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FLUID DYNAMICS AND THE CALCULUS HORIZONTAL VARIATIONS

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Abstract—A horizontal variation is a shift defined in the (t, x_i) space or in the (x_i) space by a smooth vector field φ. The corresponding laws of transport for vector or tensor fields and for vector or tensor measures are investigated. This is used to characterize the (possibly non-smooth) solutions of the dynamical equations of a nonhomogeneous compressible barotropic inviscid fluid, in Euler variables, as the critical points of some real functionals on infinite dimensional manifolds.

1. INTRODUCTION

When Lagrange's variables are used in formulating the dynamical equations of an inviscid fluid, some variational characterization of the solutions is easily derived. This simply consists in the adaptation of Hamilton's principle of least action to what actually constitutes a frictionless mechanical system with infinite degree of freedom (as a standard reference, see [1]). The question is much less clear when the dynamical equations are expressed under the Euler form, though a certain number of studies have been devoted to this subject in recent decades. The present paper proposes an answer based on what may be called the *calculus of horizontal variations*.

Let us first precise the mechanical setting. It is assumed that the conditions under which the fluid evolves allows one to eliminate the temperature in order to obtain a relation between the pressure p, the density ρ and, possibly, the time t, which holds during any possible motion. But homogeneity of the fluid is not assumed: the said relation may be written as

$$f(\kappa, t, \rho, p) = 0 \tag{1.1}$$

containing a parameter κ, of arbitrary mathematical nature, whose value is a constant for each element of the fluid. For instance κ may refer to the fixed temperature of the considered element (isothermic evolution) or to the specific entropy (isentropic evolution) but may also account for the chemical nature, possibly different in various parts of the medium.

The distributed extraneous forces are of the gravity sort, i.e. the density vector of these forces relative to the fluid mass has the form grad U, where U denotes a given function of the space coordinates x_1, x_2, x_3, possibly dependent also on t.

Boundary conditions will not be investigated in this paper; one may suppose, for instance, that the fluid is confined by walls with prescribed motion.

The above assumptions make the fluid belong to the class of frictionless mechanical systems submitted to, possibly time-dependent, potential forces. This precisely makes sense when Lagrange's description of motions is used; to this end, every particle of the fluid is labelled by a value of the triplet $\lambda = (\lambda^1, \lambda^2, \lambda^3)$, ranging over some subset Λ of R^3. A *placement* of the fluid, relative to the inertial orthonormal frame $0x_1x_2x_3$ is a $\mathscr{C}^1$ mapping $\pi : \lambda \to x$, where x stands for the triplet $(x_1, x_2, x_3) \in R^3$. A *motion* is a chain $t \to \pi^t$ of such placements. Putting $\sigma = \rho^{-1}$, one expresses the mass conservation by

$$\sigma = (\mu'_\lambda)^{-1} \frac{Dx}{D\lambda} \tag{1.2}$$

where $Dx/D\lambda$ denotes the Jacobian determinant of π, and μ'_λ a given function of λ, namely the density of the mass measure relative to the Lebesgue measure $d\lambda$ on Λ.

For brevity, *the case of an incompressible fluid, where* p *does not appear explicitly in*

relation (1.1), *will be left for a further paper* Suppose this relation solved under the form $p = p(t, \kappa, \sigma)$ and put

$$P(t, \kappa, \sigma) = \int p(t, \kappa, \sigma)\mathrm{d}\sigma, \tag{1.3}$$

an expression defined up to the addition of an arbitrary function of t and κ Then the functional

$$\pi \to -\mathcal{U}_{\mathrm{int}}(t, \pi) = -\int_{\Lambda} P(t, \kappa, \sigma)\mu'_{\lambda}(\lambda)\mathrm{d}\lambda \tag{1.4}$$

appears as the (time-dependent) potential of the internal forces of the system; this means that, if an arbitrary smooth chain of placements $\tau \to \pi^{\tau}$ is considered at a fixed t, with σ related to π^{τ} by (1.2), the corresponding work done by these forces equals the (signed) increase of the real function $\tau \to \mathcal{U}_{\mathrm{int}}(t, \pi^{\tau})$.

Similarly, the functional

$$-\mathcal{U}_{\mathrm{ext}}(t, \pi) = -\int_{\Lambda} U(t, \pi(\lambda))\mu'_{\lambda}(\lambda)\mathrm{d}\lambda$$

figures as the (time-dependent) potential of the extraneous mass-forces

There is considered on the other hand the general expression of the kinetic energy

$$\mathcal{E}_c(t, \pi, \dot{\pi}) = \int_{\Lambda} \tfrac{1}{2}[\dot{\pi}(t, \lambda)]^2\mu'_{\lambda}(\lambda)\mathrm{d}\lambda \tag{1.6}$$

where $\dot{\pi}$ stands for $\partial\pi/\partial t$, the velocity vector of the particle λ

In such a framework, easy calculation yields Hamilton's property, namely that the solutions $t \to \pi$ of the equations of fluid dynamics over some time interval $[t^1, t^2]$ are the extremals of the *action* functional

$$\mathcal{A}(t^1, t^2) = \int_{t^1}^{t^2} [\mathcal{E}_c + \mathcal{U}_{\mathrm{int}} + \mathcal{U}_{\mathrm{ext}}]\mathrm{d}t \tag{1.7}$$

Let us pass on now to Euler's description of motion Denoting by $u(t, x) = \dot{\pi}(t, \pi^{-1}(t, x))$ the velocity field, with $u_i(t, x)$ as components relative to the considered inertial orthonormal frame, we define the differential operator

$$d_u = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}, \tag{1.8}$$

acting on differentiable functions of t and x; it is sometimes called the drag-derivation along u Then the dynamical equations take the form

$$d_u u = -\sigma \operatorname{grad} p + \operatorname{grad} U, \tag{1.9}$$

while the conservation of mass is expressed by

$$d_u \sigma = -\sigma \operatorname{div} u \tag{1.10}$$

As for the conservation of κ for each particle, it may symbolically be written as

$$d_u \kappa = 0, \tag{1.11}$$

even if the mathematical nature of κ do not let the r.h.s of (1.8) make sense These three relations, joined with (1.1), constitute a complete set of equations for the unknown functions $(t, x) \to u, \sigma, p, \kappa$ Assigning a variational meaning to this set of equations has been the subject of

some papers; as typical references, see [1, 2], where a list of anterior papers may be found; more recently [3] These are mainly developed in the line initiated by J. W Herivel and C C Lin, by introducing the Lagrange multipliers associated with some relations treated as constraints However, Lin's constraint of "the conservation of particle" seems somewhat unclear Much light was thrown on the question by Casal[4] This author expresses Hamilton's principle by taking as the unknown t-dependent element, instead of the placement π, the inverse mapping π^{-1}: $x \to \lambda$; since t and x are usually called the Euler variables, the result may indeed be said a variational statement of the Eulerian sort. Similar ideas, developed under the assumption of stationary motions, yield variational properties of the classical Kelvin–Bateman type; see in particular [5].

A common drawback of all these approaches is that they do not properly provide variational characterizations of the velocity field u but of some generalized velocity potential or stream function in the style of Clebsch's representation of the flow Apparently, this feature is connected with the use of the conventional calculus of variations, which cannot directly handle discontinuous fields; in contrast, the possibility of discontinuous velocity fields is inherent to the absence of viscosity in the fluid The conventional variational procedures consist in adding to the investigated functions some variations which have to be smooth functions of t and x; that leaves invariant the locus of possible discontinuities, while such a locus is precisely an unknown of the problem

The present study overcomes this fail, thanks to what we propose to call the *calculus of horizontal variations* Non smooth solutions will be handled so that our results include as special cases some variational characterization known to hold for the slipstream surfaces or free boundaries in inviscid flows[6, 7]. Recall that the fluid is not supposed homogeneous; one may accept that its density ρ abruptly drops to zero beyond some unknown surface

Section 2 introduces horizontal variations as the transport of scalar, vector or tensor fields by some ideal continuous media in motion over the given region of R^n Such an abstract continuum, called a *carrier*, is considered only to take profit of the usual language of kinematics; it is entirely distinct from the proper fluid under study and the real variable τ ordering its chain of placements should not be mistaken with the time t of dynamics The reader familiar with differential geometry will observe a close connection between our calculations and the operations called the *Lie derivations* associated with given vector fields (see [8, 9]) But it appears that the formalism we use, with the advantage of referring only to the common kinematical background of fluid mechanists, is also better adapted to subsequent practice For instance, in *numerical computation*, τ might order an approximation process and, in such applications, the carrier velocity would usually have to vary with τ This is similar to some methods currently used in solving problems of optimal design for unknown domains, possibly arising from quite other subjects than mechanics; these methods amount to make a finite elements mesh be dragged along by some imagined fluid[10, 11]

The carrier velocity field, denoted by φ, plays a role analogous to that of the "test functions" in Schwartz's theory of distribution This vector field will always be supposed smooth, with compact support in some open subset of R^n If one is looking for necessary conditions, in order that some fields u, σ, κ satisfy the equations of hydrodynamics, the strongest assertions will be formulated by taking φ in $\mathscr{C}^1$; in what regards sufficient conditions, on the contrary, formulating them by means of $\mathscr{C}^\infty$ test fields constitutes a stronger statement

In contrast with φ, the fluid velocity u, the mass volume σ and the fluid "quality" κ may be discontinuous; hence the partial derivatives involved will be understood in the sense of the theory of distributions To this end, some of the expressions appearing in the equations have to be interpreted as the densities of vector or tensor measures, relatively to the Lebesgue measure

The reader unwilling to enter into the technical matter of vector and tensor measures may refer to [12], an introductory lecture where only smooth flows are considered

The two fundamental modes of transport of a vector of the Euclidean space $\mathscr{X}$ by a carrier, called *convection* and *transvection*, are investigated in Section 3 From φ being $\mathscr{C}^1$ and from the classical regularity properties of the integrals of $d\xi/d\tau = \varphi(\xi(\tau))$, it results that the carrier may be endowed with the structure of a $\mathscr{C}^1$ differential manifold, denoted by Λ, in such a way that, for every τ, the placement mapping π^τ is a $\mathscr{C}^1$ diffeomorphism of Λ into $\mathscr{X}$ The two modes of

transport respectively correspond to the images in $\mathscr{X}$ of vectors or covectors of this manifold In particular, the gradient of a scalar function which, in a classical sense, is convected by the carrier, constitutes a transvected vector field

Scalar or vector (resp covector) Radon measures on Λ are introduced in Section 4 by the classical duality procedure: these are linear functionals on the respective spaces of the continuous scalar or covector (resp vector) fields, with compact support in Λ Under the placement mapping π^τ, such fields are transformed into τ-dependent continuous fields in $\mathscr{X}$, with compact support; by transposition are defined the images of scalar, vector or covector measures; this yields in particular the concepts of a convected τ-dependent scalar measure or of a convected, resp transvected, τ-dependent vector measure in $\mathscr{X}$ Using the Radon–Nikodym theorem, one may observe that a τ-dependent vector measure in $\mathscr{X}$ is convected, resp. transvected, by the carrier Λ if and only if it can be represented (nonuniquely) as the product of a convected nonnegative scalar measure by a convected, resp transvected, vector field The example of the vector measure associated with a convected curve is developed; this also yields a reformulation of the Helmholtz theorem on vorticity in classical fluid dynamics

Divergence-free vector measures in $\mathscr{X}$ are of primary importance; it is shown in Section 5 that, for a convected vector measure, this is a τ-invariant property

