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A duality formula and a particle Gibbs sampler for continuous time Feynman-Kac measures on path spaces

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Abstract

Continuous time Feynman-Kac measures on path spaces are central in applied probability, partial differential equation theory, as well as in quantum physics. This article presents a new duality formula between normalized Feynman-Kac distribution and their mean field particle interpretations. Among others, this formula allows us to design a reversible particle Gibbs-Glauber sampler for continuous time Feynman-Kac integration on path spaces. We also provide new propagation of chaos estimates for continuous time genealogical tree based particle models with respect to the time horizon and the size of the systems. Our approach is based on a novel stochastic perturbation analysis based on backward semigroup techniques. These techniques allow to obtain sharp quantitative estimates of the convergence rate to equilibrium of particle Gibbs-Glauber samplers. To the best of our knowledge these results are the first of this kind for continuous time Feynman-Kac measures.

Keywords : Feynman-Kac formulae, interacting particle systems, genealogical trees, Gibbs-Glauber dynamics, propagation of chaos properties.

Mathematics Subject Classification : 60K35, 60H35, 37L05, 47D08.

1 Introduction

Feynman-Kac measures on path spaces are central in applied probability as well as in biology and quantum physics. They also arise in a variety of application domains such as in estimation and control theory, as well as a rare event analysis. For a detailed review on Feynman-Kac measures and their application domains we refer to the books [21, 22, 33, 36], see also the more recent articles [19, 50] on branching processes and neutron transport equations and the references therein.

Their mean field type particle interpretations is defined as a system of particles jumping a given rate uniformly onto the population. From the pure numerical viewpoint, this interacting jump transition can be interpreted as an acceptance-rejection scheme with a recycling. Feynman-Kac interacting particle models encapsulate a variety of algorithms such as the diffusion Monte Carlo used to solve Schrödinger ground states, see for instance the series of articles [11, 13, 35, 67, 53, 54] and the references therein.

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Their discrete time versions encapsulate a variety of well known algorithms such as particle filters [23] (a.k.a. sequential Monte Carlo methods in Bayesian literature [14, 21, 22, 33, 40]), the go-with-the-winner [1], as well as the self-avoiding random walk pruned-enrichment algorithm by Rosenbluth and Rosenbluth [69], and many others. This list is not exhaustive (see also the references therein). The research monographs [21, 22] provide a detailed discussion on these subjects with precise reference pointers.

The seminal article [2] by Andrieu, Doucet and Holenstein introduced a new way to combine Markov chain Monte Carlo methods with discrete generation particle methods. A variant of the method, where ancestors are resampled in a forward pass, was developed by Lindsten, Schön and Jordan in [55], and Lindsten and Schön [56]. In all of these studies, the validity of the particle conditional sampler is assessed by interpreting the model as a traditional Markov chain Monte Carlo sampler on an extended state space. The central idea is first to design a detailed encoding of the ancestors at each level in terms of random maps on integers, and then to extend the "target" measure on a sophisticated state space encapsulating these iterated random sequences.

In a more recent article [31], the authors provide an alternative and we believe more natural interpretation of these particle Markov chain Monte Carlo methods in terms of a duality formula extending the well known unbiasedness properties of Feynman-Kac particle measures on many-body particle measures. This article also provides sharp quantitative estimates of the convergence rate to equilibrium of the models with respect to the time horizon and the size of the systems. The analysis of these models, including backward particle Markov chain Monte Carlo samplers has been further developed in [27, 28].

The main objective of the present article is to extend these methodologies to continuous time Feynman-Kac measures on path spaces.

The first difficulty comes from the fact that the discrete time analysis [27, 28, 31] only applies to simple genetic type particle models, or equivalently to branching models with pure multinomial selection schemes. Thus, these results don't apply to discrete time approximation of continuous time models based on geometric type jumps, and any density type argument cannot be applied.

In contrast with their discrete time version, continuous time Feynman-Kac particle models are not described by conditionally independent local transitions, but in terms of interacting jump processes. This class of processes can be interpreted as Moran type interacting particle systems [62, 63]. They can also be seen as Nanbu type interpretation of a particular spatially homogeneous generalized Boltzmann equation [32, 61].

The analysis of continuous time genetic type particle models is not so developed as their discrete time versions. For instance, uniform convergence estimates are available for continuous time Feynman-Kac models with stable processes [33, 34, 35, 67]. Nevertheless, to the best of our knowledge, sharp estimates for path space models and genealogical tree based particle samplers in continuous time have never been discussed in the literature. These questions are central in the study the convergence to equilibrium of particle Gibbs-Glauber sampler on path spaces.

In the present article we provide a duality formula for continuous time Feynman-Kac measures on path-spaces (cf. theorem 1.1). This formula on genealogical tree based particle models that can be seen as an extension of well known unbiasedness properties of Feynman-Kac models to their many body version (defined in section 4). The second main result of the article is to design and to analyze the stability properties of a particle Gibbs-Glauber sampler of path space (cf. theorem 1.2). Our approach combines a perturbation analysis of nonlinear stochastic semigroups with propagation of chaos techniques (cf. section 3). Incidentally these techniques also provide with little efforts new uniform propagation of chaos estimates w.r.t. the time horizon (cf. corollary 3.13).

1.1 Statement of the main results

Let (X_t, V_t) be a continuous time Markov process and a bounded non negative function on some metric space (S, d_S) . We let \mathbb{P}_t be the distribution of $(X_s)_{s \leq t}$ on the set $D_t(S)$ of càdlàg paths from $[0, t]$ to S . As a rule in the further development of the article $\hat{X}_t := (X_s)_{s \leq t} \in \hat{S} := \cup_{t \geq 0} D_t(S)$ stands for the historical process of some process X_t . In this notation, we extend V_t to $D_t(S)$ by setting $\hat{V}_t(\hat{X}_t) = V_t(X_t)$.

The Feynman-Kac probability measures \mathbb{Q}_t associated with (X_t, V_t) are defined by the formula

$$d\mathbb{Q}_t := \frac{1}{Z_t} \exp \left[- \int_0^t V_s(X_s) ds \right] d\mathbb{P}_t \quad (1.1)$$

where Z_t stands for some normalizing constant. These measures can be computed in terms the occupation measures of the ancestral lines of an interacting jump process [32, 33, 35, 36]. Consider a system of N particles evolving independently as X_t with jump rate $V_t(X_t)$. At each jump time the particle jumps onto a particle uniformly chosen in the pool.

Equivalently, the N ancestral lines $\xi_t = (\xi_t^i)_{1 \leq i \leq N}$ of length t can also be seen as a system of N path-valued particles evolving independently as the historical process \hat{X}_t , with jump rate \hat{V}_t on \hat{S} .

The occupation measure of the genealogical tree is given by the empirical measures

$$m(\xi_t) := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_t^i} \quad \text{and we denote by } \mathbb{X}_t \text{ a random sample from } m(\xi_t)$$

The dual process $\zeta_t = (\zeta_t^i)_{1 \leq i \leq N}$ is also defined in terms of N the ancestral lines of length t of an interacting jump process. The main difference is that the first line at any time t is frozen and given by $\zeta_t^1 := \hat{X}_t$. The remaining $(N - 1)$ path-valued particles $\zeta_t^- := (\zeta_t^i)_{2 \leq i \leq N}$ are defined as above with a rescaled jump rate $(1 - 1/N)\hat{V}_t$, with an additional jump rate $2\hat{V}_t/N$ at which the path-particle jump onto the first frozen ancestral line.

A realization of the genealogical tree associated with $N = 3$ particles with 2 interacting jumps and the first frozen ancestral line is illustrated below:

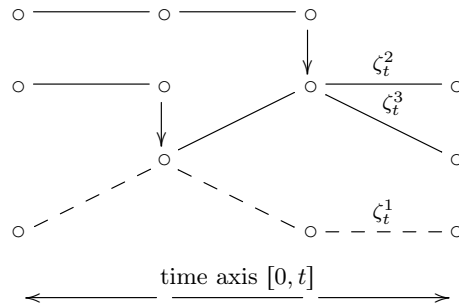


Figure 1: A genealogical tree associated with $N = 3$ particles with 2 interacting jumps. The couple of arrows stands for the interacting jumps, the dotted line represents the frozen ancestral line.

For any $N \geq 1$, we let $S_N := S^N / \mathbb{S}_N$ be the N symmetric product of S , where \mathbb{S}_N stands for the symmetric group of order N . The first main result of the article is the following duality formula.

Theorem 1.1 (Duality formula). *For any time horizon $t \geq 0$, any $N \geq 2$ and any bounded measurable function F on $\hat{S} \times D_t(\hat{S} \times \hat{S}_{N-1})$ we have*

$$\mathbb{E} \left(F(\mathbb{X}_t, \hat{\xi}_t) \exp \left[- \int_0^t m(\xi_s)(\hat{V}_s) ds \right] \right) = \mathbb{E} \left(F(\hat{X}_t, \hat{\zeta}_t) \exp \left[- \int_0^t \hat{V}_s(\hat{X}_s) ds \right] \right)$$

The proof of the above theorem is provided in section 4.2.

We consider the following regularity condition

$$(H_0) \quad \exists h > 0 \quad \text{s.t.} \quad \forall t \geq 0 \quad \forall x \in S \quad \rho(h) \mu_{t,h}(dy) \leq P_{t,t+h}(x, dy) \leq \rho(h)^{-1} \mu_{t,h}(dy) \quad (1.2)$$

for some probability $\mu_{t,h}$ on S and some constant $\rho(h) > 0$ whose value doesn't depend on the parameters (x, y) . For instance, condition (1.2) is satisfied for jump-type elliptic diffusions on compact manifolds S with a bounded jump rate.

The second main result of the article can be stated basically as follows.

Theorem 1.2 (Particle Gibbs-Glauber dynamics). *For any time horizon $t \geq 0$ the measure \mathbb{Q}_t is reversible w.r.t. the Markov transition \mathbb{K}_t on $D_t(S)$ defined for any bounded measurable function f on $D_t(S)$ and any path $x \in D_t(S)$ by the formula*

$$\mathbb{K}_t(f)(x) := \mathbb{E} \left(m(\zeta_t)(f) \mid \hat{X}_t = x \right)$$

In addition, when (H_0) is satisfied, for any probability measure μ on $D_t(S)$ we have

$$N \text{ osc}(\mathbb{K}_t(f)) \leq c (t \vee 1) \text{ osc}(f) \quad \text{and} \quad \forall n \geq 1 \quad \|\mu \mathbb{K}_t^n - \mathbb{Q}_t\|_{tv} \leq (c (t \vee 1)/N)^n$$

for some finite constant c whose value doesn't depend on the parameters (f, t, n, N) .

The proof of the above theorem is provided in section 4.3.

