Variational properties of stationary inviscid incompressible flows with possible abrupt inhomogeneity or free surface

Jean Jacques Moreau

VARIATIONAL PROPERTIES OF STATIONARY INVISCID INCOMPRESSIBLE FLOWS WITH POSSIBLE ABRUPT INHOMOGENEITY OR FREE SURFACE

J. J. MOREAU
Département de Mathématiques, Université des Sciences et Techniques du Languedoc,
34060 Montpellier Cedex, France

Abstract—The inviscid flows, possibly rotational and nonsmooth, which satisfy the equation of stationary incompressible hydrodynamics, are characterized as giving zero variation rate to some real functional when the corresponding scalar and vector fields are transported by what is called a carrier, i.e. a mobile differential manifold. This transport does not have to preserve volumes; the Bernoulli function figures as the natural unknown scalar field rather than the pressure. Inhomogeneity may be sharp, implying in particular the presence of free surfaces. The key mathematical concept is that of a divergence-free vector measure convected by a carrier. For easier handling of this concept, some versions of the main variational statement are derived, involving vector potentials and stream functions in two or three dimensions; axially symmetric flows are also considered.

1. INTRODUCTION

THE TECHNIQUE of "horizontal variations" for characterizing the solutions to some field equations has been presented in a former paper [1], devoted to barotropic fluid flows, possibly with abrupt inhomogeneity. The considered flows were not *a priori* assumed stationary and the main result of the said paper may be seen as a transcript of Hamilton's principle into the formalism of Euler's variables. For brevity, the incompressible case was left aside (except in some introductory example we shall generalize below, Prop. 6.d); actually, in the traditional spirit of analytical dynamics, the results may readily be adapted to incompressible fluids, by retaining only volume-preserving variations.

In contrast, the present paper is entirely devoted to *stationary flows*, and thus bears no direct connection with Hamilton's principle. The fluid is assumed incompressible but the considered variations do not have to preserve volume; this should be an advantage in possible computational applications.

Recall that by horizontal variation we mean the transport of the investigated mathematical objects by an imagined continuous medium A, called a *carrier*, in motion over the concerned region of space. Such a medium, or moving differential manifold, is essentially distinct from the material fluid under study; the real variable ordering its displacements is denoted by τ, not to be mistaken with the time of dynamics. The carrier motion need not be stationary with regard to τ. Some of the calculations made in the sequel are readily equivalent to what is called *Lie derivation* in differential geometry (see e.g. [11, 12]). However we think it more effective to confine the exposition to the language of the elementary kinematics of continua. This language makes clear, in particular, that the characterized objects are the critical points of real functionals defined in some infinite-dimensional manifolds consisting of possible positions of the carrier. Hence approximation procedures may be seen as walks toward these points. This is formally similar to the equilibrium problem of an elastic medium. The corresponding numerical computation will possibly take as an unknown the deformation of some finite-element mesh. Procedures of this sort are currently used in the treatment of free-boundary or optimal-design problems [2, 3]. Codes involving such deformations have also been developed to the end of computing large deformations of materials [18, 19].

Resorting to moving-mesh procedures for the treatment of such essentially "Eulerian" problems as those of stationary hydrodynamics looks at first glance uneconomic. Actually this might prove effective when singular solutions are expected. If the type of singularity is *a priori* known (for instance discontinuity across some unknown surface) a singularity

of this type will be placed in the initial approximant; computation will have to transport this singularity to the right place, while deforming it. Of course the device should prove all the more efficient as the starting point is closer to the expected solution

About the *second variation* of the investigated functionals and how some stability requirements give to measures the preeminence over other sorts of distributions, see [13] In a different way, the use of carriers might also prove effective for the study of unstationary flows: the evolutive fields, possibly nonsmooth, will be determined as resulting from reference fields by some transport; then the unknown is the carrier motion with τ equal to the proper time of dynamics.

The present paper is entirely restricted to characterizing the solutions of field equations in some open set Ω, without attention to boundary conditions To that end the considered carrier will be supposed *compact,* in the sense that the velocity field φ defining its motion has a compact support in Ω. On the other hand, φ is assumed very smooth; this is an essential feature of the calculus of horizontal variations: the smoother is the carrier motion, the more irregular may be the investigated fields In particular, the density function ρ of the fluid may present discontinuities and possibly drop to zero beyond some unknown surface; this accounts for a *liquid flow with free surface in the presence of a mass-less atmosphere.* We thereby generalize some long-known variational characterization of irrotational flows with free boundary or slipstream surface [4–6]

For the reader's convenience, preliminary Sections 2 and 3 recall the concepts and main formulas of the horizontal variation technique, with reference to [1] for proofs

Since irregular flows are to be dealt with, hydrodynamical equations must first be given a weak form, in terms of Schwartz distributions, so as to embody in particular the jump conditions across possible discontinuity surfaces That is the object of Section 4 The Lebesgue measure in Ω is denoted by l; the density field ρ of the fluid is supposed to belong to $\mathcal{L}^\infty_{\mathrm{loc}}$ and the velocity field $\mathbf{u}$ to $\mathcal{L}^2_{\mathrm{loc}}$ Then volume and mass conservations are expressed by formulating that the *vector measures* $\mathbf{u}l$ and $\rho\mathbf{u}l$ have zero divergences in the sense of distributions The momentum equation is formulated in the same style, with a vector density of extraneous force relatively to the mass measure supposed to have the form grad U

The propositions to come will put forward, instead of the pressure field p, the "Bernoulli function":

$$b = p + \tfrac{1}{2}\rho u_i u_i - \rho U$$

Section 5 recalls the mechanical relevance of this scalar field As a consequence of the dynamical equations, the vector measure $b\mathbf{u}l$ is divergence-free; the connections of this with *energy transfers* is stressed, when the considered domain surrounds some part of space in which other mechanical processes take place

Then comes in Section 6 the main variational statement: ρ, b, $\mathbf{u}$ constitute a solution of the dynamical equations in the open set Ω if and only if the integral

$$\mathcal{B} = \int_\Omega (\tfrac{1}{2}\rho u_i u_i + \rho U + b)\, \mathrm{d}l$$

is stationary when the *scalar fields* ρ, b and the *vector measure* $\mathbf{u}l$ are convected by every smooth compact carrier If one restricts the carrier to be *isochoric,* i e , div $\varphi = 0$, one obtains a statement eliminating b, which generalizes the introductory examples of [1] and [7]

The conditions div $(\mathbf{u}l) = 0$ and div $(\rho\mathbf{u}l) = 0$ of volume and mass conservations are placed *a priori*; this does not constitute a constraint, in the customary sense of the calculus of variations, since, as recalled in Section 3, these conditions are automatically preserved under the above transport; *they do not induce any restriction on the considered carriers*

In the two-dimensional case, which is the object of Section 7, the easiest way of handling divergence-free vector measures is to have each of them derive from a *stream function* ψ. This is always possible in a simply connected domain; for general Ω, some

global necessary and sufficient conditions are indicated After this it is established that, if the scalar field ψ is convected by some carrier, the corresponding vector measure is itself convected This yields a simple version of the general variational statement

The three-dimensional case is considered similarly in Section 8; if Ω is not simple enough, some global conditions have to be satisfied, in order that the divergence-free vector measure $\mathbf{u}l$ derive from a *vector potential* $\mathbf{\Pi}$ It is established that, when the vector field $\mathbf{\Pi} \in \mathcal{L}^1_{\text{loc}}$ is *transvected* by some carrier, in the sense recalled in Section 2, the corresponding vector measure (i e the curl, in the sense of distributions, of the vector measure $\mathbf{\Pi}l$) is convected This generates variants of the statements of Section 6

A more practical way of handling divergence-free vector measures in three dimensions is developed in Section 9 It consists in assuming $\mathbf{\Pi}$ under the form ψ grad θ, where ψ and θ are scalar fields This is a vector potential of the divergence-free vector measure (grad $\psi \times$ grad $\theta)l$, where the gradients are possibly understood in some weak sense This vector measure is shown to be convected by the carrier if such are the scalar fields ψ and θ The corresponding transcript of the main variational statement proves specially useful for the treatment of *axially symmetric flows,* which are subjects of the final Section 10 Then θ is taken equal to the azimuth angle, while ψ is a function of the two variables z and r, actually Stokes's stream function, traditional in the study of such flows Investigation of the flow is then restricted to some *open* meridian half-plane

List of notations

$\mathcal{D}^k(\Omega, \mathbb{R})$ (resp $\mathcal{D}(\Omega, \mathbb{R})$)	the totality of the real functions in the open subset Ω of $\mathcal{X}$, which are continuously differentiable up to order k at least (resp indefinitely differentiable) and whose support relative to Ω is compact
$\mathcal{D}'^k(\Omega, \mathbb{R})$ (resp $\mathcal{D}'(\Omega, \mathbb{R})$)	the corresponding spaces of Schwartz's distributions, i.e the dual spaces of the above
$\mathcal{D}^k(\Omega, \mathbf{X})$, $\mathcal{D}'^k(\Omega, \mathbf{X})$ (resp $\mathcal{D}(\Omega, \mathbf{X})$, $\mathcal{D}'(\Omega, \mathbf{X})$)	similar spaces consisting of vector fields and of vector distributions
l	Lebesgue's measure in $\mathcal{X}$
$\mathcal{L}^p(\Omega, l; \mathbb{R})$ (resp $\mathcal{L}^p(\Omega, l; \mathbf{X})$) $1 \leqslant p < +\infty$,	the totality of the l-measurable real functions f (resp vector fields $\mathbf{u}$) in Ω such that $\int_\Omega \lvert f \rvert^p \, dl < +\infty$ (resp $\int_\Omega \lvert\mathbf{u}\rvert^p \, dl < +\infty$, with $\lvert\cdot\rvert$ denoting the norm in $\mathbf{X}$)
$\mathcal{L}^\infty(\Omega, l; \mathbb{R})$ (resp $\mathcal{L}^\infty(\Omega, l; \mathbf{X})$)	the totality of the l-measurable real functions f (resp vector fields $\mathbf{u}$) in Ω such that $\lvert f \rvert$ (resp $\lvert\mathbf{u}\rvert$) is bounded, with the possible exception of a l-negligible subset of Ω
$\mathcal{L}^p_{\text{loc}}(\Omega, l; \mathbb{R})$ (resp $\mathcal{L}^p_{\text{loc}}(\Omega, l; \mathbf{X})$, $1 \leqslant p \leqslant +\infty$	the totality of the real functions (resp vector fields) in Ω whose restriction to every compact subset K of Ω belongs to $\mathcal{L}^p(K, l; \mathbb{R})$ resp $\mathcal{L}^p(K, l; \mathbf{X})$
M	an open half-plane, i.e not containing its edge Z
$\mathbf{M}$	the two-dimensional linear space of the vectors of M
m	the two-dimensional Lebesgue measure in M
r	the distance to Z of the generic point of M
$\mathcal{X}$	n-dimensional Euclidean point space
$\mathbf{X}$	the Euclidean vector space associated with the above, i e the totality of the free vectors of $\mathcal{X}$
Λ	an n-dimensional differential manifold
Λ'_λ	the tangent linear space to Λ at the point λ
Λ'^*_λ	the dual of the above, i e the cotangent space to Λ at λ
π	a diffeomorphism of Λ into $\mathcal{X}$
π'_λ	the tangent mapping to π at the point λ (a linear bijection of Λ'_λ onto $\mathbf{X}$)

2 FORMULAS FOR HORIZONTAL VARIATIONS

Let Ω denote an open subset of the n-dimensional Euclidean space $\mathcal{X}$ By *horizontal variation* in Ω, we mean the transport of some investigated mathematical objects by a

continuous medium Λ, conceived in the abstract, called a *carrier*; the flow of this imagined medium is defined by giving its velocity field φ, a $\mathcal{C}^k$ vector field in Ω, $1 \leqslant k \leqslant \infty$.