Section 6 is devoted to second order tensor fields Classically, on the $\mathscr{C}^1$ manifold Λ, there are defined four sorts of such fields, according to the ways of combining covariance and contravariance In contrast, the Euclidean structure of $\mathscr{X}$ is conventionally used to identify vectors and covectors, yielding a single sort of second order tensor field If one starts with a τ-independent second order tensor field in Λ, one obtains as its image under the placement mapping π^τ some τ-dependent tensor field in $\mathscr{X}$ which will be said doubly convected, doubly transvected or transported in some mixed way by the carrier Λ, according to the sort of tensor considered in Λ Symmetrically, the constant second order tensor field g of the Euclidean metric of $\mathscr{X}$ (or "unit" tensor field) is the image under π^τ of some τ-dependent doubly covariant tensor field γ^τ in Λ; this is the second order tensor field associated with the metric induced on Λ by its imbedding in $\mathscr{X}$ Calculating the τ-derivative of γ^τ, then taking the image of this tensor in $\mathscr{X}$ leads to the strain rate formula of the classical kinematics of continua

Then comes in Section 7, the concept of a tensor measure in $\mathscr{X}$ or in Λ; in this definition again, the duality procedure is applied, generating by transposition the image under π^τ of, say, a doubly contravariant tensor measure in Λ This is a τ-dependent tensor measure in the Euclidean space $\mathscr{X}$, said doubly convected by the carrier Λ The trace of such a Euclidean tensor measure $\mathrm{d}T$ is a scalar measure, whose total will be called the *trace integral* of $\mathrm{d}T$. Proposition 7.1 states what may be viewed as the main formula of the calculus of horizontal variations, a corollary of which is the following variational characterization of tensor measures with zero divergence in the sense of distribution theory: Let $\mathrm{d}T$ be a *bounded symmetric tensor measure* in the Euclidean space $\mathscr{X}$ and let $\mathrm{d}T^\tau$ denote the τ-dependent tensor measure, reducing to $\mathrm{d}T$ for $\tau = 0$, doubly convected by the carrier with velocity field φ (a $\mathscr{C}^1$ vector field, with compact support) *The trace integral of* $\mathrm{d}T^\tau$ *has, for every* φ, *a zero* τ*-derivative at* $\tau = 0$ *if and only if the divergence of* $\mathrm{d}T$ *is zero*

There is a formal similarity between this result and some developments of Souriau[13] in General Relativity; but, instead of what we call horizontal variations, this author considers variations of the non-Euclidean spacetime metric In contrast, the present paper definitely places itself in the framework of classical dynamics, with its immutable Euclidean metric; our technique of horizontal variations may be viewed as a natural extension of the conventional *virtual work* procedure

Concerning the expression of the second τ-derivative of the trace integral and the condition for it to be nonnegative see [13]; this tends to explain, in that connection, the preeminence of measures over other sorts of distributions

As we are to deal with non-smooth flows, the equations of fluid dynamics are first written under a 4-dimensional divergence form in the sense of distributions theory; this is the object of Section 8, where it is recalled that such a writing includes the case of slipstream surfaces, free surfaces and also shock waves However, realistic shock waves do not fall into the scope of this paper, as the assumptions made preclude any entropy jump

The expected variational property is established in Section 9 The functional under con-

sideration is nothing but the Eulerian 4-dimensional transcript of the action integral (1 7); hence the statement may be considered as an adaptation of Hamilton's principle to Euler's variables The relationship with Hamilton's principle is more precisely investigated in [16].

The closest analogue in literature to horizontal variations may be seen in what Drobot and Rybarski[15] call *hydromechanical variations* By this these authors mean some infinitesimal alterations of a $\mathscr{C}^1$ vector field p, which are constructed from infinitesimal shift fields $x \to \delta x$ similar to our φ, with concomitant infinitesimal variations of $p(x)$ The essential difference from the concepts introduced in the present paper is that, if the differential shift of Drobot and Rybarski is integrated so as to realize a finite evolution of what we call a carrier, the resulting alteration of p is not holonomic, i e the carrier may eventually resume its initial placement without restoring the initial vector field; hence the variational statement in [15] is only differential In contrast, our Propositions 9 5 and 10 3 make the considered functionals appear as real functions defined on some manifolds; the solutions of hydrodynamical equation are the critical points of these functions From the computational standpoints this may suggest some stepwise processes or "walks" in the manifold, tending to approach the critical points

The final Section 10 is devoted to the variational characterization of the stationary solutions of the equations of inviscid fluids; a variational property of the horizontal sort is shown to hold in that case also, not directly connected with Hamilton's principle

2 THE CALCULUS OF HORIZONTAL VARIATIONS

Let us consider, as an introductory example, a standard variational problem By Ω is denoted an open subset of R^n; for every $y \in \mathscr{C}^1(\mathrm{cl}\,\Omega, R)$ (i e y is a continuously differentiable function defined on the closure of Ω, with real values) satisfying some prescribed boundary conditions and, for unbounded Ω, some conditions at infinity, there is defined the functional

$$I(y) = \int_\Omega L(x, y(x), \operatorname{grad} y(x))\, dx; \tag{2 1}$$

here x stands for $(x_1, x_2, \ldots, x_n)$ and dx for the n-dimensional Lebesgue measure; L denotes a given element of $\mathscr{C}^2(\mathrm{cl}\,\Omega \times R \times R^n, R)$

The conventional calculus of variations derives necessary conditions for y to make I an extremum, by studying the real function $\tau \to I(y + \tau\delta y)$; the variable τ ranges over a neighbourhood of zero in R and δy denotes an arbitrary continuously differentiable real function whose support relative to Ω is compact. In what regards the surface $G(0) \subset R^{n+1}$ which constitutes the graph of y, this amounts to make it compete with a family of surfaces $G(\tau)$; every point $(x, y(x) + \tau\delta y(x))$ of $G(\tau)$ results from the corresponding point $(x, y(x))$ of $G(0)$ by the "vertical" displacement $\tau\,\delta y(x)$ For this reason we shall say that the classical algorithm consists in *a calculus of vertical variations*

A more general way of inserting $G(0)$ into a chain of nearby surfaces would be to define, on some neighbourhood of $G(0)$ in R^{n+1}, a vector field Φ, and to call $G(\tau)$ the image of $G(0)$ under the geometric transform $\exp \tau\Phi$ generated by this vector field Here the mechanist will rather use the language of the kinematics of continua: if the real variable τ is identified with the time, the mapping $\exp \tau\Phi$ represents the displacement from the instant 0 to the instant τ for every particle of a $(n + 1)$-dimensional fluid in stationary motion, with Φ as velocity field. We shall call a *carrier* such a fluid, imagined only to take advantage of familiar kinematical formulas The classical calculus of variations amounts to choose the vector field Φ under the special form $\Phi(x, y) = (0, \delta y(x))$, namely "vertical" in $R^n \times R$, and constant relatively to the y coordinate

Our calculus of *horizontal* variations consists in assuming for Φ the special form $\Phi(x, y) = (\varphi(x), 0)$ where φ denotes a n-dimensional $\mathscr{C}^1$ vector field. The $(n + 1)$-dimensional carrier defined thereby leaves each hyperplane $y = \text{const}$ invariant; its action inside such a hyperplane consists in the flow of the n-dimensional carrier admitting φ as velocity field The surface

$$G(\tau) = (\exp \tau\Phi)G(0)$$

in $R^n \times R$ is the graph of the function y^τ defined on the subset $(\exp \tau\varphi)(\mathrm{cl}\,\Omega)$ of R^n by the

following composition of mappings

$$y^\tau = y \circ \exp(-\tau\varphi) \tag{2.2}$$

For a given value of τ, this function may be seen as a *rearrangement* of the function $y^\circ = y$

The carrier in R^n with φ as velocity field will constantly be considered in the sequel As the definition (2.2) of y^τ is equivalently expressed by the implication

$$x^\tau = (\exp \tau\varphi)x \Rightarrow y^\tau(x) = y(x), \tag{2.3}$$

the interpretation of τ as the time makes y^τ appear as a real function *convected* by the carrier, i.e. to each particle of this carrier corresponds a value of y^τ independent of τ.

As we restrict ourselves to a class of functions y verifying some prescribed boundary conditions, the $\mathscr{C}^1$ vector field φ will in the sequel be chosen with *compact support* in the open set Ω; then $\exp \tau\varphi$ constitutes a one-parameter group of self-mappings of $c1\Omega$, leaving invariant some neighbourhood of the boundary, and some neighbourhood of infinity if Ω unbounded For brevity, the corresponding carrier will be said a *compact carrier* in Ω.

A necessary condition for y, satisfying the prescribed boundary conditions, to make I an extremum is that the real function $\tau \to I(y^\tau)$ have a zero derivative at $\tau = 0$ for every φ chosen in the above manner. The standard way of calculating such a derivative would consist in transforming the integral $I(y^\tau)$ by the change of variable (2.3), after which τ-derivation is commuted with x-integration. The kinematical formalism makes it easier; similarly to (1.8), let us define, for every differentiable function f of τ and x, the symbol

$$\delta_\varphi f = \frac{\partial f}{\partial \tau} + \varphi_i \frac{\partial f}{\partial x_i} \tag{2.4}$$

This is the τ-derivative of the value that f associates with a particle of the carrier; it is zero in particular when f is convected

Let us denote by g_i^τ the components of grad y^τ relative to our orthonormal frame; if y^τ is convected by the carrier, a reasoning based on the chain rule allows one to calculate the τ-derivative, at $\tau = 0$, of the real function $\tau \to g_i^\tau(x^\tau)$, namely

$$\delta_\varphi g_i = -g_j \frac{\partial \varphi_j}{\partial x_i}. \tag{2.5}$$

See [16]; actually a more efficient reasoning will be developed in Section 3. Then

$$\delta_\varphi L(x, y^\tau, \operatorname{grad} y^\tau) = \frac{\partial L}{\partial x_i} \delta_\varphi x_i + \frac{\partial L}{\partial g_i} \delta_\varphi g_i = \frac{\partial L}{\partial x_i} \varphi_i - \frac{\partial L}{\partial g_j} g_j \frac{\partial \varphi_i}{\partial x_i}$$

Finally, applying to the carrier the classical dilatation formula, one has

$$\frac{\delta}{\delta\tau} \int_\Omega L(x, y^\tau, \operatorname{grad} y^\tau) \mathrm{d}x = \int_\Omega (\delta_\varphi L + L \operatorname{div} \varphi) \mathrm{d}x$$

By making $\tau = 0$, thus $y^\tau = y$, we obtain the desired necessary condition for y to extremize I

$$\int_\Omega \left(\frac{\partial L}{\partial x_i} \varphi_i - \frac{\partial L}{\partial g_j} g_j \frac{\partial \varphi_i}{\partial x_i} + L \frac{\partial \varphi_i}{\partial x_i} \right) \mathrm{d}x = 0. \tag{2.6}$$

This is to hold for every continuously differentiable vector field φ, with compact support in Ω As usual, one performs an integration by part and finally concludes that, everywhere in Ω,

$$\frac{\partial y}{\partial x_i} \left[\frac{\partial}{\partial x_j} \left(\frac{\partial L}{\partial g_j} \right) - \frac{\partial L}{\partial y} \right] = 0 \tag{2.7}$$

This was not unexpected: (2.7) means that g satisfies either the classical Euler condition for extremality or the condition grad $y = 0$. In fact in any open set where the latter holds, y is a constant, thus unaltered by the considered rearrangements.

One reaches a less trivial conclusion if another variational problem is considered: that of *extremizing* I(y) *over a class of functions resulting from each other by convection along isochoric carriers*. Then (2.6) is to hold only for vector fields with compact support and zero divergence. By a classical reasoning this is equivalent to the existence of a real function q on Ω such that the left member in (2.7) equals $\partial q/\partial x_i$.

