



HAL
open science

GENERALIZED k -FRACTIONAL CONFORMABLE INTEGRALS AND RELATED INEQUALITIES

Feng Qi, Siddra Habib, Shahid Mubeen, Muhammad Nawaz Naeem

► **To cite this version:**

Feng Qi, Siddra Habib, Shahid Mubeen, Muhammad Nawaz Naeem. GENERALIZED k -FRACTIONAL CONFORMABLE INTEGRALS AND RELATED INEQUALITIES. AIMS Mathematics, 2019, 4 (3), pp.343–358. <10.3934/math.2019.3.343>. <hal-01788916v2>

HAL Id: hal-01788916

<https://hal.science/hal-01788916v2>

Submitted on 19 Jan 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY 4.0 - Attribution - International License



Research article

Generalized k -fractional conformable integrals and related inequalities

Feng Qi^{1,2,3,*}, Siddra Habib⁴, Shahid Mubeen⁵, and Muhammad Nawaz Naeem⁶

¹ College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, China

² School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin 300387, China

³ Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, China

⁴ Department of Mathematics, Government College University Faisalabad, Pakistan

⁵ Department of Mathematics, University of Sargodha, Sargodha, Pakistan

⁶ Department of Mathematics, Government College University Faisalabad, Pakistan

* **Correspondence:** qifeng618@gmail.com

Abstract: In the paper, the authors introduce the generalized k -fractional conformable integrals, which are the k -analogues of the recently introduced fractional conformable integrals and can be reduced to other fractional integrals under specific values of the parameters involved. Hereafter, the authors prove the existence of k -fractional conformable integrals. Finally, the authors generalize some integral inequalities to ones for generalized k -fractional conformable integrals.

Keywords: left k -fractional conformable integral; right k -fractional conformable integral; fractional integral inequality

Mathematics Subject Classification: Primary 26A33; Secondary 26D10, 35A23, 47A63

1. Introduction

Conformable derivatives are nonlocal fractional derivatives. They can be called fractional since we can derive up to arbitrary order. However, since in the community of fractional calculus nonlocal fractional derivatives only are used to be called fractional, we prefer to replace conformable fractional by conformable (as a type of local fractional). Conformable derivatives and other types of local fractional derivatives or modified conformable derivatives in [7] can gain their importance by the ability of using them to generate more generalized nonlocal fractional derivatives with singular kernels (see [4, 23, 27]). Fractional calculus is the study of derivatives and integrals of non-integer order and is the generalized form of classical derivatives and integrals. It is as dated as classical calculus, but it acquires more importance in recent two decades, this is due to its applications in

various fields such as physics, biology, fluid dynamics, control theory, image processing, signal processing, and computer networking. See [5, 11–18, 25, 26, 31, 32, 34, 58, 63–66]. In recent years, the research has been proceeded to generalize the existing inequalities through innovative ideas and approaches of fractional calculus. One of the most popular approaches among researchers is the use of fractional integral operators. Due to their potentials to be expended for the existence of nontrivial and positive solutions of several classes of fractional differential equations, the integral inequalities involving fractional integrals are considerably important.

A large bulk of existing literature consists of generalizations of numerous inequalities via fractional integral operators and their applications [9, 37, 42, 59, 62]. Mubeen and Iqbal [38] contributed the ongoing research by presenting the improved version of generalized Grüss type integral inequalities for k -Riemann–Liouville fractional integrals. Agarwal et al. [8] obtained certain Hermite–Hadamard type inequalities for generalized k -fractional integrals. Set et al. [52] presented an integral identity and generalized Hermite–Hadamard type inequalities for Riemann–Liouville fractional integral. Mubeen et al. [39] established integral inequalities of Ostrowski type for k -fractional Riemann–Liouville integrals. Sarikaya and Budak [50] utilized local fractional integrals to derive a generalized inequality. Khan et al. [35] produced some important generalized inequalities for a finite class of positive decreasing functions for fractional conformable integrals. Jleli et al. [28] determined a Hartman–Winter type inequality involving fractional derivative with respect to another function. In the papers [6, 24, 56, 57, 61] and closely related references therein, there are more information on this topic.

The main object of this paper is to develop a new notion “generalized k -fractional conformable integral” which is the generalized form of fractional operators reported in [27]. Hereafter, we also generalize some integral inequalities given in [35] for a finite class of positive and decreasing functions to ones involving our newly introduced k -fractional conformable integrals. For details of those inequalities, their applications, and their stability, we refer readers to [2, 3, 33, 36, 54, 55].

2. Notations

The notion of left and right fractional conformable derivatives for a differentiable function f , introduced by Abdeljawad [1], can be expressed as

$$\mathfrak{I}_{a^+}^\alpha f(t) = (t - a)^{1-\alpha} f'(t) \quad \text{and} \quad \mathfrak{I}_{b^-}^\alpha f(t) = (b - t)^{1-\alpha} f'(t).$$

Correspondingly, left and right fractional conformable integrals for $0 < \alpha < 1$ can be represented by

$$\mathcal{H}_{a^+}^\alpha f(t) = \int_a^t \frac{f(x)}{(x - a)^{1-\alpha}} dx \quad \text{and} \quad \mathcal{H}_{b^-}^\alpha f(t) = \int_t^b \frac{f(x)}{(b - x)^{1-\alpha}} dx.$$

Let $\Gamma(z)$ for $\Re(z) > 0$ denote the classical gamma function [43, 45]. The left and right fractional conformable integral (LFCI and RFCI) operators of order $\beta \in \mathbb{C}$ for $\Re(\beta) > 0$ can be defined [27] respectively by

$${}^\beta \mathcal{H}_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left[\frac{(x - a)^\alpha - (t - a)^\alpha}{\alpha} \right]^{\beta-1} \frac{f(t)}{(t - a)^{1-\alpha}} dt$$

and

$${}^\beta \mathcal{H}_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left[\frac{(b - x)^\alpha - (b - t)^\alpha}{\alpha} \right]^{\beta-1} \frac{f(t)}{(b - t)^{1-\alpha}} dt.$$

