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Duality Characterization of Strain Tensor Distributions in an Arbitrary Open Set

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Submitted by J. L. Lions

1. INTRODUCTION

Let E denote a Euclidean linear space with dimension ν ; in mechanical applications $\nu = 2$ or 3 . Strictly speaking, the space of elementary geometry and classical physics has no proper linear structure; it is rather an *affine* or *homogeneous* Euclidean space; but the choice of an artificial origin allows to identify it with the associated Euclidean linear space E , bringing notational simplification. The notation $S_2(E)$, or shortly S_2 , will refer to the second symmetric power of E , i.e. the subspace of $E \otimes E$ consisting of the symmetric tensors.

Let Ω be an arbitrary open subset of E . If $u \in \mathcal{C}^1(\Omega, E)$, i.e. u is a continuously differentiable vector field on Ω , let us denote by u_i the components of u relative to some orthonormal basis of E and, calling x_j the components of the generic point x of Ω , let us write as $u_{i,j}$ the partial derivative $\partial u_i / \partial x_j$. Classically, the tensor field $e \in \mathcal{C}^0(\Omega, S_2)$ with components

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

is called the *deformation* of u ; notation $e = \text{def } u$. When u constitutes the *velocity field* of a continuous medium occupying at the considered instant, the region Ω , this tensor is the *time-rate of deformation* of the medium. The same mathematics are used in the approximate treatment of the small deviations of a continuous medium from some reference state; then $u(x)$ represents the *infinitesimal displacement* of the element of the medium which, in the reference state, occupies the position x and, at this point, the tensor $\text{def } u$ measures the *infinitesimal strain* of the medium.

Classically, in order that a given $e \in \mathcal{C}^2(\Omega, S_2)$ derive in the above way from some vector field u , certain *compatibility conditions* are required; a well known necessary condition is

$$e_{ij,kl} + e_{kl,ij} = e_{ik,jl} + e_{jl,ik}.$$

This condition is sufficient when Ω is *simply connected* but, for a multiply connected Ω , the nullity of some line integrals involving e , relative to a system of fundamental circuits of Ω must be written in addition.

Let us observe now that, according to the common trend in the modern theory of partial differential equations, the solutions of boundary value problems in continuum mechanics, are looked for in such functional spaces that e does not belong to $\mathcal{C}^2(\Omega, S_2)$ anymore; e is allowed instead to be a class of equivalent locally integrable tensor fields, relative to some measure, or it is a priori treated as a measure or even more generally as an element of $\mathcal{D}'(\Omega, S_2)$ the space of *symmetric tensor distributions* on Ω . Then the aforementioned line integrals become meaningless.

The purpose of this paper is to characterize the symmetric tensor distributions (resp. the symmetric tensor fields) which have the form $\text{def } u$, by some *orthogonality property* in the sense of the duality between the space \mathcal{D}' of Schwartz's distributions on Ω and the space \mathcal{D} of the \mathcal{C}^∞ functions whose support relative to Ω is compact. The scalar product of this duality is denoted by $\langle \cdot, \cdot \rangle$; in the case where e is properly a tensor field, i.e. when its components e_{ij} are real functions instead of distributions, the same brackets will be used to denote the scalar product relative to the Lebesgue measure $d\omega$ of Ω , namely, for $s \in \mathcal{D}(\Omega, S_2)$

$$\langle e_{ij}, s_{ij} \rangle = \int_{\Omega} e_{ij} s_{ij} d\omega$$

with the usual sommation convention.

For $s \in \mathcal{D}(\Omega, S_2)$, the *divergence* is the vector field $\text{div } s \in \mathcal{D}(\Omega, E)$ with components

$$(\text{div } s)_i = s_{ij,j}.$$

THEOREM 1. *Let $e \in \mathcal{D}'(\Omega, S_2)$ (resp. $e \in \mathcal{C}^1(\Omega, S_2)$); for the existence of $u \in \mathcal{D}'(\Omega, E)$ (resp. $u \in \mathcal{C}^2(\Omega, E)$) such that $e = \text{def } u$ it is necessary and sufficient that, for every $s \in \mathcal{D}(\Omega, S_2)$ with zero divergence, one has $\langle e_{ij}, s_{ij} \rangle = 0$.*

In the particular case of a simply connected Ω and $e \in \mathcal{C}^2(\Omega, S_2)$ the same characterization is established in [1] under the name of *Donati's theorem*.

