\lambda_i\,y_1+\sum_{i=2}^n y_{w'(i)}$ where w' is a permutation of $\{2,\dots,n\}$. We then have

$\langle \lambda', Y' \rangle = \langle \lambda, w' \cdot Y' \rangle$. By the induction hypothesis $w' = w_{\lambda'} w_{Y'} \in W_{\lambda'} W_{Y'}$. We extend $w_{\lambda'}$ and $w_{Y'}$ to $w_\lambda \in W_\lambda$ and $w_Y \in W_Y$ by having them fix 1 in both cases. With the permutation $w_0 = (1\ i) \in W_\lambda$, we have $w = (1\ i)\, w_\lambda\, w_Y \in W_\lambda\, W_Y$. □

A.2. **Type B_n ($\mathfrak{so}(2\,n+1, \mathbf{C})$) and C_n ($\mathfrak{sp}(n, \mathbf{C})$).**

Proposition A.6. *Property Killing-max is verified in the case of the root systems B_n and C_n.*

Proof. Recall that B_n is the root system of $\mathfrak{so}(2n+1, \mathbf{C})$. The positive Weyl chamber is defined by the condition

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$$

The Weyl group is $W = S_n \rtimes \{\pm 1\}^n$; its elements are called "signed permutations". It is straightforward to see that the sign changes in $w \cdot Y$ strictly diminish $\langle \lambda, Y \rangle$ unless if negative terms in $w \cdot Y$ are in front of $\lambda_i = 0$.

More precisely, if $w \cdot Y$ has strictly negative terms in positions where $\lambda_i > 0$, then $\langle \lambda, w \cdot Y \rangle < \langle \lambda, w_0 w \cdot Y \rangle \leq \langle \lambda, Y \rangle$ where w_0 changes the negative signs in $w \cdot Y$ into positive ones.

Thus, if (A.1) holds, all negative terms in $w \cdot Y$ are in front of $\lambda_i = 0$. Then $w_0 \in W_\lambda$ and $\langle \lambda, w \cdot Y \rangle = \langle \lambda, w_0\, w \cdot Y \rangle$. All the terms of $w_0 w \cdot Y$ are non-negative and the result for $\mathfrak{a}_n$ applies.

To conclude, it suffices to recall that C_n is the root system for $\mathfrak{sp}(n, \mathbf{C})$. We have $W(C_n) = W(B_n)$, the only difference is in the relative length of roots ([7, p. 227]). □

A.3. **Type D_n ($\mathfrak{so}(2\,n, \mathbf{C})$).** Suppose that $\lambda, Y \in \overline{\mathfrak{a}^+}$ are singular. The Weyl group W is composed by permutations and the sign changes **by pairs**, *i.e.* of two terms simultaneously. The positive Weyl chamber $\mathfrak{a}^+$ is given by the condition

$$\lambda_1 > \lambda_2 > \ldots > \lambda_{n-1} > |\lambda_n|.$$

Lemma A.7 (The "max principle" for $W(D_n)$)**.** *Let $\lambda,\ Y \in \overline{\mathfrak{a}^+}$. Suppose that the block of λ_1 in λ has length $1 \leq j_0 < n$. Suppose also that $\min\{k : (w \cdot Y)_k = y_1\} > j_0$ or that $\{k : (w \cdot Y)_k = y_1\} = \emptyset$. Then $\langle \lambda, w \cdot Y \rangle < \langle \lambda, Y \rangle$.*

Proof. Suppose λ and Y are as in the statement of the lemma. If y_1 appears in $w \cdot Y$ then $\langle \lambda, w \cdot Y \rangle < \langle \lambda, Y \rangle$ by Lemma A.3 so we can assume that only $-y_1$ appears.

Using the standard property of the Weyl group over $\mathfrak{a}_n$, $\langle \lambda, w \cdot Y \rangle \leq \langle \lambda, w_0\, w \cdot Y \rangle$ where $w_0 \in S_n$ re-orders the entries of $w \cdot Y$ in decreasing order. The last entry of $w_0\, w \cdot Y$ has to be $-y_1$.