Example

Let us take as I the Dirichlet integral

$$I(y) = \frac{1}{2}\int_{\Omega} \text{grad}^2 y \, dx \tag{2.8}$$

The above condition for the extremality of I *relative to isochoric carriers* reduces now to

$$\exists q: \quad \Delta y \text{ grad } y = \text{grad } q$$

where Δ denotes the Laplace operator. In the special case $n = 2$, one recognizes here the dynamical equation satisfied by the *stream function of an incompressible, homogeneous inviscid fluid in stationary motion*.

The use of a stream function in hydrodynamics is limited to 2-dimensional problems, plane or axisymmetric (a result similar to the above holds for this latter case). But it must be observed that the stream function y appears in (2.8) by its gradient only, a vector field whose law of transport by the carrier is defined in (2.5). The fluid velocity u, at each point, is the vector deduced from grad y by rotation through $\pi/2$; in view of (2.5) and keeping into account the condition div $\varphi = 0$, one obtains the corresponding law of transport by the carrier

$$\delta_{\varphi} u_k = u_l \frac{\partial \varphi_k}{\partial x_l} \tag{2.9}$$

The functional (2.8) is now replaced by

$$\mathcal{E}(u) = \frac{1}{2}\int_{\Omega} u^2 dx; \tag{2.10}$$

the vanishing of the τ-derivative of this functional when u is transported according to (2.9) makes sense for every number of dimensions. One finds that it yields the dynamical vector equation for the stationary flow of an incompressible homogeneous inviscid fluid[16].

The condition div u for the fluid flow to be isochoric is introduced *a priori* as a side constraint. An essential feature is that the *transport* (2.9), if the carrier is isochoric, *preserves it* (see Section 5). Therefore, the considered variational device effectively characterizes the solutions of the dynamical equation as critical points of the real functional (2.10) on a certain infinite dimensional manifold; unlike some would-be variational statements, it does not consist of a merely local play of δ symbols. Thus, some numerical procedures altering step by step an initial solenoidal vector field in order to construct by successive approximations a solution of hydrodynamics may be conceived. In that respect, the requirement of using only isochoric carriers may be awkward; removing this drawback will be subject of further papers, where inhomogeneous fluids with discontinuous density and free surfaces will also be considered.

About terminology, let us observe that the word "horizontal" is currently used in literature with other meanings; for instance Edelen[9] call *horizontal* some vector fields in an event space which, in the terminology of Section 9, we should refer to as *isochronous*.

3. CONVECTED AND TRANSVECTED VECTORS

Let us recall that the geometric object to which reference is made, when describing a motion in classical kinematics, is not properly a Cartesian coordinate frame but an equivalence class of

such frames, any two of them related in a time-independent way This amounts to say that the reference object actually consists of a tri-dimensional Euclidean point-space, in which the said frames constitute various Cartesian coordinate systems Such a reference space will in the sequel be denoted by $\mathscr{X}$ and its number n of dimensions will possibly differ from 3; in fact, the cases $n=2$ and $n=4$ are also of use in the hydrodynamical applications in view The coordinates of an element x of $\mathscr{X}$, relative to some chosen *orthonormal* frame are denoted by x_i, $i=1,2,\ \ ,n$ The linear space associated with $\mathscr{X}$ is denoted by X; to the above frame in $\mathscr{X}$ corresponds an orthonormal base in X and the components of an element v of X are denoted by v_i If f is a function defined on some neighbourhood of a point of $\mathscr{X}$, the partial derivatives of f relative to the respective coordinates x_i are denoted by $f_{,i}$; they are the components of the vector grad $f \in X$, the *Euclidean gradient* of f at the considered point

Let Ω be an open subset of $\mathscr{X}$ and let $\varphi \in \mathscr{C}^1(\Omega, X)$ be the velocity field of a *carrier* in the sense of Section 2; one may suppose that φ depends also on the formal time τ, i.e. the carrier flow need not be stationary, in which case φ will be assumed $\mathscr{C}^1$ in τ and x jointly The concept of a moving vector, associated with a given particle of such a continuous medium as this carrier and *convected* by it, is classical Roughly speaking, up to a τ-independent infinitely large magnification ratio, such a moving vector v is determined by a pair of infinitely close particles of the carrier, hence the following rate of change

$$\frac{\delta v_i}{\delta \tau} = \varphi_{i\,k} v_k \tag{3 1}$$

This is merely the *equation of variations* associated with the system of differential eqns (3 2) below

Let us place things in a more rigorous setting Writing the general integral of the system of n differential equations

$$\frac{\delta x_i}{\delta \tau} = \varphi_i(\tau, x_1, \ \ , x_n) \tag{3 2}$$

under the form

$$x_i = \pi_i(\tau, \lambda^1, \ \ , \lambda^n) \tag{3 3}$$

with $\lambda^1, \ \ , \lambda^n$ as integration constants, amounts to labelling every particle of the carrier by some value of $(\lambda^1, \ \ , \lambda^n) \in R^n$ and interpreting (3 3) as the equations of the placement mapping From the assumption $\varphi \in \mathscr{C}^1$ and from the classical facts concerning the dependence of the integrals of a system of differential equations upon initial conditions, it ensues that the values of $\lambda^1, \ \ , \lambda^n$ may be assigned to the various integrals of (3 2) in such a way that (3 3) is, at least locally, a $\mathscr{C}^1$-diffeomorphism We shall finally restate this as follows:

The carrier may be endowed with the structure of a n-dimensional $\mathscr{C}^1$-differential manifold in which (λ^i) constitute some admissible coordinate system

Let us denote by Λ this manifold, with λ as general element; then, for every τ in an open real interval containing zero, (3 3) defines the placement mapping

$$\lambda \to x = \pi(\tau, \lambda) \tag{3 4}$$

as a $\mathscr{C}^1$-diffeomorphism of Λ into $\mathscr{X}$ With every $\lambda \in \Lambda$ is associated the *tangent space* Λ'_λ to Λ at this point, a n-dimensional linear space

In what concerns the Euclidean space $\mathscr{X}$, its tangent space is X, the same linear space at every point; thus the *tangent mapping* to $\pi(\tau, \ )$ at the point λ, denoted by $\pi'(\tau, \lambda)$, is a linear isomorphism of Λ'_λ onto X Saying that the moving element $\tau \to v(\tau)$ of X is a vector associated with the particle λ of the carrier Λ and *convected* by it, means in this context that $v(\tau)$ equals the image under $\pi'(\tau, \lambda)$ of some τ-independent element of Λ'_λ

To every $\lambda \in \Lambda$ also corresponds the *cotangent space* Λ'^*_λ, i.e. the dual of Λ'_λ. Since X is a Euclidean linear space, we agree as usual to identify it with its dual. Thus the contragredient mapping $\pi'^*_*(\tau, \lambda)$ of $\pi'(\tau, \lambda)$ (i.e. its inverse transpose) constitutes an isomorphism of Λ'^*_λ onto X. We shall say that a moving element $\tau \to w(\tau)$ of X is a *transvected vector* associated with the particle λ of the carrier Λ, if it equals the image under $\pi'^*_*(\tau, \lambda)$ of a τ-independent element of Λ'^*_λ. Equivalently, for every convected v associated with the particle λ, the Euclidean scalar product $v(\tau) \cdot w(\tau)$ is a τ-independent real number. By comparison with (3.1), this easily yields the following rate of change

$$\frac{\delta w_i}{\delta \tau} = - w_k \varphi_{k\,i} \tag{3.4}$$

It must be kept in mind that, here as well as in (3.1), the partial derivatives $\varphi_{k\,i}$ of the components of φ are to be evaluated at the point $x = \pi(\tau, \lambda)$ of $\mathscr{X}$.

A real function y of τ and x clearly is convected by the carrier, in the sense defined in Section 2, if it equals the image under π of a τ-independent real function η defined on the manifold Λ, i.e. for every fixed τ, y equals the composite mapping $\eta \circ \pi^{-1}$. Such image functions are $\mathscr{C}^1$ in $\mathscr{T}$ if and only if η is $\mathscr{C}^1$ in Λ; in that case the $\mathscr{C}^1$-differential geometry of Λ associates with η, at every $\lambda \in \Lambda$, the element $\nabla\eta(\lambda)$ of Λ'^*_λ, the gradient (or differential) of the real function η at the point λ. In other words, the element $\nabla\eta(\lambda)$ consists in the linear mapping from Λ'_λ into R, tangent at the point λ to the $\mathscr{C}^1$ mapping $\eta: \Lambda \to R$. Using the chain rule for tangent mappings and the definition of the inverse transpose, one obtains that the Euclidean gradient of y at the point $x = \pi(\tau, \lambda)$ equals the image of $\nabla\eta(\lambda)$ under $\pi'^*_*(\tau, \lambda)$. *Thus, $\tau \to \operatorname{grad} y(\tau, \pi(\tau, \lambda))$ constitutes a transvected moving vector.* In view of (3.4) this makes a proof of (2.5).

A *vector field* v defined, for every $\tau \in I$, on some subset of $\mathscr{X}$ will be said convected, resp. transvected, by the carrier if, for every $\lambda \in \Lambda$ such that $\pi(\tau, \lambda)$ belongs to the domain of v, the moving vector $v(\tau, \pi(\tau, \lambda))$ is convected, resp. transvected. Here again, such moving objects of the geometry of $\mathscr{X}$ may be considered as the images of τ-independent objects of the geometry of Λ. In fact, when dealing with a $\mathscr{C}^1$-manifold such as Λ, instead of a Euclidean space, the concept of a vector field splits into two. There may be considered a *vector field* or a *covector field*; the former is an assignment associating with every λ in a certain subset of Λ, an element of the corresponding tangent space Λ_λ, the latter an element of the cotangent space Λ^*_λ. Visibly, a τ-dependent vector field in $\mathscr{X}$ is convected by Λ, resp. transvected, if it is the image of a τ-independent vector field, resp. covector field, in Λ.

As soon as an admissible coordinate system (λ^i) is chosen in the $\mathscr{C}^1$-differential manifold Λ, some bases are automatically induced in the respective linear spaces Λ'_λ and Λ'^*_λ; then the above fields may be described by their components, which are real functions of $(\lambda^1, \lambda^2, \ldots, \lambda^n)$. Admissible changes of coordinates in Λ are, by definition, continuously differentiable transforms; this entails that the continuity of a vector, resp. covector, field, defined as the continuity of its component functions, actually constitutes a property independent of the peculiar choice of admissible coordinates in Λ. The same is true for the concept of the *support set* of a field, i.e. the complement of the greatest open subset throughout which this field vanishes.

The following will be of use in next Section. We shall denote by $\mathscr{H}(\Gamma, \Lambda')$, resp. $\mathscr{H}(\Gamma,$ the set of the continuous vector, resp. covector, fields defined on the subset Γ of Λ, whose support relative to Γ is compact. Under the natural definitions of addition and of the multiplication by scalar constants, these two sets are linear spaces. If K is a compact subset of Γ, we denote by $\mathscr{H}(\Gamma, K; \Lambda')$, resp. $\mathscr{H}(\Gamma, K; \Lambda'^*)$, the linear subspaces of the above consisting of the fields whose support is contained in K. The use of some admissible coordinate system (λ^i) in Λ permits to define on each of the linear spaces $\mathscr{H}(\Gamma, K; \Lambda')$ and $\mathscr{H}(\Gamma, K; \Lambda'^*)$ the *supremum norms*, which make of them some Banach spaces. The definition of these norms is affected by the $\mathscr{C}^1$ changes of coordinates; such changes actually replace them by some topologically equivalent norms. This means that, in the setting of the $\mathscr{C}^1$-differential structure of Λ, the linear spaces $\mathscr{H}(\Gamma, K; \Lambda')$ and $\mathscr{H}(\Gamma, K; \Lambda'^*)$ are respectively endowed with some definite "Banachizable" topologies, independent of the choice of any coordinate system.