For any given time horizon $t \geq 0$, the integral operator \mathbb{K}_t is the probability transition of a discrete generation Markov chain $\hat{X}_t^{(n)}$ taking values in the path space $D_t(S)$ and indexed by the integer parameter $n \in \mathbb{N}$. For any given $x \in D_t(S)$ and $z = (z_s)_{s \leq t} \in D_t(\hat{S} \times \hat{S}_{N-1})$, we summarize the transition of the particle Gibbs sampler graphically as follows:

$$\left\{ \begin{array}{l} \hat{X}_t^{(n)} = x \\ \hat{\zeta}_t^{(n)} = z \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \hat{X}_t^{(n+1)} = \bar{x} \sim m(z_t) \\ \hat{\zeta}_t^{(n)} = z \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \hat{X}_t^{(n+1)} = \bar{x} \\ \hat{\zeta}_t^{(n+1)} = \bar{z} \sim (\hat{\zeta}_t \mid \hat{X}_t = \bar{x}) \end{array} \right\}.$$

A realization of the transition $\hat{X}_t^{(n)} \rightsquigarrow \hat{X}_t^{(n+1)}$ for a genealogical tree with $N = 3$ ancestral lines is illustrated by the following schematic diagram:

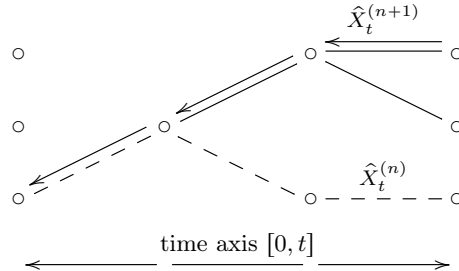


Figure 2: A realization of the transition $\hat{X}_t^{(n)} \rightsquigarrow \hat{X}_t^{(n+1)}$ of a particle Gibbs sampler on an genealogical tree with $N = 3$ ancestral lines. The dotted and plain lines account together for the three paths in $\hat{\zeta}_t^{(n+1)}$, the dotted line represents $\hat{X}_t^{(n)}$, and the sequence of arrows stands for the selected ancestral line $\hat{X}_t^{(n+1)}$.

1.2 Illustrations and comments

This section gives some comments on the impact of the above results on some application domain areas. We also provide a detailed discussion on some numerical aspects of the particle Gibbs-Glauber dynamics introduced above as well as some comparisons with existing literature on interacting particle systems.

- As mentioned in the introduction, the Feynman-Kac measures (1.1) and their mean field particle interpretations appear in wide variety of applications including in biology, physics, as well as in signal processing and mathematical finance.

Continuous time models arise when the process X_t is derived from physical or natural evolution principles, such as continuous time signals in target tracking filtering problems [71], stochastic population dynamics describing species competition and populations growths [51], Langevin gradient-type diffusions including their overdamped versions describing the evolution of a particle in a fluid [52], as well as Brownian fluctuations of atomic structures in molecular chemistry [49], and many others.

The potential function V_t depends on the problem at hand. In nonlinear filtering, it represents the log-likelihood of the robust optimal filter. In population dynamics, V_t can be interpreted as a killing rate of a branching process. In statistical physics and quantum mechanics, it represents the ground state energy (a.k.a. local energy) of a physical system, including molecular and atomic systems. It is clearly out of the scope of the present article to enter into the details of all of these models. For a more thorough discussion on these application domain areas, we refer to the books [21, 22, 33, 36] and the reference therein.

In most cases we are mainly interested in computing the final-time marginal of the Feynman-Kac measures (1.1). For instance, in nonlinear filtering these measures represent the robust optimal filter, while the path space measures represents the full conditional distributions of the random trajectories of the signal w.r.t. the observation process. Thus, they also solve the smoothing problem by estimating the signal states at any given time using observations from larger time intervals. In signal processing literature, the interacting particle system ξ_t discussed above is also known as a particle filter on path space. In this situation, the Particle Gibbs-Glauber dynamics presented in theorem 1.2 allows to improve the precision of these filtering/smoothing approximations by sampling sequentially a series of particle filters on path space with frozen trajectories.

Apart from few notable exceptions such as for linear-Gaussian models in Kalman-Bucy filtering theory and for the harmonic oscillator in the spectral theory of Schrödinger operators, the flow of final-time marginal measures has no finite recursion and cannot be solved analytically. To illustrate our results, we have chosen to describe another rather simple Feynman-Kac model arising in molecular dynamics, and more precisely in the calculation of free energy computations:

Let π_β be some collection of Boltzmann-Gibbs probability measures

$$\pi_\beta(dx) = \frac{1}{\mathcal{Z}_\beta} e^{-\beta H(x)} \lambda(dx) \quad \text{with the normalizing constant} \quad \mathcal{Z}_\beta := \int e^{-\beta H(x)} \lambda(dx).$$

In the above display, β stands for non negative parameter and H some non negative function on some differentiable manifold S equipped with some volume measure $\lambda(dx)$. We also let Y_t^β be some stochastic process with some generator L_β s.t. $\pi_\beta L_\beta = 0$. In other words, π_β is an invariant measure of the process Y_t^β .

Observe that for any sufficiently regular function f and any smooth increasing function $t \mapsto \beta_t$ we have the evolution equation

$$\begin{aligned} \partial_t \pi_{\beta_t}(f) &= \partial_t \beta_t [\pi_{\beta_t}(f) \pi_{\beta_t}(H) - \pi_{\beta_t}(fH)] \\ &= \pi_{\beta_t}(L_{\beta_t}(f)) + \pi_{\beta_t}(f) \pi_{\beta_t}(V_t) - \pi_{\beta_t}(fV_t) \quad \text{with} \quad V_t(x) := \partial_t \beta_t H(x) \end{aligned}$$

The above equation shows that $\pi_{\beta_t} = \eta_t$ coincides with the terminal time marginal η_t of the Feynman-Kac measure (1.1) as soon as $X_t := Y_t^{\beta_t}$ and $X_0 \sim \eta_0 = \pi_{\beta_0}$ (cf. for instance (2.4)). In addition, we easily check the free energy formula

$$\mathcal{Z}_{\beta_t}/\mathcal{Z}_{\beta_0} = Z_t = \mathbb{E} \left(\exp \left[- \int_0^t V_s(X_s) ds \right] \right) = \exp \left[- \int_0^t \eta_s(V_s) ds \right]$$

In physics literature the above formula is often referred as the Jarzynski formula [46, 47], see also [53, 54, 68]. For a detailed proof of the above assertion we refer to section 2.6.2 in [22], section 23.5 and chapter 27 in [36]. In Quantum Monte Carlo literature, the particle system ξ_t discussed above is also known as the population Monte Carlo algorithm and the particles ξ_t^i are often referred as walkers or replica. The quantity

$$\int_0^t V_s(X_s) ds = \int_0^t \partial_s \beta_s H(X_s) ds$$

represents the out-of-equilibrium virtual work of the system on the time horizon t . In this interpretation, the Feynman-Kac measure on path space (1.1) represents the distribution of the out-of-equilibrium random trajectories of the system. In this situation, the Particle Gibbs-Glauber dynamics presented in theorem 1.2 allows to improve these Boltzmann-Gibbs approximations by sampling sequentially a series of population Monte Carlo algorithms on path space with frozen trajectories.

- In some particular instances, the random paths of the process X_t can be sampled exactly on any time discretization mesh. This class of models includes linear-Gaussian and geometric-type Brownian models, as well as some piecewise deterministic processes and some classes of one-dimensional jump-diffusion processes [7, 8, 9, 10, 12]. Discretization-free simulation procedures for general diffusion processes based on sequential importance sampling techniques have also been developed in [43]. In this context, the interacting jump particle systems discussed in this article, including the particle Gibbs-Glauber dynamics can be sampled perfectly using conventional Poisson thinning techniques (a.k.a. Gillespie's algorithm [44]). The resulting particle sampler provides an estimate of the marginal of the Feynman-Kac measures (1.1) on the random paths w.r.t. any time discretization mesh.

- More generally, the simulation of the random trajectories of X_t requires to discretize the time parameter. For a more thorough discussion on the time discretization of stochastic processes we refer to the seminal book by Kloeden and Platen [48].

This additional level of approximation may also corrupt some statistical properties of the continuous time process. For instance, the reversible properties of overdamped Langevin diffusions are lost for any Euler-Maryuama discretization of the underlying diffusion. In this context, a Metropolis-Hastings type adjustment (a.k.a. MALA) is required to recover the reversibility property w.r.t. some prescribed target invariant measure [66]. From the physical viewpoint, the random paths simulated by MALA algorithms are based on *auxiliary non physical rejection-type transitions* so that they lose their initial physical interpretation. Therefore, in physics and statistics, the unadjusted Langevin algorithm (a.k.a. ULA) is often preferred to describe the "true" random trajectories of the system. Under appropriate global Lipschitz conditions on the gradient of the confinement potential function several bias-type estimates can be found in [20, 42].

In the same vein, the sampling of the particle Gibbs-Glauber dynamics described in theorem 1.2 requires some Euler-type discretization as soon as the underlying process X_t cannot be directly sampled. In this situation, one natural strategy is to consider the discrete time version of the Feynman-Kac measures \mathbb{Q}_t defined as in (1.1) by replacing X_t by some discrete time approximation

(see for instance chapter 5 in [22] and the references therein). In this context, several discrete time approximations of the particle Gibbs-Glauber dynamics discussed above can be designed using the discrete time particle Gibbs samplers discussed in [2, 31]. In contrast with MALA algorithms the reversible-type properties of the resulting Gibbs samplers in discrete time are preserved w.r.t. to the discrete-time version of the target Feynman-Kac measures. In addition, these discrete time approximations are not based on any type of auxiliary Metropolis-Hasting rejection so that they preserve their physical interpretations.

Several bias-type estimates between continuous and discrete time Feynman-Kac measures can be found in [22, 29, 30]. Most of these estimates are concerned with the time discretization of the terminal-time marginal of the Feynman-Kac measures (1.1), including uniform estimates w.r.t. the time horizon. The extension of these results to path space models remains an important open research question.

- The interacting particle systems discussed in the present article differ from nonlinear and interacting diffusion processes arising in fluid mechanics and granular flows [4, 5, 59, 60, 72, 73]. In this context, the interaction mechanism is encapsulated in the drift of diffusion-type particles. One common feature of these interacting processes is the nonlinearity of the distribution flow associated with these stochastic processes.

One natural idea is to interpret the mean field particle systems associated with these processes as a stochastic perturbation of a nonlinear process. This interpretation allows to enter the stability properties of the nonlinear process into the convergence analysis of these particle algorithms. This technique has been developed in [25, 26, 33] for discrete time Feynman-Kac models and further extended in [67] to continuous time models. Theorem 3.7 in the present article also provides a novel backward stochastic perturbation formula which simplifies the stability analysis of these models and provides sharp propagation of chaos estimates.