Díaz and Pariguan [19] generalized the classical Pochhammer symbol $(\lambda)_n$, the classical gamma function $\Gamma(z)$, and the classical beta function $B(u, v)$ respectively as

$$(\lambda)_{n,k} = \begin{cases} 1, & n = 0; \\ \lambda(\lambda + k) \cdots (\lambda + (n-1)k), & n \in \mathbb{N}, \end{cases} \quad \Gamma_k(x) = \lim_{n \rightarrow \infty} \frac{n! k^n (nk)^{x/k-1}}{(x)_{n,k}},$$

and

$$B_k(u, v) = \frac{1}{k} \int_0^1 t^{u/k-1} (1-t)^{v/k-1} dt.$$

See also [40, 41, 44, 46, 47]. It is not difficult to see that the k -gamma function $\Gamma_k(x)$ and the k -beta function $B_k(u, v)$ satisfy

$$\Gamma_k(x) = \int_0^\infty u^{x-1} e^{-u^k/k} du, \quad \Gamma(x) = \lim_{k \rightarrow 1} \Gamma_k(x), \quad \Gamma_k(x) = k^{x/k-1} \Gamma\left(\frac{x}{k}\right), \quad \Gamma_k(x+k) = x \Gamma_k(x),$$

and

$$B_k(u, v) = \frac{1}{k} B\left(\frac{u}{k}, \frac{v}{k}\right), \quad B_k(u, v) = \frac{\Gamma_k(u) \Gamma_k(v)}{\Gamma_k(u+v)}.$$

3. Generalized k -fractional conformable integrals

In this section, we introduce the generalized left and right k -fractional conformable integrals which generalize the Riemann–Liouville fractional integrals [49, p. 44], Hadamard fractional integrals [10], Katugampola fractional integrals [29], and generalized fractional integrals [51].

Definition 3.1. Let f be a continuous function on a finite real interval $[a, b]$. Then the generalized left and right k -fractional conformable integrals (k -FCI) of order $\beta \in \mathbb{C}$ for $\Re(\beta) > 0$ are respectively defined as

$${}^\beta_k \mathcal{H}_{a^+}^\alpha f(x) = \frac{1}{k \Gamma_k(\beta)} \int_a^x \left[\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha} \right]^{\beta/k-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt$$

and

$${}^\beta_k \mathcal{H}_{b^-}^\alpha f(x) = \frac{1}{k \Gamma_k(\beta)} \int_x^b \left[\frac{(b-x)^\alpha - (b-t)^\alpha}{\alpha} \right]^{\beta/k-1} \frac{f(t)}{(b-t)^{1-\alpha}} dt,$$

where $k > 0$ and $\alpha \in \mathbb{R} \setminus \{0\}$.

Some features of those concepts defined in Definition 3.1, such as the semi-group property, the derivative of functions, the Laplace transforms of functions using this derivative and the solution of IVP, can be found in [1].

Theorem 3.1. Let $f \in L_1[a, b]$, $\alpha \in \mathbb{R} \setminus \{0\}$, and $k > 0$. Then both ${}^\beta_k \mathcal{H}_{a^+}^\alpha f(x)$ and ${}^\beta_k \mathcal{H}_{b^-}^\alpha f(x)$ exist for all $x \in [a, b]$ and $\Re(\beta) > 0$.

Proof. Let $\Delta' = [a, b] \times [a, b]$ and $P' : \Delta' \rightarrow \mathbb{R}$ such that

$$P'(x, t) = [(x-a)^\alpha - (t-a)^\alpha]^{\beta/k-1} (t-a)^{\alpha-1}.$$

It is clear that $P' = P'_+ + P'_-$, where

$$P'_+(x, t) = \begin{cases} [(x-a)^\alpha - (t-a)^\alpha]^{\beta/k-1} (t-a)^{\alpha-1}, & a \leq t \leq x \leq b; \\ 0, & a \leq x \leq t \leq b \end{cases}$$

and

$$P'_-(x, t) = \begin{cases} [(t-a)^\alpha - (x-a)^\alpha]^{\beta/k-1} (x-a)^{\alpha-1}, & a \leq t \leq x \leq b; \\ 0, & a \leq x \leq t \leq b. \end{cases}$$

Since P' is measurable on Δ' , we can write

$$\int_a^b P'(x, t) dt = \int_a^x P'(x, t) dt = \int_a^x [(x-a)^\alpha - (t-a)^\alpha]^{\beta/k-1} (t-a)^{\alpha-1} dt = \frac{\alpha k}{\beta} (x-a)^{\alpha\beta/k}.$$

Therefore, we obtain

$$\begin{aligned} \int_a^b \left[\int_a^b P'(x, t) |f(x)| dx \right] dx &= \int_a^b |f(x)| \left[\int_a^b P'(x, t) dt \right] dx = \frac{\alpha k}{\beta} \int_a^b (x-a)^{\alpha\beta/k} |f(x)| dx \\ &\leq \frac{\alpha k}{\beta} (b-a)^{\alpha\beta/k} \int_a^b |f(x)| dx = \frac{\alpha k}{\beta} (b-a)^{\alpha\beta/k} \|f(x)\|_{L_1[a,b]} < \infty. \end{aligned}$$

So, by Tonelli's theorem for iterated integrals [21, p. 147], the function $Q' : \Delta' \rightarrow \mathbb{R}$ such that $Q'(x, t) = P'(x, t)f(x)$ is integrable over Δ' . Hence, by Fubini's theorem, it follows that $\int_a^b P'(x, t)f(x) dx$ is integrable over $[a, b]$ as a function of $t \in [a, b]$. This implies that ${}^\beta_k \mathcal{H}_{a^+}^\alpha f(x)$ exists.

The existence of the right k -fractional conformable integral ${}^\beta_k \mathcal{H}_{b^-}^\alpha f(x)$ can be proved in a similar manner. The proof of Theorem 3.1 is complete. \square

4. Inequalities for generalized k -fractional conformable integrals

Fractional integral inequalities have been analyzed for many useful purposes. One of the most useful applications of such inequalities is the existence of nontrivial solutions of fractional differential equations. Many applications find in the literature for the existence of nontrivial solution eigenvalue problems by inequalities, see [42, 62]. Generalizing pre-existing inequalities by applying fractional integral operators is becoming a popular trend in the research field, see, for example, [22, 46, 48].

In this section, we present some k -analogues of inequalities in [53, 59, 60] for generalized k -fractional conformable integrals.

Theorem 4.1. *Let $h(x)$ be a continuous increasing function and $\{g_i, 1 \leq i \leq n\}$ be a sequence of continuous positive decreasing functions on the interval $[a, b]$. Let $a < x \leq b$, $\eta > 0$, $\xi \geq \gamma_p > 0$ for $1 \leq p \leq n$. Then the left k -FCI operator ${}^\beta_k \mathcal{H}_{a^+}^\alpha$ satisfies the inequality*

$$\frac{{}^\beta_k \mathcal{H}_{a^+}^\alpha (\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x))}{{}^\beta_k \mathcal{H}_{a^+}^\alpha (\prod_{i=1}^n g_i^{\gamma_i}(x))} \geq \frac{{}^\beta_k \mathcal{H}_{a^+}^\alpha (h^\eta(x) \prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x))}{{}^\beta_k \mathcal{H}_{a^+}^\alpha (h^\eta(x) \prod_{i=1}^n g_i^{\gamma_i}(x))}. \quad (4.1)$$