One is naturally induced to define for other geometric objects some laws of transport by a carrier

First may be considered a τ-dependent scalar (unsigned) measure dm^τ, defined on some τ-dependent subset C^τ of $\mathscr{X}$. This measure will be said *convected* by the carrier Λ if it equals the image under π of some τ-independent measure $d\mu$ defined on a subset Γ of Λ

As a counter-example, such is not in general the case for the Lebesgue measure dx on C^τ; it equals the image under π of a τ-dependent measure $d\xi^\tau$ in Λ. Relatively to the Lebesgue measure $d\lambda$ associated with the (λ^i) coordinates in Λ, the measure $d\xi^\tau$ is known to admit a density function ξ'_λ equal to the Jacobian determinant of (3.3). A classical calculation yields the τ-derivative of this determinant, for a fixed λ; equivalently may be considered the function x'_λ of τ and x defined on C^τ as the image of ξ'_λ. Then the above τ-derivative may be viewed as the drag-derivative of $x_\lambda{}'$ along the carrier Λ; we shall denote it by $\delta_\varphi x'_\lambda$, in accordance with the notation used in the previous Sections, although, in the present case, x'_λ may fail to be a differentiable function of τ and x. With this notation, the classical result takes the form

$$\delta_\varphi x'_\lambda = x'_\lambda \operatorname{div} \varphi \tag{4.1}$$

Turning back to the measure dm^τ, let us consider the case where it admits a density function m'_x relatively to the Lebesgue measure of $\mathscr{X}$. From (4.1) one easily deduces that dm^τ is convected by the carrier if and only if

$$\delta_\varphi m'_x = - m'_x \operatorname{div} \varphi \tag{4.2}$$

holds almost everywhere. If the carrier is a material continuum with dm^τ as mass measure, one recognizes here the classical equation of mass conservation

Let us consider now a *vector measure* dv; this may be seen as a measure with values in X instead of R or, equivalently, as a collection of n scalar measures dv_i which constitute the *components* of dv relative to the base chosen in X. But the *duality* way of defining a vector measure serves our purpose much better. From this standpoint[17], a vector measure dv on a locally compact subset C of $\mathscr{X}$ is, by definition, a linear scalar functional on the linear space $\mathscr{K}(C, X)$ of the continuous vector fields with compact support in C, possessing the following continuity property: *the restriction of this functional to the (Banach) subspace $\mathscr{K}(C, K; X)$, consisting of the elements of $\mathscr{K}(C, X)$ whose support in contained in the compact subset K of C, is continuous for every K.* The value given to this functional by a vector field $w \in \mathscr{K}(C, X)$ will indifferently be written as $\ll w \ dv \gg$, or $\int w \ dv$, or $\ll w_i, dv_i \gg$, or $\int w_i dv_i$, with some natural definition of the n scalar measures dv_i, the *components* of dv relative to the chosen orthonormal base in X.

The duality definition of vector measures applies as well when it is considered, instead of an Euclidean space, a n-dimensional $\mathscr{C}^1$-differential manifold such as the carrier Λ. Let Γ be a locally compact subset of Λ; a *vector measure* on Γ is, by definition, a linear scalar functional on the linear space $\mathscr{K}(\Gamma, \Lambda'^*)$ of the continuous *covector* fields on Γ with compact support, possessing this continuity property: for every compact subset K of Γ, the restriction of the functional to the subspace $\mathscr{K}(\Gamma, K; \Lambda'^*)$ is continuous. Symmetrically, a *covector measure* consists in a linear scalar functional on $\mathscr{K}(\Gamma, \Lambda')$ with the similar continuity property

If $d\psi$ is, for instance, a vector measure and θ an element of $\mathscr{K}(\Gamma, \Lambda'^*)$, the value given by θ to the linear functional $d\psi$ will be denoted $\ll \theta, d\psi \gg$ or $\int < \theta, d\psi >$; here the symbol $< \ . \ >$ refers to the bilinear duality form of the tangent and cotangent spaces Λ'_λ, Λ'^*_λ at any point λ of Λ. As soon as an admissible coordinate system is chosen in Λ, the covector field θ may be described by its component functions θ_i; on the other hand there may be defined n scalar measures $d\psi^i$, the components of $d\psi$, such that the above is also expressed under the form $\ll \theta_i, d\psi^i \gg$ or $\int < \theta_i, d\psi^i >$

For every τ in the considered interval, the placement mapping $\pi(\tau, \ )$ operates on vector or covector fields in the way defined in Section 3. This visibly establishes a linear isomorphism between $\mathscr{K}(\Gamma, \Lambda')$, resp. $\mathscr{K}(\Gamma, \Lambda'^*)$ and the similar space $\mathscr{K}(C^\tau, X)$, with $C^\tau = \pi(\tau, \Gamma)$; recall that X, as a Euclidean linear space, has been identified with its dual. For every compact subset K of

Γ, the subspaces $\mathscr{K}(\Gamma, K; \Lambda'^*)$, resp $\mathscr{K}(\Gamma, K; \Lambda')$, and $\mathscr{K}(C^\tau, \pi(\tau, K); X)$ correspond to each other in the above isomorphism; moreover this correspondence is bicontinuous. The definition of the image under $\pi(\tau,)$ of a vector, resp covector, measure on the subset Γ of Λ follows automatically: this image constitutes a τ-dependent vector measure on the subset C^τ of $\mathscr{X}$.

The τ-dependent vector measures in $\mathscr{X}$ resulting in the above way from τ-independent vector, resp. covector, measures in Λ are said convected, resp transvected, by the carrier Λ

Observe on the other hand that, in view of some n-dimensional extension of the Radon–Nikodym theorem [17], every vector measure $d\psi$ on the subset Γ of Λ may be represented as follows: there exists (nonuniquely) a positive scalar measure $d\mu$ on Γ and a bounded $d\mu$-measurable vector field ψ'_μ such that, for every $\theta \in \mathscr{K}(\Gamma, \Lambda'^*)$, one has

$$\int < \theta, d\psi > = \int < \theta, \psi'_\mu > d\mu \tag{4.3}$$

One writes $d\psi = \psi'_\mu d\mu$ and the vector field ψ'_μ (defined up to an arbitrary alteration in a $d\mu$-negligible subset of Γ) is called *the density of the vector measure* $d\psi$ *relatively to the scalar measure* $d\mu$. Similarly, every covector measure in Λ may be represented by a positive scalar measure $d\mu$, together with a bounded $d\mu$-measurable covector field as density.

Consequently, a τ-dependent vector measure dv on the subset $\pi(\tau, \Gamma)$ of $\mathscr{X}$ is convected, resp. transvected, by Λ if and only if there exists (nonuniquely) some convected positive scalar measure dm^τ and some convected, resp. transvected, bounded dm^τ-measurable vector field v'_m such that $dv = v'_m dm^\tau$ for every τ in the considered interval.

The representation of a vector or covector measure through a positive scalar measure and a density field may be used to make (4.3) meaningful for more general fields θ than the continuous ones with compact support. In fact (4.3) makes sense as soon as the scalar function $< \theta, \psi'_\mu >$ is $d\mu$-integrable. Such fields θ will be said integrable relatively to $d\psi$. As the components of ψ'_μ belongs to $L^\infty(d\mu)$, this amounts to saying that the components of θ belong to $L^1(d\mu)$. The vector (resp. covector) measure $d\psi$ will be said *bounded* if all the bounded continuous fields are integrable relatively to it; equivalently the positive scalar measure $d\mu$ is bounded.

Let us observe, on the other hand, that when a vector measure, such as $dv = v'_m dm$ is considered in R^n, this makes sense to write the integral relatively to dv of a scalar function f, a symbol whose value is an element of R^n. Taking in particular as f the indicator of a dm-measurable subset of R^n, one defines the dv-measure of this set, with value in R^n thereby connecting the present theory with the conventional aspect of measures, namely σ-additive functions of sets. But it should be stressed that the concepts of the integral of a scalar function or of the measure of a set *do not make sense anymore when it is considered a vector measure in the manifold* Λ. This is connected with the observation that, in such a manifold, there is no possibility of addition for the elements of the tangent spaces at different points.

Example 1

Let us consider an *absolutely continuous curve* in Λ, i.e. a mapping γ of some compact real interval $[r_1, r_2]$ into Λ, defined by n coordinate functions $r \to \lambda^i = \gamma^i(r)$ which are absolutely continuous on $[r_1, r_2]$. There equivalently exist n Lebesgue-integrable functions $r \to \gamma'^i_r(r)$ such that

$$\gamma^i(r) = \gamma^i(r_1) + \int_{r_1}^{r} \gamma'^i_r(\rho) d\rho.$$

It may be checked that these properties are preserved under any admissible change of coordinates in Λ and that the $\gamma'^i_r(r)$ constitute the components of an element $\gamma'_\lambda(r)$ of $\Lambda'_{\lambda(r)}$, independent of the peculiar choice of a coordinate system.

Let us define the vector measure $d\gamma$ on Λ as the functional assigning to every $\theta \in \mathscr{K}(\Lambda, \Lambda'^*)$ the real quantity

$$\ll \theta, d\gamma \gg = \int_{r_1}^{r_2} < \theta(\gamma(r)), \gamma'_r(r) > dr; \tag{4.4}$$

in accordance with our previous notations, this quantity may equivalently be written as

$$\int_{r_1}^{r_2} \theta_i(\gamma(r))\,\gamma_r^i(r)\,dr = \int_{r_1}^{r_2} \theta_i(\gamma(r))\,d\gamma^i(r) = \int_{r_1}^{r_2} <\theta(\gamma(r)),\, d\gamma(r)>$$

The image in $\mathscr{X}$ of the vector measure $d\gamma$ under the placement mapping $\pi(\tau, \ )$ equals the vector measure dc^τ associated in the same way with the absolutely continuous curve $c^\tau : r \to \pi(\tau, \gamma(r))$ For every continuous vector field v on $\mathscr{X}$, the value of $\ll v, dc^\tau \gg$ is nothing but the *circulation* or *work* of v along c^τ For τ ranging over some real interval, c^τ constitutes a moving curve dragged along by the carrier Λ and dc^τ gives an example of a vector measure convected by Λ Equivalently, for every continuous vector field v which is transvected by Λ, the said circulation is a constant with respect to τ; as a trivial instance, one may take $v = \text{grad } y$, where y is a convected scalar function

Remark

It is for brevity that we have restricted ourselves to absolutely continuous curves; more generally, rectifiable curves may be considered, i e mappings of $[r_1, r_2]$ into $\mathscr{X}$ or into Λ which are continuous with bounded variation

Example 2

Let $n = 3$ and the carrier consist of an inviscid fluid performing a barotropic motion with velocity field u, supposed $\mathscr{C}^1$; here τ will be the time t of dynamics Classically, a local version of the Kelvin–Helmholtz theorem may be formulated as follows: *if ρ denotes the density of the fluid, a function of t and x, and $\omega = \text{curl } u$, the vector field $\rho^{-1}\omega$ is convected* Actually, the bearing of ρ to the situation is only that the mass measure $dm = \rho dx$ is convected The property may equivalently be formulated by saying that *the vector measure $\rho^{-1}\omega\rho dx = \omega dx$ is convected* This vector measure is simply the curl, in the sense of distribution theory, of the vector measure $u dx$; eliminating ρ from the formulation clarifies the statement

5 DIVERGENCE-FREE VECTOR MEASURE

Definition

A vector measure dv defined in some open subset Ω of $\mathscr{X}$ will be said divergence-free (or solenoïdal) in Ω if $\int \text{grad } f \ dv = 0$ for every $\mathscr{C}^1$ (equivalently $\mathscr{C}^\infty$) scalar function f whose support relative to Ω is compact.