We shall base the proof of Theorem 1 on the following, which refers to a more common situation.

THEOREM 0. *Let $g \in \mathcal{D}'(\Omega, E)$ (resp. $g \in \mathcal{C}^0(\Omega, E)$); for the existence of $f \in \mathcal{D}'(\Omega, \mathbf{R})$ (resp. $f \in \mathcal{C}^1(\Omega, E)$) such that $g = \text{grad } f$, it is necessary and sufficient that, for every $v \in \mathcal{D}(\Omega, E)$ with zero divergence, one has $\langle g_i, v_i \rangle = 0$.*

The version $g \in \mathcal{C}^0(\Omega, E)$ of this statement is of current use in the Calculus of Variations or in Hydrodynamics (see some proofs in [2], [3] or [4]). According to a remark of J. L. Lions reported in [5], the case $g \in \mathcal{D}'(\Omega, E)$ may be related

to De Rham's theory of the homology of *currents* on a differential manifold [6]. For the reader's convenience, we give an original proof of Theorem 0 in the Annex. Since we do not make use of polyhedral decompositions of Ω , this proof makes actually Theorem 0 stand apart from De Rham's deep results about the homology of currents of arbitrary order.

An early state of the present study was inserted in the multigraphed seminar report [4]. Similar methods are also used in [7] to establish the following result whose main mechanical relevance lies in the theory of plates (then $\nu = 2$): for $e \in \mathcal{D}'(\Omega, S_2)$ to be a tensor of second derivatives, i.e. $e_{ij} = f_{,ij}$ with $f \in \mathcal{D}'(\Omega, \mathbf{R})$, it is necessary and sufficient that $\langle e_{ij}, s_{ij} \rangle = 0$ for every $s \in \mathcal{D}(\Omega, S_2)$ such that $\text{div div } s = 0$.

Remark. The formulation of Theorem 1, in its version concerning a proper tensor field e , assumes $e \in \mathcal{C}^1(\Omega, S_2)$; this is another improvement on Donati's theorem which was restricted to $e \in \mathcal{C}^2(\Omega, S_2)$. But, by comparison with Theorem 0, where $g \in \mathcal{C}^0(\Omega, E)$, this raises the question of the regularity properties of the distribution u when the distribution $e = \text{def } u$ is a \mathcal{C}^0 tensor field. As a part of some regularity study made in [8], it is established that such an u is a continuous vector field; but to the difference with the situation of Theorem 0, the assumption $\text{def } u \in \mathcal{C}^0(\Omega, S_2)$ is not found to imply the differentiability of u , which would allow to understand the writing $e = \text{def } u$ in the sense of the elementary partial derivatives.

2. ORTHOGONALITY AND CLOSEDNESS

Elementarily, if $v \in \mathcal{D}(\Omega, E)$, the condition $\text{div } v = 0$ is equivalent to the fact that $\langle g_i, v_i \rangle = 0$ holds for every $g = \text{grad } f$, $f \in \mathcal{D}'(\Omega, \mathbf{R})$ (a connected remark is that the linear mappings $\text{grad}: \mathcal{D}'(\Omega, \mathbf{R}) \rightarrow \mathcal{D}'(\Omega, E)$ and $-\text{div}: \mathcal{D}(\Omega, E) \rightarrow \mathcal{D}(\Omega, \mathbf{R})$ are the transpose of each other). Therefore, Theorem 0 expresses that the range of the linear mapping $\text{grad}: \mathcal{D}'(\Omega, \mathbf{R}) \rightarrow \mathcal{D}'(\Omega, E)$ is a subset of $\mathcal{D}'(\Omega, E)$ equal to its biorthogonal in the sense of the dual pair $\mathcal{D}'(\Omega, E), \mathcal{D}(\Omega, E)$; this equivalently means that the considered range is a *closed linear subspace* of $\mathcal{D}'(\Omega, E)$. Consequently, for the proof of Theorem 0, one cannot expect to reduce the case $g \in \mathcal{D}'(\Omega, E)$ to the elementary situation $g \in \mathcal{C}^0(\Omega, E)$ by any approximation or regularization technique, since the use of approximations would precisely rest on this closedness.