In order to prevent in dynamical problems any confusion with the physical time, the variable, ranging in some real interval I, which indexes the evolution of Λ is denoted by τ instead of t. Possibly φ will depend on τ, i.e. the carrier flow may not be stationary, provided φ is supposed $\mathcal{C}^k$ jointly with regard to τ and x in $I \times \Omega$.

In the sections to come we shall only consider *compact* carriers in Ω, i.e. *the support of the vector field* $x \dashrightarrow \varphi(\tau, x)$ *is contained in a* τ*-constant compact subset of* Ω. Consequently every particle of Λ which, at some τ, happens to have a position outside of this compact set remains at rest.

From the velocity field being $\mathcal{C}^k$ jointly in τ and x, it classically follows that the transition (or "deplacement") of Λ in $\mathcal{X}$ between every two instants τ_1 and τ_2 is a $\mathcal{C}^k$ diffeomorphism. Equivalently, the medium Λ is endowed with the structure of $\mathcal{C}^k$ differential manifold, the points of which are its particles, such that for every τ the *placement mapping* π_τ: $\lambda \longrightarrow \pi(\tau, \lambda)$ from Λ into $\mathcal{X}$ is a $\mathcal{C}^k$ diffeomorphism. It is precisely a $\mathcal{C}^k$ diffeomorphism of Λ *onto* Ω if the carrier is compact in Ω. In a classical way, every object of the $\mathcal{C}^k$ differential geometry of Λ is transformed under such a (τ-dependent) diffeomorphism into a similar (τ-dependent) object of the differential geometry of Ω; in fact the latter constitutes a differential submanifold of $\mathcal{X}$.

In particular a τ-dependent *real function* y^τ: $\Omega \longrightarrow \mathbb{R}$ may equal the image under π_τ of some τ-constant real function η: $\Lambda \longrightarrow \mathbb{R}$, i.e. $\forall \tau \in I$, $\forall x \in \Omega$: $y_\tau(x) = \eta(\pi_\tau^{-1}(x))$. According to the usual vocabulary of continuum mechanics, in such a case y^τ is said *convected* by the carrier Λ. This is equivalent to the vanishing, for every $\tau \in I$, of its *drag-derivative* δ_φ along the carrier Λ. In general, for every function $(\tau, x) \longrightarrow y(\tau, x)$ from $I \times \Omega$ into $\mathbb{R}$, this drag-derivative is by definition

$$\delta_\varphi y(\tau, x) = \partial_1 \eta(\tau, \pi_\tau^{-1}(x)), \tag{2.1}$$

where η denotes the function defined in $I \times \Lambda$ by

$$\eta(\tau, \lambda) = y(\tau, \pi_\tau(\lambda)) \tag{2.2}$$

and $\partial_1 \eta$ the partial derivative of this function relatively to its first argument. Observe that the existence of the drag-derivative does not require of y to be differentiable in Ω; if the latter holds, a well-known elementary formula relates $\delta_\varphi y$ to the partial derivative $\partial y / \partial \tau$.

Similarly, a τ-dependent *vector field* in Ω, i.e. a mapping $\mathbf{u}^\tau$ of Ω into the linear space $\mathbf{X}$ associated with $\mathcal{X}$, is said *convected* by Λ if it equals, for every $\tau \in I$, the image under π_τ of some τ-constant vector field of the differential manifold Λ. If the vector $\mathbf{u}^\tau(x)$ is defined by its components $u_i(\tau, x)$ relative to some orthonormal Cartesian frame of $\mathcal{X}$, such a mode of transport is found characterized by the condition

$$\delta_\varphi u_i(\tau, x) = \varphi_{i\,j}(\tau, x) u_j(\tau, x); \tag{2.2}$$

here δ_φ denotes as above the drag-derivative along Λ of the considered real functions; by $\varphi_{i\,j}$ are denoted the partial derivatives of the components of the velocity field φ, relative to the said Cartesian frame.

Recall that a vector field of the differential manifold Λ (or cross-section of the tangent fiber bundle) is by definition an assignment associating with every $\lambda \in \Lambda$ an element of the linear space Λ'_λ, tangent to the manifold at this point.

Symmetrically, there may be considered a *covector field*, associating with every λ an element of the *cotangent space* Λ'^*_λ, the dual of Λ'_λ. Every diffeomorphism π_τ yields as image of a covector field in Λ a covector field in Ω; here one has to recall that, due to the Euclidean structure of $\mathcal{X}$, the common practice identifies the tangent and cotangent spaces at every point of $\mathcal{X}$ with the single Euclidean linear space $\mathbf{X}$. Hence, a τ-dependent vector field $(\tau, x) \longrightarrow \mathbf{v}^\tau(x)$ in Ω may happen to equal for every τ the image under π_τ of

some τ-constant *covector* field of Λ; we refer to this situation by saying that the vector field $\mathbf{v}^\tau$ is *transvected* by the carrier Λ. Equivalently, for every $\mathbf{u}^\tau$ *convected*, the Euclidean scalar product $\mathbf{u}^\tau(\pi_\tau(\lambda)) \cdot \mathbf{v}^\tau(\pi_\tau(\lambda))$, for each particle λ of Λ, does not depend on τ, i.e. the real function $\mathbf{u}^\tau \cdot \mathbf{v}^\tau$ is convected. In view of (2.2) this may be used to derive the following characterization of a transvected vector field

$$\delta_\varphi v_i(\tau, x) = -\varphi_{j,i}(\tau, x) v_j(\tau, x) \tag{2.3}$$

3. THE TRANSPORT OF SCALAR OR VECTOR MEASURES

Measures are most easily introduced into the geometry of the differential manifold Λ by the *duality* method.

From this standpoint, a *scalar measure* (nonnecessarily positive) on Λ is a real linear functional, satisfying some known continuity requirements (sketched in [1]; for detailed exposition, see e.g. [8, 9]) on the linear space $\mathcal{D}^0(\Lambda, \mathbb{R})$ of the continuous real functions with compact support in Λ. Under every diffeomorphism, such an object possesses a naturally defined image. For instance, the Lebesgue measure l is a scalar measure on the manifold constituted by the open subset Ω of $\mathcal{X}$; its image under π_τ^{-1} is a τ-dependent scalar measure on Λ, say μ^τ. The *dilatation formula* of the classical kinematics of continua amounts to the fact [10] that the mapping $\tau \to \mu^\tau$ of the interval I into the topological linear space $\mathcal{D}'^0(\Lambda, \mathbb{R})$ of the scalar measures on Λ admits as derivative a scalar measure on Λ, whose image under π^τ equals div φl. Let us develop in that line *the drag-derivation rule for integrals*, to be used in further sections of this paper.

Proposition 3.a. *Let γ denote a function from $I \times \Omega$ into $\mathbb{R}$, such that, for every τ in some neighborhood H of τ_0 in I, the Lebesgue integral*

$$f(\tau) = \int_\Omega \gamma(\tau, x)\, \mathrm{d}l(x)$$

exists. Suppose that, for $(\tau, x) \in H \times \Omega$ (with the possible exception of a Lebesgue-negligible subset), the drag-derivative (2.1) *exists and that, for every $\tau \in H$, the function $x \to |\delta_\varphi \gamma(\tau, x)|$ is dominated by some τ-constant l-integrable function in Ω. Then $\delta_\varphi \gamma(\tau_0, \cdot)$ is Lebesgue integrable in Ω and the function f possesses at $\tau = \tau_0$ a derivative equal to*

$$f'(\tau_0) = \int_\Omega (\delta_\varphi \gamma(\tau_0, x) + \gamma(\tau_0, x) \varphi_{i,i}(\tau_0, x))\, \mathrm{d}l(x). \tag{3.1}$$

The proof simply consists in interpreting π_τ as a change of variables, which transforms $f(\tau)$ into an integral over Λ. Let us denote by (ξ^i) some admissible coordinates in this manifold and by $J(\tau, \lambda)$ the Jacobian determinant of π_τ at the point $\lambda \in \Lambda$ when this mapping is expressed through (ξ^i) coordinates; then

$$f(\tau) = \int_\Lambda \gamma(\tau, \pi_\tau(\lambda)) J(\tau, \lambda)\, \mathrm{d}\xi(\lambda)$$

where ξ denotes the measure induced on Λ by the n-dimensional Lebesgue measure of the (ξ^i) variables. The neighborhood H of τ_0 may be taken as a compact subinterval of I; on the other hand, the assumptions made about the velocity field φ entail that the partial derivative $\partial J/\partial \tau$ exists and is continuous throughout $H \times \Lambda$, with compact support. This allows one to apply Lebesgue's dominated convergence theorem to the integrand $[\eta(\tau, \lambda) J(\tau, \lambda) - \eta(\tau_0, \lambda) J(\tau_0, \lambda)](\tau - \tau_0)^{-1}$, with η defined in (2.2) and τ assuming an arbitrary sequence of values converging to τ_0. Finally, use the definition (2.1) of the drag-derivative, together with the classical dilatation formula

$$\frac{1}{J} \frac{\partial J}{\partial \tau}(\tau, \lambda) = \operatorname{div} \varphi(\tau, \pi(\tau, \lambda)) \tag{3.2}$$

Let us consider similarly on the manifold Λ a *vector* (resp. *covector*) *measure*, say θ, defined as a real linear functional, satisfying certain continuity requirements (see e.g. [1]), on the linear space of the continuous *covector* (resp. *vector*) fields with compact support in Λ. By a multidimensional version of the Lebesgue-Nikodym theorem, this definition is proved equivalent to the existence of a (nonunique) nonnegative scalar measure ν together with a locally ν-integrable vector (resp. covector) field θ'_ν such that $\theta = \theta'_\nu \nu$; one calls θ'_ν the *density* of θ relative to the scalar measure ν.

Here again we are in the presence of objects whose images under every diffeomorphism are readily defined. On the other hand, in the Euclidean manifold Ω, the concepts of vector and of covector measures merge into a single one. For instance, a τ-dependent vector measure on Ω, say $\mathbf{v}^\tau$, may happen to equal the image under π_τ of some τ-constant vector (resp. covector) measure on Λ; in that case we shall say that $\mathbf{v}^\tau$ is *convected* (resp. *transvected*) by the carrier Λ. Equivalently there exists a nonnegative scalar measure n^τ on Ω, convected by Λ, and a convected (resp. transvected) vector field $\mathbf{v}'^\tau_n$ in Ω such that, for every τ, one has $\mathbf{v}^\tau = \mathbf{v}'^\tau_n n^\tau$.