We underline that the stochastic perturbation techniques discussed above and in the present article differs from the log-Sobolev functional techniques [57, 58], entropy dissipation approaches [15, 17], as well as gradient flows in Wasserstein metric spaces, optimal transportation inequalities [6, 15, 16, 64, 65] and the more recent variational approach [3] currently used in the analysis of gradient type flow interacting diffusions.

In this connection, we mention that the backward perturbation analysis developed in the present article relies on weak Taylor expansions of the evolution semigroup of Feynman-Kac measures. We project to extend these expansions to nonlinear diffusions in a forthcoming article.

The duality formula and the particle Gibbs-Glauber dynamics introduced in this article open up a whole new avenue of research questions.

Recall that the Feynman-Kac measures (1.1) can be interpreted as the distribution of the random paths of a non absorbed particle evolving as X_t and killed at rate V_t . This class of models are often referred as particle models in absorbing medium with soft obstacles [24, 33, 35]. A natural research project is to extend this framework to absorbing medium with hard obstacles [38, 39, 74].

Another important question is to extend the Taylor expansions of the Gibbs sampler developed in [31] to continuous time models. One possible route is to combine the weak Taylor expansions developed in [37] for particle approximating measures with the backward analysis developed in the present article.

We mention that the perturbation analysis developed in [31] allows to destimate the \mathbb{L}_p -decays rates to equilibrium in terms of the norm of integral operators. In this connection, one important question is to quantify with more precision the exponential convergence rates to equilibrium of the Particle Gibbs-Glauber dynamics stated in theorem 1.2.

1.3 Basic notation and preliminary results

Let $\mathcal{B}(S)$ be the Banach space of bounded functions f on S equipped with the uniform norm $\|f\| := \sup_{x \in S} |f(x)|$. Also let $\text{Osc}(S) \subset \mathcal{B}(S)$ be the subset of functions f with unit oscillations; that is s.t. $\text{osc}(f) := \sup_{x,y} |f(x) - f(y)| \leq 1$.

We also let $\mathcal{M}(S)$ the set of finite signed measures on S , $\mathcal{M}_+(S) \subset \mathcal{M}(S)$ the subset of positive measures and $\mathcal{P}(S) \subset \mathcal{M}_+(S)$ the subset of probability measures. Given a random measure μ on S we write $\mathbb{E}(\mu)$ the first moment measure given by

$$\mathbb{E}(\mu) : f \in \mathcal{B}(S) \mapsto \mathbb{E}(\mu)(f) = \mathbb{E}(\mu(f)) \quad \text{with} \quad \mu(f) = \int \mu(dx) f(x)$$

The total variation norm on the set $\mathcal{M}(S)$ is defined by

$$\|\mu\|_{tv} := \sup \{ |\mu(f)| : f \in \text{Osc}(S) \} \quad (1.3)$$

1.3.1 Integral operators

For any bounded positive integral operator $Q(x, dy)$ and any $(\mu, f, x) \in (\mathcal{M}(S) \times \mathcal{B}(S) \times S)$ we define by $\mu Q \in \mathcal{M}(S)$ and $Q(f) \in \mathcal{B}(S)$ by the formulae

$$(\mu Q)(dy) := \int \mu(dx) Q(x, dy) \quad \text{and} \quad Q(f)(x) := \int Q(x, dy) f(y)$$

By Fubini theorem we have $\mu Q f := \mu(Q(f)) = (\mu Q)(f)$. We also write Q^n the n iterate of Q defined by the recursion $Q^n(f) = Q(Q^{n-1}(f)) = Q^{n-1}(Q(f))$.

When $Q(1) > 0$ we let \bar{Q} be the Markov operator

$$\bar{Q} : f \in \mathcal{B}(S) \mapsto \bar{Q}(f) := Q(f)/Q(1) \in \mathcal{B}(S)$$

We also let ϕ be the mapping from $\mathcal{P}(S)$ into itself defined by

$$\phi(\eta) = \eta Q^\eta \quad \text{with} \quad Q^\eta := \frac{Q}{\eta Q(1)} \implies \eta Q^\eta(1) = 1 \quad \text{and} \quad \phi(\delta_x)(f) = \bar{Q}(f)(x) \quad (1.4)$$

Notice that

$$Q^\eta(1) = \mu Q^\eta(1) Q^\mu(1) \implies (\mu Q^\eta(1))^{-1} = \eta Q^\mu(1)$$

1.3.2 Taylor expansions

Observe that for any $\eta, \nu \in \mathcal{P}(S)$ we have the decomposition

$$\phi(\nu) - \phi(\eta) = \eta Q^\nu(1) \times (\nu - \eta) \partial_\eta \phi$$

with the first order operator

$$\partial_\eta \phi : f \in \mathcal{B}(S) \mapsto \partial_\eta \phi(f) = Q^\eta [f - \phi(\eta)(f)] \in \mathcal{B}(S) \implies \eta \partial_\eta \phi = 0 = \partial_\eta \phi(1) \quad (1.5)$$

Also observe that

$$\partial_\eta \phi(f)(x) = Q^\eta(1)(x) \int \eta(dy) Q^\eta(1)(y) (\bar{Q}(f)(x) - \bar{Q}(f)(y)) \quad (1.6)$$

$$\implies \|\partial_\eta \phi\| \leq \|Q^\eta(1)\| \text{osc}(\bar{Q}(f)) \quad \text{and} \quad \|\phi(\nu) - \phi(\eta)\|_{tv} \leq [\|Q^\nu(1)\| \wedge \|Q^\eta(1)\|] \text{osc}(\bar{Q}(f))$$

More generally, using the identity

$$\frac{1}{x} = \sum_{0 \leq k < n} (1-x)^k + \frac{(1-x)^n}{x} \quad (1.7)$$

which is valid for any $x > 0$ and $n \geq 1$, we check the Taylor with remainder expansion

$$\phi(\nu) = \phi(\eta) + \sum_{1 \leq k \leq n} \frac{1}{k!} (\nu - \eta)^{\otimes k} \partial_\eta^k \phi + \frac{1}{(n+1)!} (\nu - \eta)^{\otimes (n+1)} \bar{\partial}_{\nu, \eta}^{n+1} \phi \quad (1.8)$$

In the above display, $\partial_\eta^k \phi$ stand for the collection of integral operators

$$\partial_\eta^k \phi(f) := (-1)^{k-1} k! \left[Q^\eta(1)^{\otimes (k-1)} \otimes \partial_\eta \phi(f) \right] \quad \text{and} \quad \bar{\partial}_{\nu, \eta}^{n+1} \phi := \eta Q^\nu(1) \partial_\eta^{n+1} \phi$$

For any $\mu, \eta \in \mathcal{P}(S)$ we have the decomposition

$$\partial_\eta \phi(f) = Q^\eta[f - \phi(\eta)f] = \mu Q^\eta(1) (\partial_\mu \phi(f) + Q^\mu(1) [\phi(\mu) - \phi(\eta)](f))$$

1.3.3 Carré du champ operators

The carré du champ operator associated with some the generator L acting on an algebra of functions $\mathcal{D}(S) \subset \mathcal{B}(S)$ is defined by the quadratic form

$$(f, g) \in \mathcal{D}(S)^2 \mapsto \Gamma_L(f, g) = L(fg) - fL(g) - gL(f) \in \mathcal{B}(S)$$

When $f = g$ sometimes we write $\Gamma_L(f)$ instead of $\Gamma_L(f, f)$. We also recall the Cauchy-Schwartz inequality

$$|\Gamma_L(f, g)| \leq \sqrt{\Gamma_L(f, f) \Gamma_L(g, g)} \quad \text{and} \quad \Gamma_L(cf) = c^2 \Gamma_L(f) \quad (1.9)$$

The above inequality yields the estimate

$$\Gamma_L(f + g) = \Gamma_L(f) + \Gamma_L(g) + 2\Gamma_L(f, g) \leq \left[\sqrt{\Gamma_L(f)} + \sqrt{\Gamma_L(g)} \right]^2 \quad (1.10)$$

Let L^d be some bounded jump-type generator of the following form

$$L^d(f)(u) = \lambda(u) \int (f(v) - f(u)) J(u, dv)$$

for some bounded rate function λ and some Markov transition J on S . In this case, we have

$$\Gamma_{L^d}(f, g)(u) = \int L^d(u, dv) (\delta_v - \delta_u)^{\otimes 2} (f \otimes g)$$

We consider the n -th order operators

$$\Gamma_{L^d}^{(n)}(f_1, \dots, f_n)(u) := \int L^d(u, dv) (\delta_v - \delta_u)^{\otimes n} (f_1 \otimes \dots \otimes f_n) \quad (1.11)$$

We also have the carré du champ formula

$$(\eta Q^\mu(1))^2 \Gamma_L(Q^\eta(1), \partial_\eta \phi(f)) = \Gamma_L(Q^\mu(1), \partial_\mu \phi(f)) + [\phi(\mu) - \phi(\eta)](f) \Gamma_L(Q^\mu(1)) \quad (1.12)$$

for any $f \in \mathcal{D}(S)$ as soon as $Q^\eta(1), \partial_\eta \phi(f) \in \mathcal{D}(S)$.