We first assume $n = 2$, or $n \geq 3$ and $j_0 \leq n-2$. As $y_n \geq -y_i$ for all $i < n$, we can suppose that the $(n-1)$-entry is $-y_i$. Using the element w_1 of the Weyl group which changes signs and permutes the last two entries, we have $\langle \lambda, w_1\, w_0\, w \cdot Y \rangle - \langle \lambda, w_0\, w \cdot Y \rangle = (\lambda_{n-1} + \lambda_n)\,(y_1 + y_i) \geq 0$. It is easy to check that the last inequality is strict if $n = 2$. Finally, by another application of Lemma A.3, $\langle \lambda, w_1\, w_0\, w \cdot Y \rangle < \langle \lambda, Y \rangle$ and the result follows.

We next handle the case $j_0 \geq n-1$, with $n \geq 3$. Let $\lambda = (a, \ldots, a, b)$ with $b \in (-a, a]$. and $n \geq 3$. We will show that $\Delta = \langle \lambda, Y \rangle - \langle \lambda, w_0\, w \cdot Y \rangle > 0$. If $-y_n$ appears in $w_0 w \cdot Y$, we have, using $\sum_{i \neq 1,n} a\, y_i \geq \sum_{i \neq 1,n} a\,(\pm y_i)$,

$$\Delta = \langle \lambda, Y \rangle - \langle \lambda, w_0 w \cdot Y \rangle \geq a\, y_1 + b\, y_n - [a\,(-y_n) + b\,(-y_1)] = (a+b)(y_1 + y_n) > 0$$

where we used the hypothesis $b \neq -a$ and the fact that $y_1 + y_n > 0$ (otherwise $-y_n = y_1$ appears in $w_0 w \cdot Y$). If $-y_n$ does not appear in $w_0 w \cdot Y$, another $-y_k$ appears among the $n-1$ first entries of $\sigma w \cdot Y$. This time, we obtain $\Delta \geq (a+b)\, y_1 + a\,(y_k - y_n) + a\, y_k + b\, y_n > 0$, where we used $y_1 > 0$ (as $Y \neq 0$), the hypothesis $a + b > 0$, and the inequalities $y_k \geq y_n$, $a\, y_k \geq |b\, y_n|$. □

Lemma A.8. *Suppose $\lambda = a\,(1, \ldots, 1)$, $a > 0$ and $Y \in \overline{\mathfrak{a}^+}$. Then* (A.1) *is satisfied if and only if $w \in W_\lambda\, W_Y$.*

Proof. Note that $\langle \lambda, Y \rangle = (n-1)\, a\, b\, x + a\, b\, x = n\, a\, b\, x$. The only way that $\langle \lambda, w \cdot Y \rangle = n\, a\, b\, x$ is if $w \cdot Y = Y$ *i.e.* $w \in W_Y = W_\lambda$. □

Proposition A.9. *Property Killing-max is verified in the case of the root system D_n.*

Proof. We proceed by induction on $n \geq 2$. Given Lemma A.8, if both λ and $Y \in \mathbf{R}\,(1, 1 \ldots, 1, -1)$ then there is nothing to prove. Given the symmetry of the problem, if $\lambda \in \mathbf{R}\,(1, 1 \ldots, 1, -1)$ and $Y \notin \mathbf{R}\,(1, 1 \ldots, 1, -1)$, we can switch their roles and suppose that $\lambda \notin \mathbf{R}\,(1, 1 \ldots, 1, -1)$.

The base case $n = 2$, in which, by Lemma A.8, we can assume that $\lambda = (\lambda_1, \lambda_2) \notin \mathbf{R}(1, -1)$, is clear by inspection.

