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## Some Calculative Properties of Linear Viscosity

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This note is concerned with incompressible homogeneous fluids. Let  $u_i$ ,  $i = 1, 2, 3$ , denote the components of the fluid velocity  $\mathbf{u}$ , relative to some orthonormal frame, and  $u_{i,j}$  their partial derivatives. The classical law of Newtonian viscosity consists in the following expression for the viscous stress tensor at every point

$$V_{ij} = \mu(u_{i,j} + u_{j,i}), \quad (1)$$

with  $\mu > 0$ , the viscosity coefficient. Equivalently, let a portion of the fluid be limited by a smooth surface  $S$ , with  $\mathbf{v}$  as normal unit vector directed toward the said portion; the distribution of contact forces exerted, across  $S$ , by the fluid portion upon the material beyond admits the surface density

$$\mathbf{T}(\mathbf{v}) = -p\mathbf{v} + \mu \frac{d\mathbf{u}}{d\mathbf{v}} + \mu \text{grad } \mathbf{v} \cdot \mathbf{u}; \quad (2)$$

here  $p$  denotes the fluid pressure and the grad operation is understood with the vector  $\mathbf{v}$  treated as a constant. We shall refer to  $\mathbf{T}$  as the *tension vector* at the considered point of  $S$ .

However, some textbooks in engineering or elementary physics, starting with the simple example of a parallel shear flow, illegitimately induce as general, instead of (2), the following expression for the tension on  $S$

$$\mathbf{T}'(\mathbf{v}) = -p\mathbf{v} + \mu \frac{d\mathbf{u}}{d\mathbf{v}}. \quad (3)$$

This would involve, for the viscous stress tensor, the expression

$$V'_{ij} = \mu u_{i,j} \quad (4)$$

which violates the principles of mechanics in two respects: this tensor is not symmetric and it is related to the fluid motion in a way which depends on the reference frame.

It happens that this incorrect law of viscosity generates the same Navier–Stokes equations as the correct one and yields also correct results in a certain number of topics, with the advantage of shorter computation (see e.g. [3]). Our purpose here is to list some of these topics; for brevity, the proofs, which are easy, have been omitted; the reader may find them, with some other developments, in [10], a Seminar report.

Meanwhile, another incorrect law of viscosity will be considered, namely

$$\mathbf{T}''(\mathbf{v}) = -p\mathbf{v} + \mu(\text{curl } \mathbf{u}) \times \mathbf{v} \quad (5)$$

equivalent to (2) and (3) in the aforesaid case of a parallel shear flow orthogonal to  $\mathbf{v}$ . The corresponding viscous stress tensor would be

$$V''_{ij} = \mu(u_{i,j} - u_{j,i}) \quad (6)$$

yielding again the exact Navier–Stokes equations.

Observe that (cf. R. Berker [1, 2])

$$\mathbf{T}(\mathbf{v}) = 2\mathbf{T}'(\mathbf{v}) - \mathbf{T}''(\mathbf{v}). \quad (7)$$

### Resultant of the Tension on a Closed Surface

Let  $S$  denote a closed bounded and oriented surface, with piecewise continuous normal. Let the velocity field  $\mathbf{u}$  and its first order derivatives be defined and continuous, at least in a one-side neighborhood of  $S$ . Then the three expressions  $\mathbf{T}(\mathbf{v})$ ,  $\mathbf{T}'(\mathbf{v})$  and  $\mathbf{T}''(\mathbf{v})$  of the tension yield the same total on  $S$ . Concerning the respective values of the total moment of the tensions on  $S$ , one has

$$\begin{aligned} \iint_S \mathbf{x} \times \mathbf{T}(\mathbf{v}) \, d\sigma - \iint_S \mathbf{x} \times \mathbf{T}'(\mathbf{v}) \, d\sigma &= \iint_S \mathbf{x} \times \mathbf{T}''(\mathbf{v}) \, d\sigma - \iint_S \mathbf{x} \times \mathbf{T}'(\mathbf{v}) \, d\sigma \\ &= \mu \iint_S \mathbf{u} \times \mathbf{v} \, d\sigma. \end{aligned} \quad (8)$$

Observe that  $S$  is not supposed to constitute the whole boundary of a domain in which  $\mathbf{u}$  would have to be everywhere defined. In particular,  $S$  may enclose some fluid part with immersed bodies.

In the special case where  $S$  is the boundary of a rigid body immersed in the fluid,  $\mathbf{u}$  classically equals, at every point of this boundary, the velocity of the corresponding particle of the body; finally, the last member of (8) is found equal to  $-2\mu\mathcal{V}\boldsymbol{\omega}$  where  $\mathcal{V}$  denotes the volume of the body and  $\boldsymbol{\omega}$  its angular velocity vector.