1.3.4 Empirical measures

We fix some integer $N \geq 2$ and for any $2 \leq i < j \leq N$ and $x = (x^i)_{1 \leq i \leq N} \in S_N$ we set

$$\begin{aligned} x^{-i} &= (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^N) \in S_{N-1} \\ x^{-\{i,j\}} &= (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{j-1}, x^{j+1}, \dots, x^N) \in S_{N-2} \end{aligned}$$

For any $2 \leq i \leq N$ and $x = (x^i)_{1 \leq i \leq N} \in S_N$ we consider the functions

$$\begin{aligned} \varphi_{x^{-i}} : u \in S &\mapsto \varphi_{x^{-i}}(u) = (x^1, \dots, x^{i-1}, u, x^{i+1}, \dots, x^N) \in S^N \\ m : x \in S_N &\mapsto m(x) = \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{x^i} \in \mathcal{P}(S) \end{aligned} \quad (1.13)$$

Let $X = (X^i)_{1 \leq i \leq N}$ be N independent random samples from some distribution $\eta \in \mathcal{P}(S)$. Using (1.8) we have the first order expansion

$$\phi(m(X)) - \phi(\eta) = (m(X) - \eta) \partial_\eta \phi - \eta(Q^{m(X)}(1)) (m(X) - \eta)(Q^\eta(1)) (m(X) - \eta) \partial_\eta \phi$$

Several estimates can be derived from the above decomposition. For instance using Cauchy-Schwartz inequality we have the bias estimate

$$\log(Q(1)(x)/Q(1)(y)) \leq q \implies N |\mathbb{E}[\phi(m(X))(f)] - \phi(\eta)(f)| \leq e^q \text{osc}(\bar{Q}(f))$$

2 A brief review on Feynman-Kac measures

2.1 Evolution semigroups

Consider the flow of Feynman-Kac measures $(\gamma, \eta) : t \in \mathbb{R}_+ := [0, \infty[\mapsto (\gamma_t, \eta_t) \in (\mathcal{M}_+(S) \times \mathcal{P}(S))$ defined for any $f \in \mathcal{B}(S)$ by the formulae

$$\eta_t(f) = \gamma_t(f)/\gamma_t(1) \quad \text{with} \quad \gamma_t(f) := \mathbb{E}(f(X_t)Z_t(X)) \quad (2.1)$$

In the above display, $Z_t(X)$ stands for the exponential weight

$$Z_t(X) := \exp \left[- \int_0^t V_s(X_s) ds \right] \implies \log \mathbb{E}(Z_t(X)) = - \int_0^t \eta_s(V_s) ds$$

This shows that

$$\bar{Z}_t(X) := Z_t(X)/\mathbb{E}(Z_t(X)) = \exp \left[- \int_0^t \bar{V}_s(X_s) ds \right] \quad \text{with} \quad \bar{V}_t := V_t - \eta_t(V)$$

We also consider the Feynman-Kac semigroup

$$Q_{s,t}(f)(x) = \mathbb{E}(f(X_t) Z_{s,t}(X) | X_s = x) \quad \text{with} \quad Z_{s,t}(X) := Z_t(X)/Z_s(X) \quad (2.2)$$

When $V = 0$ the semigroup $Q_{s,t}$ resumes to the Markov semigroup $P_{s,t}$ of the reference process X_t .

The mathematical model defined above is called the Feynman-Kac model associated with the reference process and the potential function (X_t, V_t) .

We further assume that the (infinitesimal) generators L_t of X_t are well defined on some common sub-algebra $\mathcal{D}(S) \subset \mathcal{B}(S)$, and for any $s < t$ we have $Q_{s,t}(\mathcal{B}(S)) \subset \mathcal{D}(S)$.

We let $\mathcal{V}_t(f) = V_t f$ the multiplication operator on $\mathcal{B}(S)$. We also let $L_t = L_t^c + L_t^d$ be the decomposition of the generator L_t in terms of a diffusion-type operator L_t^c and a bounded jump-type generator of the following form

$$L_t^d(f)(u) = \lambda_t(u) \int (f(v) - f(u)) J_t(u, dv)$$

for some bounded rate function λ_t and some Markov transition J_t on S .

In this notation, for any $f \in \mathcal{D}(S)$ and $s \leq t$ we have

$$\partial_t \gamma_t(f) = \gamma_t(L_t^V(f)) \quad \text{with} \quad L_t^V = L_t - \mathcal{V}_t \implies \gamma_t = \gamma_s Q_{s,t} \quad (2.3)$$

The semigroup associated with the normalized Feynman-Kac measures η_t is given for any $s \leq t$ by the formula

$$\eta_t = \phi_{s,t}(\eta_s) := \frac{\eta_s Q_{s,t}}{\eta_s Q_{s,t}(1)} \implies \partial_t \eta_t(f) = \Lambda_t(\eta_t)(f) := \eta_t(L_t^V(f)) + \eta_t(V_t) \eta_t(f) \quad (2.4)$$

with the collection of functional linear operators

$$\Lambda_t(\eta) : f \in \mathcal{D}(S) \mapsto \Lambda_t(\eta)(f) := \eta(L_t^V(f)) + \eta(V_t) \eta(f) \in \mathbb{R}$$

Finally we recall that $\eta_t = \text{Law}(\bar{X}_t)$ can be interpreted as the law of a nonlinear Markov process \bar{X}_t associated with the collection of generators L_{t,η_t} defined for any $(\eta, f, x) \in (\mathcal{P}(S), \mathcal{D}(S) \times S)$ by

$$L_{t,\eta}(f)(x) = L_t(f)(x) + V(x) \int (f(y) - f(x)) \eta(dy) \implies \Lambda_t(\eta) = \eta L_{t,\eta} \quad (2.5)$$

2.2 Path space measures

Consider a Feynman-Kac model $(\gamma'_t, \eta'_t, Q'_{s,t}, \mathbb{Q}'_t, \dots)$ associated with some auxiliary Markov process X'_t on some metric space $(S', d_{S'})$, and some bounded non negative potential functions V'_t on S' . Also let L'_t be the generator of X'_t defined on some common sub-algebra $\mathcal{D}(L') \subset \mathcal{B}(S')$.

Assume that the process X_t discussed in (2.1) is the historical process

$$X_t := (X'_s)_{s \leq t} \in S := \cup_{t \geq 0} D_t(S') \quad \text{and} \quad V_t(X_t) := V'_t(X'_t) \implies \eta_t = \mathbb{Q}'_t \quad (2.6)$$

In this situation, the generator L_t and the domain $\mathcal{D}(S)$ of the historical process can be defined in two different ways:

The more conventional approach is to consider cylindrical functions

$$f(X_t) = \varphi(X'_{s_1 \wedge t}, \dots, X'_{s_n \wedge t})$$

that only depend on a finite collection of time horizons $s_i \leq s_{i+1}$, with $1 \leq i < n$, and some bounded functions φ from $(S')^n$ into \mathbb{R} . The regularity of the "test" function φ depends on the process at hand. For jump-type processes, no additional regularity is required. For diffusion-type processes the function is often required to be compactly supported and twice differentiable.

Another elegant and more powerful approach is to use the functional Itô calculus theory introduced by B. Dupire in an unpublished article [41], and further developed in [18, 45]. This path-dependent stochastic calculus allows to consider more general functions such as running integrals or running maximum of the process X'_t . It also allows to consider diffusion-type processes with a drift and a diffusion term that depends on the history of the process.

The path space S is equipped with a time-space metric d_S so that (S, d_S) is a complete and separable metric space (cf. for instance proposition 1.1.13 and theorem 1.1.15 in [70]). The smoothness properties of continuous function f on S are defined in terms of time and space functional derivatives. Thus, for diffusion-type historical processes X_t , the generator L_t is defined on functions $f \in \mathcal{D}(L)$ which are differentiable w.r.t. the time parameter and, as before twice differentiable with compactly supported derivatives (cf. for instance theorem 1.3.1 in [70]).

It is clearly not the scope of the article to describe in full details the above functional Itô calculus. We refer the reader to the article [45] and the Ph.D thesis of Saporito [70].

In the further development of the article we shall use these ideas back and forth. We already mention that the mean field particle interpretation of the Feynman-Kac measures associated with an historical process coincides with the genealogical tree-based particle evolution of the marginal model.

2.3 Some regularity conditions

This section discusses in some details the two main regularity conditions used in the further development of the article.

Firstly, observe that the semigroup $P_{s,t}$ associated with the historical process $X_t = (X'_s)_{s \leq t}$ discussed in (2.6) never satisfies the regularity condition (H_0) stated in (1.2). Nevertheless it may happen that the semigroup $P'_{s,t}$ associated with X'_t satisfies condition (H_0) . In this situation, to avoid repetition or unnecessary long discussions we simply say that (H'_0) is met.

We also use the following weaker conditions:

$$(H_1) \quad \exists \alpha < \infty \quad \exists \beta > 0 \quad \text{s.t.} \quad \forall s \leq t \quad \text{osc}(\overline{Q}_{s,t}(f)) \leq \alpha e^{-\beta(t-s)} \text{osc}(f)$$

$$(H_2) \quad \exists q < \infty \quad \text{s.t.} \quad \forall s \leq t \quad \forall x, y \in S \quad \log(Q_{s,t}(1)(x)/Q_{s,t}(1)(y)) \leq q$$

As before when the semigroup $\overline{Q}'_{s,t}$ and $Q'_{s,t}$ of Feynman-Kac model associated with some parameters (X'_t, V'_t) satisfy condition (H'_i) , to avoid repetition or unnecessary long discussions we simply say that (H'_i) is met. We recall that

$$(H_0) \implies (H_1) \implies (H_2)$$

The proof of the l.h.s. assertion can be found in [34]. In this context, the parameters (α, β) don't depend on the measure $\mu_{t,h}$ discussed in (1.2). To check the second we observe that

$$\log(Q_{s,t}(1)(x)/Q_{s,t}(1)(y)) = \int_s^t [\phi_{s,u}(\delta_y)(V_u) - \phi_{s,u}(\delta_x)(V_u)] du \quad (2.7)$$

This implies that

$$(H_1) \implies (H_2) \quad \text{with} \quad q = \alpha \beta^{-1} \text{osc}(V) \quad \text{with} \quad \text{osc}(V) = \sup_{t \geq 0} \text{osc}(V_t)$$

Using (1.6) we also have

$$\begin{aligned} (H_2) &\implies \|\partial_\eta \phi_{s,t}(f)\| \leq e^q \text{osc}(f) \quad (\text{since } \text{osc}(\overline{Q}_{s,t}(f)) \leq \text{osc}(f)) \\ (H_1) &\implies \|\partial_\eta \phi_{s,t}(f)\| \leq r e^{-\beta(t-s)} \text{osc}(f) \quad \text{with} \quad r = \alpha e^q \quad \text{and} \quad q = \alpha \beta^{-1} \text{osc}(V) \end{aligned} \quad (2.8)$$

We return to the historical process $X_t = (X'_s)_{s \leq t}$ discussed in (2.6). In this case, for any $x_s = (x'(u))_{u \leq s} \in D_s(S')$ we have

$$Q_{s,t}(f)(x_s) = Q'_{s,t}(f')(x'_s)$$

in the above display, f and f' stand for some bounded measurable functions on the path space $D_t(S')$ and on S' such that

$$\forall y_t = (y'(u))_{u \leq t} \in D_t(S') \quad f(y_t) = f'(y'_t)$$

This implies that

$$(H'_1) \implies (H_2) \quad \text{is met with } q = \alpha\beta^{-1} \text{ osc}(V) \quad \text{and} \quad \|\partial_\eta \phi_{s,t}(f)\| \leq e^q \text{ osc}(f) \quad (2.9)$$

2.4 Forward and backward equations

Proposition 2.1. *For any $s \leq t$ and $\eta \in \mathcal{P}(S)$ we have the Gelfand-Pettis forward and backward differential equations*

$$\partial_t \phi_{s,t}(\eta_s) = \Lambda_t(\phi_{s,t}(\eta_s)) \quad \text{and} \quad \partial_s \phi_{s,t}(\eta) = -\Lambda_s(\eta) \partial_\eta \phi_{s,t} \quad (2.10)$$