In other words dv possesses a zero divergence relatively to Ω in the sense of distribution theory

Suppose dv convected by the carrier Λ; if the scalar function f in the above definition is viewed as convected by the carrier, its gradient is transvected, hence yields a τ-independent integral with respect to dv Therefore:

Proposition 5 *If a convected vector measure is divergence-free for some value of τ, so it is for every τ*

Remark

The same sort of reasoning proves that, if the convected vector measure dv possesses, in the sense of distribution theory, a divergence which happens to be a measure, this measure is convected Incidentally, one could agree to restrict oneself to carriers with $\mathscr{C}^\infty$ velocity fields; such a carrier may be endowed with the structure of a $\mathscr{C}^\infty$ differential manifold, independent of τ, involving the concept of *convected distributions* Then it comes that every convected vector distribution admits as divergence a convected scalar distribution; this will not be used in the present paper

Example

Let c denote, as in Section 4, Example 1, an absolutely continuous curve For every $\mathscr{C}^1$ scalar function f, one elementarily has

$$\int \text{grad } f \ dc = f(b) - f(a)$$

with $a = c(r_1)$ and $b = c(r_2)$, the ends of the curve. This means that the divergence of the vector measure dc, in the sense of Schwartz's distributions, equals $\delta_a - \delta_b$, the difference of the Dirac measures at the respective ends. If the curve c is convected by the carrier Λ, this divergence measure is clearly convected; it is zero if the curve is a cycle, i.e. $a = b$. The vector measure dc has also a zero divergence *relative to* Ω if the ends of the curve c in $\mathscr{X}$ do not lie in Ω.

6 TENSOR FIELDS

Vector or covector fields in the $\mathscr{C}^1$-differential manifold Λ are special cases of *tensor fields*: such a field, defined on a subset $\Gamma \subset \Lambda$, is a correspondence assigning to every $\lambda \in \Gamma$ an element of some tensor product space whose factors are copies of the linear spaces Λ'_λ or Λ'^*_λ. This may be, for instance, an element of $\Lambda'_\lambda \otimes \Lambda'_\lambda$ in which case the tensor field is said *doubly contravariant*; a *doubly covariant* tensor field admits as values some elements of $\Lambda'^*_\lambda \otimes \Lambda'^*_\lambda$; two sorts of *mixed* tensor fields of the second order are also to be considered, whose values are elements of $\Lambda'_\lambda \otimes \Lambda'^*_\lambda$ or $\Lambda'^*_\lambda \otimes \Lambda'_\lambda$, respectively.

For every τ in the considered interval, the placement mapping $\pi(\tau, \ )$ sends Λ into $\mathscr{X}$, while the linear isomorphisms $\pi'(\tau, \lambda)$ and $\pi'^*(\tau, \lambda)$ maps Λ'_λ and Λ'^*_λ onto X; this yields the definition of the images under $\pi(\tau, \ )$ of any of the above second order tensor fields; these images are second order τ-dependent tensor fields in $\mathscr{X}$. Since the Euclidean linear space X has been identified with its dual, one finds in $\mathscr{X}$ only one sort of second order tensor field instead of four. Such a τ-dependent tensor field in $\mathscr{X}$ is said *doubly convected* by the carrier if it equals the image, in the above sense, of some doubly contravariant τ-independent tensor field on a subset of Λ; it is said *doubly transvected* if it equals the image of some doubly covariant tensor field; two mixed modes of transport could also be considered.

To give a fundamental example of images of tensor fields, let us start with the constant tensor field g in $\mathscr{X}$, associated with the Euclidean metric. Relatively to some orthonormal frame, its components equal the Kronecker symbol δ_{ij} but, if arbitrary curvilinear coordinates (l^i) are used in $\mathscr{X}$, this tensor field is represented by doubly covariant components $g_{ij} = g_{ji}$ which are functions of $(l^1, l^2, \ , l^n)$. If a $\mathscr{C}^1$ curve is considered in $\mathscr{X}$, its length equals the integral $\int ds$, with

$$ds^2 = g_{ij} dl^i dl^j \tag{6.1}$$

Consider now the carrier Λ and any of its placements $\pi(\tau, \ )$; let us denote by $\gamma^\tau(\lambda)$ the element of $\Lambda'^*_\lambda \otimes \Lambda'^*_\lambda$ whose image equals the element g of $X \otimes X$. For every $\mathscr{C}^1$ curve Γ in Λ, the image $\pi(\tau, \Gamma)$ is a $\mathscr{C}^1$ curve in $\mathscr{X}$, whose Euclidean length could be calculated as above; but equivalently, the placement mapping may be interpreted as a change of variables, yielding the expression of this length as an integral $\int d\sigma$ along Γ, with

$$d\sigma^2 = \gamma^\tau_{ij} d\lambda^i d\lambda^j \tag{6.2}$$

This expresses the metric induced on the $\mathscr{C}^1$ differential manifold Λ by its embedding $\pi(\tau, \ )$ into the Euclidean space $\mathscr{X}$.

A standard problem in the kinematics of continua is to calculate the rate of change of the Euclidean length of the curve $\pi(\tau, \Gamma)$ when Γ is a τ-independent curve in Λ. In view of (6.2), this requires, for every point λ of the curve, the expression of the τ-derivatives $\delta\gamma^\tau_{ij}/\delta\tau$, i.e. the components of the τ-derivative of the element $\gamma^\tau(\lambda)$ of the linear space $\Lambda'^*_\lambda \otimes \Lambda'^*_\lambda$. To this end, one may consider an arbitrary pair of τ-independent elements α, β of Λ'_λ and the doubly contracted product

$$< \alpha, \gamma^\tau, \beta > = \alpha^i \gamma^\tau_{ij} \beta^j \tag{6.3}$$

Then

$$\frac{\delta}{\delta\tau} < \alpha, \gamma, \beta > = < \alpha, \frac{\delta}{\delta\tau} \gamma^\tau(\lambda), \beta > \tag{6.4}$$

The images a^τ and b^τ of α and β under the tangent mapping $\pi'(\tau, \lambda)$ are some moving vectors *convected* by the carrier In view of the definition of images, one has

$$< \alpha, \gamma^\tau, \beta > = < a^\tau, g, b^\tau > = a^\tau \cdot b^\tau \tag{6.5}$$

where the dot refers to the Euclidean scalar product in X The rate of change of a^τ and b^τ in X is given by (3.1), which involves the use of orthonormal Cartesian coordinates; hence

$$\frac{\delta}{\delta\tau}(a^\tau \cdot b^\tau) = \frac{\delta}{\delta\tau}(a_i^\tau b_i^\tau) = (\varphi_{ij} + \varphi_{ji}) a_i^\tau b_j^\tau \tag{6.6}$$

Classically, for every differentiable vector field such as φ, the tensor D whose components, relative to orthonormal Cartesian coordinates, are expressed by

$$D_{ij} = \tfrac{1}{2}(\varphi_{ij} + \varphi_{ji})$$

is denoted by def φ By reapproaching (6.3), (6.4), (6.5) and (6.6) one obtains the classical following result, which justifies the name of *deformation rate* given to the tensor D

Proposition 6 *The image of the element* $\delta\gamma^\tau(\lambda)/\delta\tau$ *of* $\Lambda_\lambda'^* \otimes \Lambda_\lambda'^*$ *equals the value of the tensor field* 2 def φ *at the point* $\pi(\tau, \lambda)$ *of* $\mathcal{X}$.

7 TENSOR MEASURES; THE MAIN FORMULA

Tensor measures in the $\mathscr{C}^1$-differential manifold Λ may be defined by the duality way, as was done in Section 4 for vector measures For brevity, we restrict ourselves to second order doubly contravariant tensor measures

Similarly to the notation used in Section 3, let us denote by $\mathscr{K}(\Gamma, \Lambda'^* \otimes \Lambda'^*)$ the linear space consisting of the continuous doubly covariant tensor fields defined on the (locally compact) subset Γ of Λ and whose support relative to Γ is compact If K is a compact subset of Γ, let $\mathscr{K}(\Gamma, K; \Lambda'^* \otimes \Lambda'^*)$ denote the linear subspace of the above consisting of the fields whose support is contained in K Using some admissible coordinate system in Λ, one defines on such a subspace the supremum norm, hence a topology which, similarly to what has been observed in the case of vector fields, does not depend on the peculiar choice of the coordinate system By definition, a doubly contravariant tensor measure $d\Theta$ on Γ is a scalar linear functional on the space $\mathscr{K}(\Gamma, \Lambda'^* \otimes \Lambda'^*)$ with the following continuity property: for every compact subset K of Γ, the restriction of this functional to $\mathscr{K}(\Gamma, K; \Lambda'^* \otimes \Lambda'^*)$ is continuous The value that the functional $d\Theta$ assigns to some element Φ of $\mathscr{K}(\Gamma, \Lambda'^* \otimes \Lambda'^*)$ will be denoted by $\ll \Phi, d\Theta \gg$ or $\int < \Phi; d\Theta >$ When some admissible coordinate system is chosen in Λ, one may define n^2 scalar measures $d\Theta^{ij}$, the *components* of the tensor measure, such that the above value equals

$$\ll \Phi_{ji}, d\Theta^{ij} \gg = \int < \Phi_{ji}, d\Theta^{ij} > \tag{7.1}$$

As in the case of vector measures, one has the representation property: there exist (non uniquely) on Γ a positive scalar measure $d\mu$ and a *bounded* $d\mu$-measurable doubly contravariant tensor field (defined up to an arbitrary alteration over any $d\mu$-negligible subset) Θ'_μ such that $d\Theta = \Theta'_\mu d\mu$ Hence one extends the writing (7.1) to more general tensor fields Φ than the continuous ones with compact support, and one defines boundedness for the tensor measure $d\Theta$: that amounts to the boundedness of $d\mu$

The doubly contravariant tensor measure $d\Theta$ is said symmetric (resp skew-symmetric) if $\ll \Phi, d\Theta \gg = 0$ as soon as the doubly covariant tensor field Φ is skew-symmetric (resp symmetric) Equivalently, for every i and j, the scalar measures $d\Theta^{ij}$ and $d\Theta^{ji}$ as equal (resp are the negative of each other); equivalently also the values of the tensor field Θ'_μ are symmetric tensors (resp skew-symmetric) except possibly in a $d\mu$-negligible subset

The same procedure applies to the definition of tensor measures in the Euclidean space $\mathcal{X}$, and yields a natural definition for the image in $\mathcal{X}$ of every tensor measure in Λ, under every placement $\pi(\tau, \ )$

A τ-dependent tensor measure $\mathrm{d}T^\tau$ on a τ-dependent subset C^τ of $\mathscr{X}$ is said *doubly convected* by the carrier Λ if, for every τ in the considered interval, it equals the image of a τ-independent doubly contravariant tensor measure on some subset of Λ. There equivalently exist a convected scalar positive measure $\mathrm{d}m^\tau$ and a doubly convected tensor field T'^τ_m such that $\mathrm{d}T^\tau = T'^\tau_m \mathrm{d}m^\tau$.

Let us recall now that the *trace* of a second order tensor properly makes sense for a tensor of the mixed sort. But, when one starts with a Euclidean linear space as X, the identification of this space with its dual enables the four sorts of second order tensors to merge into a single one; there only remain four ways of representing a given tensor relatively to some base and these four ways merge into a single one if the base is orthonormal.

When orthonormal coordinates are used in the Euclidean space $\mathscr{X}$, the component measures $\mathrm{d}T^{ij}$ of the tensor measure $\mathrm{d}T$ may serve to define the *trace measure* of $\mathrm{d}T$ as the scalar measure

$$\mathrm{tr}\,\mathrm{d}T = \mathrm{d}T^{ii} = T'^{ii}_\mu \,\mathrm{d}\mu = (\mathrm{tr}\, T'_\mu)\mathrm{d}\mu$$

independent of the peculiar choice of orthonormal coordinates. If arbitrary coordinates are used in $\mathscr{X}$, one has to introduce the metric tensor g, as in Section 6, and write

$$\mathrm{tr}\,\mathrm{d}T = g_{ij}\mathrm{d}T^{ij} = \langle g; \mathrm{d}T \rangle$$

where the symbol $\langle\ ;\ \rangle$ refers to the doubly contracted tensor product.