Assume the result true for $n-1$. As explained above, we may assume that $\lambda \notin \mathbf{R}(1,\dots,1,-1)$. By Lemma A.7, the equality (A.1) implies that "$\max\lambda$ meets $\max Y$". As in the case A_n, it follows that there exist permutations $\sigma \in W_\lambda$ and $\gamma \in W_Y$ such that $(\sigma w\gamma \cdot Y)_1 = y_1$. We consider $\tilde\lambda_1 = (\lambda_2,\lambda_3,\dots,\lambda_n)$, $\tilde Y_1 = (Y_2,\dots,Y_n)$ and $\tilde w_1 = \sigma w\gamma|_{\tilde{\mathfrak{a}}}$ where $\tilde{\mathfrak{a}} = \{(x_2,\dots,x_n)|\ X = (x_i)_{i\geq 1} \in \mathfrak{a}\}$ and we use the induction hypothesis or Lemma A.8 depending on the situation. □

A.4. **Type F_4.** We use Helgason [15] and some simple facts about the Weyl group $W = W(F_4)$ from [2]. We consider the simple roots $\alpha_1 = \mathbf{e}_2 - \mathbf{e}_3$, $\alpha_2 = \mathbf{e}_3 - \mathbf{e}_4$, $\alpha_3 = \mathbf{e}_4$ and $\alpha_4 = (\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4)/2$ and the corresponding reflections $s_{\alpha_i} = s_i$. It follows that $\mathfrak{a}^+ = \{(x_1,x_2,x_3,x_4)\colon x_1 > x_2 + x_3 + x_4,\ x_2 > x_3 > x_4 > 0\}$.

Denote $\alpha_{12} = \mathbf{e}_1 - \mathbf{e}_2$ and $s_{12} = s_{\alpha_{12}}$. Note that $\alpha_{12} = \alpha_2 + 2\,\alpha_3 + \alpha_4$ is a positive root. It is easy to check that

$$s_3\, s_4\, s_{12} = s_2\, s_3\, s_4 \tag{A.2}$$

by inspection or using [2, Table 1] on the basis $(\mathbf{e}_i)$.

Let $X = (x_1,x_2,x_3,x_4)$ with $x_1 \geq x_2 \geq x_3 \geq x_4 \geq 0$, *i.e.* $X \in \overline{\mathfrak{a}^+(B_4)}$. We define $W_X^{B_4} \subset W(B_4)$ as the subgroup generated by a subset of the symmetries $s \in \{s_{12}, s_1, s_2, s_3\}$ such that $s(X) = X$.

Lemma A.10. *Let $\lambda \in \overline{\mathfrak{a}^+(B_4)}$. Then $W_\lambda^{B_4} \subset W_\lambda$.*

Proof. Clear from the definition of $W_\lambda^{B_4}$. □

Let α, β, γ denote the three sets of roots of F_4 defined in [2, p. 85], with $\alpha = (\pm\mathbf{e}_i)_{i=1}^4$. Let δ, $\eta \in \{\alpha,\beta,\gamma\}$ and $W_{\delta\eta} = \{w \in W\colon w(\delta) = \eta\}$. By [2], we have $W = W_{\alpha\alpha} \cup W_{\alpha\beta} \cup W_{\alpha\gamma}$. In order to describe the action of $w \in W$, we define $w_0^\alpha = id$, $w_0^\beta = s_3\, s_4$ and $w_0^\gamma = s_4$. Then, by [2, Table 1], we have $w_0^\delta(\alpha) = \delta$ with $\delta \in \{\alpha,\beta,\gamma\}$.

The following result is proven in [2]. Recall that $W(B_4)$ is the group of signed permutations of 4 elements.

Lemma A.11. *Let $\delta \in \{\alpha,\beta,\gamma\}$ and $w \in W_{\alpha\delta}$. There exists $\sigma \in W(B_4)$ such that if $Y = \sum_{i=1}^4 y_i\,\mathbf{e}_i$, then $w \cdot Y = \sum_{i=1}^4 y_{\sigma(i)}\, w_0^\delta(\mathbf{e}_i)$. Equivalently, $(w_0^\delta)^{-1}\, w$ is a signed permutation with respect to the basis $(\mathbf{e}_i)$.*

Proposition A.12. *Property Killing-max is verified in the case of the root system F_4.*

Proof. Suppose that $\lambda = \sum_{i=1}^4 \lambda_i\,\mathbf{e}_i$, $Y = \sum_{i=1}^4 y_i\,\mathbf{e}_i \in \overline{\mathfrak{a}^+(F_4)}$ are singular. Our objective is to solve the equation (A.1). We will assume from now on that (A.1) holds. We consider three cases $w \in W_{\alpha\delta}$, where $\delta = \alpha,\beta,\gamma$.