In accordance with the definition of differential operators in the theory of Schwartz distributions, a vector measure $\mathbf{v}$ on Ω is said *divergence free* if

$$\int_\Omega (\operatorname{grad} a) \cdot \mathrm{d}\mathbf{v} = 0$$

holds for every $a \in \mathcal{D}(\Omega, \mathbb{R})$, or equivalently for every $a \in \mathcal{D}^1(\Omega, \mathbb{R})$. This notion may actually be developed without any Euclidean metric. In fact, in the differential manifold Λ, for every $\mathcal{C}^1$ real function α, with compact support, the gradient (or "differential") of α constitutes a $\mathcal{C}^0$ *covector* field with compact support; hence the integral of this gradient field with regard to any vector measure θ on Λ makes sense. By definition, the value of the integral is a real number essentially preserved when the considered objects are replaced by their respective images under any diffeomorphism. Therefore:

Proposition 3 b. *If a τ-dependent vector measure $\mathbf{v}^\tau$ on Ω, convected by Λ, happens to be divergence-free for some $\tau \in I$, the same holds for every τ.*

Let us end this section by considering the special case where some τ-dependent vector measure $\mathbf{v}^\tau$ in Ω possesses a density, say $\mathbf{v}'^\tau_l$, relative to the Lebesgue measure l. As before, let us denote by μ^τ the image of l under π_τ^{-1}; then the image of $\mathbf{v}^\tau$ under π_τ^{-1} equals $\theta'^\tau_\mu \mu^\tau$, where θ'^τ_μ denotes the vector field image of $\mathbf{v}'^\tau_l$ under π_τ^{-1}. Now, let us use again the τ-constant scalar measure ξ defined on Λ by means of the Lebesgue measure of some admissible coordinates (ξ^i), and the corresponding Jacobian determinant $J^\tau(\lambda) = J(\tau, \lambda)$, a continuous real function on $I \times \Lambda$. Since $\mu^\tau = J^\tau \xi$, the vector measure $\theta'^\tau_\mu \mu^\tau$ is τ-constant if and only if the vector field $J^\tau \theta'^\tau_\mu$ is τ-constant, except possibly on some ξ-negligible (equivalently μ^τ-negligible) subset of Λ. Therefore, the τ-dependent vector measure $\mathbf{v}^\tau = \mathbf{v}'^\tau_l l$ on Ω is convected by Λ if and only if the vector field defined in Ω (up to the possible exception of Lebesgue-negligible set) by

$$x \longrightarrow J^\tau(\pi_\tau^{-1}(x)) \mathbf{v}'^\tau_l$$

is convected by Λ. In view of (2.2) and (3.2) this is finally found equivalent to the assertion that the drag-derivatives of the Cartesian components v'_i of $\mathbf{v}'^\tau$, exist and satisfy

$$\delta_\varphi v'_i = \varphi_{i,j} v'_j - \varphi_{j,j} v'_i, \tag{3.3}$$

with the possible exception of a Lebesgue-negligible subset of $I \times \Omega$.

4 THE DYNAMICAL EQUATIONS

In all the sequel $\mathcal{X}$ is the Euclidean n-dimensional space defined by some inertial reference frame (practically n equals 2 or 3); as before we shall denote by $\mathbf{X}$ the associated linear space, i.e. the set of the free vectors of $\mathcal{X}$.

Let Ω be an open subset of $\mathcal{X}$, in which the *stationary flow* of some *incompressible*, possibly inhomogeneous, *inviscid fluid* is observed. The pressure p and the density ρ are real functions in Ω, as well as the components u_i, relative to some orthonormal base, of the velocity $\mathbf{u}$. The density of distributed extraneous forces (usually gravity forces) relative to the mass measure is assumed to have the form grad U, where U denotes a given real function.

In the elementary case where all these functions are $\mathcal{C}^1$, the dynamical equations of the fluid read

$$\rho u_j u_{i\,j} = -p_{,i} + \rho U_i, \tag{4.1}$$

to be joined with the kinematical conditions of volume and mass conservation

$$u_{j\,j} = 0, \tag{4.2}$$

$$(\rho u_j)_{,j} = 0 \tag{4.3}$$

As we are to deal with possibly nonsmooth flows, these three relations will first be given a form with extended meaning. Under (4.3) the left-hand side of (4.1) becomes $(\rho u_j u_i)_{,j}$, which is the i^{th} component of the divergence vector of the tensor field with components $\rho u_j u_i$. In order to translate this into the language of Schwartz distributions, let us denote as before by l the Lebesgue measure in Ω and suppose

$$\rho \in \mathcal{L}^{\infty}_{\text{loc}}(\Omega, l; \mathbb{R}), \qquad \mathbf{u} \in \mathcal{L}^{2}_{\text{loc}}(\Omega, l; \mathbf{X}), \qquad p \in \mathcal{L}^{1}_{\text{loc}}(\Omega, l; \mathbb{R}) \tag{4.4}$$

The given components U_i of the mass density of extraneous forces are usually very smooth; it suffices here to suppose them in $\mathcal{L}^{1}_{\text{loc}}(\Omega, l; \mathbb{R})$. Under assumptions (4.4), the functions $\rho u_i u_j$ and p constitute the densities, relative to l, of some measures in Ω. Then, the following relations, involving the partial derivatives of these measures, in the sense of Schwartz distributions in the open set Ω,

$$(u_j u_i l)_{,j} = -(pl)_{,i} + U_i l, \tag{4.5}$$

are meaningful and, for the $\mathcal{C}^1$ case, equivalent to (4.1). Similarly (4.2) and (4.3) become

$$(u_j l)_{,j} = 0, \tag{4.6}$$

$$(\rho u_j l)_{,j} = 0, \tag{4.7}$$

expressing that the *vector measures* $\mathbf{u}l$ (the "volume current") and $\rho\mathbf{u}l$ (the "mass current") are divergence-free in Ω.

Undoubtedly (4.5), (4.6) and (4.7) are the conditions to be satisfied by the flow whenever (4.4) holds.

For instance, let Σ denote a surface separating two open subsets Ω^+ and Ω^- of Ω and supposed to possess a continuous normal unit vector $\mathbf{n}$, directed toward Ω^+. For a real function f which is $\mathcal{C}^1$ in Ω^+ and Ω^- and admits unilateral limits f^+ and f^- at every point of Σ, the gradient of the scalar measure fl (a vector distribution whose components are the partial derivatives of fl in the sense of the distributions in Ω) classically equals the sum of the two following vector measures: the measure (grad f)l, diffused in Ω, and the measure $(f^+ - f^-)\mathbf{n}\sigma$, concentrated on Σ, with σ denoting the scalar measure "area" on this surface. Let us apply this by taking respectively as f the functions u_i, ρ, p of a flow assumed smooth in Ω^+ and Ω^-, with Σ as a locus of discontinuity. In that case, (4.5) (4.6) (4.7) turn out equivalent to

(i) (4.1), (4.2) and (4.3) holding in Ω^+ and Ω^-;

(ii) the vanishing of the corresponding measure concentrated on Σ.

Therefore (4.7) entails that $\rho^+ u_j^+ n_j^+$ and $\rho^- u_j^- n_j^-$ equal the same real function, say a, defined in Σ; in view of that, (4.5) yields

$$a(u_i^+ - u_i^-) = -(p^+ - p^-)n_i.$$

As (4.6) entails $(u_i^+ - u_i^-)n_i = 0$ one first obtains

$$p^+ - p^- = 0 \tag{4.8}$$

and finally

$$a(u_i^+ - u_i^-) = 0.$$

This means that either $\mathbf{u}^+ = \mathbf{u}^-$, i.e. the velocity has no jump on Σ, or $a = 0$, i.e. $\mathbf{u}^+$ and $\mathbf{u}^-$ are tangential to Σ.

The above applies in particular when ρ is supposed to vanish throughout Ω^+; then (4.5) yields that p equals a constant, say p_0, in this region (assumed to be connected). This accounts for *an atmosphere of negligible density, while* Σ *describes the free surface of some liquid occupying* Ω^-; on this surface (4.8) reduces to the classical condition $p = p_0$. In such situations the liquid motion is the proper object of the study, with Σ *a priori* unknown; the velocity $\mathbf{u}$ at every point of the atmosphere has only to satisfy $u_{i\,i} = 0$ for consistency and the simplest is to imagine $\mathbf{u} = 0$ all over Ω^+.

Concerning the general use of (4.5), let us observe that the pressure is a mechanically meaningful function, and not only the scalar measure pl. For instance, when investigating how the hydrodynamical efforts are distributed over some physical boundary, one has to determine p as a function on this surface and not only as an element of $\mathcal{L}^1_{\mathrm{loc}}(\Omega, l; \mathbb{R})$. Even when discussing the flow inside the open region Ω, one may have to check the feasibility of a solution by comparing p with the vaporization pressure of the liquid (see e.g. [14]). However, in many studies aimed at determining only the fluid motion, the constraint of volume conservation for every part of the material is treated as unconditional; then p may be seen as the *reaction associated with this constraint,* actually frictionless. In the spirit of traditional mechanics it is wished to eliminate such an unknown from the calculation. To that end, instead of (4.5), we shall write

$$\operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}l) - \rho(\operatorname{grad} U)l \in \operatorname{grad} \mathcal{D}'(\Omega, \mathbb{R}) \tag{4.9}$$

Here the right-hand side denotes the totality of the vector distributions in Ω which equal the gradients of elements of $\mathcal{D}'(\Omega, \mathbb{R})$. About ρ and $\mathbf{u}$, we make here the same assumptions as in (4.4). But, without further study of the left-hand side in (4.9) there is no reason to assert that the scalar distribution (defined up to the addition of a constant distribution) admitting it as gradient has the form pl, $p \in \mathcal{L}^1_{\mathrm{loc}}(\Omega, l; \mathbb{R})$. Therefore (4.9) constitutes a *weaker formulation* of hydrodynamics than (4.5).

Let us finish this section by recalling some facts to be used in the sequel about the subspace $\operatorname{grad} \mathcal{D}'(\Omega, \mathbb{R})$ of $\mathcal{D}'(\Omega, \mathbf{X})$. If $\mathbf{g} \in \mathcal{D}'(\Omega, \mathbf{X})$ is a gradient, the partial derivatives of the scalar distributions g_i which constitute its components relative to some orthonormal Cartesian frame trivially satisfy

$$g_{i\,j} - g_{j\,i} = 0, \tag{4.10}$$

i.e. $\mathbf{g}$ has zero curl in Ω. In the special case where Ω equals a product of coordinate intervals (possibly the whole of $\mathbb{X}$) condition (4.10) conversely implies $\mathbf{g} \in \operatorname{grad} \mathcal{D}'(\Omega, \mathbb{R})$ (cf. [15], Chap. 2). This more generally holds if Ω is "simply connected" but, with arbitrary Ω, (4.10) is not sufficient for $\mathbf{g}$ to be a gradient. Let us denote as before by $\langle \cdot, \cdot \rangle$ the duality bilinear form between elements of $\mathcal{D}(\Omega, \mathbb{R})$ and $\mathcal{D}'(\Omega, \mathbb{R})$

Proposition 4 a *The vector distribution* $\mathbf{g}$ *is a gradient if and only if*

$$\langle g_i, \varphi_i \rangle = 0 \tag{4.11}$$

holds for every vector field $\varphi \in \mathcal{D}(\Omega, \mathbf{X})$ *with zero divergence*

This ensues, as a very special case, from De Rham's homology theory of *currents* on differential manifolds [16]; for a more elementary proof, see [17], Annex When in particular $\mathbf{g}$ is a *vector measure,* the bracket in (4.11) may be written as an integral; this specially holds with $\mathbf{g} = \mathbf{g}'_l l$ where $\mathbf{g}'_l$ is a *vector field* belonging to $\mathcal{L}^1_{\mathrm{loc}}(\Omega, l; \mathbf{X})$. Finally in the common case where $\mathbf{g}'_l$ is a *continuous vector field,* the above condition may equivalently (through the use of mollifiers) be replaced by the vanishing of the *circulation* of this vector field along every closed curve in Ω; this is a well-known characterization of the continuous vector fields which are, in the elementary sense, the gradients of scalar fields.