Similarly, Theorem 1, expresses that *the mapping* $\text{def}: \mathcal{D}'(\Omega, E) \rightarrow \mathcal{D}'(\Omega, S_2)$ *has a closed range.*

Mechanically, such orthogonality properties are meaningful in the framework of the virtual work, or virtual power, method. A modern formalization of this method consists in the following (cf. [9], [10], [11]): with every possible position of the mechanical system under consideration are associated two linear spaces \mathcal{V} and \mathcal{F} ; the elements of the former constitute, in a general sense, the possible

values of the velocity of the system if it happens to pass through the considered position; the elements of the latter constitute, formally, the values of the various forces that the system may experience in that position. These spaces are placed in duality by the bilinear form *power*, denoted $\langle \cdot, \cdot \rangle$. Such a mathematical pattern may usually be applied in several different ways to a given mechanical situation; they correspond to different ways of expressing the information about physical forces in term of elements of the linear space \mathcal{F} (cf. [9], Sect. 3.j). In particular, when dealing with a continuous medium filling the open set Ω as above, it may be convenient to make \mathcal{V} consist of strain tensor fields (resp. tensor distributions). Then \mathcal{F} consists of stress tensor fields, with the known expression of the power (optionally the pairing bilinear form may be taken as the negative of this power to avoid some minus signs in solid mechanics). In such a framework, the fact that only the elements of \mathcal{V} with the form $e = \text{def } u$ are feasible has to be treated as a constraint; the kinematical condition of it may be written $e \in I$, where I is a linear subspace of \mathcal{V} . In existential studies it is usually important that I be exactly the orthogonal of the subspace J of \mathcal{F} consisting of the self-equilibrated stress fields; in other words the constraint $e \in I$ may be formally described as frictionless, some element of J being considered as the associated *reaction*. As Theorem 1 makes use of the smallest reasonable space of self-equilibrated stresses, namely the \mathcal{C}^∞ symmetric tensor fields with compact supports and zero divergence, this Theorem appears as a general tool for checking that a pair I, J of mutually orthogonal linear subspaces of \mathcal{V} and \mathcal{F} has effectively been constructed. In fact \mathcal{V} is practically always a subspace of $\mathcal{D}'(\Omega, S_2)$ (frequently some Sobolev space).

Remark. Results similar to Theorem 0 and Theorem 1 may easily be obtained by replacing the space \mathcal{D}' of the distributions in Ω , by the algebraic dual \mathcal{D}^* of \mathcal{D} . This consists in straightforward algebraic arguments (cf. [4]) but does not seem to have any mechanical relevance.

3. PROOF OF THEOREM 1

The definition of the derivatives of a distribution (resp. an integration by parts) immediately yields that if $e = \text{def } u$, $u \in \mathcal{D}'(\Omega, E)$ (resp. $u \in \mathcal{C}^2(\Omega, E)$), the element e of $\mathcal{D}'(\Omega, S_2)$ (resp. of $\mathcal{C}^1(\Omega, S_2)$) possesses the property:

(P) For every $s \in \mathcal{D}(\Omega, S_2)$ with zero divergence one has $\langle e_{ij}, s_{ij} \rangle = 0$.

To prove the converse implication we first establish two lemmata.