If $w \in W_{\alpha\alpha}$, we note that $\mathfrak{a}^+(F_4) \subset \mathfrak{a}^+(B_4)$. Lemma A.11, Proposition A.6 and Lemma A.10 imply that $w \in W_\lambda^{B_4}\, W_Y^{B_4} \subset W_\lambda\, W_Y$.

In the case $w \in W_{\alpha\beta}$, we use $w_0 = w_0^\beta = s_3\, s_4$. If $\lambda = \sum_{i=1}^4 \lambda_i\,\mathbf{e}_i \in \overline{\mathfrak{a}^+(F_4)}$ then $\lambda' = w_0^{-1}\cdot\lambda = \sum_{i=1}^4 \lambda'_i\,\mathbf{e}_i$ with $\lambda'_1 \geq \lambda'_2 \geq \lambda'_3 \geq \lambda'_4 \geq 0$ since

$$\lambda' = \frac{1}{2}[(\lambda_1+\lambda_2+\lambda_3-\lambda_4)\,\mathbf{e}_1 + (\lambda_1+\lambda_2-\lambda_3+\lambda_4)\,\mathbf{e_2} + (\lambda_1-\lambda_2+\lambda_3+\lambda_4)\,\mathbf{e}_3 + (\lambda_1-\lambda_2-\lambda_3-\lambda_4)\,\mathbf{e}_4]. \tag{A.3}$$

Using (A.1), Lemma A.11 and the standard property of the Killing form for B_4, we have

$$\langle\lambda, Y\rangle = \langle\lambda, w\cdot Y\rangle = \langle w_0^{-1}\cdot\lambda, w_0^{-1}\, w\cdot Y\rangle \leq \langle w_0^{-1}\cdot\lambda, Y\rangle = \langle\lambda, w_0\cdot Y\rangle \leq \langle\lambda, Y\rangle.$$

This means that $\langle\lambda', w_0^{-1}\, w\cdot Y\rangle = \langle\lambda', Y\rangle$ and therefore that $w \in w_0\, W_{\lambda'}^{B_4}\, W_Y^{B_4}$ by Proposition A.6.

We reason similarly if $w \in W_{\alpha\gamma}$, with $w_0 = w_0^\gamma = s_4$ and

$$\lambda' = s_4(\lambda) = \frac{1}{2}[(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\,\mathbf{e}_1 + (\lambda_1+\lambda_2-\lambda_3-\lambda_4)\,\mathbf{e}_2 + (\lambda_1-\lambda_2+\lambda_3-\lambda_4)\,\mathbf{e}_3 + (\lambda_1-\lambda_2-\lambda_3+\lambda_4)\,\mathbf{e}_4].$$

It therefore follows that $w \in w_0\, W_{\lambda'}^{B_4}\, W_Y^{B_4}$ with $\lambda' = w_0^{-1}\cdot\lambda$.

It is important to note that a feature of both cases $w \in W_{\alpha\beta}$ and $w \in W_{\alpha\gamma}$ implies that the respective w_0 satisfy $\langle\lambda, Y\rangle = \langle\lambda, w_0\cdot Y\rangle$. It follows that these cases do not occur if $\alpha_4 \notin \Sigma_\lambda \cup \Sigma_Y$. Indeed, using the formula $s_i(X) = X - 2\,\frac{\alpha_i(X)}{\|\alpha_i\|^2}\alpha_i$, we have for $w_0^\beta = s_3\, s_4$ and for $w_0^\gamma = s_4$,

$$\begin{aligned}\langle\lambda, Y\rangle - \langle\lambda, s_3\, s_4\, Y\rangle &= 2\,\alpha_4(\lambda)\,\alpha_4(Y) + 2\,\alpha_3(\lambda)\,\alpha_3(Y) + 2\,\alpha_3(\lambda)\,\alpha_4(Y)\\ \langle\lambda, Y\rangle - \langle\lambda, s_4\, Y\rangle &= 2\,\alpha_4(\lambda)\,\alpha_4(Y)\end{aligned} \tag{A.4}$$