5 THE BERNOULLI FUNCTION

In the elementary case where the considered functions are differentiable, eqn (4.1) is classically transformed into

$$\rho u_j (u_{i\,j} - u_{j\,i}) = -b_{,i} + (\tfrac{1}{2}u^2 - U)\rho_{,i}, \tag{5.1}$$

with u denoting the magnitude of $\mathbf{u}$ and where

$$b = p + \tfrac{1}{2}\rho u^2 - \rho U \tag{5.2}$$

is the *Bernoulli function* As (4.2) and (4.3) yield $u_i \rho_{,i} = 0$, (5.1) readily implies

$$u_i b_{,i} = 0, \tag{5.3}$$

meaning that b equals a constant along every streamline The classical Bernoulli theorem concerns a connected region where it is assumed that ρ is a constant and $u_{i\,j} - u_{j\,i} = 0$, i.e curl $\mathbf{u} = 0$; then, (5.1) shows that b equals a constant throughout this region.

The essential feature of the function b is its connection with *energy transfers* Assume the dimension equal to 3; the following embodies some commonly encountered formulas about the energy balance of hydraulic machines

First observe that, in view of $u_{i\,i} = 0$, (5.3) becomes

$$\operatorname{div} b\mathbf{u} = 0 \tag{5.4}$$

Hence the vector field $b\mathbf{u}$ has zero flux across the boundary of every bounded region where our set of equations is satisfied.

More generally, suppose these equations satisfied in an open region Ω, surrounding some compact part K of $\mathcal{X}$, with possible exchange of fluid between Ω and K Then (5.4) implies that the flux

$$\mathcal{P} = \int_\Sigma b\mathbf{u} \cdot \nu \, \mathrm{d}\sigma \tag{5.5}$$

assumes the same value for every simple closed surface Σ drawn in Ω and surrounding K, with ν as outward normal unit It is found that $\mathcal{P}$ equals *the mechanical power transferred from* K *into* Ω; for instance, if K consists of a wind energy converter, $\mathcal{P}$ expresses the negative of the power extracted from the wind.

The writing in (5.5) requires that the vector field $b\mathbf{u}$ is smooth enough for the surface integral to make sense But the following equivalent expression may be considered in

more general instances: one constructs a $\mathcal{C}^1$ real function α vanishing in some open subset of Ω surrounding K and assuming the value 1 on Σ and beyond; then

$$\mathcal{P} = \int_\Omega b\mathbf{u} \cdot \operatorname{grad} \alpha \, dl \tag{5 6}$$

This makes sense as soon as the vector field $b\mathbf{u}$ belongs to $\mathcal{L}^1_{loc}(\Omega, l; \mathbf{X})$ Any other $\mathcal{C}^1$ function α', vanishing in some open subset of Ω surrounding K and assuming the value 1 beyond such surface as Σ confers the same value to the integral in (5 6), since $\alpha - \alpha' \in \mathcal{D}^1(\Omega, \mathbb{R})$

Similar remarks apply to the divergence-free vector measures $\mathbf{u}l$, the *volume current*, and $\rho\mathbf{u}l$, the *mass current* The two integrals

$$\mathcal{V} = \int_\Omega \mathbf{u} \cdot \operatorname{grad} \alpha \, \mathbf{d}l,$$

the *volume flux* from K, and

$$\mathcal{M} = \int_\Omega \rho\mathbf{u} \cdot \operatorname{grad} \alpha \, dl,$$

the *mass flux* from K, are independent of α whenever this function meets the same requirements as above Observe however that the preceding concept of the power transferred seems mechanically unclear if the balance of fluid exchange between K and Ω is not zero; thus we shall require $\mathcal{M} = 0$ In contrast, $\mathcal{V}$ may differ from zero if, during its transit through K, the fluid undergoes some density change resulting, for instance, from temperature alteration

What precedes emphasizes the importance of b among the functions describing the flow, on an equal footing with the pressure p. As seen in Section 4, if some free surface is present, separating the proper fluid from a massless atmosphere which occupies the region Ω^+, the pressure condition on this surface is automatically involved in the dynamical equations of the whole system, when written in terms of distributions The constant pressure p_0 of such an atmosphere is usually among the data; since $\rho = 0$ throughout Ω^+, this constant equals the value in Ω^+ of b as well

We have adopted (4 5) as the general form of the dynamical equations; introducing b instead of p yields the alternative writing

$$(\rho u_j u_i l)_{,j} = -[(b - \tfrac{1}{2}\rho u^2 + \rho U)l]_{,i} + \rho U_{,i} l \tag{5 7}$$

with b assumed to belong to $\mathcal{L}^1_{loc}(\Omega, l; \mathbb{R})$

6 MAIN VARIATIONAL STATEMENT

For simplicity, it will be assumed *in all subsequent sections* that $\int_\Omega |U| \, dl < +\infty$; otherwise some covering argument should be associated with the forthcoming variational statements

In this section the dimension of $\mathcal{X}$ is arbitrary

Proposition 6.a *Let a vector field* $\mathbf{u} \in \mathcal{L}^2(\Omega, l; \mathbf{X})$, *two scalar fields* $b \in \mathcal{L}^1(\Omega, l; \mathbb{R})$, $\rho \in \mathcal{L}^\infty(\Omega, l; \mathbb{R})$ *be defined in an open subset* Ω *of* $\mathcal{X}$ *(*l *denotes as before the Lebesgue measure in* Ω*) These elements satisfy the dynamical equation* (5 7) *in* Ω *if and only if, for every compact* $\mathcal{C}^1$ *carrier in* Ω, *the* τ*-derivative of the following functional vanishes at* $\tau = 0$:

$$\mathcal{B}(\tau) = \int_\Omega (\tfrac{1}{2}\rho^\tau u_i^\tau u_i^\tau + \rho^\tau U + b^\tau) \, dl \tag{6 1}$$

where b^τ, ρ^τ, reducing to b and ρ for $\tau = 0$, are scalar fields convected by the carrier and $\mathbf{u}^\tau$, reducing to $\mathbf{u}$ for $\tau = 0$, is a vector field such that the vector measure $\mathbf{u}^\tau l$ is convected

In this statement the velocity field φ of the carrier may equivalently be assumed τ-constant and restricted to belong to $\mathcal{D}(\Omega, \mathbf{X})$

Proof One easily checks (cf [1], Prop 9 4) that the integral $\mathcal{B}(\tau)$ makes sense for every τ in some compact neighborhood H of zero in $\mathbb{R}$ In order to apply Proposition 3 a, we first have to show that the integrand in (6 1) possesses a drag-derivative δ_φ In fact b^τ and ρ^τ have zero drag-derivatives by hypothesis and

$$\delta_\varphi(\rho^\tau U) = \rho^\tau \delta_\varphi U = \rho^\tau U_i \varphi_i \tag{6 2}$$

On the other hand, by using (3 3), one obtains

$$\delta_\varphi(\tfrac{1}{2}\rho^\tau u_i^\tau u_i^\tau) = \rho^\tau u_i^\tau u_j^\tau \varphi_{i\,j} - \rho^\tau u_i^\tau u_i^\tau \varphi_{j\,j} \tag{6 3}$$

That the right-hand sides in (6 2) and (6 3) have absolute values dominated, for $\tau \in H$, by some τ-constant l-integrable functions is readily established by the same reasoning as in [1], proof of Prop 9 4 Then Proposition 3 a above yields

$$\mathcal{B}'(0) = \int_\Omega [\rho u_i u_j \varphi_{i\,j} - \rho u_i u_i \varphi_{j\,j} + \rho U_i \varphi_i + (\tfrac{1}{2}\rho u_i u_i + \rho U + b)\varphi_{j\,j}]\, \mathrm{d}l$$

$$= \int_\Omega [\rho u_i u_j \varphi_{i\,j} + (\rho U + b - \tfrac{1}{2}\rho u_j u_j)\varphi_{i\,i} + \rho U_i \varphi_i]\, \mathrm{d}l$$

If $\langle \cdot, \cdot \rangle$ represents the duality bilinear form between Schwartz distributions and elements of $\mathcal{D}(\Omega, \mathbb{R})$, with u denoting as before the magnitude of $\mathbf{u}$, this writes down equivalently as

$$\mathcal{B}'(0) = \langle \rho u_i u_j l, \varphi_{i\,j} \rangle + \langle (\rho U + b - \tfrac{1}{2}\rho u^2) l, \varphi_{i\,i} \rangle + \langle \rho U_i l, \varphi_i \rangle$$

$$= -\langle (\rho u_i u_j l)_{,j}, \varphi_i \rangle - \langle [(\rho U + b - \tfrac{1}{2}\rho u^2) l]_{,i}, \varphi_i \rangle + \langle \rho U_i l, \varphi_i \rangle$$

The vanishing of this expression for every $\varphi \in \mathcal{D}(\Omega, \mathbf{X})$ (equivalently for every $\varphi \in \mathcal{D}^1(\Omega, \mathbf{X})$, due to the special form of the considered distributions) is equation (5 7)

Remark 6 b The conditions $(u_i l)_{,i} = 0$ and $(\rho u_i l)_{,i} = 0$ of volume and mass conservations will be additionally imposed to the fields involved in this proposition This does not constitute a *constraint* in the customary sense of the calculus of variations since, as observed in Section 3, these properties are automatically satisfied by all competing elements, without restricting the variation procedure in any way

Starting with some field set $(\mathbf{u}^1, b^1, \rho^1)$ verifying these conditions, one may consider the totality of the field sets in Ω resulting from this one by the transport, in the way prescribed in the proposition, by arbitrary $\mathcal{C}^1$ (resp $\mathcal{C}^\infty$) compact carriers in Ω This constitutes, roughly speaking, an infinite-dimensional manifold and Prop 6 a characterizes the elements of this manifold satisfying (5 7) (if any) as the *critical points* of the functional (6 1) The author's prospect about the use of this fact in numerical computation has been described in the introductory section

Proving the *existence* of such critical points remains today out of sight This is very similar to the existence problem in large deformation elasticity (see e g [11]), a problem essentially unsolved to-date, though numerical methods are effective

In Section 5, the equations of hydrodynamics have been shown to imply that the vector measure $b\mathbf{u}l$ is divergence-free This property also is preserved under the considered transport since, by assumption, the scalar field b and the vector measure $\mathbf{u}l$ are convected by the carrier Hence the manifold constructed as above from $(\mathbf{u}^1, b^1, \rho^1)$ as starting element can contain critical points of $\mathcal{B}$ only if div $(b^1 \mathbf{u}^1 l) = 0$ (supposedly $b^1 \mathbf{u}^1 \in \mathcal{L}^1_{\mathrm{loc}}(\Omega, l; \mathbf{X})$)

In Section 5 there was also considered the case where Ω surrounds some compact part K of $\mathcal{X}$ in which other mechanical processes take place. This put forward the fluxes $\mathcal{P}$, $\mathcal{V}$, $\mathcal{M}$ of the respective divergence-free vector measures $b\mathbf{u}l$, $\mathbf{u}l$, $\rho\mathbf{u}l$. Here as in (5.6) one may consider for every τ

$$\mathcal{P}^\tau = \int_\Omega b^\tau \mathbf{u}^\tau \cdot \operatorname{grad} \alpha^\tau \, dl$$

and similar expressions $\mathcal{V}^\tau$ and $\mathcal{M}^\tau$, with $\alpha^\tau \in \mathcal{C}^1(\Omega, \mathbb{R})$ vanishing in some open subset of Ω surrounding K and assuming the value 1 beyond some closed surface enclosing the whole. These features of α^τ are preserved under the transport by the carrier when this real function is supposed convected; then grad α^τ is a transvected continuous vector field with compact support in Ω. *Hence $\mathcal{P}^\tau$, $\mathcal{V}^\tau$, $\mathcal{M}^\tau$ are τ-constant if the corresponding vector measures are convected by some carrier* (nonnecessarily compact in Ω).