In addition, for any mapping ϕ of the form (1.4) we also have

$$\partial_t \phi(\phi_{s,t}(\eta)) = \Lambda_t(\eta) \partial_{\phi_{s,t}(\eta)} \phi \quad \text{and} \quad \partial_s \phi(\phi_{s,t}(\eta)) = -\Lambda_s(\eta) \partial_{\phi_{s,t}(\eta)} \phi \quad (2.11)$$

Proof. The l.h.s. assertion in (2.10) is a direct consequence of (2.4). Applying these decompositions to $\phi_{s,t}$, for any $s+h \leq t$ we find that

$$\begin{aligned} & \phi_{s+h,t}(\eta + [\phi_{s,s+h}(\eta) - \eta]) \\ &= \phi_{s+h,t}(\eta) + [\phi_{s,s+h}(\eta) - \eta] \circ \partial_\eta \phi_{s+h,t} + \frac{1}{\phi_{s,s+h}(\eta) Q^\eta(1)} \frac{1}{2} [\phi_{s,s+h}(\eta) - \eta]^{\otimes 2} \circ \partial_\eta^2 \phi_{s+h,t} \end{aligned}$$

On the other hand we have

$$\phi_{s,s+h}(\eta) = \eta + \Lambda_s(\eta) h + O(h^2) \quad \text{and} \quad \phi_{s,s+h}(\eta) Q^\eta(1) = 1 + O(h)$$

This yields the backward evolution formula

$$h^{-1} [\phi_{s+h,t}(\eta) - \phi_{s,t}(\eta)] \xrightarrow{h \rightarrow 0} \partial_s \phi_{s,t}(\eta) = -\Lambda_s(\eta) \partial_\eta \phi_{s,t}$$

For any mapping ϕ of the form (1.4) we also have

$$\begin{aligned} & \phi(\phi_{s+h,t}(\eta)) - \phi(\phi_{s,t}(\eta)) = (\phi_{s+h,t}(\eta) - \phi_{s,t}(\eta)) \partial_{\phi_{s,t}(\eta)} \phi \\ & + \frac{1}{2} (\phi_{s+h,t}(\eta) - \phi_{s,t}(\eta))^{\otimes 2} \partial_{\phi_{s,t}(\eta)}^2 \phi + \frac{1}{\phi_{s+h,t}(\eta) Q^{\phi_{s,t}(\eta)}(1)} \frac{1}{3} (\phi_{s+h,t}(\eta) - \phi_{s,t}(\eta))^{\otimes 3} \circ \partial_{\phi_{s,t}(\eta)}^3 \phi \end{aligned}$$

Arguing as above we check (2.11). This ends the proof of the proposition. \blacksquare

2.5 Mean field particle systems

Let $\mathcal{B}(S_N) \subset \mathcal{B}(S^N)$ be the subset of symmetric functions on S^N , and $\mathcal{B}(S \times S_{N-1}) \subset \mathcal{B}(S^N)$ be the set of functions F on S^N symmetric with respect to the last $(N-1)$ arguments. Also let $\mathcal{D}(S^N) \subset \mathcal{B}(S^N)$ be the set of functions $F \in \mathcal{B}(S^N)$ s.t. for any $x \in S_N$ we have

$$F_{x^{-i}} := F \circ \varphi_{x^{-i}} \in \mathcal{D}(S)$$

with the functions $\varphi_{x^{-i}}$ and the set $\mathcal{D}(S)$ introduced in (1.13) and (2.3).

Also let $\mathcal{D}(S_N) \subset \mathcal{B}(S_N)$, resp. $\mathcal{D}(S \times S_{N-1}) \subset \mathcal{B}(S \times S_{N-1})$ the trace of $\mathcal{D}(S^N)$ on $\mathcal{B}(S_N)$, resp. $\mathcal{B}(S \times S_{N-1})$.

Definition 2.2. The N -mean field particle interpretation of the nonlinear process discussed in (2.5) is defined by the Markov process $\xi_t = (\xi_t^i)_{1 \leq i \leq N} \in S_N$ with generators \mathcal{G}_t given for any $F \in \mathcal{D}(S_N)$ and any $x = (x^i)_{1 \leq i \leq N} \in S_N$ by

$$\mathcal{G}_t(F)(x) = \sum_{1 \leq i \leq N} L_{t,m(x)}(F_{x^{-i}})(x^i) \quad (2.12)$$

We let $\mathcal{F} := (\mathcal{F}_t)_{t \geq 0}$, with $\mathcal{F}_t = \sigma(\xi_u : u \leq t)$ be the filtration generated by the mean field particle model defined in (2.12).

We let $D([0, T], S_N)$ be the set of function $F : (t, x) \in ([0, T] \times S_N) \mapsto F_t(x) \in \mathbb{R}$ with a bounded derivative w.r.t. the first argument and s.t. $F_t \in D(S_N)$. For any $F \in D([0, T], S_N)$, and any $T \geq 0$, we have

$$dF_t(\xi_t) = [\partial_t F_t + \mathcal{G}_t(F_t)](\xi_t) dt + d\mathcal{M}_t(F)$$

In the above display \mathcal{M}_t stands for a martingale random field on $D([0, T], S_N)$ with angle bracket defined for any functions $F, G \in D([0, T], S_N)$ and any time horizon $t \in [0, T]$ by the formula

$$\partial_t \langle \mathcal{M}(F), \mathcal{M}(G) \rangle_t = \Gamma_{\mathcal{G}_t}(F_t, G_t)(\xi_t)$$

Choosing functions of the form

$$F_t(x) = m(x)(f_t) \quad \text{and} \quad G_t(x) = m(x)(g_t) \implies \Gamma_{\mathcal{G}_t}(F_t, G_t)(\xi_t) = m(\xi_t) \Gamma_{L_{t,m(\xi_t)}}(f_t, g_t) \quad (2.13)$$

we also check that the occupation measure $m(\xi_t) \in \mathcal{P}(S)$ satisfies the stochastic equation

$$dm(\xi_t)(f_t) = [m(\xi_t)(\partial_t f_t) + \Lambda_t(m(\xi_t))(f_t)] dt + \frac{1}{\sqrt{N}} dM_t(f) \quad (2.14)$$

with a martingale random field M_t on $D([0, T], S)$ with angle brackets by the formula

$$\begin{aligned} & \partial_t \langle M(f), M(g) \rangle_t \\ &= m(\xi_t) (\Gamma_{L_t}(f_t, g_t)) + \int m(\xi_t)(du) m(\xi_t)(dv) V_t(u) (f_t(v) - f_t(u))(g_t(v) - g_t(u)) \end{aligned}$$

With a slight abuse of notation we also write M_t the extension of the random field M_t to \mathcal{F} -predictable functions $D([0, T], S)$.

In the further development of the article we write (M_t^c, \mathcal{M}_t^c) and (M_t^d, \mathcal{M}_t^d) the continuous and the discontinuous part of the martingales (M_t, \mathcal{M}_t) ; as well as

$$L_{t,\eta} = L_{t,\eta}^c + L_{t,\eta}^d \quad \text{with} \quad L_{t,\eta}^c = L_t^c$$

The angle bracket of \mathcal{M}_t^d is given for any functions $F, G \in D([0, T], S_N)$ and any time horizon $t \in [0, T]$ by the formula

$$\begin{aligned} & \partial_t \langle \mathcal{M}^d(F), \mathcal{M}^d(G) \rangle_t \\ &= \sum_{1 \leq i \leq N} \int \left[F_{t,\xi_t^{-i}}(v) - F_t(\xi_t) \right] \left[G_{t,\xi_t^{-i}}(v) - G_t(\xi_t) \right] \left[V_t(\xi_t^i) m(\xi_t)(dv) + \lambda_t(\xi_t^i) J_t(\xi_t^i, dv) \right] \end{aligned}$$

Definition 2.3. Let $\zeta_t = (\zeta_t^i)_{1 \leq i \leq N} \in (S \times S_{N-1})$ be the Markov process with initial condition $\zeta_0 = \xi_0$ and generators \mathcal{H}_t defined for any $F \in \mathcal{D}(S \times S_{N-1})$ and $x = (x^i)_{1 \leq i \leq N} \in (S \times S_{N-1})$ by

$$\begin{aligned} & \mathcal{H}_t(F)(x) \\ &= L_t(F_{x^{-1}})(x^1) + \sum_{2 \leq i \leq N} \left(L_t(F_{x^{-i}})(x^i) + V_t(x^i) \int (F_{x^{-i}}(u) - F(x)) m_{x^1}(x^{-\{1,i\}})(du) \right) \end{aligned}$$

with the empirical probability measures

$$m_{x^1}(x^{-\{1,i\}}) := \left(1 - \frac{2}{N}\right) m(x^{-\{1,i\}}) + \frac{2}{N} \delta_{x^1}$$

Theorem 2.4. Given the historical process $\widehat{\zeta}_t^1$ the process $(\zeta_s^-)_{s \leq t}$ coincides with the $(N-1)$ -mean field interpretation (2.12) of the Feynman-Kac model $(\eta_s^-)_{s \leq t}$ defined as in (2.1) and (2.12) by replacing (L_s, V_s) by (L_s^-, V_s^-) , with the jump generator

$$L_s^-(f)(u) = L_s(f)(u) + \frac{2}{N} V_s(u) (f(\zeta_s^1) - f(u)) \quad \text{and} \quad V_s^- := \left(1 - \frac{1}{N}\right) V_s$$

Proof. By construction, the generators \mathcal{G}_s^- of the process $(\zeta_s^-)_{s \leq t}$ given $\widehat{\zeta}_t^1$ are defined for any $s \leq t$, any $F \in \mathcal{D}(S_{N-1})$ and any $x = (x^i)_{1 \leq i < N} \in S_{N-1}$ by the formula

$$\mathcal{G}_s^-(F)(x) = \sum_{1 \leq i < N} L_s(F_{x^{-i}})(x^i) + \sum_{1 \leq i < N} V_s(x^i) \int (F_{x^{-i}}(u) - F(x)) m_{\zeta_s^1}(x^{-i})(du)$$

Observe that for any $x = (x^i)_{1 \leq i < N} \in S_{N-1}$ and $y \in S$ we have

$$\begin{aligned} & \int (F_{x^{-i}}(u) - F(x)) m_y(x^{-i})(du) \\ &= \left(1 - \frac{1}{N}\right) \int (F_{x^{-i}}(u) - F(x)) m(x)(du) + \frac{2}{N} (F_{x^{-i}}(y) - F(x)) \end{aligned}$$

This implies that

$$\begin{aligned} & \sum_{1 \leq i < N} V_s(x^i) \int (F_{x^{-i}}(u) - F(x)) m_y(x^{-\{i\}})(du) \\ &= \sum_{1 \leq i < N} V_s^-(x^i) \int (F_{x^{-i}}(u) - F(x)) m(x)(du) + \frac{2}{N} \sum_{1 \leq i < N} V_s(x^i) (F_{x^{-i}}(y) - F(x)) \end{aligned}$$

We conclude that

$$\mathcal{G}_s^-(F)(x) = \sum_{1 \leq i < N} \left[L_s^-(F_{x^{-i}})(x^i) + V_s^-(x^i) \int (F_{x^{-i}}(u) - F(x)) m(x)(du) \right]$$

This ends the proof of the theorem. ■

3 Perturbation analysis

3.1 Semigroup estimates

We consider a collection of generators L_t^ϵ and potential functions V_t^δ of the form

$$L_t^\epsilon = L_t + \epsilon \bar{L}_t \quad \text{and} \quad V_t^\delta = V_t + \delta \bar{V}_t \quad \text{with} \quad \epsilon, |\delta| \in [0, 1]$$

In the above display, \bar{V}_t stands for some uniformly bounded function and \bar{L}_t a bounded generator of an auxiliary jump type Markov process of the form

$$\bar{L}_t(f)(x) = \lambda(x) \int (f(y) - f(x)) K_t(x, dy)$$

for some jump rate function $\lambda(x)$ and some Markov transitions $K_t(x, dy)$ such that

$$\lambda_1 \leq \lambda(x) \leq \lambda_2 \quad \text{and} \quad \varpi_1 \kappa_t(dy) \leq K_t(x, dy) \leq \varpi_2 \kappa_t(dy)$$

In the above display, λ_i, ϖ_i stands for some positive parameters and κ_t some probability measures.