Proof. Under given conditions, we have

$$[h^\eta(\rho) - h^\eta(\tau)][g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)] \geq 0.$$

Let us define a function

$${}^{\beta}_k\mathcal{J}_{a^+}^{\alpha}(x, \rho, \tau) = \frac{1}{k\Gamma_k(\beta)} \left[\frac{(x-a)^{\alpha} - (\tau-a)^{\alpha}}{\alpha} \right]^{\beta/k-1} \frac{\prod_{i=1}^n g_i^{\gamma_i}(\tau)}{(\tau-a)^{1-\alpha}} [h^{\eta}(\rho) - h^{\eta}(\tau)] [g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)]. \quad (4.2)$$

Accordingly, the function ${}^{\beta}_k\mathcal{J}_{a^+}^{\alpha}(x, \rho, \tau)$ is positive for all $\tau \in (a, b]$. Integrating on both sides of the above equation (4.2) with respect to τ from a to x shows

$$\begin{aligned} 0 &\leq \int_a^x {}^{\beta}_k\mathcal{J}_{a^+}^{\alpha}(x, \rho, \tau) d\tau = \frac{1}{k\Gamma_k(\beta)} \int_a^x \left[\frac{(x-a)^{\alpha} - (\tau-a)^{\alpha}}{\alpha} \right]^{\beta/k-1} \\ &\quad \times \prod_{i=1}^n g_i^{\gamma_i}(\tau) [h^{\eta}(\rho) - h^{\eta}(\tau)] [g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)] \frac{d\tau}{(\tau-a)^{1-\alpha}} \\ &= h^{\eta}(\rho) \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] + g_p^{\xi-\gamma_p}(\rho) \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(h^{\eta}(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\ &\quad - h^{\eta}(\rho) g_p^{\xi-\gamma_p}(\rho) \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] - \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(h^{\eta}(x) \prod_{i \neq p}^n g_i^{\gamma_i}(x) \right) \right]. \end{aligned} \quad (4.3)$$

Multiplying on both sides of the relation (4.3) by

$$\frac{1}{k\Gamma_k(\beta)} \left[\frac{(x-a)^{\alpha} - (\rho-a)^{\alpha}}{\alpha} \right]^{\beta/k-1} \frac{\prod_{i=1}^n g_i^{\gamma_i}(\rho)}{(\rho-a)^{1-\alpha}}. \quad (4.4)$$

and integrating on both sides with respect to ρ from a to x give

$$\begin{aligned} 0 &\leq \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(h^{\eta}(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\ &\quad - \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(h^{\eta}(x) \prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]. \end{aligned} \quad (4.5)$$

Dividing on both sides of (4.5) by

$${}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(h^{\eta}(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]$$

leads to (4.1). The proof of Theorem 4.1 is complete. \square

Corollary 4.1. Let $\{g_i, 1 \leq i \leq n\}$ be a sequence of continuous positive decreasing functions on the interval $[a, b]$. Let $a < x \leq b$, $\eta > 0$, $\xi \geq \gamma_p > 0$ for $1 \leq p \leq n$. Then the left k -FCI operator ${}^{\beta}_k\mathcal{H}_{a^+}^{\alpha}$ satisfies the inequality

$$\frac{{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right)}{{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right)} \geq \frac{{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right)}{{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right)}. \quad (4.6)$$

Proof. This can be derived from taking $h(x) = x - a$ in Theorem 4.1. The proof of Corollary 4.1 is complete. \square

Corollary 4.2. Let $\{g_i, 1 \leq i \leq n\}$ be a sequence of continuous positive decreasing functions on the interval $[a, b]$. Let $a < x \leq b$, $\eta > 0$, $\xi \geq \gamma_p > 0$ for $1 \leq p \leq n$. Then the left k -FCI operator ${}^{\beta}_k\mathcal{H}_{a^+}^{\alpha}$ satisfies the inequality

$$\begin{aligned} & \left[{}^{\theta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \\ & + \left[{}^{\theta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\ & \geq \left[{}^{\theta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \\ & + \left[{}^{\theta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]. \end{aligned} \quad (4.7)$$

Proof. Multiplying on both sides of the relation

$$\begin{aligned} 0 & \leq \int_a^x {}^{\beta}_k\mathfrak{S}_{a^+}^{\alpha}(x, \rho, \tau) d\tau = \frac{1}{k\Gamma_k(\beta)} \int_a^x \left[\frac{(x-a)^{\alpha} - (\tau-a)^{\alpha}}{\alpha} \right]^{\beta/k-1} \\ & \times \prod_{i=1}^n g_i^{\gamma_i}(\tau) [(\rho-a)^{\eta} - (\tau-a)^{\eta}] [g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)] \frac{d\tau}{(\tau-a)^{1-\alpha}} \\ & = (\rho-a)^{\eta} \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] + g_p^{\xi-\gamma_p}(\rho) \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\ & - (\rho-a)^{\eta} g_p^{\xi-\gamma_p}(\rho) \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] - \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i \neq p}^n g_i^{\gamma_i}(x) \right) \right] \end{aligned}$$

by

$$\frac{1}{k\Gamma_k(\theta)} \left[\frac{(x-a)^{\alpha} - (\rho-a)^{\alpha}}{\alpha} \right]^{\theta/k-1} \frac{\prod_{i=1}^n g_i^{\gamma_i}(\rho)}{(\rho-a)^{1-\alpha}} \quad (4.8)$$

and integrating on both sides with respect to ρ from a to x arrive at

$$\begin{aligned} 0 & \leq \left[{}^{\theta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \\ & + \left[{}^{\theta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\ & - \left[{}^{\theta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \\ & - \left[{}^{\theta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]. \end{aligned} \quad (4.9)$$

Dividing on both sides of (4.9) by

$$\left[{}^{\theta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] + \left[{}^{\theta}_k\mathcal{H}_{a^+}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{a^+}^{\alpha} \left((x-a)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]$$

leads to (4.7). The proof of Corollary 4.2 is complete. \square

Corollary 4.3. *Let $h(x)$ be a continuous increasing function and $\{g_i, 1 \leq i \leq n\}$ be a sequence of continuous positive decreasing functions on the interval $[a, b]$. Let $a < x \leq b, \eta > 0, \xi \geq \gamma_p > 0$ for $1 \leq p \leq n$. Then the left k -FCI operator ${}^\beta_k\mathcal{H}_{a^+}^\alpha$ satisfies the inequality*

$$\begin{aligned} & \left[{}^\theta_k\mathcal{H}_{a^+}^\alpha \left(h^\eta(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \left[{}^\beta_k\mathcal{H}_{a^+}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \\ & + \left[{}^\theta_k\mathcal{H}_{a^+}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \left[{}^\beta_k\mathcal{H}_{a^+}^\alpha \left(h^\eta(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\ & \geq \left[{}^\theta_k\mathcal{H}_{a^+}^\alpha \left(h^\eta(x) \prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \left[{}^\beta_k\mathcal{H}_{a^+}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \\ & + \left[{}^\theta_k\mathcal{H}_{a^+}^\alpha \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \left[{}^\beta_k\mathcal{H}_{a^+}^\alpha \left(h^\eta(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]. \end{aligned} \tag{4.10}$$