Thus $\langle\lambda, Y\rangle \neq \langle\lambda, w_0\cdot Y\rangle$ if $\alpha_4 \notin \Sigma_\lambda \cup \Sigma_Y$ and $w \in W_{\alpha\beta}$ or $w \in W_{\alpha\gamma}$. We showed above that in the case $w \in W_{\alpha\alpha}$, formula (A.1) implies that $w \in W_\lambda\, W_Y$. The Proposition is thus proven for $\alpha_4 \notin \Sigma_\lambda \cup \Sigma_Y$.

It remains to treat the cases $\alpha_4 \in \Sigma_\lambda$ or $\alpha_4 \in \Sigma_Y$. By symmetry of the problem (A.1), it is sufficient to treat the case $\alpha_4 \in \Sigma_\lambda$, for any singular Y. We assume henceforth that $\alpha_4 \in \Sigma_\lambda$.

We showed above that in the case $w \in W_{\alpha\alpha}$, formula (A.1) implies that $w \in W_\lambda W_Y$.

If $w \in W_{\alpha\gamma}$, we have $w_0 = w_0^\gamma = s_4$ and therefore $\lambda' = s_4 \cdot \lambda = \lambda$ since $\alpha_4 \in \Sigma_\lambda$. Since $s_4 \in W_\lambda$, we have $w \in s_4 W_{\lambda'}^{B_4} W_Y^{B_4} = s_4 W_\lambda^{B_4} W_Y^{B_4} \subset W_\lambda W_Y$.

Suppose that $w \in W_{\alpha\beta}$ and recall that $w_0 = w_0^\beta = s_3\, s_4$. By (A.4), we have the following two cases:

(A) $\alpha_3(\lambda) = 0$ or (B) $\alpha_3(\lambda) \neq 0$, $\alpha_3(Y) = 0$ and $\alpha_4(Y) = 0$.

In the case (A), we have $w_0^{-1} \cdot \lambda = \lambda$ *i.e.* $\lambda' = \lambda$ and $s_3\, s_4 \in W_\lambda$. Therefore, we have $w \in s_3\, s_4 W_{\lambda'}^{B_4} W_Y^{B_4} = s_3\, s_4\, W_\lambda^{B_4} W_Y^{B_4} \subset W_\lambda W_Y$.

In the case (B), we compute using (A.3), $\lambda' = (\lambda_2 + \lambda_3, \lambda_2 + \lambda_4, \lambda_3 + \lambda_4, 0)$, where $\lambda_4 > 0$. We will be using s_3 defined by $s_3(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4)$. Note that $s_3 \cdot Y = Y$ since $y_4 = \alpha_3(Y) = 0$, and that s_3 commutes with s_1 and s_{12}. We consider the following mutually exclusive cases (B1)–(B4):

(B1) $\Sigma_\lambda = \{\alpha_4\}$: in that case, $W_{\lambda'}^{B_4} = \{id, s_3\}$ and $w \in s_3\, s_4 W_{\lambda'}^{B_4} W_Y^{B_4} \subset W_Y$.

(B2) $\Sigma_\lambda = \{\alpha_1, \alpha_4\}$, *i.e.* $\lambda_2 = \lambda_3 > \lambda_4 > 0$: in that case, $W_{\lambda'}^{B_4} = \{id, s_1, s_3, s_1\, s_3 = s_3\, s_1\} \times \{id, s_3\}$. Since s_1 commutes with s_3 and s_4, we have $w \in \{id, s_1\}\, s_3\, s_4\, \{id, s_3\}\, W_Y^{B_4} \subset W_\lambda W_Y$.

(B3) $\Sigma_\lambda = \{\alpha_2, \alpha_4\}$, *i.e.* $\lambda_3 = \lambda_4 > 0$: in that case, $W_{\lambda'}^{B_4} = \{id, s_{12}, s_{12}\} \times \{id, s_3\}$. Using (A.2), we find that $w \in s_3\, s_4 \{id, s_{12}\} \times \{id, s_3\}\, W_Y^{B_4} \subset W_\lambda W_Y$.