We let $P_{s,t}^\epsilon$ be the transition semigroup of the process with generator L_t^ϵ . In this notation, we have the following technical lemma.

Lemma 3.1. *Assume that $P_{s,t}$ satisfies (H_0) for some parameters h and $\rho(h) > 0$ and some probability measures $\mu_{t,h}$. In this situation, for any $\epsilon \in [0, 1]$ and $t \geq 0$ there exists some probability measures $\mu_{t,h}^\epsilon$ such that*

$$\rho_\epsilon(h) \mu_{t,h}^\epsilon(dy) \leq P_{t,t+h}^\epsilon(x, dy) \leq \rho_\epsilon(h)^{-1} \mu_{t,h}^\epsilon(dy) \quad (3.1)$$

with the parameters

$$\begin{aligned} \rho_\epsilon(h) &:= \rho(h) \left(e^{-\epsilon\lambda_2 h} + (1 - e^{-\epsilon\lambda_2 h}) \varpi_2 \right) \min \left((\lambda_1/\lambda_2)(\varpi_1/\varpi_2), e^{-\epsilon(\lambda_2 - \lambda_1)h} \right) \\ &\geq \rho(h) \min \left((\lambda_1/\lambda_2)(\varpi_1/\varpi_2), e^{-(\lambda_2 - \lambda_1)h} \right) \end{aligned}$$

The proof of the above lemma is provided in the appendix on page 26.

We consider the Feynman-Kac semigroup $Q_{s,t}^{\delta,\epsilon}$ be defined as $Q_{s,t}$ by replacing V_t by V_t^δ and X_t by a Markov process with generator L_t^ϵ .

Also let $\phi_{s,t}^{(\delta,\epsilon)}$ be defined as $\phi_{s,t}$ by replacing $Q_{s,t}$ by $Q_{s,t}^{\delta,\epsilon}$, and set

$$L_t^{\delta,\epsilon} = \epsilon \bar{L}_t - \delta \bar{V}_t \quad \text{and} \quad L_{t,\eta}^{\delta,\epsilon} = \epsilon \bar{L}_t - \delta(\bar{V}_t - \eta(\bar{V}_t))$$

Theorem 3.2. *For any $|\epsilon|, |\delta| \in [0, 1]$ and any $s \leq t$ we have the semigroup perturbation formulae*

$$Q_{s,t}^{\delta,\epsilon} - Q_{s,t} = \int_s^t Q_{s,u}^{\delta,\epsilon} L_u^{\delta,\epsilon} Q_{u,t} du = \int_s^t Q_{s,u} L_u^{\delta,\epsilon} Q_{u,t}^{\delta,\epsilon} du \quad (3.2)$$

In addition, for any $\eta \in \mathcal{P}(S)$ we have

$$\phi_{s,t}^{\delta,\epsilon}(\eta) - \phi_{s,t}(\eta) = \int_s^t \phi_{s,u}^{\delta,\epsilon}(\eta) L_{u,\phi_{s,u}^{\delta,\epsilon}(\eta)}^{\delta,\epsilon} \partial_{\phi_{s,u}^{\delta,\epsilon}(\eta)} \phi_{u,t} du = \int_s^t \phi_{s,u}(\eta) L_{u,\phi_{s,u}(\eta)}^{\delta,\epsilon} \partial_{\phi_{s,u}(\eta)} \phi_{u,t}^{\delta,\epsilon} du$$

Proof. We check (3.2) the fact that

$$\partial_u(Q_{s,u}^{\delta,\epsilon} Q_{u,t}) = Q_{s,u}^{\delta,\epsilon} (L_u^\epsilon - L_u - \delta \bar{V}_u) Q_{u,t} = \epsilon Q_{s,u}^{\delta,\epsilon} \bar{L}_u Q_{u,t} - \delta Q_{s,u}^{\delta,\epsilon} \bar{V}_u Q_{u,t}$$

and

$$\partial_u(Q_{s,u} Q_{u,t}^{\delta,\epsilon}) = -\epsilon Q_{s,u} \bar{L}_u Q_{u,t}^{\delta,\epsilon} + \delta Q_{s,u} \bar{V}_u Q_{u,t}^{\delta,\epsilon}$$

The perturbation analysis of the normalized semigroups $\phi_{s,t}^{\delta,\epsilon}$ is slightly more involved.

Let $\Lambda_t^{\delta,\epsilon}$ be defined as Λ_t by replacing (L_t, V_t) by $(L_t^\epsilon, V_t^\delta)$. Notice that

$$h^{-1} \left[\phi_{t,t+h}^{\delta,\epsilon}(\eta) - \eta \right] = \Lambda_t^{\delta,\epsilon}(\eta) + O(h)$$

For any given $s \leq t$, we consider the interpolating maps $u \in [s, t] \mapsto \Delta_{s,u,t}^{\delta,\epsilon}$ defined by

$$\Delta_{s,u,t}^{\delta,\epsilon} := \phi_{u,t} \circ \phi_{s,u}^{\delta,\epsilon}$$

On the other hand, for any $s \leq u \leq u+h \leq t$ we have the decomposition

$$\begin{aligned} & \Delta_{s,u+h,t}^{\delta,\epsilon}(\eta) - \Delta_{s,u,t}^{\delta,\epsilon}(\eta) \\ &= \phi_{u+h,t} \left(\phi_{s,u+h}^{\delta,\epsilon}(\eta) \right) - \phi_{u,t} \left(\phi_{s,u+h}^{\delta,\epsilon}(\eta) \right) + \phi_{u,t} \left(\phi_{s,u+h}^{\delta,\epsilon}(\eta) \right) - \phi_{u,t} \left(\phi_{s,u}^{\delta,\epsilon}(\eta) \right) \\ &= -\Lambda_u(\phi_{s,u}^{\delta,\epsilon}(\eta)) \left(\partial_{\phi_{s,u}^{\delta,\epsilon}(\eta)} \phi_{u,t} \right) h \\ & \quad + \phi_{u,t} \left(\phi_{s,u}^{\delta,\epsilon}(\eta) + \left[\phi_{s,u+h}^{\delta,\epsilon}(\eta) - \phi_{s,u}^{\delta,\epsilon}(\eta) \right] \right) - \phi_{u,t} \left(\phi_{s,u}^{\delta,\epsilon}(\eta) \right) + O(h^2) \end{aligned}$$

This implies that

$$\begin{aligned} & h^{-1} \left[\Delta_{s,u+h,t}^{\delta,\epsilon}(\eta) - \Delta_{s,u,t}^{\delta,\epsilon}(\eta) \right] \\ &= -\Lambda_u(\phi_{s,u}^{\delta,\epsilon}(\eta)) \partial_{\phi_{s,u}^{\delta,\epsilon}(\eta)} \phi_{u,t} + h^{-1} \left[\phi_{s,u+h}^{\delta,\epsilon}(\eta) - \phi_{s,u}^{\delta,\epsilon}(\eta) \right] \partial_{\phi_{s,u}^{\delta,\epsilon}(\eta)} \phi_{u,t} + O(h) \end{aligned}$$

We conclude that

$$\partial_u \Delta_{s,u,t}^{\delta,\epsilon}(\eta) = \left[\Lambda_u^{\delta,\epsilon}(\phi_{s,u}^{\delta,\epsilon}(\eta)) - \Lambda_u(\phi_{s,u}^{\delta,\epsilon}(\eta)) \right] \partial_{\phi_{s,u}^{\delta,\epsilon}(\eta)} \phi_{u,t}$$

On the other hand, we have

$$\left[\Lambda_t^{\delta,\epsilon}(\eta) - \Lambda_t(\eta) \right] (f) = \epsilon \eta(\bar{L}_t(f)) - \delta \eta(f(\bar{V}_t - \eta(\bar{V}_t)))$$

By symmetry arguments, this ends the proof of the theorem. ■

Corollary 3.3. *For any $s \leq t$ and any $\eta \in \mathcal{P}(S)$ we have the estimates*

$$\begin{aligned} (H_1) & \implies \|\phi_{s,t}^{\delta,\epsilon}(\eta) - \phi_{s,t}(\eta)\|_{tv} \leq c(\epsilon + \delta) \\ (H_2) & \implies \|\phi_{s,t}^{\delta,\epsilon}(\eta) - \phi_{s,t}(\eta)\|_{tv} \leq c(\epsilon + \delta)(t - s) \end{aligned}$$

for some finite constant c whose value doesn't depend on the parameters (s, t, η) , nor on (ϵ, δ) .