Proof. Multiplying on both sides of the relation (4.3) by (4.8) and integrating on both sides with respect to ρ from a to x derive

$$\begin{aligned} 0 & \leq \left[{}^\theta_k\mathcal{H}_{a^+}^\alpha \left(h^\eta(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \left[{}^\beta_k\mathcal{H}_{a^+}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \\ & + \left[{}^\theta_k\mathcal{H}_{a^+}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \left[{}^\beta_k\mathcal{H}_{a^+}^\alpha \left(h^\eta(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\ & - \left[{}^\theta_k\mathcal{H}_{a^+}^\alpha \left(h^\eta(x) \prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \left[{}^\beta_k\mathcal{H}_{a^+}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \\ & - \left[{}^\theta_k\mathcal{H}_{a^+}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \left[{}^\beta_k\mathcal{H}_{a^+}^\alpha \left(h^\eta(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]. \end{aligned} \tag{4.11}$$

Dividing on both sides of (4.11) by

$$\left[{}^\theta_k\mathcal{H}_{a^+}^\alpha \left(h^\eta(x) \prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \left[{}^\beta_k\mathcal{H}_{a^+}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] + \left[{}^\theta_k\mathcal{H}_{a^+}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \left[{}^\beta_k\mathcal{H}_{a^+}^\alpha \left(h^\eta(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]$$

reveals (4.10). The proof of Corollary 4.3 is complete. \square

Theorem 4.2. *Let $\{g_i, 1 \leq i \leq n\}$ be a sequence of continuous positive decreasing functions on the interval $[a, b]$. Let $a < x \leq b, \eta > 0, \xi \geq \gamma_p > 0$ for $1 \leq p \leq n$. Then the right k -FCI operator ${}^\beta_k\mathcal{H}_{b^-}^\alpha$ satisfies the inequality*

$$\frac{{}^\beta_k\mathcal{H}_{b^-}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right)}{{}^\beta_k\mathcal{H}_{b^-}^\alpha \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right)} \geq \frac{{}^\beta_k\mathcal{H}_{b^-}^\alpha \left((b-x)^\eta \prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right)}{{}^\beta_k\mathcal{H}_{b^-}^\alpha \left((b-x)^\eta \prod_{i=1}^n g_i^{\gamma_i}(x) \right)}. \tag{4.12}$$

Proof. Under given conditions, we have

$$[(b - \rho)^\eta - (b - \tau)^\eta][g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)] \geq 0.$$

Let us define a function

$$\begin{aligned} & {}_k^{\beta}\mathfrak{S}_{b^-}^\alpha(x, \rho, \tau) \\ &= \frac{1}{k\Gamma_k(\beta)} \left[\frac{(b-x)^\alpha - (b-\tau)^\alpha}{\alpha} \right]^{\beta/k-1} \frac{\prod_{i=1}^n g_i^{\gamma_i}(\tau)}{(b-\tau)^{1-\alpha}} [(b-\rho)^\eta - (b-\tau)^\eta][g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)]. \end{aligned} \quad (4.13)$$

Consequently, the function ${}_k^{\beta}\mathfrak{S}_{b^-}^\alpha(x, \rho, \tau)$ is positive for all $\tau \in (a, b]$. Integrating on both sides of the above equation (4.13) with respect to τ from x to b gives

$$\begin{aligned} 0 &\leq \int_x^b {}_k^{\beta}\mathfrak{S}_{b^-}^\alpha(x, \rho, \tau) d\tau = \frac{1}{k\Gamma_k(\beta)} \int_x^b \left[\frac{(b-x)^\alpha - (b-\tau)^\alpha}{\alpha} \right]^{\beta/k-1} \\ &\quad \times \prod_{i=1}^n g_i^{\gamma_i}(\tau) [(b-\rho)^\eta - (b-\tau)^\eta][g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)] \frac{d\tau}{(b-\tau)^{1-\alpha}} \\ &= (b-\rho)^\eta \left[{}_k^{\beta}\mathcal{H}_{b^-}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] + g_p^{\xi-\gamma_p}(\rho) \left[{}_k^{\beta}\mathcal{H}_{b^-}^\alpha \left((b-x)^\eta \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\ &\quad - (b-\rho)^\eta g_p^{\xi-\gamma_p}(\rho) \left[{}_k^{\beta}\mathcal{H}_{b^-}^\alpha \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] - \left[{}_k^{\beta}\mathcal{H}_{b^-}^\alpha \left((b-x)^\eta \prod_{i \neq p}^n g_i^{\gamma_i}(x) \right) \right]. \end{aligned} \quad (4.14)$$

Multiplying the relation (4.14) by

$$\frac{1}{k\Gamma_k(\beta)} \left[\frac{(b-x)^\alpha - (b-\rho)^\alpha}{\alpha} \right]^{\beta/k-1} \frac{\prod_{i=1}^n g_i^{\gamma_i}(\rho)}{(b-\rho)^{1-\alpha}} \quad (4.15)$$

and integrating on both sides with respect to ρ from x to b produce

$$\begin{aligned} 0 &\leq \left[{}_k^{\beta}\mathcal{H}_{b^-}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \left[{}_k^{\beta}\mathcal{H}_{b^-}^\alpha \left((b-x)^\eta \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\ &\quad - \left[{}_k^{\beta}\mathcal{H}_{b^-}^\alpha \left((b-x)^\eta \prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \left[{}_k^{\beta}\mathcal{H}_{b^-}^\alpha \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]. \end{aligned} \quad (4.16)$$

Dividing on both sides of (4.16) by

$${}_k^{\beta}\mathcal{H}_{b^-}^\alpha \left((b-x)^\eta \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \left[{}_k^{\beta}\mathcal{H}_{b^-}^\alpha \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]$$

yields (4.12). The proof of Theorem 4.2 is complete. □

Corollary 4.4. Let $\{g_i, 1 \leq i \leq n\}$ be a sequence of continuous positive decreasing functions on the interval $[a, b]$. Let $a < x \leq b, \eta > 0, \xi \geq \gamma_p > 0$ for $1 \leq p \leq n$. Then the right k -FCI operator ${}_k^{\beta}\mathcal{H}_{b^-}^\alpha$

satisfies the inequality

$$\begin{aligned}
& \left[{}^{\theta}_k \mathcal{H}_{b^-}^{\alpha} \left((b-x)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \left[{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \\
& + \left[{}^{\theta}_k \mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left((b-x)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\
& \geq \left[{}^{\theta}_k \mathcal{H}_{b^-}^{\alpha} \left((b-x)^{\eta} \prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \\
& + \left[{}^{\theta}_k \mathcal{H}_{b^-}^{\alpha} \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \left[{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left((b-x)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right].
\end{aligned} \tag{4.17}$$