LEMMA 1. If the element e of $\mathcal{D}'(\Omega, S_2)$ (resp. of $\mathcal{C}^1(\Omega, S_2)$) possesses the property P, one has, for every $v \in \mathcal{D}(\Omega, E)$ with zero divergence and for every values of i and k

$$\langle e_{ij,k} - e_{jk,i}, v_j \rangle = 0. \quad (3.1)$$

Proof. Equality (3.1) is equivalent to

$$\langle e_{ij}, v_{j,k} \rangle - \langle e_{jk}, v_{j,i} \rangle = 0. \tag{3.2}$$

Using Kronecker's δ one has

$$\begin{aligned} \langle e_{ij}, v_{j,k} \rangle &= \langle e_{lm}, \delta_{il} v_{m,k} \rangle \\ \langle e_{jk}, v_{j,i} \rangle &= \langle e_{lm}, \delta_{km} v_{l,i} \rangle. \end{aligned}$$

As e_{lm} is symmetric, the left member of (3.2) may then be written as

$$\frac{1}{2} \langle e_{lm}, \delta_{il} v_{m,k} + \delta_{im} v_{l,k} - \delta_{km} v_{l,i} - \delta_{ki} v_{m,i} \rangle.$$

For fixed i and k , let us denote by s_{lm} the second factor in this bracket; this defines the components of an element s of $\mathcal{D}(\Omega, S_2)$ whose divergence is

$$\begin{aligned} s_{lm,m} &= \delta_{il} v_{m,km} + \delta_{im} v_{l,km} - \delta_{km} v_{l,im} - \delta_{ki} v_{m,im} \\ &= \delta_{il} (v_{m,m})_{,k} + v_{l,ki} - v_{l,ik} - \delta_{ki} (v_{m,m})_{,i} = 0. \quad \blacksquare \end{aligned}$$

LEMMA 2. *Let x_j denote the components of the generic point of E relative to the chosen orthonormal basis. If the element e of $\mathcal{D}'(\Omega, S_2)$ (resp. of $\mathcal{C}^1(\Omega, S_2)$) possesses the property P, one has, for every $v \in \mathcal{D}(\Omega, E)$ with zero divergence, and for every value of i*

$$\langle x_j (e_{ij,k} + e_{ik,j} - e_{kj,i}), v_k \rangle = 0. \tag{3.3}$$

Proof. Equality (3.3) is equivalent to

$$\langle e_{ij}, (x_j v_k)_{,k} \rangle + \langle e_{ik}, (x_j v_k)_{,j} \rangle - \langle e_{kj}, (x_j v_k)_{,i} \rangle = 0$$

whose left member may also be written as (recall that ν denotes the dimension of E)

$$\begin{aligned} &\langle e_{ij}, \delta_{jk} v_k + x_j v_{k,k} \rangle + \langle e_{ik}, \delta_{jj} v_k + x_j v_{k,j} \rangle - \langle e_{kj}, \delta_{ji} v_k + x_j v_{k,i} \rangle \\ &= \langle e_{ij}, v_j \rangle + \langle e_{ik}, \nu v_k + x_j v_{k,j} \rangle - \langle e_{ki}, v_k \rangle - \langle e_{kj}, x_j v_{k,i} \rangle \\ &= \langle e_{lm}, \delta_{li} \delta_{mk} (\nu v_k + x_j v_{k,j}) - \delta_{lk} \delta_{mj} x_j v_{k,i} \rangle \\ &= \langle e_{lm}, \nu \delta_{li} v_m + \delta_{li} x_j v_{m,j} - x_m v_{l,i} \rangle. \end{aligned}$$

As e_{lm} is symmetric, the second factor in the latter bracket may be replaced by the symmetrized expression (disregarding the factor $\frac{1}{2}$)

$$\nu \delta_{li} v_m + \nu \delta_{mi} v_l + \delta_{li} x_j v_{m,j} + \delta_{mi} x_j v_{l,j} - x_m v_{l,i} - x_l v_{m,i}.$$

For a fixed i let us denote by s_{lm} this expression; it defines the components of an element s of $\mathcal{D}(\Omega, S_2)$ whose divergence may be calculated as follows:

$$\begin{aligned} s_{lm,m} &= \nu \delta_{li} v_{m,m} + \nu \delta_{mi} v_{l,m} + \delta_{li} \delta_{jm} v_{m,j} + \delta_{li} x_j v_{m,jm} + \delta_{mi} \delta_{jm} v_{l,j} \\ &\quad + \delta_{mi} x_j v_{l,jm} - \delta_{mm} v_{l,i} - x_m v_{l,im} - \delta_{lm} v_{m,i} - x_l v_{m,im} \\ &= \nu v_{l,i} + v_{l,i} + x_j v_{l,ji} - \nu v_{l,i} - x_m v_{l,im} - v_{l,i} = 0. \quad \blacksquare \end{aligned}$$