### Local Tension on a Translating Boundary

Let  $S$  be a surface portion; we make the same assumption for  $\mathbf{u}$  as above. If  $\mathbf{u}$  has the same value at every point of  $S$  (for instance, if  $S$  is part of the boundary of a translating immersed body), then  $\mathbf{T}(\mathbf{v}) = \mathbf{T}'(\mathbf{v}) = \mathbf{T}''(\mathbf{v})$  at every regular point of  $S$ ; the equality  $\mathbf{T} = \mathbf{T}''$  under such circumstances has already been pointed out by R. Berker [1, 2].

### Dissipated Power

In the correct theory of viscosity the following expression is found for the *power dissipated by viscosity* in a fluid portion  $D$

$$\mathcal{D}(u) = 2\mu \iiint_D e_{ij} e_{ij} \, d\tau, \quad (9)$$

with  $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ . If the incorrect laws (4) and (6) are used, the following expressions

$$\mathcal{D}'(u) = \mu \iiint_D u_{i,j} u_{i,j} \, d\tau \quad (10)$$

$$\mathcal{D}''(u) = \mu \iiint_D (\text{curl } \mathbf{u})^2 \, d\tau \quad (11)$$

are respectively obtained. In the special case where  $\mathbf{u}$  vanishes on the boundary of  $D$ , these three expressions are equal, a fact already observed by J. Serrin [8].

### Boundary Value Problem for Stokes Flows

As the three expressions of the viscous stress yield the same Navier–Stokes equations, they lead in particular to the same approximate equations in the case of negligible acceleration, namely

$$\mu u_{i,jj} = p_{,i}, \quad u_{i,i} = 0. \quad (12)$$

Flows obeying these equations are called Stokes flows; the most classical boundary problem concerning them consists in prescribing the distribution of the values of  $\mathbf{u}$  on the boundary of some domain  $D$ ; furthermore, if the domain is unbounded, the limit of  $\mathbf{u}$  at infinity is prescribed (see [6] for the precise statement of this condition at infinity). The unique solution of this problem is classically characterized by the property that it minimizes the functional (9) among the vector fields  $\mathbf{u}$  satisfying the boundary conditions and the constraint  $u_{i,i} = 0$ . This fact (see e.g. [4, 9]) is of use for the existential study of the problem as well as for numerical computation.

Now we find that the functional  $\mathcal{D}'$  (and also, with less advantage,  $\mathcal{D}''$ ) may equivalently be used in this variational statement, bringing noticeable simplification to calculations and numerical procedures.

Of course, the replacement of  $\mathcal{D}$  by  $\mathcal{D}'$  or  $\mathcal{D}''$  is no more valid when boundary problems involving free surfaces are considered.

### A Problem of Optimal Design

Several papers [3, 5, and 7] have been recently devoted to the following: Let  $B$  a fixed body immersed in an unbounded Stokes flow with velocity  $\mathbf{V}$  at infinity. If  $\mathbf{V}$  as well as the volume of  $B$  are prescribed, the problem is to determine the shape of  $B$  in order to minimize the drag  $F$  experienced by this body. Using the asymptotic structure at infinity of the fluid velocity  $\mathbf{u}$  [6], one establishes that

$$F \cdot \mathbf{V} = \mathcal{D}(\mathbf{u}) = \mathcal{D}'(\mathbf{u}) = \mathcal{D}''(\mathbf{u})$$

and the problem equivalently amounts to minimizing  $\mathcal{D}$ ,  $\mathcal{D}'$  or  $\mathcal{D}''$ . The simplest is to make use of  $\mathcal{D}'$ .

On the other hand it has been found by O. Pironneau [7] that, for the body of optimal shape, the scalar expression  $|d\mathbf{u}/d\mathbf{v}|$  assumes a constant value at all the regular points of the body surface  $S$ . We equivalently replace this property by: (i) the norm of the tensor  $e_{ij}$  assumes a constant value; (ii) the norm of the vector  $\text{rot } \mathbf{u}$  assumes a constant value. The latter has been used by J. M. Bourot [3] in some numerical procedures.

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### Abstract

Several topics are discussed in which the classical law of Newtonian viscosity may equivalently be replaced by other relations.

### Résumé

On présente plusieurs questions à l'égard desquelles la loi classique de la viscosité peut être remplacée par d'autres relations sans altérer les résultats.

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