3.2 Particle stochastic flows

For any $t \geq 0$, we let $\Delta m(\xi_t)$ be the random jump occupation measure

$$\Delta m(\xi_t) := m(\xi_t) - m(\xi_{t-}) = \Delta M_t = M_t - M_{t-}$$

with the martingale random field M_t defined in (2.14). In this notation, we have

$$N^{n-1} \partial_t \mathbb{E} \left[(\Delta m(\xi_t))^{\otimes n} (f_t^{(1)} \otimes \dots \otimes f_t^{(n)}) \mid \mathcal{F}_{t-} \right] = m(\xi_{t-}) \Gamma_{L_{m(\xi_{t-})}^d}^{(n)} (f_t^{(1)}, \dots, f_t^{(n)}) \quad (3.3)$$

with the operators $\Gamma_{L_{m(\xi_{t-})}^d}^{(n)}$ defined in (1.11). When $n = 2$ the above formula resumes to

$$\begin{aligned} \partial_t \mathbb{E} [\Delta m(\xi_t)(f_t) \Delta m(\xi_t)(g_t) \mid \mathcal{F}_{t-}] &= \frac{1}{N} m(\xi_{t-}) \Gamma_{L_{m(\xi_{t-})}^d} (f_t, g_t) \\ &= \partial_t \langle M_t^d(f), M_t^d(g) \rangle_t = \partial_t \langle \mathcal{M}_t^d(F), \mathcal{M}_t^d(G) \rangle_t \quad \text{with } (F, G) \text{ defined in (2.13)} \end{aligned}$$

Definition 3.4. For any $t \geq s$ and $n \geq 1$, we consider the integral random operators

$$\Delta^n \phi_{s,t}(m(\xi_s)) := N^{n-1} \frac{1}{n!} (\Delta m(\xi_s))^{\otimes n} \bar{\partial}_{m(\xi_{s-}) + \Delta m(\xi_s), m(\xi_{s-})}^n \phi_{s,t}$$

and their first variational measure

$$\Upsilon_{m(\xi_{s-})}^n \phi_{s,t} := \partial_s \mathbb{E} [\Delta^n \phi_{s,t}(m(\xi_s)) \mid \mathcal{F}_{s-}]$$

Choosing $n = 1$ we have

$$\Delta \phi_{s,t}(m(\xi_s)) := \Delta^1 \phi_{s,t}(m(\xi_s)) = \phi_{s,t}(m(\xi_s)) - \phi_{s,t}(m(\xi_{s-}))$$

Arguing as in the proof of (2.8) and using (3.3), for any collection of functions $f^{(n)} \in \text{Osc}(S)$ we have the estimate

$$N^{n-1} \partial_s \mathbb{E} \left[\Delta \phi_{s,t}(m(\xi_s))^{\otimes n} (f_t^{(1)} \otimes \dots \otimes f_t^{(n)}) \mid \mathcal{F}_{s-} \right] \leq e^{nq} \|\lambda + V\| \quad (3.4)$$

Proposition 3.5. For any $t \geq s$ and $n \geq 1$, we have

$$\Delta^n \phi_{s,t}(m(\xi_s)) = \frac{N^{n-1}}{n!} (\Delta m(\xi_s))^{\otimes n} \partial_{m(\xi_{s-})}^n \phi_{s,t} + \frac{1}{N} \Delta^{n+1} \phi_{s,t}(m(\xi_s)) \quad (3.5)$$

In addition, for any $f \in \mathcal{B}(S)$ we have

$$\begin{aligned} &\Upsilon_{m(\xi_{s-})}^n \phi_{s,t}(f) \\ &= (-1)^{n-1} m(\xi_{s-}) \Gamma_{L_{m(\xi_{s-})}^d}^{(n)} \left(Q_{s,t}^{m(\xi_{s-1})}(1), \dots, Q_{s,t}^{m(\xi_{s-1})}(1), \partial_{m(\xi_{s-})} \phi_{s,t}(f) \right) + \frac{1}{N} \Upsilon_{m(\xi_{s-})}^{(n+1)} \phi_{s,t}(f) \end{aligned} \quad (3.6)$$

Proof. We have

$$\begin{aligned} \Delta^{n+1} \phi_{s,t}(m(\xi_s)) &= N^n \left[\Delta \phi_{s,t}(m(\xi_s)) - \sum_{1 \leq k \leq n} \frac{1}{k!} (\Delta m(\xi_s))^{\otimes k} \partial_{m(\xi_{s-})}^k \phi_{s,t} \right] \\ &= N \Delta^n \phi_{s,t}(m(\xi_s)) - \frac{N^n}{n!} (\Delta m(\xi_s))^{\otimes n} \partial_{m(\xi_{s-})}^n \phi_{s,t} \iff (3.5) \end{aligned}$$

This implies that

$$\begin{aligned} \partial_s \mathbb{E} [\Delta^n \phi_{s,t}(m(\xi_s)) \mid \mathcal{F}_{s-}] &:= \Upsilon_{m(\xi_{s-})}^n \phi_{s,t} \\ &= \frac{N^{n-1}}{n!} \partial_s \mathbb{E} \left[(\Delta m(\xi_s))^{\otimes n} \partial_{m(\xi_{s-})}^n \phi_{s,t} \mid \mathcal{F}_{s-} \right] + \frac{1}{N} \partial_s \mathbb{E} [\Delta^{n+1} \phi_{s,t}(m(\xi_s)) \mid \mathcal{F}_{s-}] \end{aligned}$$

This ends the proof of the proposition. \blacksquare

Lemma 3.6. *For any $n \geq 1$ and $s \leq t$ we have the almost sure uniform estimates*

$$(H_2) \implies \|\Upsilon_{m(\xi_{s-})}^n \phi_{s,t}\|_{tv} \leq 2^{n-1} e^{(n+1)q} \|\lambda + V\| \quad (3.7)$$

The detailed proof of the above estimate is provided in the appendix, on page 27.

In the further development of this section, for any given time horizon t and any $f \in \mathcal{B}(S)$ we let

$$s \in [0, t] \mapsto \mathcal{M}_s^d(\phi_{\cdot,t}(m(\cdot)))(f)$$

be the martingale $s \in [0, t] \mapsto \mathcal{M}_s^d(F)$ associated with the function

$$(s, x) \in [0, t] \times S_N \mapsto F(s, x) = \phi_{s,t}(m(x))(f)$$

We also denote by

$$s \in [0, t] \mapsto M_s^c(\partial_{m(\xi_{\cdot})} \phi_{\cdot,t}(f)) \quad , \quad \text{resp.} \quad M_s^c(Q_{\cdot,t}^{m(\xi_{\cdot})}(1))$$

the martingale $M_s^c(f)$ associated with the \mathcal{F} -predictable bounded function

$$(s, x) \in [0, t] \times S \mapsto f_s(x) = \partial_{m(\xi_{s-})} \phi_{s,t}(f)(x) \quad , \quad \text{resp.} \quad f_s(x) = Q_{s,t}^{m(\xi_{s-})}(1)(x)$$

We are now in position to state and to prove the main result of this section.

Theorem 3.7. *For any time horizon $t \geq 0$ and any $f \in \mathcal{B}(S)$ the interpolating function*

$$s \in [0, t] \mapsto \phi_{s,t}(m(\xi_s))(f) \in \mathbb{R}$$

satisfies the stochastic differential equation

$$\begin{aligned} d\phi_{s,t}(m(\xi_s))(f) &= \frac{1}{\sqrt{N}} dM_s^c(\partial_{m(\xi_{\cdot})} \phi_{\cdot,t}(f)) + d\mathcal{M}_s^d(\phi_{\cdot,t}(m(\cdot)))(f) \\ &\quad + \frac{1}{N} \Upsilon_{m(\xi_{s-})}^2 \phi_{s,t}(f) ds - \frac{1}{N} m(\xi_s) \Gamma_{L_s^c} \left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)} \phi_{s,t}(f) \right) ds \end{aligned}$$

Proof. Observe that

$$dm(\xi_s) = \Lambda_s(m(\xi_s)) ds + \frac{1}{\sqrt{N}} dM_s^c + \Delta m(\xi_s) - \underbrace{\mathbb{E}(\Delta m(\xi_s) \mid \mathcal{F}_{s-})}_{\ll ds}$$

Using Itô formula and the backward formula (2.10) we have

$$\begin{aligned} d\phi_{s,t}(m(\xi_s))(f) &= -\Lambda_s(m(\xi_s)) (\partial_{m(\xi_s)} \phi_{s,t}(f)) ds + [\phi_{s,t}(m(\xi_{s-}) + dm(\xi_s)) - \phi_{s,t}(m(\xi_{s-}))](f) \\ &= \frac{1}{\sqrt{N}} dM_s^c(\partial_{m(\xi_{\cdot})} \phi_{\cdot,t}(f)) + d\mathcal{M}_s^d(\phi_{\cdot,t}(m(\cdot)))(f) \\ &\quad + \frac{1}{2N} (dM_s^c \otimes dM_s^c) \partial_{m(\xi_s)}^2 \phi_{s,t}(f) + \partial_s \mathbb{E} [\Delta \phi_{s,t}(m(\xi_s))(f) - \Delta m(\xi_s) \partial_{m(\xi_{s-})} \phi_{s,t}(f) \mid \mathcal{F}_{s-}] ds \end{aligned}$$

This ends the proof of the theorem. ■

Next corollary is a direct consequence of the recursion (3.6).

Corollary 3.8. *For any $t \geq 0$ and any $f \in \mathcal{B}(S)$ we have the almost sure formula*

$$\begin{aligned} & \phi_{s,t}(m(\xi_s))(f) - \phi_{0,t}(m(\xi_0))(f) \\ &= \frac{1}{\sqrt{N}} M_s^c (\partial_{m(\xi_s)} \phi_{\cdot,t}(f)) + \mathcal{M}_s^d (\phi_{\cdot,t}(m(\cdot)))(f) \\ & \quad - \frac{1}{N} \int_0^s m(\xi_u) \Gamma_{L_u, m(\xi_u)} \left(Q_{u,t}^{m(\xi_u)}(1), \partial_{m(\xi_u)} \phi_{u,t}(f) \right) du + \frac{1}{N^2} \int_0^s \Upsilon_{m(\xi_{u-})}^3 \phi_{u,t}(f) du \end{aligned} \tag{3.8}$$

Choosing $s = t$ and taking the expectation in (3.8) we obtain the following result.

Corollary 3.9. *For any $t \geq 0$ and $f \in \mathcal{D}(S)$ we have the formula*

$$\begin{aligned} & \mathbb{E}(m(\xi_t)(f)) - \mathbb{E}(\phi_{0,t}(m(\xi_0))(f)) \\ &= -\frac{1}{N} \int_0^t \mathbb{E} \left[m(\xi_s) \Gamma_{L_s, m(\xi_s)} \left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)} \phi_{s,t}(f) \right) \right] ds + \frac{1}{N^2} \int_0^t \mathbb{E} \left[\Upsilon_{m(\xi_{s-})}^3 \phi_{s,t}(f) \right] ds \end{aligned}$$

3.3 Some non asymptotic estimates

Theorem 3.10. *For any time horizon $t \geq 0$ and any function $f \in \text{Osc}(S)$ we have*

$$\begin{aligned} (H_1) & \implies |\mathbb{E}(m(\xi_t)(f)) - \eta_t(f)| \leq c/N \\ (H_2) & \implies |\mathbb{E}(m(\xi_t)(f)) - \eta_t(f)| \leq c t/N \end{aligned}$$

for some finite constant c whose value doesn't depend on the parameters (t, N) .