Let us now complete the proof of Theorem 1. We suppose that $e \in \mathcal{D}'(\Omega, S_2)$ possesses the property P. Then, by Lemma 1 and Theorem 0, for every i and k , there exists $f_{ik} \in \mathcal{D}'(\Omega, \mathbf{R})$, defined up to an additive constant in every connected component of Ω , such that

$$f_{ik,j} = e_{ij,k} - e_{jk,i}. \quad (3.4)$$

This implies

$$(f_{ik} + f_{ki})_{,j} = e_{ij,k} - e_{jk,i} + e_{ki,j} - e_{ji,k} = 0.$$

Thus the distribution $f_{ik} + f_{ki}$ equals a constant in every connected component of Ω ; let us take profit of the arbitrary constants in the construction of f_{ik} to make that

$$f_{ik} + f_{ki} = 0 \quad (3.5)$$

Then the proof of Theorem 1 consists in establishing the existence of $u \in \mathcal{D}'(\Omega, E)$ such that

$$u_{i,k} = e_{ik} + f_{ik}.$$

Now

$$e_{ik} + f_{ik} = (x_j(e_{ij} + f_{ij}))_{,k} - x_j(e_{ij,k} + f_{ij,k}); \quad (3.6)$$

the first term in the right hand member constitutes in fact the ik component of a tensor distribution deriving from some element of $\mathcal{D}'(\Omega, E)$ in the required way; it remains to show the same for the second term, i.e. in view of Theorem 0, to prove that for every $v \in \mathcal{D}(\Omega, E)$ with zero divergence

$$\langle x_j(e_{ij,k} + f_{ij,k}), v_k \rangle = 0.$$

This results from Lemma 2, since, in view of (3.4)

$$e_{ij,k} + f_{ij,k} = e_{ij,k} + e_{ik,j} - e_{kj,i}.$$

Let us finally adapt this reasoning to the alternative assumption $e \in \mathcal{C}^1(\Omega, S_2)$. In that case Lemma 1 and Theorem 0 imply the existence of $f_{ik} \in \mathcal{C}^1(\Omega, \mathbf{R})$

satisfying (3.4) and (3.5). Lemma 2 and Theorem 0 ensure that $e_{ik} + f_{ik}$ as it is written in (3.6) has the form $u_{i,k}$, with $u \in \mathcal{C}^1(\Omega, E)$; actually, since e_{ik} and f_{ik} belong to $\mathcal{C}^1(\Omega, E)$ one has $u \in \mathcal{C}^2(\Omega, E)$. ■

4. DEGREE OF INDETERMINATION OF u

The study of the degree of indetermination of u corresponding to some e satisfying the conditions of Theorem 1 consists in describing the kernel of the linear mapping def .

THEOREM 2. *Let $u \in \mathcal{D}'(\Omega, E)$; the distribution $\text{def } u \in \mathcal{D}'(\Omega, S_2)$ is zero if and only if, for every connected component of Ω , there exist some constant $V \in E$ and $F \in E \wedge E$ (i.e. F is an antisymmetric tensor of order two) such that the restriction of u to this component equals the restriction of the vector field $x \mapsto w(x)$ defined by*

$$w_k(x) = V_k + F_{kj}x_j. \tag{4.1}$$

Proof. Clearly (4.1) implies $\text{def } w = 0$. Conversely, let $u \in \mathcal{D}'(\Omega, E)$ such that

$$\text{def } u = 0. \tag{4.2}$$

By partial derivation this yields

$$\begin{aligned} u_{i,jk} + u_{j,ik} &= 0 \\ u_{j,ki} + u_{k,ji} &= 0 \\ u_{k,ij} + u_{i,kj} &= 0. \end{aligned}$$

Adding the two last equalities and subtracting the first one, one obtains $2u_{k,ij} = 0$. Hence the restriction of the distribution $u_{k,i}$ to a connected component ω of Ω is a constant F_{ki} , and $F_{ki} + F_{ik} = 0$ by (4.2). Interpreting the function $x \mapsto F_{kj}x_j$ as a distribution on ω , one has

$$(u_k - F_{kj}x_j)_{,i} = u_{k,i} - F_{ki} = 0.$$

Therefore the restriction to ω of the distribution $u_k - F_{kj}x_j$, equals a constant, say V_k . ■

5. ANNEX: PROOF OF THEOREM 0

Clearly the theorem concerns each connected component of Ω separately. Hence we suppose in all the sequel that Ω is a connected open subset of E .