Remark 6.c. The carrier in Prop. 6.a is not supposed to comply with the incompressibility condition imposed on another account to the material fluid. In that respect, the scalar function b in the functional $\mathcal{B}$ plays a role roughly analogous to that of a Lagrange multiplier. On the contrary, by restricting ourselves to *isochoric* carriers, i.e. div $\varphi = 0$, we shall now obtain a statement which eliminates b (equivalently the pressure), and thus characterizes the solutions $\mathbf{u}$, ρ of the dynamical condition (4.9). This generalizes a result of [7], an introductory lecture to the calculus of horizontal variations.

Proposition 6.d. *The elements $\mathbf{u} \in \mathcal{L}^2(\Omega, l; \mathbf{X})$ and $\rho \in \mathcal{L}^\infty(\Omega, l; \mathbb{R})$ satisfy the dynamical condition* (4.9) *in Ω if and only if, for every compact isochoric $\mathcal{C}^1$ (equivalently $\mathcal{C}^\infty$) carrier in Ω, the τ-derivative of the following functional vanishes at $\tau = 0$:*

$$B(\tau) = \int_\Omega (\tfrac{1}{2}\rho^\tau u_i^\tau u_i^\tau + \rho^\tau U) \, dl,$$

where the scalar field ρ^τ and the vector field $\mathbf{u}^\tau$, reducing to ρ and $\mathbf{u}$ for $\tau = 0$, are convected by the carrier.

Proof. Since the carrier is isochoric, the Lebesgue measure l is convected, hence the vector measure $\mathbf{u}^\tau l$ is convected [equivalently, the last term in (3.3) vanishes]. Then the same calculation as in the proof of Prop. 6.a yields

$$B'(0) = \langle -(\rho u_i u_j l)_{,j} - [(\rho U - \tfrac{1}{2}\rho u^2) l]_{,i} + \rho U_{,i} l, \varphi_i \rangle.$$

For every φ in $\mathcal{D}(\Omega, \mathbf{X})$ (equivalently in $\mathcal{D}^1(\Omega, \mathbf{X})$ satisfying $\varphi_{i,i} = 0$), one has

$$\langle [(\rho U - \tfrac{1}{2}\rho u^2) l]_{,i}, \varphi_i \rangle = -\langle (\rho U - \tfrac{1}{2}\rho u^2) l, \varphi_{i,i} \rangle = 0.$$

Therefore, Prop. 4.a shows that $B'(0)$ vanishes for every such φ if and only if (4.9) holds.

7. STREAM FUNCTION OF A PLANE FLOW

In this section, the Euclidean space $\mathcal{X}$ is supposed two-dimensional and, for technical simplicity, oriented. That the vector measure $\mathbf{u}l$ or, more generally, some vector distribution $\mathbf{v} \in \mathcal{D}'(\Omega, \mathbf{X})$ has zero divergence in the open subset Ω may equivalently be expressed in terms of the partial derivatives of the two scalar distributions v_1, v_2 which constitute its components relative to some orthonormal Cartesian frame, by

$$v_{1,1} - (-v_2)_{,2} = 0. \tag{7.1}$$

In view of the facts recalled at the end of Section 4, if Ω is *simply connected*, this is equivalent to the existence of a scalar distribution $\Psi \in \mathcal{D}'(\Omega, \mathbb{R})$ such that

$$v_1 = \Psi_{,2}, \qquad -v_2 = \Psi_{,1}, \tag{7.2}$$

i e by using the two-dimensional alternator symbol ϵ_{ij},

$$v_i = \epsilon_{ij}\Psi_{,j}. \tag{7.3}$$

This relation is invariant under any change of the Cartesian frame, as long as it remains orthonormal and positive

With arbitrary Ω, (7.1) is no more sufficient for the existence of Ψ related to $\mathbf{v}$ in the above way The necessary and sufficient condition provided by Prop 4 a becomes here

$$\langle -v_2, \varphi_1 \rangle + \langle v_1, \varphi_2 \rangle = 0,$$

holding for every $\boldsymbol{\varphi} \in \mathcal{D}(\Omega, \mathbf{X})$ with zero divergence Since, in this two-dimensional case, the rotation of the considered vectors through $\pi/2$ exchanges the operators "div" and "curl," this condition is equivalent to $\langle v_i, \gamma_i \rangle = 0$ holding for every $\boldsymbol{\gamma} \in \mathcal{D}(\Omega, \mathbf{X})$ with zero curl Now $\boldsymbol{\gamma}$ having a compact support in the open subset Ω of $\mathcal{X}$, its extension to the whole of $\mathcal{X}$ by zero in $\mathcal{X} \setminus \Omega$ has also a zero curl; this elementarily implies the existence of a $\mathcal{C}^\infty$ real function α such that $\boldsymbol{\gamma} = \operatorname{grad} \alpha$; observe that α has not necessarily a compact support relative to Ω: this function is only a constant in every connected component $\Omega \setminus \operatorname{supp} \boldsymbol{\gamma}$.

If it is additionally supposed, as in the previous sections, that $\mathbf{v} = \mathbf{u}l$, with $\mathbf{u} \in \mathcal{L}^2_{\mathrm{loc}}(\Omega, l; \mathbf{X})$, one finds $\Psi = \psi l$, where ψ is a function belonging to $\mathcal{L}^p_{\mathrm{loc}}(\Omega, l; \mathbb{R})$ for every $p \in [1, \infty]$ (see e g [15], Chap 6). In the traditional case where $\mathbf{u}$ is more specially a continuous vector field, ψ becomes a $\mathcal{C}^1$ real function in Ω defined up to the addition of an arbitrary constant (we naturally suppose Ω connected) classically called the *stream function* of $\mathbf{u}$ and $u_1 = \psi_{,2}$, $u_2 = -\psi_{,1}$ hold in the sense of elementary partial derivation. The above conditions concerning $\mathbf{v}$ are equivalent in that case to $\mathbf{u}$ having a zero flux across every closed curve drawn in Ω

Proposition 7 a. Suppose that Ω is an arbitrary open subset of the two-dimensional Euclidean space $\mathcal{X}$ Let a vector measure $\mathbf{v}$ in Ω derive from a stream function $\psi \in \mathcal{L}^1_{\mathrm{loc}}(\Omega, l; \mathbb{R})$ in the sense that (7.3) holds in Ω, with $\Psi = \psi l$ Let a τ-dependent vector measure $\mathbf{v}^\tau$ and a τ-dependent scalar function ψ^τ be convected by a $\mathcal{C}^1$ carrier in Ω and respectively reduce to $\mathbf{v}$ and ψ for $\tau = \tau_0$. Then, ψ^τ is, for every τ, the stream function of the vector measure $\mathbf{v}^\tau$

We shall base the proof on the following:

Lemma 7 b. Let $\mathbf{w}^\tau$ be a τ-dependent $\mathcal{C}^1$ vector field in Ω, transvected by a $\mathcal{C}^2$ carrier Λ. Then the scalar measure

$$\zeta^\tau = \epsilon_{ij} w^\tau_{j\,i} l \tag{7.4}$$

is convected by the carrier

Establishing this amounts to check that the measure ζ^τ possesses a τ-constant integral over every moving compact subset of Ω which is convected, i.e. which equals the image under $\pi(\tau, \cdot)$ of some τ-constant subset Δ of the carrier Λ. One may equivalently restrict oneself to the case of a convected subset D^τ of Ω whose boundary ∂D^τ is a piecewise-$\mathcal{C}^1$ curve; by choosing on this curve the adequate orientation, the Stokes-Riemann integral formula yields

$$\int_{D^\tau} \epsilon_{ij} w^\tau_{j\,i}\, \mathrm{d}l = \int_{\partial D^\tau} w^\tau_i\, \mathrm{d}x_i. \tag{7.5}$$

Now, for every τ, the curve ∂D^τ equals the image under π_τ of the τ-constant curve $\partial\Delta$ of the manifold Λ, admitting a piecewise-$\mathcal{C}^1$ parametrization, say $r \to \lambda(r)$, $r \in [r_1, r_2]$

Putting $x^\tau(r) = \pi_\tau(\lambda(r))$, one obtains a piecewise-$\mathscr{C}^1$ parametrization of ∂D^τ, and thus expresses the right-hand member of (7.5) under the form

$$\int_{r_1}^{r_2} \mathbf{w}^\tau(x^\tau(r)) \cdot \frac{\mathrm{d}x^\tau}{\mathrm{d}r}\,\mathrm{d}r$$

Now $\mathrm{d}x^\tau/\mathrm{d}r$ constitutes, for every fixed r, a moving vector convected by Λ; since $\mathbf{w}$ has been supposed transvected the above expression is τ-constant, q.e.d.

Proof of Proposition 7.a. That ψ^τ constitutes the stream function of the vector measure $\mathbf{v}^\tau$, i.e. that (7.3) is satisfied with $\Psi^\tau = \psi^\tau l$, means, according to the definition of derivatives in the theory of distributions, that for every vector field $\mathbf{w} \in \mathscr{D}(\Omega, \mathbf{X})$ one has

$$\langle v_i^\tau, w_i \rangle = \langle \epsilon_{ij}\Psi^\tau{}_{,j}, w_i \rangle = -\langle \Psi^\tau, \epsilon_{ij} w_{i\,j} \rangle \tag{7.6}$$

Since, in the present case, the considered distributions are measures the same equivalently holds for every $\mathbf{w}^\tau$ in the space $\mathscr{D}^1(\Omega, \mathbf{X})$ of the $\mathscr{C}^1$ vector fields with compact support in Ω and the integral notation may be used, under which (7.6) becomes

$$\int w_i^\tau \,\mathrm{d}v_i^\tau = -\int \epsilon_{ij} w_{i\,j}^\tau \psi^\tau \,\mathrm{d}l \tag{7.7}$$

Observe that the property $\psi^\tau \in \mathscr{L}^1_{\mathrm{loc}}(\Omega, l; \mathbb{R})$ is preserved under the convection of ψ^τ by the $\mathscr{C}^2$ carrier Λ. As for the law of dependence of $\mathbf{w}^\tau$ on τ, let us suppose this vector field *transvected* by Λ. Clearly if, for $\tau = \tau_\bullet$, the element $\mathbf{w}^\tau$ runs through the whole of $\mathscr{D}^1(\Omega, \mathbf{X})$, so it does also for every τ. This proves Prop. 7.a since, in view of Lemma 7.b, both members of (7.7) are τ-constant.

This allows us to transcript Prop. 6.a as follows.

Proposition 7.c. *Let a vector field* $\mathbf{u} \in \mathscr{L}^2(\Omega, l; \mathbf{X})$ *derive from a stream function* ψ, *let* $b \in \mathscr{L}^1(\Omega, l; \mathbb{R})$ *and* $\rho \in \mathscr{L}^\infty(\Omega, l; \mathbb{R})$. *These elements constitute a solution of the dynamical equation* (5.7) *in* Ω *if and only if, for every compact* $\mathscr{C}^1$ *carrier in* Ω, *the* τ*-derivative of the following functional vanishes at* $\tau = 0$:

$$\mathscr{B}(\tau) = \int_\Omega (\tfrac{1}{2}\rho^\tau (\mathrm{grad}\ \psi^\tau)^2 + \rho^\tau U + b^\tau)\,\mathrm{d}l$$

where ψ^τ, ρ^τ, b^τ, *reducing to* ψ, ρ, b *for* $\tau = 0$, *are scalar fields convected by the carrier.*

Equivalently, the velocity field of the carrier may be assumed τ*-constant and restricted to belong to* $\mathscr{D}(\Omega, \mathbf{X})$.