If, in addition, the tensor measure is *bounded*, one may consider the *trace integral*

$$\int \mathrm{tr}\,\mathrm{d}T = \int \langle g; \mathrm{d}T \rangle$$

Then it comes what may be seen as the main formula of the calculus of horizontal variations:

Proposition 7.1. Let $\mathrm{d}T$ be a bounded second order tensor measure on an open subset Ω of $\mathscr{X}$. Let $\mathrm{d}T^\tau$ denote the tensor measure doubly convected by some compact carrier in Ω which equals $\mathrm{d}T$ for $\tau = 0$. The real function $\tau \to \int \mathrm{tr}\,\mathrm{d}T^\tau$ possesses at $\tau = 0$ a derivative equal to

$$\int 2\langle \mathrm{def}\,\varphi; \mathrm{d}T \rangle = 2 \ll \mathrm{def}\,\varphi; \mathrm{d}T \gg \tag{7.2}$$

where φ denotes the velocity field of the carrier.

Proof. Let us introduce, as in Section 6, the doubly covariant symmetric tensor field γ^τ on the carrier Λ whose image in $\mathscr{X}$ equals the constant tensor field g of the Euclidean metric. By assumption, $\mathrm{d}T^\tau$ is the image of a τ-independent doubly contravariant tensor measure on Λ, namely $\mathrm{d}\Theta = \Theta'_\mu \mathrm{d}\mu$, where Θ'_μ is a τ-independent doubly contravariant tensor field and $\mathrm{d}\mu$ a τ-independent positive scalar measure. The definition of images makes that

$$\int \mathrm{tr}\,\mathrm{d}T^\tau = \int \langle g; \mathrm{d}T \rangle = \ll g, \mathrm{d}T \gg = \ll \gamma^\tau, \mathrm{d}\Theta \gg = \int \langle \gamma^\tau; \Theta'_\mu \rangle \,\mathrm{d}\mu \tag{7.3}$$

Provided that the last integration in the above commutes with τ-derivation, one has

$$\frac{\delta}{\delta\tau} \int \langle \gamma^\tau; \Theta'_\mu \rangle \,\mathrm{d}\mu = \int \langle \frac{\delta\gamma^\tau}{\delta\tau}; \Theta'_\mu \rangle \,\mathrm{d}\mu = \int \langle \frac{\delta\gamma^\tau}{\delta\tau}; \mathrm{d}\Theta \rangle \tag{7.4}$$

In view of Proposition 6, this yields (7.2). In order to justify the commutation of $\delta/\delta\tau$ with $\int$ let us make use of some admissible coordinates (λ^i) in the manifold Λ. For τ ranging through a compact neighbourhood k of zero in $\mathbb{R}$, Proposition 6 entails that the components $\delta\gamma_{ij}/\delta\tau$ are continuous functions of $\tau, \lambda^1, \ldots, \lambda^n$, with compact support in $k \times R^n$, hence bounded. Then some standard argument applies, involving the mean value theorem for the real functions

$\tau \to \gamma_{ij}(\tau, \lambda)$ and Lebesgue's dominated convergence theorem for the integral of $\tau^{-1}(\gamma_{ij}^{\tau} - \gamma_{ij}^{0})$ relatively to the measure $\Theta_{\mu}^{\prime ij}\, \mathrm{d}\mu$.

Corollary 7.2 Let the tensor measure $\mathrm{d}T$ *in above Proposition be symmetric; the function* $\tau \to \int \mathrm{tr}\, \mathrm{d}T^{\tau}$ *possesses at* $\tau = 0$ *a zero derivative for any compact carrier in* Ω *if and only if* $\mathrm{d}T$ *has a zero divergence in* Ω *in the sense of Schwartz's distributions.*

Proof Thanks to the symmetry of $\mathrm{d}T$, the expression (7.2) of the said derivative takes the form

$$\ll \varphi_{i\,j} + \varphi_{j\,i}, \mathrm{d}T_{ij} \gg = 2 \ll \varphi_{j\,i}, \mathrm{d}T_{ij} \gg = -2 \ll \varphi_{j}, \mathrm{d}T_{ij\,i} \gg$$

Here the notation $\mathrm{d}T_{ij\,i}$ refers to partial derivatives of the measure $\mathrm{d}T_{ij}$ in the sense of Schwartz's distributions. Since measures are distributions of order zero, these derivatives are distributions of order one; therefore the last member in the above equality makes sense for every φ_j which is $\mathscr{C}^1$, with compact support in Ω. That is zero for every such φ_j (or equivalently, for every φ_j in the more restricted class of $\mathscr{C}^{\infty}$ vector fields with compact support) if and only if the vector distribution in Ω with components $\mathrm{d}T_{ij\,i}$, namely the divergence of $\mathrm{d}T$, is zero.

8 INVISCID FLUID

When the considered fields are smooth enough, the dynamical equations of an inviscid fluid are classically written under the form

$$\rho(u_{i\,t} + u_j u_{i\,j}) = -p_{\,i} + f_i \tag{8.1}$$

to be joined with the equation of mass conservation

$$\rho_{\,t} + (\rho u_i)_{\,i} = 0 \tag{8.2}$$

Here the orthonormal cartesian frame (x_i) is supposed inertial; by $_{\,t}$ and $_{\,i}$ are respectively denoted the partial derivatives with respect to the time t and the x_i coordinate; the vector field f is the volume density of extraneous force.

The following 4-dimensional transcription of (8.1) and (8.2) is similar to what is commonly done in Relativity theory. Let us rename x_0 the variable t and let us agree for all the sequel that, while *Latin subscripts range through the set* $\{1, 2, 3\}$, *Greek subscripts range through the set* $\{0, 1, 2, 3\}$. We shall denote by $\mathfrak{X}$ the 4-dimensional Euclidean space where (x_α) constitute the cartesian coordinates of the point denoted by $\mathfrak{x}$, relative to some orthonormal frame. Let us put the conventions

$$u_0 = 1 \tag{8.3}$$

$$f_0 = 0 \tag{8.4}$$

so that (u_α) and (f_α) are the components of some 4-dimensional vector fields $\mathfrak{u}$ and $\mathfrak{f}$.

By $\Pi_{\alpha\beta}$ we denote the components of the "space" projector, i.e. $\Pi_{ij} = \delta_{ij}$ (Kronecker symbol) and $\Pi_{0\alpha} = \Pi_{\alpha 0} = 0$. Then (8.1) and (8.2) are condensed into the equivalent form

$$(\rho u_{\alpha} u_{\beta} + \Pi_{\alpha\beta} p)_{\,\beta} = f_{\alpha} \tag{8.5}$$

In the left member appear the components of the 4-dimensional divergence vector of the tensor field $\rho \mathfrak{u} \otimes \mathfrak{u} + \Pi p$.

One of the advantages of this writing lies in its possible extension to non-smooth cases; then the divergences or the gradients shall be understood in the sense of distributions. More precisely, let us recall that assimilating a real function F on some open subset Ψ of (for instance) R^4 with a Schwartz distribution on Ψ is an abuse of language. What actually

constitutes a distribution is the scalar measure equal to the product of the Lebesgue measure $d\Xi$ on Ψ by the function F, which has to belong to $L^1_{loc}(d\Xi)$ The most elementary non-smooth situation presents itself when F has the following form: there exists a hypersurface Σ dividing Ψ into two regions Ψ^- and Ψ^+; this hypersurface is supposed to admit a continuous normal unit vector $\mathfrak{N}$, directed from Ψ^- toward Ψ^+; one assumes that F is $\mathscr{C}^1$ in Ψ^- and Ψ^+, with unilateral limits F^- and F^+ at every point of Σ Then using $\mathscr{C}^\infty$ test functions, with compact support in Ψ, and performing some easy integrations by parts, one obtains:

Lemma 8 *The gradient of the scalar measure* $F d\Xi$ *in the sense of the distributions on* Ψ *equals the sum of the two* 4*-dimensional vector measures* grad $F\, d\Xi$, *diffused in* Ψ, *and* $(F^+ - F^-)\mathfrak{N} d\Sigma$ ($d\Sigma$: *the area measure of* Σ), *concentrated on* Σ

Equivalently the components constitute the partial derivatives of $F\, d\Xi$ with respect to the x_α coordinates

Let us apply that to fluid dynamics *A moving surface* $t \to \Sigma_t$ is considered in the space $\mathscr{X}$ and the hypersurface Σ in $\mathfrak{X}$ consists of the points with coordinates (t, x_1, x_2, x_3) such that $(x_1, x_2, x_3) \in \Sigma_t$ The normal unit vector $\mathfrak{N}$ to Σ at any point is supposed non-parallel to the x_0 axis; therefore some vector $\mathfrak{N}'$ parallel to $\mathfrak{N}$, with the same direction, has the components $(-G, n_1, n_2, n_3)$, where (n_i) denote the three components of the normal unit vector to the corresponding surface Σ_t in $\mathscr{X}$, directed toward the + region; the real number G is readily found to constitute the *normal speed* of the moving surface Σ_t at the considered point (counted as positive when Σ_t progresses toward the + region)

Suppose that Σ_t is a surface of discontinuity of the fields ρ, u, p We are to show that interpreting (8 5) in terms of distributions accounts at the same time for the equations (8 1) and (8 2), satisfied in the conventional sense in the regions Ψ^- and Ψ^+, and for the classical *jump conditions* across Σ_t To that end we introduce the Lebesgue measure $d\Xi$ in Ψ (i e the time-volume measure $dt\, dx$) and, instead of (8.5), we state as dynamical equations

$$(\rho u_\alpha u_\beta d\Xi + \Pi_{\alpha\beta} p\, d\Xi)_\beta = de_\alpha \tag{8.6}$$

where the partial derivatives are to be understood in the sense of Schwartz's distributions in the open subset Ψ of $\mathfrak{X}$ The measures de_α with $de_0 = 0$, will be called the components of the *extraneous pulsion*; we expect them to be the sum of the diffuse measures $f_\alpha d\Xi$, representing the distributed extraneous forces, and possibly of some measures db_α concentrated on Σ, with $db_0 = 0$

These equations express that, in the open subset Ψ of the 4-dimensional Euclidean space $\mathfrak{X}$, the second order tensor measure $(\rho \mathfrak{u} \otimes \mathfrak{u} + \Pi p) d\Xi$ admits the four dimensional vector measure $d\mathfrak{e}$ as divergence By Lebesgue's decomposition theorem, such an equality of measures is equivalent to the equality of the absolutely continuous parts on one hand, i.e. to (8 5) being satisfied throughout the regions Ψ^- and Ψ^+, and, on the other hand, to the equality of the singular parts, concentrated on Σ. We now explicit these singular parts in two usual situations of fluid dynamics

1° *Slipstream surface or material boundary*

One supposes that the moving surface Σ_t separates two material parts with preserved identity That means that, in Ψ, the 4-dimensional vector lines of $\mathfrak{u}$ do not cross the hypersurface Σ, hence

$$N_\beta u_\beta^+ = N_\beta u_\beta^- = 0, \tag{8.7}$$

where N_β denote the components of $\mathfrak{N}$. Then Lemma 8 yields as the singular part of equation (8.6)

$$\Pi_{\alpha\beta}(p^+ - p^-) N_\beta d\Sigma = db_\alpha$$

For $\alpha = 0$, both members are essentially zero; for $\alpha = i \in \{1, 2, 3\}$ this reduces to

$$(1 + G^2)^{-\frac{1}{2}}(p^+ - p^-) n_i\, d\Sigma = db_i \tag{8.8}$$

In the usual case where no extraneous pulsion is concentrated on the separating surface, this equality simply consists in $p^+ - p^- = 0$, i e the classical condition of *continuity of the pressure across a slipstream surface*

Actually the above also applies to the case where all the considered fields vanish in the region Ψ^-; this region may be a vacuum, but such a formalism can also account for the case where the matter possibly contained in Ψ^- is declared to be no part of the study (i e it constitutes another "system"). The meaning of (8 8) is made clearer if one refers to the calculation of any integral relative to the area measure $d\Sigma$ of the hypersurface Σ, through the cutting of this hypersurface into strips by hyperplanes $x_0 = t$; this introduces the area measure $d\Sigma_t$ of the surface Σ_t in the 3-dimensional space X and, by a classical use of orthogonal projection,

$$(1 + G^2)^{-\frac{1}{2}} d\Sigma = d\Sigma_t dt;$$

in the more precise setting of the general theory of integration, this is called a *desintegration* of the measure $(1 + G^2)^{-\frac{1}{2}} d\Sigma$. One may similarly assume that the measure db_i admits a desintegration of the form $db_i = dc_i^t dt$, where dc_i^t is a measure concentrated on Σ_t (such a desintegration involves the assumption that the extraneous actions on Σ_t are not of the *percussion* sort). Then, for the said situation of "nothingness in Ψ^-", (8 8) becomes equivalent to

$$p^+ n_i d\Sigma_t = dc_i^t$$

for every t. That is just the property through which the concept of pressure is introduced in the traditional exposition of fluid mechanics.