Proof. Multiplying the relation (4.14) by

$$\frac{1}{k\Gamma_k(\theta)} \left[\frac{(b-x)^{\alpha} - (b-\rho)^{\alpha}}{\alpha} \right]^{\theta/k-1} \frac{\prod_{i=1}^n g_i^{\gamma_i}(\rho)}{(b-\rho)^{1-\alpha}} \tag{4.18}$$

and integrating on both sides with respect to ρ from x to b procure

$$\begin{aligned}
0 & \leq \left[{}^{\theta}_k \mathcal{H}_{b^-}^{\alpha} \left((b-x)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \left[{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \\
& + \left[{}^{\theta}_k \mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left((b-x)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\
& - \left[{}^{\theta}_k \mathcal{H}_{b^-}^{\alpha} \left((x-a)^{\eta} \prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \\
& - \left[{}^{\theta}_k \mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left((b-x)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right].
\end{aligned} \tag{4.19}$$

Dividing on both sides of (4.19) by

$$\left[{}^{\theta}_k \mathcal{H}_{b^-}^{\alpha} \left((b-x)^{\eta} \prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] + \left[{}^{\theta}_k \mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left((b-x)^{\eta} \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]$$

demonstrates (4.17). The proof of Corollary 4.4 is complete. \square

Theorem 4.3. Let $h(x)$ be a continuous increasing function and $\{g_i, 1 \leq i \leq n\}$ be a sequence of continuous positive decreasing functions on the interval $[a, b]$. Let $a < x \leq b$, $\eta > 0$, $\xi \geq \gamma_p > 0$ for $1 \leq p \leq n$. Then the right k -FCI operator ${}^{\beta}_k \mathcal{H}_{b^-}^{\alpha}$ satisfies the inequality

$$\frac{{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right)}{{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right)} \geq \frac{{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left(h^{\eta}(x) \prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right)}{{}^{\beta}_k \mathcal{H}_{b^-}^{\alpha} \left(h^{\eta}(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right)}. \tag{4.20}$$

Proof. Under given conditions, we have

$$[h^\eta(\rho) - h^\eta(\tau)][g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)] \geq 0.$$

Let us define a function

$${}^\beta_k \mathcal{J}_{b^-}^\alpha(x, \rho, \tau) = \frac{1}{k\Gamma_k(\beta)} \left[\frac{(b-x)^\alpha - (b-\tau)^\alpha}{\alpha} \right]^{\beta/k-1} \frac{\prod_{i=1}^n g_i^{\gamma_i}(\tau)}{(b-\tau)^{1-\alpha}} [h^\eta(\rho) - h^\eta(\tau)][g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)]. \quad (4.21)$$

Thus, the function ${}^\beta_k \mathcal{J}_{b^-}^\alpha(x, \rho, \tau)$ is positive for all $\tau \in (a, b]$. Integrating on both sides of the above equation (4.21) with respect to τ from x to b results in

$$\begin{aligned} 0 &\leq \int_x^b {}^\beta_k \mathcal{J}_{b^-}^\alpha(x, \rho, \tau) d\tau = \frac{1}{k\Gamma_k(\beta)} \int_x^b \left[\frac{(b-x)^\alpha - (b-\tau)^\alpha}{\alpha} \right]^{\beta/k-1} \\ &\quad \times \prod_{i=1}^n g_i^{\gamma_i}(\tau) [h^\eta(\rho) - h^\eta(\tau)][g_p^{\xi-\gamma_p}(\tau) - g_p^{\xi-\gamma_p}(\rho)] \frac{d\tau}{(b-\tau)^{1-\alpha}} \\ &= h^\eta(\rho) \left[{}^\beta_k \mathcal{H}_{b^-}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] + g_p^{\xi-\gamma_p}(\rho) \left[{}^\beta_k \mathcal{H}_{b^-}^\alpha \left(h^\eta(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\ &\quad - h^\eta(\rho) g_p^{\xi-\gamma_p}(\rho) \left[{}^\beta_k \mathcal{H}_{b^-}^\alpha \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] - \left[{}^\beta_k \mathcal{H}_{b^-}^\alpha \left(h^\eta(x) \prod_{i \neq p}^n g_i^{\gamma_i}(x) \right) \right]. \end{aligned} \quad (4.22)$$

Multiplying the relation (4.22) by (4.15) and integrating on both sides with respect to ρ from x to b yield

$$\begin{aligned} 0 &\leq \left[{}^\beta_k \mathcal{H}_{b^-}^\alpha \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \left[{}^\beta_k \mathcal{H}_{b^-}^\alpha \left(h^\eta(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\ &\quad - \left[{}^\beta_k \mathcal{H}_{b^-}^\alpha \left(h^\eta(x) \prod_{i \neq p}^n g_i^{\gamma_i} g_p^\xi(x) \right) \right] \left[{}^\beta_k \mathcal{H}_{b^-}^\alpha \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]. \end{aligned} \quad (4.23)$$

Dividing on both sides of (4.23) by

$${}^\beta_k \mathcal{H}_{b^-}^\alpha \left(h^\eta(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \left[{}^\beta_k \mathcal{H}_{b^-}^\alpha \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]$$

leads to (4.20). The proof of Theorem 4.3 is complete. \square

Corollary 4.5. Let $h(x)$ be a continuous increasing function and $\{g_i, 1 \leq i \leq n\}$ be a sequence of continuous positive decreasing functions on the interval $[a, b]$. Let $a < x \leq b$, $\eta > 0$, $\xi \geq \gamma_p > 0$ for

$1 \leq p \leq n$. Then the right k -FCI operator ${}^{\beta}_k\mathcal{H}_{b^-}^{\alpha}$ satisfies the inequality

$$\begin{aligned}
 & \left[{}^{\theta}_k\mathcal{H}_{b^-}^{\alpha} \left(h^{\eta}(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \\
 & + \left[{}^{\theta}_k\mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{b^-}^{\alpha} \left(h^{\eta}(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\
 & \geq \left[{}^{\theta}_k\mathcal{H}_{b^-}^{\alpha} \left(h^{\eta}(x) \prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \\
 & + \left[{}^{\theta}_k\mathcal{H}_{b^-}^{\alpha} \left(\prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{b^-}^{\alpha} \left(h^{\eta}(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right].
 \end{aligned} \tag{4.24}$$