Let c denote an integration path in Ω , i.e. a continuous mapping $t \mapsto \xi(t)$ of the real interval $[0, 1]$ into Ω , with bounded variation. For every $\varphi \in \mathcal{C}^0(\Omega, \mathbf{R})$

and every coordinate x_i of the generic element x of Ω , one classically defines the integral $\int_c \varphi dx_i$. Equivalently may be written the vector integral $\int_c \varphi dx$, an element of E independent of the choice of the basis in E . The functional

$$c_i: \varphi \mapsto \int_c \varphi dx_i$$

is a fortiori defined for every $\varphi \in \mathcal{D}(\Omega, \mathbf{R})$ and constitutes a distribution on Ω (actually a measure). We shall use the same symbol c to denote the path and the vector functional

$$c: \varphi \mapsto \int_c \varphi dx;$$

this functional is an element of $\mathcal{D}'(\Omega, E)$ whose the $c_i \in \mathcal{D}'(\Omega, \mathbf{R})$ are the components relative to the chosen basis. The support of the distribution c is contained in the image of $[0, 1]$ under the mapping ξ , thus it is a compact subset of Ω . Then c may trivially be extended as a distribution on the whole of E , with the same support; we shall also denote by c this element of $\mathcal{D}'(E, E)$.

Let us put $\xi(0) = a$, $\xi(1) = b$, respectively called the origin and the extremity of c . If $a = b$ we shall say that c is a *circuit*.

LEMMA 3. Denoting by ϵ_a and ϵ_b the Dirac measures at the points a and b , one has

$$\operatorname{div} c = \epsilon_a - \epsilon_b. \quad (5.1)$$

In particular, if c is a circuit, its divergence is zero.

Proof. Using an orthonormal basis in E , we have, for every $\varphi \in \mathcal{D}(\Omega, \mathbf{R})$,

$$\begin{aligned} \langle c_{i,i}, \varphi \rangle &= -\langle c_i, \varphi_{,i} \rangle = -\int_c \varphi_{,i} dx_i \\ &= \varphi(a) - \varphi(b) = \langle \epsilon_a - \epsilon_b, \varphi \rangle. \quad \blacksquare \end{aligned}$$

LEMMA 4. If an element g of $\mathcal{D}'(\Omega, E)$ yields $\langle g_i, v_i \rangle = 0$ for every $v \in \mathcal{D}(\Omega, E)$ with zero divergence, one has

$$g_{i,j} = g_{j,i}. \quad (5.2)$$

Proof. For every $\varphi \in \mathcal{D}(\Omega, \mathbf{R})$

$$\begin{aligned} \langle g_{i,j}, \varphi \rangle - \langle g_{j,i}, \varphi \rangle &= -\langle g_i, \varphi_{,j} \rangle + \langle g_j, \varphi_{,i} \rangle \\ &= \langle g_k, -\delta_{ki}\varphi_{,j} + \delta_{kj}\varphi_{,i} \rangle. \end{aligned}$$

Now, for fixed i and j , the vector field v with components

$$v_k = -\delta_{ki}\varphi_{,j} + \delta_{kj}\varphi_{,i}$$

has the divergence

$$v_{k,k} = -\delta_{ki}\varphi_{,jk} + \delta_{kj}\varphi_{,ik} = -\varphi_{,ji} + \varphi_{,ij} = 0. \quad \blacksquare$$

LEMMA 5. *Let Π denote an open parallelootope in E ; if $g \in \mathcal{D}'(\Pi, E)$ satisfies (5.2), there exists $f \in \mathcal{D}'(\Pi, \mathbf{R})$ such that $g = \text{grad } f$.*