By the above development one sees that our mathematical setting, based on Schwartz distributions in some *open* subset Ψ of the 4-dimensional space $\mathfrak{X}$ (Ψ may be the whole of $\mathfrak{X}$ or, more usually, some open strip $t^1 < x_0 < t^2$) does not preclude the treatment of boundaries.

2° *Shock waves*

Extracting similarly through Lemma 8, the singular part of (8 6) in the case where the moving surface $t \to \Sigma_t$ is a shock wave, yields for $\alpha = 0$

$$\rho^+(u^+ \cdot n - G) = \rho^-(u^- \cdot n - G)$$

which is the classical condition of mass conservation, and for $\alpha = i \in \{1, 2, 3\}$

$$\rho^+ u^+ (u^+ \cdot n - G) = \rho^- u^- (u^- \cdot n - G)$$

i e the momentum conservation.

We do not have here to dwell on this case, for realistic shock waves do not fall into the setting of this paper. In fact the entropy jump across a shock wave contradicts the assumption of the same relation $p = p(\kappa, \sigma)$ to hold on the whole of Ψ.

Remark

The formulation of classical dynamics sketched in this Section may be extended to much more general situations than hydrodynamics, up to include such singular systems as, for instance, discrete systems of mass points. Let us recall on the other hand that making $\alpha = 0$ in (8 6), with $de_0 = 0$, yields the conservation of mass. Actually, by removing the assumption $de_0 = 0$, one may account for the exchange of matter between the considered "system" and the extraneous world.

9 THE VARIATIONAL STATEMENT

The purpose presented in Section 1 was the variational characterization of the solutions of (1 1), (1 9), (1 10), (1 11). As we are to include non-smooth solutions, (1 9) and (1 10) will

actually be replaced by (8.6); recall that, by definition,

$$u_0 = 1 \tag{9.1}$$

everywhere in the given open subset Ψ of $\mathfrak{X}$. For $\alpha = 0$, (8.6) expresses the mass conservation and may be read as

$$\mathrm{Div}\,(\sigma^{-1}\mathbf{u}\,\mathrm{d}\Xi) = 0 \tag{9.2}$$

where Div denotes the divergence of the considered 4-dimensional vector measure, and $\sigma = \rho^{-1}$. For $\alpha = i \in \{1, 2, 3\}$, (8.6) becomes

$$(\sigma^{-1}u_i u_\beta \mathrm{d}\Xi)_{,\beta} + (p\,\mathrm{d}\Xi)_{,i} = \sigma^{-1}U_i\mathrm{d}\Xi \tag{9.3}$$

since the extraneous pulsion now consists of the gravity-like distributed forces. On the other hand, with P defined in (1.3), the compressibility law (1.1) is taken under the form

$$p = \frac{\partial P}{\partial \sigma}(x_0, \kappa, \sigma) \tag{9.4}$$

(precluding the case of incompressible fluids). Then, by the elimination of p, (9.3) reduces to

$$(\sigma^{-1}u_i u_\beta \mathrm{d}\Xi)_{,\beta} + \left(\frac{\partial P}{\partial \sigma}\mathrm{d}\Xi\right)_{,i} = \sigma^{-1}U_i\mathrm{d}\Xi. \tag{9.5}$$

In our 4-dimensional language, (1.11) reads: κ is a constant on each vector line of $\mathbf{u}$. (9.6)

Carriers will be considered, whose 4-dimensional velocity field, with compact support in Ψ, satisfies

$$\varphi_0 = 0. \tag{9.7}$$

Such carriers preserve every hyperplane $x_o = \mathrm{const.}$; hence we call them *isochronous* carriers.

Only (9.5) will properly be the object of the variational property in view, which acquires most of its interest from this preliminary assertion:

Proposition 9.1. *Suppose that $\mathbf{u}^\tau$ is a τ-dependent 4-dimensional vector field convected by an isochronous carrier; if condition* (9.1) *is satisfied for some value of τ, then it is satisfied for every τ.*

Suppose in addition that the τ-dependent function κ^τ, defined in $\mathfrak{X}$, with values in an arbitrary set, is convected by the carrier; if conditions (9.6) *is satisfied for some τ, then it is satisfied for every τ.*

Suppose finally that σ^τ is a τ-dependent real function related to the carrier in such a way that the scalar measure $(\sigma^\tau)^{-1}\mathrm{d}\Xi$ is convected (as before, $\mathrm{d}\Xi$ denotes the Lebesgue measure in $\mathfrak{X}$); if condition (9.2) *is satisfied for some τ, then it is satisfied for every τ.*

Proof If $\mathbf{u}^\tau$ is convected, one has

$$\delta_\varphi u_0^\tau = u_\alpha^\tau \varphi_{0\,\alpha}$$

which is zero in view of (9.7); that establishes the first statement.

The second statement is made obvious by observing that, if $\mathbf{u}^\tau$ is convected, its vector lines are the images under π^τ of τ-independent lines in the carrier manifold Λ; now the assumption about κ^τ means that this function is the image under π^τ of some function defined on Λ in a τ-independent way.

If the scalar measure $(\sigma^\tau)^{-1}\mathrm{d}\Xi$ is convected, as well as the vector field $\mathbf{u}^\tau$, the vector measure $(\sigma^\tau)^{-1}\mathrm{d}\Xi$ is convected; then the last statement follows from Proposition 5.

Two assumptions are now made for all the sequel:

Hypothesis 9 2 *It is supposed that, in the whole region* Ψ, *the mass density of extraneous force, with components* $g_\alpha = \Pi_{\alpha\beta} U_\beta$, *is bounded*

The above assumption, which holds in practical situations, implies that the function U is Lipschitz in Ψ with regard to the variables (x_1, x_2, x_3), uniformly in x_0

On the other hand:

Hypothesis 9 3 *It is supposed that, for all the considered values of* x_0, κ *and* σ, *the function* $p(x_0, \kappa, \sigma)$ *is bounded*

Hence by (9 4) the function P is uniformly Lipschitz with regard to the variable σ We should concede that, for realistic compressibility laws, the boundedness of p only holds under some *minoration of* σ Therefore, Hypothesis 9 3 amounts to assuming that such a minoration holds for the considered solution of the dynamical equations

The above hypotheses secure the convergence of the integrals to be involved in the sequel, namely:

Proposition 9 4 *Let us consider a compact isochronous carrier in the open subset* Ψ *of* $\mathfrak{E}$ *Let* u^τ, κ^τ *and* σ^τ *be transported by this carrier in the way defined in Proposition* 9 1

If the integral

$$\int_\Psi \frac{1}{2} u_i^\tau u_i^\tau (\sigma^\tau)^{-1} \mathrm{d}\Xi \tag{9 8}$$

makes sense for $\tau = 0$, *then it makes sense for every* τ *in some neighbourhood of zero in* R

Under Hypothesis 9 2, *the same is true for the integral*

$$\int_\Psi (\sigma^\tau)^{-1} U \mathrm{d}\Xi \tag{9 9}$$

Under Hypothesis 9 3, *the same is true for the integral*

$$\int_\Psi P(x_0, \kappa^\tau, \sigma^\tau)(\sigma^\tau)^{-1} \mathrm{d}\Xi \tag{9 10}$$

Proof The law of transport defined in Proposition 9 1 means that $(\sigma^\tau)^{-1}\mathrm{d}\Xi$ is the image under the placement mapping π^τ of some τ-independent nonnegative measure $\mathrm{d}\mu$ in the carrier manifold Λ Then the three above integrals are identical to the integrals, relative to $\mathrm{d}\mu$, of the functions defined in Λ by composing the respective integrands with the mapping π^τ Let (λ^α) denote an admissible coordinate system in Λ and $\pi_\beta^\tau(\lambda)$ the Cartesian coordinates of $\pi^\tau(\lambda)$ in $\mathfrak{E}$

By assumption, u^τ is the image of some τ-independent vector field ω in Λ, with components ω^α; hence

$$u_i^\tau(\pi^\tau(\lambda)) = \frac{\partial \pi_i^\tau(\lambda)}{\partial \lambda^\alpha} \omega^\alpha(\lambda). \tag{9 11}$$

Then the assertion concerning the integral (9 8) ensues from the boundedness of $\partial \pi_i^\tau / \partial \lambda^\bullet$; in fact this function is continuous in Λ, with compact support

As for the integral (9 9), one makes use of the majoration of $U(\pi^\tau(\lambda))$ resulting from Hypothesis 9 2 and from the fact that the first coordinate π_0^τ of π^τ is a constant with regard to τ

The assertion concerning the integral (9 10) is similarly proved by observing that

$$\sigma^\tau(\pi^\tau(\lambda)) = k(\lambda) \det \frac{\partial \pi_\beta^\tau}{\partial \lambda^\alpha} \tag{9 12}$$

where k denotes a τ-independent real function in Λ, and that τ-independent values of x_0 and κ^τ correspond to each $\lambda \in \Lambda$

Now comes the variational statement:

Proposition 9 5 *Let a* 4*-dimensional vector field* u, *a scalar function* σ *and a function* κ *be defined in some open subset* Ψ *of* $\mathfrak{X}$, *supposed to satisfy* (9 1), (9 2) *and* (9 6) *and to make the integrals* (9 8), (9 9) *and* (9 10) *meaningful These elements constitute a solution of* (9 5) *in* Ψ *if and only if, for every isochronous compact carrier in* Ψ, *the* τ*-derivative of the following functional vanishes at* $\tau = 0$

$$A(\tau) = \int_{\Psi} \left[\frac{1}{2} u_i^\tau u_i^\tau + P(x_0, \kappa^\tau, \sigma^\tau) + U \right] (\sigma^\tau)^{-1} \mathrm{d}\Xi, \tag{9 13}$$

where u^τ, σ^τ, κ^τ *are transported by the carrier in the way defined in Proposition* 9 1 *and reduce to the above for* $\tau = 0$

Proof As in the proof of Proposition 9 4, let us transform $A(\tau)$ into an integral on the carrier manifold Λ, relative to the τ-independent measure $\mathrm{d}\mu$ Observing that $A(\tau)$ equals the sum of the integrals (9 8), (9 9) and (9 10), we shall perform the τ-derivation on these three terms separately