Proof. Multiplying the relation (4.22) by (4.18) and integrating on both sides with respect to ρ from x to b give

$$\begin{aligned}
 0 & \leq \left[{}^{\theta}_k\mathcal{H}_{b^-}^{\alpha} \left(h^{\eta}(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \\
 & + \left[{}^{\theta}_k\mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{b^-}^{\alpha} \left(h^{\eta}(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right] \\
 & - \left[{}^{\theta}_k\mathcal{H}_{b^-}^{\alpha} \left(h^{\eta}(x) \prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \\
 & - \left[{}^{\theta}_k\mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{b^-}^{\alpha} \left(h^{\eta}(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right].
 \end{aligned} \tag{4.25}$$

Dividing on both sides of (4.25) by

$$\left[{}^{\theta}_k\mathcal{H}_{b^-}^{\alpha} \left(h^{\eta}(x) \prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] + \left[{}^{\theta}_k\mathcal{H}_{b^-}^{\alpha} \left(\prod_{i \neq p}^n g_i^{\gamma_i} g_p^{\xi}(x) \right) \right] \left[{}^{\beta}_k\mathcal{H}_{b^-}^{\alpha} \left(h^{\eta}(x) \prod_{i=1}^n g_i^{\gamma_i}(x) \right) \right]$$

concludes (4.24). The proof of Corollary 4.5 is complete. \square

5. Conclusions

In this paper, we have presented the left and right k -fractional conformable integrals and generalized some important integral inequalities to ones for our newly introduced k -FCI operators related to a finite sequence of positive and decreasing functions. Our work produces k -analogues of many pre-existing results in the literature. Further, many special cases for other integral operators can be derived from our generalizations. The results obtained can be employed to confirm the existence of nontrivial solutions of fractional differential equations of different classes. The k -FCI operators in this paper are different from those introduced by Katugampola [30] as their kernels depend on the boundary points a and b and need a different convolution theory under conformable Laplace. Our k -fractional conformable integrals in this paper generalize well-known fractional integral operators such as Caputo integral operators [49,

p. 44], Riemann–Liouville integral operators [49, p. 44], Hadamard integral operators [10], and their k -analogues.

Finally we state that possible future works can be in proving new inequalities in the frame of new generalized integrals. The integrals correspond to certain fractional derivatives with nonsingular kernels, for example. See the papers [4, 23].

Remark 5.1. This paper is a slightly revised version of the preprint [20].

Acknowledgments

The authors would like to thank anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. **279** (2015), 57–66; available online at <https://doi.org/10.1016/j.cam.2014.10.016>.
2. T. Abdeljawad, R. P. Agarwal, J. Alzabut, F. Jarad, and A. Özbekler, *Lyapunov-type inequalities for mixed non-linear forced differential equations within conformable derivatives*, J. Inequal. Appl. 2018, **2018**:143, 17 pp. Available online at <https://doi.org/10.1186/s13660-018-1731-x>.
3. T. Abdeljawad and J. Alzabut, *On Riemann–Liouville fractional q -difference equations and their application to retarded logistic type model*, Math. Methods Appl. Sci. **41** (2018), no. 18, 8953–8962; available online at <https://doi.org/10.1002/mma.4743>.
4. T. Abdeljawad, F. Jarad, and J. Alzabut, *Fractional proportional differences with memory*, Eur. Phys. J. Special Topics **226** (2017), 3333–3354; available online at <https://doi.org/10.1140/epjst/e2018-00053-5>.
5. J. Alzabut, S. Tyagi, and S. Abbas, *Discrete fractional-order bam neural networks with leakage delay: existence and stability results*, Asian J. Control **22** (2020), no. 1, 1–13; available online at <https://doi.org/10.1002/asjc.1918>.
6. M. Al-Refai and T. Abdeljawad, *Fundamental results of conformable Sturm-Liouville eigenvalue problems*, Complexity **2017**, Article ID 3720471, 7 pp. Available online at <https://doi.org/10.1155/2017/3720471>.
7. D. R. Anderson and D. J. Ulness, *Newly defined conformable derivatives*, Adv. Dyn. Syst. Appl. **10** (2015), no. 2, 109–137.
8. P. Agarwal, M. Jleli, and M. Tomar, *Certain Hermite–Hadamard type inequalities via generalized k -fractional integrals*, J. Inequal. Appl. **2017**, Paper No. 55, 10 pp. Available online at <https://doi.org/10.1186/s13660-017-1318-y>.

9. R. P. Agarwal and A. Özbekler, *Lyapunov type inequalities for mixed nonlinear Riemann–Liouville fractional differential equations with a forcing term*, J. Comput. Appl. Math. **314** (2017), 69–78; available online at <https://doi.org/10.1016/j.cam.2016.10.009>.
10. A. A. Kilbas, *Hadamard-type fractional calculus*, J. Korean Math. Soc. **38** (2001), no. 6, 1191–1204.
11. D. Băleanu, R. P. Agarwal, H. Khan, R. A. Khan, and H. Jafari, *On the existence of solution for fractional differential equations of order $3 < \delta_1 \leq 4$* , Adv. Difference Equ. **2015**, 2015:362, 9 pp. Available online at <https://doi.org/10.1186/s13662-015-0686-1>.
12. D. Băleanu, R. P. Agarwal, H. Mohammadi, and S. Rezapour, *Some existence results for a nonlinear fractional differential equation on partially ordered Banach spaces*, Bound. Value Probl. **2013**, 2013:112, 8 pp. Available online at <https://doi.org/10.1186/1687-2770-2013-112>.
13. D. Băleanu, H. Khan, H. Jafari, R. A. Khan, and M. Alipour, *On existence results for solutions of a coupled system of hybrid boundary value problems with hybrid conditions*, Adv. Difference Equ. **2015**, 2015:318, 14 pp. Available online at <https://doi.org/10.1186/s13662-015-0651-z>.
14. D. Băleanu and O. G. Mustafa, *On the global existence of solutions to a class of fractional differential equations*, Comput. Math. Appl. **59** (2010), no. 5, 1835–1841; available online at <https://doi.org/10.1016/j.camwa.2009.08.028>.
15. D. Băleanu, O. G. Mustafa, and R. P. Agarwal, *An existence result for a superlinear fractional differential equation*, Appl. Math. Lett. **23** (2010), no. 9, 1129–1132; available online at <https://doi.org/10.1016/j.aml.2010.04.049>.
16. D. Băleanu, O. G. Mustafa, and R. P. Agarwal, *On the solution set for a class of sequential fractional differential equations*, J. Phys. A **43** (2010), no. 38, 385209, 7 pp. Available online at <https://doi.org/10.1088/1751-8113/43/38/385209>.
17. A. Bolandtalat, E. Babolian, H. Jafari, *Numerical solutions of multi-order fractional differential equations by Boubaker polynomials*, Open Phys. **14** (2016), no. 1, 226–230; available online at <https://doi.org/10.1515/phys-2016-0028>.
18. A. Debbouche and V. Antonov, *Finite-dimensional diffusion models of heat transfer in fractal mediums involving local fractional derivatives*, Nonlinear Stud. **24** (2017), no. 3, 527–535.
19. R. Díaz and E. Pariguan, *On hypergeometric functions and Pochhammer k -symbol*, Divulg. Mat. **15** (2007), no. 2, 179–192.
20. S. Habib, S. Mubeen, M. N. Naeem, and F. Qi, *Generalized k -fractional conformable integrals and related inequalities*, AIMS Math. **4** (2019), no. 3, 343–358; available online at <https://doi.org/10.3934/Math.2019.3.343>.
21. P. R. Halmos, *Measure Theory*, D. Van Nostrand Company, Inc., New York, N. Y., 1950.
22. C.-J. Huang, G. Rahman, K. S. Nisar, A. Ghaffar, and F. Qi, *Some inequalities of the Hermite–Hadamard type for k -fractional conformable integrals*, Austral. J. Math. Anal. Appl. **16** (2019), no. 1, Article 7, 9 pages; available online at <http://ajmaa.org/cgi-bin/paper.pl?string=v16n1/V16I1P7.tex>.