(B4) $\Sigma_\lambda = \{\alpha_1, \alpha_2, \alpha_4\}$, *i.e.* $\lambda_2 = \lambda_3 = \lambda_4 > 0$: in that case, $W_{\lambda'}^{B_4} = \{id, s_{12}, s_1, s_{12}s_1, s_1 s_{12}, s_1 s_{12} s_1\} \times \{id, s_3\}$.

Similarly as in (B2) and (B3), we verify that $s_3\, s_4 W_{\lambda'}^{B_4} \subset W_\lambda W_Y$. For example, $s_3\, s_4 (s_1 s_{12} s_1) = s_1\, s_3\, s_4 s_{12} s_1 = s_1\, s_2\, s_3\, s_4\, s_1 = s_1\, s_2\, s_1\, s_3\, s_4 \in W_\lambda W_Y$. Thus (A.1) implies that $w \in s_3\, s_4\, W_{\lambda'}^{B_4} W_Y^{B_4} \subset W_\lambda W_Y$. □

A.5. **Type G_2.** The Cartan space is given by $\mathfrak{a}(G_2) = \{H_{A,B} = (A, B, A-B, 0, B-A, -B, -A) \mid A, B \in \mathbf{R}\}$ and two simple positive roots are $\alpha(H_{A,B}) = A - B$ and $\beta(H_{A,B}) = B - (A - B) = 2B - A$. Consequently, the positive Weyl chamber is given by $\mathfrak{a}^+ = \{H_{A,B} \mid A > B > A - B > 0\}$.

Note that it is sufficient to work on the space $\mathfrak{a} = \{h_{A,B} = (A, B, A-B) \colon A, B \in \mathbf{R}\}$ which is isomorphic to $\mathfrak{a}(G_2)$. We will work on this space $\mathfrak{a}$ from now on. Observe also that the Weyl group W is generated by s_α which interchanges the first two entries and changes the sign of the third and $s_\beta = (2, 3)$, so it is included in $S_3 \rtimes \{1, -1\}^3$. This inclusion is strict: the group W has 12 elements and $S_3 \rtimes \{1, -1\}^3$ has $6 \times 2^3 = 48$ elements.

Proposition A.13. *Property Killing-max is verified in the case of the root system G_2.*

Proof. Given that the root system is of rank 2, we only need to consider three cases of singular λ and Y:

(C1) $\alpha(\lambda) = \alpha(Y) = 0$: We have $\lambda = (l, l, 0)$, $Y = (y, y, 0)$, l, $y > 0$ and $\langle \lambda, w \cdot Y \rangle = \langle \lambda, Y \rangle = 2\, l\, y$. It follows that 0 in Y cannot change position in $w \cdot Y$ and no y can become $-y$, so $w \cdot Y = Y$ and $w \in W_Y$.

(C2) $\alpha(\lambda) = \beta(Y) = 0$: We have $\lambda = (l, l, 0)$, $Y = (2\, y, y, y)$, l, $y > 0$ and $\langle \lambda, w \cdot Y \rangle = \langle \lambda, Y \rangle = 3\, l\, y$. Then no minus sign is possible in the first two terms of $w \cdot Y$ and $2\, y$ cannot go to the third position. Consequently, using the fact that $(h_{A,B})_3 = (h_{A,B})_1 - (h_{A,B})_2$, we find that $w \cdot Y = (2\, y, y, y) = Y$ (so $w \in W_Y$) or $w \cdot Y = (y, 2\, y, -y) = s_\alpha Y$, which implies that $s_\alpha w \in W_Y$ and $w \in s_\alpha W_Y \subset W_\lambda W_Y$.

(C3) $\beta(\lambda) = \beta(Y) = 0$: We have $\lambda = (2\, l, l, l)$, $Y = (2\, y, y, y)$, l, $y > 0$. Then $2\, y$ must remain in the first position in $w \cdot Y$ and no sign change can happen, thus $w \cdot Y = Y$ and $w \in W_Y$.

□

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