When u_i^τ is evaluated at the point $\pi^\tau(\lambda)$ of $\mathfrak{X}$, with λ fixed in Λ, one has by (3 1)

$$\frac{\delta}{\delta\tau}\left(\frac{1}{2} u_i^\tau u_i^\tau\right) = \varphi_{i\beta} u_\beta^\tau u_i^\tau$$

where the partial derivatives $\varphi_{i\alpha}$ of the carrier velocity field are evaluated at the point $\pi^\tau(\lambda)$ of $\mathfrak{X}$ The right hand member is continuous on Λ with compact support, thus bounded Then the same argument as in the Proof of Proposition 7 1 allows one to commute $\mathrm{d}\mu$-integration with τ-derivation; by finally expressing the result as an integral in Ψ, one obtains

$$\frac{\delta}{\delta\tau} \int_{\Psi} \frac{1}{2} u_i^\tau u_i^\tau (\sigma^\tau)^{-1} \mathrm{d}\Xi = \int_{\Psi} \varphi_{i\beta} u_\beta^\tau u_i^\tau (\sigma^\tau)^{-1} \mathrm{d}\Xi \tag{9 14}$$

Similarly, for every λ, the expression $\kappa^\tau(\pi^\tau(\lambda))$ is a constant with regard to τ, while the convection of the measure $(\sigma^\tau)^{-1}\mathrm{d}\Xi$ entails (Section 3)

$$\frac{\delta \sigma^\tau(\pi^\tau(\lambda))}{\delta\tau} = \sigma^\tau \varphi_{\alpha\alpha} = \sigma^\tau \varphi_{ii}$$

since the carrier is isochronous; therefore

$$\frac{\delta}{\delta\tau} P(x_0, \kappa^\tau, \sigma^\tau) = \frac{\partial P}{\partial \sigma} \sigma^\tau \varphi_{ii}$$

In view of Hypothesis 9 3, that is bounded, allowing again for the commutation of τ-derivation with $\mathrm{d}\mu$-integration Hence

$$\frac{\delta}{\delta\tau} \int_{\Psi} P(x_0, \kappa^\tau, \sigma^\tau)(\sigma^\tau)^{-1} \mathrm{d}\Xi = \int_{\Psi} \varphi_{ii} \frac{\partial P}{\partial \sigma}(x_0, \kappa^\tau, \sigma^\tau) \mathrm{d}\Xi \tag{9 15}$$

Lastly, for every λ,

$$\frac{\delta}{\delta\tau} U(\pi^\tau(\lambda)) = U_i(\pi^\tau(\lambda)) \frac{\partial \pi_i^\tau(\lambda)}{\partial \tau} = U_i \varphi_i,$$

thus, due to Hypothesis 9 2,

$$\frac{\delta}{\delta\tau} \int_{\Psi} (\sigma^\tau)^{-1} U \mathrm{d}\Xi = \int_{\Psi} \varphi_i (\sigma^\tau)^{-1} U_i \mathrm{d}\Xi \tag{9 16}$$

By assumption, the functions φ_i, $i \in \{1, 2, 3\}$, are $\mathscr{C}^1$ in Ψ, with compact support; in view of the definition of the partial derivatives in Schwartz's distributions theory, the derivative of $A(\tau)$ at $\tau = 0$ finally reads as

$$\frac{\delta A}{\delta \tau} = < -(\sigma^{-1} u_i u_\beta \mathrm{d}\Xi)_\beta - \left(\frac{\partial P}{\partial \sigma}\mathrm{d}\Xi\right)_i + \sigma^{-1} U_i, \varphi_i >$$

Its vanishing for every such φ_i is (9.5)

Remark 9.6

It is equivalent to restrict the above to the functions φ_i belonging to $\mathscr{D}(\Psi)$, i.e. the $\mathscr{C}^\infty$ functions with compact support in Ψ.

Remark 9.7

The conditions (9.1), (9.2) and (9.6) imposed in this variational statement do not constitute *constraints* in the usual sense of the calculus of variations; in fact Proposition 9.1 secures that these conditions are automatically satisfied by all the competing elements.

Remark 9.8

The type of calculation used above in expressing the τ-derivative of (9.8) could also yield a proof of Proposition 7.1. Conversely, the result would directly follow from Proposition 7.1 if one additionally assumed that the measure of Ψ with respect to $(\sigma^\tau)^{-1}\mathrm{d}\Xi$ is finite. In fact

$$\int_\Psi \frac{1}{2} u_i^\tau u_i^\tau (\sigma^\tau)^{-1}\mathrm{d}\Xi = \frac{1}{2}\int_\Psi u_\alpha^\tau u_\alpha^\tau (\sigma^\tau)^{-1}\mathrm{d}\Xi - \frac{1}{2}\int_\Psi (\sigma^\tau)^{-1}\mathrm{d}\Xi;$$

under the assumptions made in Proposition 9.5, the last integral does not depend on τ while the first one on the right side is nothing but the trace integral of the tensor measure $u^\tau \otimes u^\tau (\sigma^\tau)^{-1}\mathrm{d}\Xi$; since u^τ is convected, as well as the scalar measure $(\sigma^\tau)^{-1}\mathrm{d}\Xi$, this tensor measure is ***doubly convected***.

Remark 9.9

in [16], an introductory Seminar Report which presents the above Proposition in the case of smooth fields, it is developed that this Proposition may be viewed as a transcript of Hamilton's "principle" of the least action into the language of Euler's variables. But the method of horizontal variation has a much wider scope; in particular, the result of the Section to come is not directly connected with Hamilton's principle since the concept of a stationary motion does not make sense in the framework of usual analytical dynamics.

10. STATIONARY MOTION

For a fluid satisfying the same assumptions as in Section 9, let us *a priori* suppose that the flow is *stationary* in some open subset Ω of X, i.e. the functions u_i, σ, κ, defined in Ω, are constant with regard to t. Of course, the compressibility law (9.14) and the function U are henceforward supposed independent of t. We denote by dx the Lebesgue measure on Ω. Instead of (9.2), the mass conservation is expressed by

$$\mathrm{div}\,(\sigma^{-1} u \mathrm{d}x) = 0 \tag{10.1}$$

and, instead of (9.5), one has a dynamical equation

$$(\sigma^{-1} u_i u_j \mathrm{d}x)_j + \left(\frac{\partial P}{\partial \sigma}\mathrm{d}x\right)_i = \sigma^{-1} U_i \mathrm{d}x \tag{10.2}$$

The vector lines of u in Ω are also the trajectories of the fluid particles, hence the conservation of κ for each of these particles is now expressed by

$$\kappa \text{ is a constant on each vector line of } u \tag{10.3}$$

The following may be proved in the same way as for Proposition 9.1

Proposition 10.1. *Suppose that u^τ is a τ-dependent 3-dimensional vector field convected by some carrier and that the function κ^τ, defined in X with values in an arbitrary set is convected. If condition* (10.3) *is satisfied for some value of τ, thus it is satisfied for every τ.*

Suppose in addition that σ^τ is a τ-dependent real function related to the carrier in such a way that the scalar measure $(\sigma^\tau)^{-1}\mathrm{d}x$ is convected; if condition (10.1) *is satisfied for some value of τ, then it is satisfied for every τ.*

We make again the Hypotheses 9.2 and 9.3; similarly to Proposition (9.4), they entail:

Proposition 10.2. *Let us consider a compact carrier in the open subset Ω of X. Let u^τ, κ^τ and σ^τ be transported by this carrier in the way defined in Proposition* 10.1. *If the integrals*

$$\int_\Omega \frac{1}{2} u_i^\tau u_i^\tau (\sigma^\tau)^{-1}\mathrm{d}x, \quad \int_\Omega (\sigma^\tau)^{-1} U \mathrm{d}x, \quad \int_\Omega P(\kappa^\tau, \sigma^\tau)(\sigma^\tau)^{-1}\mathrm{d}x$$

make sense for $\tau = 0$, then they make sense for every τ in some neighbourhood of zero.

Now comes the variational statement:

Proposition 10.3. *Let a vector field u, a scalar function σ and a function κ be defined in an open subset Ω of X, supposed to satisfy* (10.1), (10.3) *and to make the three integrals of Proposition* 10.1 *meaningful. These elements constitute a solution of* (10.2) *in Ω if and only if, for every compact carrier in Ω, the τ-derivative of the following functional vanishes at $\tau = 0$*

$$B(\tau) = \int_\Omega \left[\frac{1}{2} u_i^\tau u_i^\tau + P(\kappa^\tau, \sigma^\tau) + U\right](\sigma^\tau)^{-1}\mathrm{d}x \tag{10.4}$$

where u^τ, σ^τ, κ^τ are transported by the carrier in the way defined in Proposition 10.1 *and reduce to the above for $\tau = 0$.*

The proof runs in the same way as for Proposition 9.5. One should observe that $\int_\Omega u_i u_i \sigma^{-1}\mathrm{d}x$ is nothing but the trace integral of the symmetric tensor measure $u \otimes u\sigma^{-1}\mathrm{d}x$; hence Proposition 7.1 may directly be applied to calculate its τ-derivative.

REFERENCES

[1] J. SERRIN, In *Handbuch der Physik* (Edited by S. Flugge and C. Truesdell), Vol. VIII/1, Springer-Verlag, Berlin (1959).

[2] R. L. SELINGER and G. B. WHITHAM, *Proc. Roy. Soc.* **A305**, 1 (1968).

[3] H. GOUIN, *Contribution à une étude géométrique et variationnelle des milieux continus*. Thèse Université d'Aix-Marseille I (1978).

[4] P. CASAL, *J. de Mécanique* **5**, 149 (1966).

[5] F. P. BRETHERTON, *J. Fluid Mech.* **44**, 19 (1970).

[6] P. CASAL, *C. R. Acad. Sci.* **234**, 804 (1952).

[7] P. R. GARABEDIAN and D. C. SPENCER, *J. Rat. Mech. Anal.* **1**, 359 (1952).

[8] R. HERMANN, *Geometry, Physics, and Systems*. Marcel Dekker, New York (1973).

[9] D. G. B. EDELEN, *Lagrangian Mechanics of Nonholonomic Systems*. Nordhoff, Leyden (1977).

[10] Ph. MORICE, *Proc. 3rd GAMM-Conf. Num. Meth. Fluid Mech. Cologne* 1979 (Edited by E. H. Hirschel), Vol. 2, pp. 202–210, Notes on Numerical Fluid Mechanics, Vieweg, Braunschweig/Wiesbaden (1980).

[11] J. P. ZOLESIO, *Proc. NATO-NSF Study, Iowa City* 1980 (Edited by J. Céa and Ed. Haug). Stijthoff and Nordhoff, Alphen aan den Rijn (1981).

[12] J. J. MOREAU, *Nonlinear Problems of Analysis in Geometry and Mechanics* (Edited by M. Attéia, D. Bancel and I. Gumowski), pp. 31–49, Pitman, Boston, London, Melbourne (1981).

[13] J. M. SOURIAU, *Ann. Inst. Henri Poincaré* **20**, 315 (1974).

[14] J. J. MOREAU, *Variations Horizontales Première et Seconde d'une Intégrale de Trace, Vol. 11, exp. No. 3. Travaux du Séminaire d'Analyse Convexe, U.S.T.L., Montpellier* (1981).

[15] S. DROBOT and A. RYBARSKI, *Arch. Rat. Mech. Anal.* **2**, 393 (1959).

[16] J. J. MOREAU, *Le Calcul des Variations Horizontales et l'Hydrodynamique*, Vol. 9, exp. No. 4. Travaux du Séminaire d'Analyse convexe, U.S.T.L., Montpellier (1979).

[17] N. BOURBAKI, *Intégration*, Chap. 6. Hermann, Paris (1959).