The proof simply consists in observing that in view of (7.3), the vector measure $\mathbf{v}^\tau = \mathbf{u}^\tau l$ admits ψ^τ as stream function if and only if the vector measure with components $\epsilon_{ik} v_i^\tau$ equals the gradient of $\Psi^\tau = \psi^\tau l$. Equivalently, $\epsilon_{ik} u_i^\tau$ are the components of grad ψ^τ, an element of $\mathscr{L}^2(\Omega, l; \mathbf{X})$ whose value at almost every point of Ω has the same Euclidean norm as $\mathbf{u}^\tau$.

Similarly, Prop. 6.d is transcripted into the following:

Proposition 7.d. *Let a vector field* $\mathbf{u} \in \mathscr{L}^2(\Omega, l; \mathbf{X})$ *derive from a stream function* ψ, *let* $\rho \in \mathscr{L}^\infty(\Omega, l; \mathbb{R})$. *These elements constitute a solution of the dynamical conditions* (4.9) *if and only if, for every compact* $\mathscr{C}^1$ *(equivalently* $\mathscr{C}^\infty$ *and/or* τ*-constant) isochoric carrier in* Ω, *the* τ*-derivative of the following functional vanishes at* $\tau = 0$:

$$B(\tau) = \int_\Omega (\tfrac{1}{2}\rho^\tau (\mathrm{grad}\ \psi^\tau)^2 + \rho^\tau U)\,\mathrm{d}l,$$

where, ψ^τ *and* ρ^τ, *reducing to* ψ *and* ρ *for* $\tau = 0$, *are scalar fields convected by the carrier.*

The special case of a smooth flow of incompressible homogeneous fluid served as the introductory example of horizontal variation in [1, 7].

8 VECTOR POTENTIAL

In this section, the Euclidean space $\mathcal{X}$ is supposed three-dimensional and, for technical simplicity, oriented. For every smooth vector field $\mathbf{\Pi}$ in the open subset Ω of $\mathcal{X}$, the vector field $\mathbf{u} = \operatorname{curl} \mathbf{\Pi}$ has zero divergence; $\mathbf{\Pi}$ is classically called a *vector potential* of $\mathbf{u}$. Equivalently, the vector measure $\mathbf{v} = \mathbf{u}l$ equals the curl, in the sense of distributions, of the vector measure $\Psi = \mathbf{\Pi}l$. In terms of the components of the considered vector distributions, relative to some *positive* orthonormal frame, this is expressed by

$$v_i = \epsilon_{ijk}\Psi_{kj}, \tag{8.1}$$

where ϵ_{ijk} denotes the three-dimensional alternator.

For arbitrary distributions in Ω, (8.1) is immediately found to imply $v_{it} = 0$. In the simple case where Ω is delimited by coordinate planes it is easy to establish that, conversely, this condition secures the existence of a (nonunique) vector distribution Ψ satisfying (8.1). But in general, the vanishing of v_{it} in Ω is not sufficient. The following is a special case of De Rham's homology theory of currents [16].

There exists a vector distribution Ψ satisfying (8.1) *in Ω if and only if $\langle v_i, \gamma_i \rangle$ vanishes for every vector field $\gamma \in \mathcal{D}(\Omega, \mathbf{X})$ with zero curl.*

As in Section 7 one sees that such γ are exactly the elements of $\mathcal{D}(\Omega, \mathbf{X})$ of the form grad α (then the real function α equals a constant in every connected component of $\Omega \setminus \operatorname{supp} \gamma$).

In the simple case where $\mathbf{v} = \mathbf{u}l$, with $\mathbf{u}$ a *continuous vector field*, the above condition is found equivalent to the vanishing of the flux of $\mathbf{u}$ across every smooth surface which equals a connected component of the boundary of a compact subset of Ω. This in turn is equivalent to $\operatorname{div} \mathbf{u} = 0$ holding in Ω (at least in the sense of distributions, if $\mathbf{u}$ is not $\mathcal{C}^1$) together with the vanishing of the said flux for a certain base set of smooth closed surfaces in Ω (a finite set in usual instances).

Similarly to Prop. 7 a, one has the following.

Proposition 8 a. *Suppose that Ω is an arbitrary open subset of the three-dimensional oriented Euclidean space $\mathcal{X}$. Let a vector measure $\mathbf{v}$ in Ω derive from a vector potential $\mathbf{\Pi} \in \mathcal{L}^1_{\mathrm{loc}}(\Omega, l; \mathbf{X})$, in the sense that* (8.1) *holds with $\Psi = \mathbf{\Pi}l$. Let a τ-dependent vector measure $\mathbf{v}^\tau$ be convected by some $\mathcal{C}^2$ carrier in Ω and reduce to $\mathbf{v}$ for $\tau = \tau_0$, let a τ-dependent vector field $\mathbf{\Pi}^\tau$ be transvected by the same carrier and reduce to $\mathbf{\Pi}$ for $\tau = \tau_0$. Then $\mathbf{\Pi}^\tau$ is for every τ a vector potential of $\mathbf{v}^\tau$.*

Proof. Let us first establish the statement in the special case $\mathbf{\Pi} \in \mathcal{C}^1(\Omega, \mathbf{X})$, so that $\mathbf{v} = \mathbf{u}l$, where $\mathbf{u} = \operatorname{curl} \mathbf{\Pi}$ is a continuous vector field. For every τ the transvected vector field $\mathbf{\Pi}^\tau$ is $\mathcal{C}^1$ and the convected vector measure $\mathbf{v}^\tau$ has the form $\mathbf{u}^\tau l$, with $\mathbf{u}^\tau$ a continuous vector field. We have to show that $\mathbf{u}^\tau = \operatorname{curl} \Psi^\tau$ or, equivalently, that the Stokes formula

$$\int_{C^\tau} \mathbf{\Pi}^\tau \cdot \mathrm{d}\mathbf{x} = \int_{S^\tau} \mathbf{u}^\tau \cdot \mathbf{n}\, \mathrm{d}\sigma \tag{8.2}$$

holds for every compact orientable portion S^τ of $\mathcal{C}^1$ surface, whose boundary C^τ is a piecewise-$\mathcal{C}^1$ curve, with orientation connected in the customary way with the direction of the normal unit vector $\mathbf{n}$ to S^τ. This surface may be supposed convected by the carrier, i.e. it equals for every τ the image under π_τ of some fixed $\mathcal{C}^1$ surface portion Σ in the manifold A. For the task of calculating the right-hand side of (8.2) one would choose a $\mathcal{C}^1$ parametrization, say $(r, s) \to \lambda(r, s)$ of Σ, (r, s) ranging through a compact subset Δ of $\mathbb{R}^2$. Putting $x^\tau(r, s) = \pi_\tau(\lambda(r, s))$ one obtains a $\mathcal{C}^1$ parametrization of S^τ, hence

$$\int_{S^\tau} \mathbf{u}^\tau \cdot \mathbf{n}\, \mathrm{d}\sigma = \int_\Delta \mathbf{u}^\tau(x(r, s)) \cdot \left(\frac{\partial x^\tau}{\partial r} \times \frac{\partial x^\tau}{\partial s} \right) \mathrm{d}r\, \mathrm{d}s. \tag{8.3}$$

Now the partial derivatives $\partial x^\tau/\partial r$ and $\partial x^\tau/\partial s$ constitute, for every fixed $(r, s) \in \Delta$, a pair of moving vectors convected by the carrier A. That the vector measure $\mathbf{v}^\tau = \mathbf{u}^\tau l$ is

convected, has been characterized in Section 3 by introducing some coordinates (ξ^i) in the manifold Λ and the Jacobian determinant J^τ of the corresponding expression of π^τ; this results in

$$\mathbf{u}^\tau(x(t, s)) = \frac{k^\tau}{J^\tau(\lambda(t, s))},$$

where k^τ denotes some convected vector, i e the image under $\pi^\tau_t(\lambda(t, s))$ of some τ-constant element κ of the tangent space $\Lambda'_{\lambda(t,s)}$. Consequently, the integrand on the right in (8.3) equals the determinant of the components of the three elements κ, $\partial\lambda/\partial t$, $\partial\lambda/\partial s$ of this tangent space relative to the base induced by (ξ^i) coordinates; for every (t, s) in Δ, this is τ-constant, so the right-hand side in (8.2) is τ-constant. The left-hand side is also τ-constant, by the same argument as in the proof of Lemma 7 b.

This special case being established, it will play now a role similar to Lemma 7 b in the proof of Prop. 7 a. That v^τ and Ψ^τ satisfy (8.1) means that for every $\mathbf{w}^\tau \in \mathcal{D}(\Omega, \mathbf{X})$ one has

$$\langle v_i^\tau, w_i^\tau \rangle = \langle \epsilon_{ijk}\Psi^\tau_{k\,j}, w_i^\tau \rangle = -\langle \Psi_k^\tau, \epsilon_{ijk}w^\tau_{i\,j} \rangle$$

Here $\Psi^\tau = \mathbf{\Pi}^\tau l$, with $\mathbf{\Pi}^\tau \in \mathcal{L}^1_{\mathrm{loc}}(\Omega, l; \mathbf{X})$; hence the integral notation may be used, yielding

$$\int w_i^\tau \, \mathrm{d}v_i^\tau = -\int \epsilon_{ijk}w^\tau_{i\,j}\Pi^\tau_k \, \mathrm{d}l \tag{8.4}$$

to hold equivalently for every $\mathbf{w}^\tau \in \mathcal{D}^1(\Omega, \mathbf{X})$. The latter τ-dependent vector field may be chosen *transvected* so that, in view of our first step, the vector measure with components $\epsilon_{ijk}w^\tau_{i,j}l$, i e $-(\mathrm{curl}\ \mathbf{w}^\tau)l$, is *convected*. Since, by assumption, the vector field $\mathbf{\Pi}^\tau$ is transvected and the vector measure $\mathbf{v}^\tau$ convected, both members of (8.4) are τ-constant; Prop. 8 a is thus proved.

Using the above propositions to formulate a transcript of Prop. 6 a is left to the reader; the sections to come are devoted to some special cases.

9. PAIRS OF STREAM FUNCTIONS IN THREE DIMENSIONS

The three-dimensional Euclidean space $\mathcal{X}$ is supposed oriented. As soon as two scalar functions ψ and θ are smooth enough for elementary calculation to apply, one finds

$$\mathrm{grad}\ \psi \times \mathrm{grad}\ \theta = \mathrm{curl}\ (\psi\ \mathrm{grad}\ \theta) = -\mathrm{curl}\ (\theta\ \mathrm{grad}\ \psi)$$

Hence the vector field $\mathrm{grad}\ \psi \times \mathrm{grad}\ \theta$ derives from a vector potential, and thus is divergence-free. If ψ and θ are convected by a carrier, the vector fields $\psi\ \mathrm{grad}\ \theta$ and $\theta\ \mathrm{grad}\ \psi$ are transvected, permitting the use of Prop. 8 a. The two following propositions place this observation in a more general setting.