Proof. The reasoning used in [13] for \mathbf{R}^n also holds for an open parallelo-
tope. \blacksquare

LEMMA 6. *Let ω be an open subset of E and let c be an integration path in ω , with origin a and extremity b . Let $\rho > 0$ be strictly less than $\text{dist}(\text{supp } c, E \setminus \omega)$ and let β denote the open ball with center at the zero of E , with radius ρ . For every $\alpha \in \mathcal{D}(E, \mathbf{R})$ with support contained in β , the convolution products $\epsilon_a * \alpha$, $\epsilon_b * \alpha$ (i.e. the functions deduced from α by the translations of vectors a and b) and $c_i * \alpha$ define elements of $\mathcal{D}(\omega, \mathbf{R})$. For every $f \in \mathcal{D}'(\omega, \mathbf{R})$ one has*

$$\langle f_{,i}, c_i * \alpha \rangle = \langle f, \epsilon_b * \alpha \rangle - \langle f, \epsilon_a * \alpha \rangle. \quad (5.3)$$

Proof. As $\text{supp } c$ is a compact subset of ω , there exists $\rho > 0$ agreeing with the statement. The functions $\epsilon_a * \alpha$ and $\epsilon_b * \alpha$ are \mathcal{C}^∞ with supports respectively contained in $\beta + a$ and $\beta + b$, open balls in ω . In defining $c_i * \alpha$, one interprets c_i as an element of $\mathcal{D}'(E, \mathbf{R})$; the support of $c_i * \alpha$ is contained in $\text{supp } c + \text{cl } \beta$, a compact subset of ω . Let $\gamma \in \mathcal{D}(E, \mathbf{R})$ with support contained in ω and taking the value 1 at every point of $\text{supp } c + \text{cl } \beta$. The product γf is a distribution on ω with compact support; let \hat{f} denote its extension as an element of $\mathcal{D}'(E, \mathbf{R})$ with the same support. Denoting by $\langle \cdot, \cdot \rangle_E$ the pairing between $\mathcal{D}'(E, \mathbf{R})$ and $\mathcal{D}(E, \mathbf{R})$, and using Lemma 3, one writes the left member of (5.3) under the form

$$\langle \hat{f}_{,i}, c_i * \alpha \rangle_E = \langle \hat{f}, c_{i,i} * \alpha \rangle_E = \langle \hat{f}, (\epsilon_a - \epsilon_b) * \alpha \rangle_E$$

which equals the right member. \blacksquare

In the sequel such tricks as the replacement of f by \hat{f} will be implicit and, when writing a pairing bracket, we shall omit the mention of the concerned open set.

LEMMA 7. *With the same notations as in the above lemma, let us suppose that c is a circuit. If $g \in \mathcal{D}'(\Omega, E)$ yields $\langle g_i, v_i \rangle = 0$ for every $v \in \mathcal{D}(\omega, E)$ with zero divergence, one has*

$$\langle g_i, c_i * \alpha \rangle = 0.$$

Proof. In fact $c * \alpha$ is an element of $\mathcal{D}(\omega, E)$ whose divergence $c_{i,i} * \alpha$ is zero by Lemma 3. \blacksquare

Let us now complete the proof of Theorem 0. For notational simplicity, suppose a translation made, such that the origin of E belongs to Ω .

Let $g \in \mathcal{D}'(\Omega, E)$ yielding $\langle g_i, v_i \rangle = 0$ for every $v \in \mathcal{D}(\Omega, E)$ with zero divergence.

In view of Lemmata 4 and 5, there exists an open ball ω^0 with center 0, whose closure is contained in Ω , and a distribution $f^0 \in \mathcal{D}'(\omega^0, \mathbf{R})$ such that $\text{grad } f^0$ equals the restriction of g to ω^0 .