23. F. Jarad, T. Abdeljawad, and J. Alzabut, *Generalized fractional derivatives generated by a class of local proportional derivatives*, Eur. Phys. J. Special Topics **226** (2017), 3457–3471; available online at <https://doi.org/10.1140/epjst/e2018-00021-7>.
24. F. Jarad, Y. Adjabi, D. Baleanu, and T. Abdeljawad, *On defining the distributions δ^r and $(\delta^r)'$ by conformable derivatives*, Adv. Difference Equ. **2018**, Paper No. 407, 20 pp. Available online at <https://doi.org/10.1186/s13662-018-1865-7>.
25. H. Jafari, H. K. Jassim, S. P. Moshokoa, V. M. Ariyan, and F. Tchier, *Reduced differential transform method for partial differential equations within local fractional derivative operators*, Adv. Mech. Eng. **8** (2016), no. 4, 1–6; available online at <https://doi.org/10.1177/1687814016633013>.
26. H. Jafari, H. K. Jassim, F. Tchier, and D. Băleanu, *On the approximate solutions of local fractional differential equations with local fractional operators*, Entropy **18** (2016), no. 4, Paper No. 150, 12 pp. Available online at <https://doi.org/10.3390/e18040150>.
27. F. Jarad, E. Uğurlu, T. Abdeljawad, and D. Băleanu. *On a new class of fractional operators*, Adv. Difference Equ. **2017**, Paper No. 247, 16 pp. Available online at <https://doi.org/10.1186/s13662-017-1306-z>.
28. M. Jleli, M. Kirane, and B. Samet, *Hartman-Wintner-type inequality for a fractional boundary value problem via a fractional derivative with respect to another function*, Discrete Dyn. Nat. Soc. **2017**, Article ID 5123240, 8 pp. Available online at <https://doi.org/10.1155/2017/5123240>.
29. U. N. Katugampola, *New approach to a generalized fractional integral*, Appl. Math. Comput. **218** (2011), no. 3, 860–865; available online at <https://doi.org/10.1016/j.amc.2011.03.062>.
30. U. N. Katugampola, *New fractional integral unifying six existing fractional integrals*, arXiv preprint (2016), available online at <https://arxiv.org/abs/1612.08596>.
31. H. Khan, H. Jafari, D. Băleanu, R. A. Khan, and A. Khan, *On iterative solutions and error estimations of a coupled system of fractional order differential-integral equations with initial and boundary conditions*, Differ. Equ. Dyn. Syst. (2019), in press; available online at <https://doi.org/10.1007/s12591-017-0365-7>.
32. H. Khan, Y.-J. Li, W. Chen, D. Băleanu, and A. Khan, *Existence theorems and Hyers-Ulam stability for a coupled system of fractional differential equations with p -Laplacian operator*, Bound. Value Probl. **2017**, Paper No. 157, 16 pp. Available online at <https://doi.org/10.1186/s13661-017-0878-6>.
33. A. Khan, Y.-J. Li, K. Shah, and T. S. Khan, *On coupled p -Laplacian fractional differential equations with nonlinear boundary conditions*, Complexity **2017**, Article ID 8197610, 9 pp. Available online at <https://doi.org/10.1155/2017/8197610>.
34. H. Khan, Y.-J. Li, H.-G. Sun, and A. Khan, *Existence of solution and Hyers-Ulam stability for a coupled system of fractional differential equations with p -Laplacian operator*, J. Nonlinear Sci. Appl. **10** (2017), no. 10, 5219–5229.
35. H. Khan, H.-G. Sun, W. Chen, and D. Băleanu, *Inequalities for new class of fractional integral operators*, J. Nonlinear Sci. Appl. **10** (2017), no. 12, 6166–6176.