Proposition 9 a. *Let the vector fields* $\mathbf{g}$ *and* $\mathbf{h}$, *elements of* $\mathcal{L}^2_{\mathrm{loc}}(\Omega, l; \mathbf{X})$, *be curl-free in the weak sense (i e the vector measures* $\mathbf{g}l$ *and* $\mathbf{h}l$ *are curl-free in the sense of distributions in* Ω*). Then the vector measure* $\mathbf{g} \times \mathbf{h}l$ *is divergence-free in* Ω.

Proof One has to show that, for every $\alpha \in \mathcal{D}(\Omega, \mathbb{R})$,

$$\int_\Omega \alpha_{\,i}\epsilon_{ijk}g_jh_k \, \mathrm{d}l = 0 \tag{9.1}$$

Through the use of mollifiers, the vector fields $\mathbf{g}$ and $\mathbf{h}$ may arbitrarily be approximated, in the L^2 norm of some compact subset of Ω containing supp α in its interior, by curl-free vector fields $\tilde{\mathbf{g}}$ and $\tilde{\mathbf{h}}$ smooth enough for the following calculation to be valid:

$$(\epsilon_{ijk}\tilde{g}_j\tilde{h}_k)_{,i} = \epsilon_{ijk}\tilde{g}_{j,i}\tilde{h}_k + \epsilon_{ijk}\tilde{g}_j\tilde{h}_{k,i} = 0$$

This establishes (9.1), after integration by parts.

As for the transport by some $\mathscr{C}^1$ carrier, one has the following

Proposition 9.b. *If the vector fields* $\mathbf{g}^\tau$ *and* $\mathbf{h}^\tau$, *elements of* $\mathcal{L}^2_{\text{loc}}(\Omega, l; \mathbf{X})$, *are transvected, the vector measure* $\mathbf{g}^\tau \times \mathbf{h}^\tau l$ *is convected*

In fact, let us check that for every transvected vector field $\mathbf{w}^\tau \in \mathscr{D}^0(\Omega, \mathbf{X})$ the integral $\int \mathbf{w}^\tau \cdot (\mathbf{g}^\tau \times \mathbf{h}^\tau)\, dl$ is τ-constant. By using coordinates (ξ^i) in the carrier manifold A, this integral may be expressed as

$$\int (\mathbf{w}^\tau(\pi_\tau), \mathbf{g}^\tau(\pi_\tau), \mathbf{h}^\tau(\pi_\tau))\left(\frac{\partial \pi_\tau}{\partial \xi^1}, \frac{\partial \pi_\tau}{\partial \xi^2}, \frac{\partial \pi_\tau}{\partial \xi^3}\right) d\xi^1\, d\xi^2\, d\xi^3,$$

where each of the two expressions in parentheses represents the scalar triple product of three elements of $\mathbf{X}$. Using a positive orthonormal base in $\mathbf{X}$ one expresses these triple products as the determinants of the corresponding components. Then the multiplication rule of determinants transforms the above integrand into a determinant whose elements are nine scalar products such as $\mathbf{w}^\tau(\pi_\tau) \cdot (\partial \pi_\tau / \partial \xi^1)$. Since by assumption $\mathbf{w}^\tau(\pi_\tau)$, $\mathbf{g}^\tau(\pi_\tau)$, $\mathbf{h}^\tau(\pi_\tau)$ are transvected, while the three vectors $\partial \pi_\tau / \partial \xi^i$ are visibly convected, these scalar products are τ-constant. This establishes the proposition.

Observe on the other hand, as a consequence of Prop. 8.a, that *if the transvected vector fields* $\mathbf{g}^\tau$ *and* $\mathbf{h}^\tau$ *are curl-free for* $\tau = \tau_0$, *so they are for every* τ.

The simplest way of constructing such transvected curl-free vector fields is to take them equal, at least locally, to the respective gradients of convected scalar functions. These gradients may be understood in the weak sense, i.e. concerning for instance $\mathbf{g}^\tau$, one supposes the existence of a convected scalar function $\psi^\tau \in \mathcal{L}^1_{\text{loc}}(\Omega, l; \mathbb{R})$ such that for every τ and every $\alpha \in \mathscr{D}^1(\Omega, \mathbb{R})$ one has

$$\int_\Omega \alpha \mathbf{g}^\tau\, dl = -\int_\Omega \psi^\tau \operatorname{grad} \alpha\, dl.$$

In other words, the vector measure $\mathbf{g}^\tau l$ equals the gradient, in the sense of distributions, of the scalar measure $\psi^\tau l$. Concerning the regularity of ψ^τ in such a situation, see e.g. [15], Chap. 6.

The above generates various transcriptions of Prop. 6.a. For instance, the vector field $\mathbf{u}^\tau \in \mathcal{L}^2(\Omega, l; \mathbf{X})$ involved in this proposition may be taken under the form

$$\mathbf{u}^\tau = (\operatorname{grad} \psi^\tau) \times (\operatorname{grad} \theta^\tau),$$

where the scalar functions ψ^τ and θ^τ are convected by the carrier. In view of the foregoing this secures that the vector measure $\mathbf{u}^\tau l$ is divergence-free and convected; therefore the following proposition

Proposition 9.c. *Let a vector field* $\mathbf{u} \in \mathcal{L}^2(\Omega, l; \mathbf{X})$ *equal* grad $\psi \times$ grad θ *in* Ω. *Let* $b \in \mathcal{L}^1(\Omega, l; \mathbb{R})$ *and* $\rho \in \mathcal{L}^\infty(\Omega, l; \mathbb{R})$. *These elements constitute a solution of the dynamical equation* (5.7) *in* Ω *if and only if for every compact* $\mathscr{C}^1$ *(equivalently* $\mathscr{C}^\infty$*) carrier in* Ω *the* τ*-derivative of the following functional vanishes at* $\tau = 0$:

$$\mathcal{B}(\tau) = \int_\Omega [\tfrac{1}{2}\rho^\tau (\operatorname{grad} \psi^\tau \times \operatorname{grad} \theta^\tau)^2 + \rho^\tau U + b^\tau]\, dl$$

where ψ^τ, θ^τ, ρ^τ, b^τ, *reducing to* ψ, θ, ρ, b *for* $\tau = 0$, *are scalar fields convected by the carrier*

Remark 9 d In the case where ψ, θ, ρ are $\mathcal{C}^1$, the identity

$$\operatorname{div}(\rho \operatorname{grad} \psi \times \operatorname{grad} \theta) = (\operatorname{grad} \rho, \operatorname{grad} \psi, \operatorname{grad} \theta)$$

makes the condition $(\rho u_i l)_i = 0$ of mass conservation equivalent to the vanishing of the Jacobian determinant of ρ, ψ, θ; this is known to imply that the three functions ρ, ψ, θ are *dependent*, i.e. the triplet $(\rho(x), \psi(x), \theta(x))$, for x running through Ω, ranges in a strict submanifold of $\mathbb{R}^3$ evidently independent of τ when the three functions are convected by some carrier. The same remark applies to b, since the dynamical equation has been shown in Section 5 to imply $\operatorname{div} b\mathbf{u} = 0$: the three functions b, ψ, θ are dependent.

10 AXIALLY SYMMETRIC FLOWS

In this section it is supposed that the open subset Ω of the three-dimensional space $\mathcal{X}$ presents the *rotational symmetry* about some axis denoted by Z. The vector field $\mathbf{u}$ and the scalar fields p, ρ, b defining the investigated flow in Ω are a priori assumed to have the same symmetry (up to a Lebesgue-negligible subset of Ω), as well as the given scalar field U. For simplicity, we assume in addition the *mirror symmetry* of the flow, relative to any meridian plane, which amounts to say that, for (almost) every x in Ω, the vector $\mathbf{u}(x)$ is parallel to the meridian plane through x. This is commonly referred to as the *axially symmetric* case.

The dynamical equation (4.5), formulates the vanishing of some vector distribution $\mathbf{I} \in \mathcal{D}'(\Omega, \mathbf{X})$, i.e. for every test vector field $\boldsymbol{\varphi} \in \mathcal{D}(\Omega, \mathbf{X})$ the vanishing of the real number $\langle T_i, \varphi_i \rangle$, denoted in the sequel by $\langle I; \boldsymbol{\varphi} \rangle$. Now the above assumptions readily imply that $\mathbf{I}$ is axially symmetric in the sense that $\langle \mathbf{I}; \boldsymbol{\varphi} \rangle$ remains unchanged when $\boldsymbol{\varphi}$ is replaced by $\boldsymbol{\varphi}_\alpha$, the vector field resulting from $\boldsymbol{\varphi}$ by the rotation through an arbitrary angle α about Z and also when $\boldsymbol{\varphi}$ is replaced by its mirror image $\boldsymbol{\varphi}\star$ relative to a meridian plane.

With every $\boldsymbol{\varphi} \in \mathcal{D}(\Omega, \mathbf{X})$ is associated its *axially symmetric average* $\bar{\boldsymbol{\varphi}} = (\tilde{\boldsymbol{\varphi}} + \tilde{\boldsymbol{\varphi}}\star)/2$, where $\tilde{\boldsymbol{\varphi}}$ denotes the *rotational average* of $\boldsymbol{\varphi}$, i.e. the vector field defined for every $x \in \Omega$ by

$$\tilde{\boldsymbol{\varphi}}(x) = \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{\varphi}_\alpha(x)\, d\alpha$$

Visibly $\bar{\boldsymbol{\varphi}}$ is an axially symmetric element of $\mathcal{D}(\Omega, \mathbf{X})$ and, for $\mathbf{I}$ as above, $\langle \mathbf{I}; \boldsymbol{\varphi} \rangle = \langle \mathbf{I}; \bar{\boldsymbol{\varphi}} \rangle$.

Therefore, since the axial symmetry of the flow is a priori assumed, *all variational statements of Section* 6, 8, 9 *may equivalently be reformulated by restricting the considered carriers to have axially symmetric velocity fields*.

The economic way of describing the axially symmetric vector field $\boldsymbol{\varphi} \in \mathcal{D}(\Omega, \mathbf{X})$ (resp. $\boldsymbol{\varphi} \in \mathcal{D}^k(\Omega, \mathbf{X})$ with k a positive integer) consists in giving its restriction $\boldsymbol{\varphi}_M$ to some meridian half-plane M, *open*, i.e. not including its edge Z. Let us denote by ω the open subset $\Omega \cap M$ of M and by $\mathbf{M}$ the two-dimensional Euclidean linear space associated with M. The restrictions to M of all axially symmetric elements of $\mathcal{D}(\Omega, \mathbf{X})$ (resp. $\mathcal{D}^k$) constitute a linear subspace of $\mathcal{C}^\infty(\omega, \mathbf{M})$ (resp. $\mathcal{C}^k$) that we shall denote by $\mathcal{D}_a(\omega, \mathbf{M})$ (resp. $\mathcal{D}_a^k$). Clearly if $\Omega \cap Z = \emptyset$ one has $\mathcal{D}_a(\omega, \mathbf{M}) = \mathcal{D}(\omega, \mathbf{M})$ (and similarly for $\mathcal{D}_a^k$) but, when $\Omega \cap Z \neq \emptyset$, some conditions in the vicinity of Z, not to be developed here, have to be satisfied by an element of $\mathcal{C}^\infty(\omega, \mathbf{M})$ in order that it belongs to $\mathcal{D}_a(\omega, \mathbf{M})$ (and similarly for $\mathcal{D}_a^k$). The same observations and notations apply to a rotationally symmetric scalar field $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$ (resp. $\mathcal{D}^k$) and to its restriction $\varphi_M \in \mathcal{D}_a(\omega, \mathbb{R})$ (resp. $\mathcal{D}_a^k$).