As Ω is an open connected set of E , it is also arcwise connected (see e.g. [12]). With every $x \in \Omega$ let us associate an integration path c^x in Ω with origin 0 and extremity x . Let us choose ρ^x less than or equal to the radius of ω^0 , and satisfying

$$0 < \rho^x < \text{dist}(\text{supp } c^x, E \setminus \Omega).$$

Denoting by ω^x the open ball with center x and radius ρ^x , let $f^x \in \mathcal{D}'(\omega^x, \mathbf{R})$ be defined as follows: for every $\varphi \in \mathcal{D}(\omega^x, \mathbf{R})$ the translate function $\alpha = \epsilon_{-x} * \varphi$ has its support contained in ω^0 ; put

$$\langle f^x, \varphi \rangle = \langle f^0, \alpha \rangle + \langle g_i, c_i^x * \alpha \rangle.$$

This in fact defines f^x as a distribution on ω^x , since $\varphi \mapsto \alpha$ is a continuous linear mapping from $\mathcal{D}(\omega^x, \mathbf{R})$ into $\mathcal{D}(\omega^0, \mathbf{R})$ and since $\alpha \mapsto c_i^x * \alpha$ is a continuous linear mapping from $\mathcal{D}(\omega^0, \mathbf{R})$ into $\mathcal{D}(\Omega, \mathbf{R})$.

Let us prove that $\text{grad } f^x$ equals the restriction of g to ω^x .

$$\begin{aligned} \langle f_{,k}^x, \varphi \rangle &= -\langle f^x, \varphi_{,k} \rangle = -\langle f^0, \alpha_{,k} \rangle - \langle g_i, c_i^x * \alpha_{,k} \rangle \\ &= \langle f_{,k}^0, \alpha \rangle + \langle g_{i,k}, c_i^x * \alpha \rangle. \end{aligned}$$

Lemma 4 shows that $g_{i,k} = g_{k,i}$, hence, using Lemma 3, one has

$$\begin{aligned} \langle g_{i,k}, c_i^x * \alpha \rangle &= \langle g_{k,i}, c_i^x * \alpha \rangle = -\langle g_k, c_{i,i}^x * \alpha \rangle \\ &= \langle g_k, (\epsilon_x - \epsilon_0) * \alpha \rangle = \langle g_k, \varphi \rangle - \langle g_k, \alpha \rangle. \end{aligned}$$

As the restriction of g_k to ω^0 is $f_{,k}$ by construction, this yields the expected result.

It remains to invoke the piecing theorem (see e.g. [13]) to prove the existence of $f \in \mathcal{D}'(\Omega, \mathbf{R})$ whose restriction to ω^x is f^x for every $x \in \Omega$. We have to check that for any two points x and x' of Ω , if the intersection $\omega^x \cap \omega^{x'}$ is nonempty, f^x and $f^{x'}$ have the same restriction to this intersection. This amounts to check that if $y \in \omega^x \cap \omega^{x'}$ and if β is an open ball, with center 0, such that $\beta + y \subset \omega^x \cap \omega^{x'}$, the distributions f^x and $f^{x'}$ have the same restriction to $\beta + y$, or in other words that for every $\alpha \in \mathcal{D}(\beta, \mathbf{R})$ on has

$$\langle f^x, \epsilon_y * \alpha \rangle = \langle f^{x'}, \epsilon_y * \alpha \rangle. \quad (5.4)$$

By applying Lemma 6 to a rectilinear path σ with origin x and extremity y in $\omega = \omega^x$, one obtains, since $f_i^x = g_i$ in ω^x ,

$$\langle f^x, \epsilon_y * \alpha \rangle = \langle f^x, \epsilon_x * \alpha \rangle + \langle g_i, \sigma_i * \alpha \rangle.$$

Similarly, denoting by σ' a rectilinear path with origin x' and extremity y ,

$$\langle f^{x'}, \epsilon_y * \alpha \rangle = \langle f^{x'}, \epsilon_{x'} * \alpha \rangle + \langle g_i, \sigma'_i * \alpha \rangle$$

By the definition of f^x one has

$$\langle f^x, \epsilon_x * \alpha \rangle = \langle f^0, \alpha \rangle + \langle g_i, c_i^x * \alpha \rangle$$

and similarly for $f^{x'}$. Then equality (5.4) follows from Lemma 7 applied to the circuit $c^x + \sigma - \sigma' - c^{x'}$. ■

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