36. Y.-J. Li, K. Shah, and R. A. Khan, *Iterative technique for coupled integral boundary value problem of non-integer order differential equations*, Adv. Difference Equ. **2017**, Paper No. 251, 14 pp. Available online at <https://doi.org/10.1186/s13662-017-1308-x>.
37. W.-J. Liu, Q.-A. Ngô, and V. N. Huy, *Several interesting integral inequalities*, J. Math. Inequal. **3** (2009), no. 2, 201–212; available online at <https://doi.org/10.7153/jmi-03-20>.
38. S. Mubeen and S. Iqbal, *Grüss type integral inequalities for generalized Riemann–Liouville k -fractional integrals*, J. Inequal. Appl. **2016**, Paper No. 109, 13 pp. Available online at <https://doi.org/10.1186/s13660-016-1052-x>.
39. S. Mubeen, S. Iqbal, and Z. Iqbal, *On Ostrowski type inequalities for generalized k -fractional integrals*, J. Inequal. Spec. Funct. **8** (2017), no. 3, 107–118.
40. K. S. Nisar and F. Qi, *On solutions of fractional kinetic equations involving the generalized k -Bessel function*, Note Mat. **37** (2017), no. 2, 11–20; available online at <https://doi.org/10.1285/i15900932v37n2p11>.
41. K. S. Nisar, F. Qi, G. Rahman, S. Mubeen, and M. Arshad, *Some inequalities involving the extended gamma function and the Kummer confluent hypergeometric k -function*, J. Inequal. Appl. **2018**, Paper No. 135, 12 pages; available online at <https://doi.org/10.1186/s13660-018-1717-8>.
42. D. O'Regan and B. Samet, *Lyapunov-type inequalities for a class of fractional differential equations*, J. Inequal. Appl. **2015**, 2015:247, 10 pp. Available online at <https://doi.org/10.1186/s13660-015-0769-2>.
43. F. Qi and R. P. Agarwal, *On complete monotonicity for several classes of functions related to ratios of gamma functions*, J. Inequal. Appl. **2019**, Paper No. 36, 42 pages; available online at <https://doi.org/10.1186/s13660-019-1976-z>.
44. F. Qi, A. Akkurt, and H. Yildirim, *Catalan numbers, k -gamma and k -beta functions, and parametric integrals*, J. Comput. Anal. Appl. **25** (2018), no. 6, 1036–1042.
45. F. Qi and B.-N. Guo, *Integral representations and complete monotonicity of remainders of the Binet and Stirling formulas for the gamma function*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **111** (2017), no. 2, 425–434; available online at <https://doi.org/10.1007/s13398-016-0302-6>.
46. F. Qi, G. Rahman, S. M. Hussain, W.-S. Du, and Kottakkaran Sooppy Nisar, *Some inequalities of Čebyšev type for conformable k -fractional integral operators*, Symmetry **10** (2018), no. 11, Article 614, 8 pages; available online at <https://doi.org/10.3390/sym10110614>.
47. F. Qi, G. Rahman, and K. S. Nisar, *Convexity and inequalities related to extended beta and confluent hypergeometric functions*, AIMS Math. **4** (2019), no. 5, 1499–1507; available online at <https://doi.org/10.3934/Math.2019.5.1499>.
48. G. Rahman, K. S. Nisar, and F. Qi, *Some new inequalities of the Grüss type for conformable fractional integrals*, AIMS Math. **3** (2018), no. 4, 575–583; available online at <https://doi.org/10.3934/Math.2018.4.575>.
49. S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and*

Applications, Edited and with a foreword by S. M. Nikol'skii. Translated from the 1987 Russian original. Revised by the authors. Gordon and Breach Science Publishers, Yverdon, 1993.

50. M. Z. Sarikaya and H. Budak, *Generalized Ostrowski type inequalities for local fractional integrals*, Proc. Amer. Math. Soc. **145** (2017), no. 4, 1527–1538; available online at <https://doi.org/10.1090/proc/13488>.
51. M. Z. Sarikaya, Z. Dahmani, M. E. Kiris, and F. Ahmad, *(k, s)-Riemann-Liouville fractional integral and applications*, Hacet. J. Math. Stat. **45** (2016), no. 1, 77–89.
52. E. Set, M. A. Noor, M. U. Awan, and A. Gözpinar, *Generalized Hermite–Hadamard type inequalities involving fractional integral operators*, J. Inequal. Appl. **2017**, Paper No. 169, 10 pp. Available online at <https://doi.org/10.1186/s13660-017-1444-6>.
53. E. Set, M. Tomar, and M. Z. Sarikaya, *On generalized Grüss type inequalities for k-fractional integrals*, Appl. Math. Comput. **269** (2015), 29–34; available online at <https://doi.org/10.1016/j.amc.2015.07.026>.
54. K. Shah, H. Khalil, and R. A. Khan, *Upper and lower solutions to a coupled system of nonlinear fractional differential equations*, Progress Frac. Differ. Equ. Appl. **2** (2016), no. 1, 31–39; available online at <https://doi.org/10.18576/pfda/020104>.
55. K. Shah and R. A. Khan, *Study of solution to a toppled system of fractional differential equations with integral boundary conditions*, Int. J. Appl. Comput. Math. **3** (2017), no. 3, 2369–2388; available online at <https://doi.org/10.1007/s40819-016-0243-y>.
56. D.-P. Shi, B.-Y. Xi, and F. Qi, *Hermite–Hadamard type inequalities for (m, h_1, h_2) -convex functions via Riemann–Liouville fractional integrals*, Turk. J. Anal. Number Theory **2** (2014), no. 1, 23–28; available online at <https://doi.org/10.12691/tjant-2-1-6>.
57. D.-P. Shi, B.-Y. Xi, and F. Qi, *Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals of (α, m) -convex functions*, Fract. Diff. Calculus **4** (2014), no. 2, 31–43; available online at <https://doi.org/10.7153/fdc-04-02>.
58. H.-G. Sun, Y. Zhang, W. Chen, and D. M. Reeves, *Use of a variable-index fractional-derivative model to capture transient dispersion in heterogeneous media*, J. Contaminant Hydrology **157** (2014), 47–58; available online at <https://doi.org/10.1016/j.jconhyd.2013.11.002>.
59. Y.-Z. Tian, M. Fan, and Y.-G. Sun, *Certain nonlinear integral inequalities and their applications*, Discrete Dyn. Nat. Soc. **2017**, Article ID 8290906, 8 pp. Available online at <https://doi.org/10.1155/2017/8290906>.
60. M. Tunç *On new inequalities for h-convex functions via Riemann–Liouville fractional integration*, Filomat **27** (2013), no. 4, 559–565; available online at <https://doi.org/10.2298/FIL1304559T>.
61. S.-H. Wang and F. Qi, *Hermite–Hadamard type inequalities for s-convex functions via Riemann–Liouville fractional integrals*, J. Comput. Anal. Appl. **22** (2017), no. 6, 1124–1134.
62. X. Yang and K. Lo, *Lyapunov-type inequality for a class of even-order differential equations*, Appl. Math. Comput. **215** (2010), no. 11, 3884–3890; available online at <https://doi.org/10.1016/j.amc.2009.11.032>.

63. X.-J. Yang, J. A. T. Machado, and J. J. Nieto, *A new family of the local fractional PDEs*, *Fund. Inform.* **151** (2017), no. 1-4, 63–75; available online at <https://doi.org/10.3233/FI-2017-1479>.
64. X.-J. Yang, *Fractional derivatives of constant and variable orders applied to anomalous relaxation models in heat-transfer problems*, *Therm. Sci.* **21** (2017), no. 3, 1161–1171; available online at <https://doi.org/10.2298/TSCI161216326Y>.
65. X.-J. Yang, J. A. T. Machao, and D. Băleanu, *Anomalous diffusion models with general fractional derivatives within the kernels of the extended Mittag-Leffler type functions*, *Roman. Rep. Phys.* **69** (2017), Article no. 115, 20 pp.
66. X.-J. Yang, H. M. Srivastava, and J. A. T. Machado, *A new fractional derivative without singular kernel: Application to the modelling of the steady heat flow*, *Therm. Sci.* **20** (2016), no. 2, 753–756; available online at <https://doi.org/10.2298/TSCI151224222Y>.



AIMS Press

© 2019 F. Qi, S. Habib, S. Mubeen, M. N. Naeem, licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)