As for the locally integrable fields $\mathbf{u}$, ρ, b, p in Ω corresponding as above to an axially symmetric flow, it should be kept in mind that the meaningful objects are actually the measures $\mathbf{u}l$, ρl, bl, pl. Let us denote by m the two-dimensional Lebesgue measure in ω and by r the real function $x \to \operatorname{dist}(x, Z)$. By *the restriction* $\mathbf{u}_M$ *of* $\mathbf{u}$ *to* M, we shall mean an element of $\mathcal{L}^1_{\mathrm{loc}}(\omega, m; \mathbf{M})$ such that, for every $\boldsymbol{\varphi}_M \in \mathcal{D}(\omega, \mathbf{M})$ (equivalently $\boldsymbol{\varphi}_M \in \mathcal{D}^k(\omega, \mathbf{M})$) one has

$$2\pi \int_{\omega} r\mathbf{u}_M \cdot \boldsymbol{\varphi}_M \, dm = \int_{\Omega} \mathbf{u} \cdot \boldsymbol{\varphi} \, dl, \tag{10 1}$$

where $\boldsymbol{\varphi}$ denotes the axially symmetric vector field generated by $\boldsymbol{\varphi}_M$ This reflects the classical technique of reducing the triple integral of a rotationally symmetric scalar field to a double integral in some meridian half-plane Observe that, when $\Omega \cap Z \neq \emptyset$, some condition have to be satisfied by an element of $\mathcal{L}^1_{\text{loc}}(\omega, m; \mathbf{M})$ in order that it corresponds in this way to some axially symmetric element of $\mathcal{L}^1_{\text{loc}}(\Omega, l; \mathbf{X})$; this condition is automatically satisfied by every element of $\mathcal{L}^1(\omega, rm; \mathbf{M})$

Similar definitions and observations apply to the restrictions ρ_M, b_M, p_M of the locally integrable axially symmetric scalar fields, ρ, b, p

Now, let us take the axially symmetric vector field $\boldsymbol{\varphi} \in \mathcal{D}(\Omega, \mathbf{X})$ (resp $\mathcal{D}^k$) as the velocity field of a carrier; clearly the motion of this carrier leaves invariant every meridian plane and, in particular, its action in ω may be depicted as a two-dimensional carrier with $\boldsymbol{\varphi}_M$ as velocity field The convection in Ω of axially symmetric scalar fields such as p, ρ, b readily passes on to the convection of their respective restrictions to ω; the same is true for the convection (resp the transvection) of an axially symmetric vector field such as $\mathbf{u}$ But some care must be taken when expressing the convection of the three-dimensional axially symmetric *vector measure* $\mathbf{u}l$ in Ω; similarly to (10 1) it comes out that this vector measure is convected by the three-dimensional axially symmetric carrier with velocity field $\boldsymbol{\varphi}$ if and only if the two-dimensional vector measure $r\mathbf{u}_M m$ is convected by the two-dimensional carrier with velocity field $\boldsymbol{\varphi}_M$ The following makes this operation easier

Let us use in M Cartesian orthonormal coordinates, respectively z, along Z as an axis, and $r > 0$ as before Let us denote by u_z and u_r the components of $\mathbf{u}_M$ If the axially symmetric vector measure $\mathbf{u}l$ is divergence-free in Ω, the two-dimensional vector measure $r\mathbf{u}_M m$ is divergence-free in ω. When ω is simply connected, this implies in the same way as in Sect 7 the existence of $\psi_M \in \mathcal{L}^1_{\text{loc}}(\omega, m; \mathbb{R})$ such that, in the sense of distributions in ω, one has

$$u_z = \frac{1}{r} \frac{\partial \psi_M}{\partial r}, \qquad u_r = -\frac{1}{r} \frac{\partial \psi_M}{\partial z} \tag{10 2}$$

In the differentiable case, ψ_M is nothing but classical *Stoke's stream function* of the considered axially symmetric flow

This is directly connected with the results of Section 9 Let us take here for h the gradient of the azimuth angle $x \to \theta(x)$, a real function locally defined in $\mathcal{X} \setminus Z$ up to an additive constant The vector field $\mathbf{h}$ is $\mathcal{C}^\infty$ in $\mathcal{X} \setminus Z$ with zero curl On the other hand, let us take for the function ψ of Section 9 the rotationally symmetric extension of the function denoted above by ψ_M; this yields

$$\mathbf{u}l = (\operatorname{grad} \psi) \times \mathbf{h}l, \tag{10 3}$$

which turns out to be equivalent to (10 2)

Since the scalar field θ is a constant in M, it is automatically convected by any two-dimensional carrier in ω Therefore Prop 9 c yields the next proposition

Proposition 10.a. *Let the open subset Ω of $\mathcal{X}$ be rotationally symmetric about Z, as well as the potential function U of extraneous forces Let an axially symmetric flow be described as above by the meridian elements ψ_M, ρ_M, b_M, with*

$$\operatorname{grad} \psi_M \in \mathcal{L}^2\left(\omega, \frac{1}{r} dm; \mathbf{M}\right) \tag{10 4}$$

$$b_M \in \mathcal{L}^1(\omega, r \, dm; \mathbb{R}), \tag{10 5}$$

$$\rho_M \in \mathcal{L}^\infty(\omega, dm; \mathbb{R}) \tag{10 6}$$

This flow satisfies (4.5) *in* Ω *if and only if, for every carrier with velocity field* $\varphi_M \in \mathcal{D}_a(\omega, \mathbf{M})$ *(equivalently* $\mathcal{D}^1_a(\omega, \mathbf{M})$*) the* τ*-derivative of the following functional vanishes at* $\tau = 0$:

$$\mathcal{B}(\tau) = \int_\omega \left[\frac{1}{r^2} \rho^\tau_M (\operatorname{grad} \psi^\tau_M)^2 + b^\tau_M + \rho^\tau_M U \right] r \, dm,$$

where ψ^τ_M, ρ^τ_M, b^τ_M, *reducing to* ψ_M, ρ_M, b_M *for* $\tau = 0$, *are scalar fields convected by the carrier*

Remark 10.6. Supposing $\mathbf{u}$ under the form (10.2), or equivalently (10.3), involves that the vector measure $\mathbf{u}l$ is divergence-free in $\Omega \setminus Z$, not necessarily in Ω if $\Omega \setminus Z \neq \emptyset$. In fact the vector field $\mathbf{h}$ figuring in (10.3) is curl-free in $\mathcal{X} \setminus Z$ but, as an element of $\mathcal{L}^1_{loc}(\mathcal{X}, l; \mathbf{X})$, its curl in the sense of distributions equals a nonzero vector measure concentrated on Z. The following proposition shows that, in the present instance, we are freed from this hindrance thanks to assumption (10.4).

Proposition 10.c. *Let* Ω *be an open subset of the three-dimensional Euclidean space* $\mathcal{X}$, *intersecting the straight line* Z. *Every element of* $\mathcal{L}^2(\Omega, l; \mathbf{X})$ *with zero divergence in the sense of distributions in the open set* $\Omega \setminus Z$ *also has a zero divergence in* Ω

Proof. we have to show that, under the assumptions made,

$$\int_\Omega \mathbf{u} \cdot \operatorname{grad} \varphi \, dl = 0 \tag{10.7}$$

holds for every $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$. Let us choose a $\mathcal{C}^\infty$ function $f: [0, +\infty[\to [0, 1]$, taking the constant value 1 in $[0, \frac{1}{2}]$ and vanishing in $[1, +\infty[$. With every $\epsilon > 0$ let us associate the function

$$\alpha: x \to f\left[\frac{1}{\epsilon} \operatorname{dist}(x, Z) \right],$$

which belongs to $\mathcal{C}^\infty(\mathcal{X}, \mathbb{R})$, as well as $\beta = 1 - \alpha$. The left-hand side in (10.7) equals the sum of two terms. First, $\int_\Omega \mathbf{u} \cdot \operatorname{grad} (\beta\varphi) dl$ vanishes since $\beta\varphi \in \mathcal{D}(\Omega \setminus Z, \mathbb{R})$. Second, we are to show that under assumption $\mathbf{u} \in \mathcal{L}^2(\Omega, l; \mathbf{X})$, the expression

$$\int_\Omega \mathbf{u} \cdot \operatorname{grad}(\alpha\varphi) \, dl = \int_\Omega \alpha \mathbf{u} \cdot \operatorname{grad} \varphi \, dl + \int_\Omega \varphi \mathbf{u} \cdot \operatorname{grad} \alpha \, dl,$$

where φ is fixed in $\mathcal{D}(\Omega, \mathbb{R})$, tends to zero with ϵ. This is trivial in what concerns the first term on the right, since $|\alpha| < 1$ and since the Lebesgue measure of the set $K = (\operatorname{supp} \alpha) \cap (\operatorname{supp} \varphi)$ tends to zero. As for the last term, the Cauchy-Schwarz inequality yields

$$\left| \int_\Omega \varphi \mathbf{u} \cdot \operatorname{grad} \alpha \, dl \right|^2 \leq \int_K \varphi^2 u^2 \, dl \int_K (\operatorname{grad} \alpha)^2 \, dl.$$

Now, introducing the characteristic function of supp α, one has

$$\int_K \varphi^2 u^2 \, dl = \int_\Omega \chi_{\operatorname{supp}\alpha} \varphi^2 u^2 \, dl,$$

which tends to zero by Lebesgue dominated convergence; on the other hand, if D denotes the diameter of the compact set supp φ, one elementarily finds the following upper bound, independent of ϵ:

$$\int_K (\operatorname{grad} \alpha)^2 \, dl \leq 2\pi D \int_0^1 s f'^2(s) \, ds.$$

Remark 10.d. As in the preceding sections, the condition div $(\rho \mathbf{u}l) = 0$ of mass conservation has to be satisfied additionally. Due to (10.4) and (10.6), Prop. 10.c implies that this condition holds in the whole of Ω as soon as it holds in $\Omega \setminus Z$. This in turn is readily found equivalent to div $(\rho_M \mathbf{u}_M rm) = 0$, holding in the sense of two-dimensional distributions in ω. In view of (10.2), this in turn is equivalent to

$$\frac{\partial}{\partial z}\left(\rho_M \frac{\partial \psi_M}{\partial r}\right) - \frac{\partial}{\partial r}\left(\rho_M \frac{\partial \psi_M}{\partial z}\right) = 0,$$

which is meaningful in the sense of distributions in ω. In the case where ρ_M and ψ_M are smooth, this reduces to the vanishing of the Jacobian determinant of these two functions, which means that the couple (ρ_M, ψ_M) takes it values in some strict submanifold of $\mathbb{R}^2$. These properties are naturally preserved when the scalar functions ρ_M and ψ_M are convected by a carrier.

Remark 10.e. Similarly to the above remarks, if $\Omega \cap Z \neq \emptyset$, one may ask whether the dynamical equation (4.5) holding in $\Omega \setminus Z$ is sufficient for this equation to hold in Ω. If so, the assumption $\varphi_M \in \mathcal{D}_a(\omega, \mathbf{M})$ in Prop. 10.a could be replaced by $\varphi_M \in \mathcal{D}(\omega, \mathbf{M})$. By the same sort of upper bounds as in the proof of Prop. 10.c, the answer is found to be yes, if, in addition to (10.4), (10.5), and (10.6) it is supposed that $\mathbf{u} \in \mathcal{L}^4_{\mathrm{loc}}(\Omega, l; \mathbf{X})$; but we are unable to assert that some less stringent sufficient condition cannot be found.

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