The proof of the above theorem is mainly based on the decomposition presented in corollary 3.9. The estimates rely on elementary but rather technical carré du champ inequalities, and semigroup techniques. Thus, the detail of the proof is housed in the appendix, on page 27.

The first estimate stated in the above corollary extend the bias estimate obtained in [67] to time varying Feynman-Kac models. The central difference between homogeneous and time varying models lies on the fact that we cannot use h -process techniques. The latter allows to interpret the Feynman-Kac semigroups in terms of more conventional Markov semigroups.

We end this section with a some more or less direct consequences of the above estimates in the analysis of the measures discussed in theorem 2.4.

By corollary 3.3, for any $N > 1$ we have

$$(H_1) \implies \|\eta_t^- - \eta_t\|_{tv} \leq c/N \quad \text{and} \quad (H_2) \implies \|\eta_t^- - \eta_t\|_{tv} \leq c t/N$$

By (3.1), when (H_0) is satisfied, the Feynman-Kac model defined in terms of (L_s^-, V^-) satisfy the stability property (H_1) . Thus, using theorem 3.10 we readily deduce the following estimates.

Corollary 3.11. *We have almost sure and uniform estimates*

$$(H_0) \implies |\mathbb{E} \left(m(\zeta_t^-)(f) \mid \widehat{\zeta}_t^1 \right) - \eta_t(f)| \leq c/N$$

We further assume that the Feynman-Kac model is associated with the historical process $X_t = (X'_s)_{s \leq t}$ discussed in (2.6). Also assume that the transition semigroup $P'_{s,t}$ of the auxiliary process X'_t satisfies condition (H_0) ; that is (H'_0) is met. In this situation, using (2.9) we check that the Feynman-Kac model associated with the historical process X_t satisfies (H_2) . Thus, using corollary 3.10 we also deduce the following estimates.

Corollary 3.12. *Assume that the Feynman-Kac model is associated with the historical process $X_t = (X'_s)_{s \leq t}$ of the auxiliary process X'_t . In this situation, for any $N > 1$ we have almost sure and uniform estimates*

$$(H'_0) \implies |\mathbb{E} \left(m(\zeta_t^-)(f) \mid \widehat{\zeta}_t^1 \right) - \eta_t(f)| \leq c t/N$$

The above results give some information on the bias of the occupation measures. We end this section with some propagation of chaos estimate. Using (3.8), for any functions $f_i \in \text{Osc}(S)$ we have

$$\begin{aligned} & \mathbb{E} (m(\xi_t)(f_1) m(\xi_t)(f_2)) - \mathbb{E} (\phi_{0,t}(m(\xi_0))(f_1) \phi_{0,t}(m(\xi_0))(f_2)) \\ &= -\frac{1}{N} \sum_{(k,l) \in \{(1,2), (2,1)\}} \int_0^t \mathbb{E} \left[\phi_{s,t}(m(\xi_s))(f_k) m(\xi_u) \Gamma_{L_u, m(\xi_u)} \left(Q_{u,t}^{m(\xi_u)}(1), \partial_{m(\xi_u)} \phi_{u,t}(f_l) \right) \right] du \\ & \quad + \frac{1}{N} \int_0^t \mathbb{E} \left[m(\xi_u) \Gamma_{L_u, m(\xi_u)} \left(\partial_{m(\xi_u)} \phi_{u,t}(f_1), \partial_{m(\xi_u)} \phi_{u,t}(f_2) \right) \right] \\ & \quad + \int_0^t \partial_s \mathbb{E} [\Delta \phi_{s,t}(m(\xi_s))(f_1) \Delta \phi_{s,t}(m(\xi_s))(f_2)] ds \\ & \quad + \frac{1}{N^2} \sum_{(k,l) \in \{(1,2), (2,1)\}} \int_0^t \mathbb{E} \left[\phi_{s,t}(m(\xi_s))(f_k) \Upsilon_{m(\xi_{s-})}^3 \phi_{s,t}(f_l) \right] ds \end{aligned}$$

By (3.4) and using the same lines of arguments as in the proof of theorem 3.10 we check the following estimates.

Corollary 3.13. *For any time horizon $t \geq 0$ and any $f, g \in \text{Osc}(S)$ we have*

$$\begin{aligned} (H_1) & \implies |\mathbb{E} (f(\xi_t^1) g(\xi_t^2)) - \eta_t(f) \eta_t(g)| \leq c/N \\ (H_2) & \implies |\mathbb{E} (f(\xi_t^1) g(\xi_t^2)) - \eta_t(f) \eta_t(g)| \leq c t/N \end{aligned}$$

In the settings of corollary 3.12 we also check the almost sure estimate

$$(H'_0) \implies |\mathbb{E} \left(f(\zeta_t^2) g(\zeta_t^3) \mid \widehat{\zeta}_t^1 \right) - \eta_t(f) \eta_t(g)| \leq c t/N$$

We can extend the above arguments to any finite block of particles.

4 Many-body Feynman-Kac measures

4.1 Description of the models

We let \mathbb{P}_t^ξ and \mathbb{P}_t^ζ be the distribution of the historical process

$$\widehat{\xi}_t := (\xi_s)_{s \leq t} := (\xi_s^1, \dots, \xi_s^N)_{s \leq t} \quad \text{and} \quad \widehat{\zeta}_t := (\zeta_s)_{s \leq t} := (\zeta_s^1, \dots, \zeta_s^N)_{s \leq t} \in D_t(S \times S_{N-1})$$

We set

$$Z_t(\xi) := \exp \left[- \int_0^t m(\xi_s)(V_s) ds \right] \quad \text{and} \quad \bar{Z}_t(\xi) := \exp \left[- \int_0^t m(\xi_s)(\bar{V}_s) ds \right]$$

We recall for any $f \in \mathcal{B}(S)$ the unbiased property

$$\mathbb{E} (m(\xi_t)(f) \bar{Z}_t(\xi)) = \eta_t(f)$$

For any $1 \leq i \leq N$ we also consider the historical process

$$\hat{\zeta}_t^i := (\zeta_s^i)_{s \leq t} \quad \text{and the reduced particle system} \quad \zeta_t^- = (\zeta_t^2, \dots, \zeta_t^N) \in S_{N-1}$$

Definition 4.1. For any time horizon $t \in \mathbb{R}_+$, the N -many-body Feynman-Kac measures $\mathbb{Q}_t^\xi \in \mathcal{P}(D_t(S \times S_{N-1}))$ and $\mathbb{Q}_t^\zeta \in \mathcal{P}(D_t(S \times S_{N-1}))$ are defined by Radon-Nikodym the formulae

$$d\mathbb{Q}_t^\xi / d\mathbb{P}_t^\xi := \bar{Z}_t(\xi) \quad \text{and} \quad d\mathbb{Q}_t^\zeta / d\mathbb{P}_t^\zeta := \bar{Z}_t(\zeta^1) \quad (4.1)$$

4.2 A duality formula

In contrast with conventional changes of probability measures the exponential terms $\bar{Z}_t(\xi)$ and $\bar{Z}_t(\zeta^1)$ have unit mean but they are not martingales w.r.t. the laws \mathbb{P}_t^ξ and \mathbb{P}_t^ζ . We let $\mathbb{Q}_t^{\zeta^i}$ be the $\hat{\zeta}_t^i$ -marginal of \mathbb{Q}_t^ζ , with $1 \leq i \leq N$.

Theorem 4.2. For any $1 \leq i \leq N$ and any time horizon $t \geq 0$ we have

$$\mathbb{Q}_t^\xi = \mathbb{Q}_t^\zeta \quad \text{and} \quad \mathbb{Q}_t^{\zeta^i} = \mathbb{Q}_t \quad (4.2)$$

Proof. Observe that $\hat{\xi}_t := (\xi_s)_{s \leq t}$ and $\hat{\zeta}_t := (\zeta_s)_{s \leq t}$ coincide with the historical processes of processes ξ_s and ζ_s . In addition, for any $x = (x_s)_{s \leq t} \in D_t(S)$ we have

$$\hat{V}_t(x) := V_t(x_t) \implies m(\hat{\xi}_s)(\hat{V}_s) = m(\xi_s)(V_s)$$

In this case, \mathbb{Q}_t^ξ and \mathbb{Q}_t^ζ coincide with the t -time marginal of the measures of the measures $\hat{\mathbb{Q}}_t^\xi$ and $\hat{\mathbb{Q}}_t^\zeta$ defined as above by replacing (ξ_t, ζ_t, V_t) by $(\hat{\xi}_t, \hat{\zeta}_t, \hat{V}_t)$. In this situation the state space S is replaced by the space of paths $\hat{S} = \cup_{t \geq 0} D_t(S)$. In addition, the generators $(\mathcal{G}_t, \mathcal{H}_t, \mathcal{G}_t^-)$ are replaced by the generators $(\hat{\mathcal{G}}_t, \hat{\mathcal{H}}_t, \hat{\mathcal{G}}_t^-)$ of the historical processes $(\hat{\xi}_t, \hat{\zeta}_t, \hat{\zeta}_t^-)$. These generators are defined as above by replacing (S, L_t, V_t) by $(\hat{S}, \hat{L}_t, \hat{V}_t)$ where \hat{L}_t stands for the generator of the historical process $\hat{X}_t := (X_s)_{s \leq t}$. Thus, there is no loss of generality to prove (4.2) for the t -marginal probability measures $(\overline{\mathbb{Q}}_t^\xi, \overline{\mathbb{Q}}_t^\zeta)$ of $(\mathbb{Q}_t^\xi, \mathbb{Q}_t^\zeta)$.

For any $(F, x) \in (\mathcal{D}(S \times S_{N-1}) \times S_N)$ we set

$$\begin{aligned} \mathcal{L}_t(F)(x) &:= \sum_{1 \leq i \leq N} L_t(F_{x^{-i}})(x^i) \\ \mathcal{L}_t^V(F)(x) &:= \sum_{1 \leq i \leq N} L_t^V(F_{x^{-i}})(x^i) = \mathcal{L}_t(F)(x) - N m(x)(V_t) F(x) \end{aligned}$$

Observe that

$$\begin{aligned} \mathcal{G}_t(F)(x) &= \mathcal{L}_t(F)(x) + \sum_{1 \leq i \leq N} V_t(x^i) \int (F_{x^{-i}}(u) - F(x)) m(x)(du) \\ &= \mathcal{L}_t(F)(x) + \frac{1}{N} \sum_{1 \leq i \neq j \leq N} V_t(x^i) [F_{x^{-i}}(x^j) - F_{x^{-i}}(x^i)] \\ &= \mathcal{L}_t(F)(x) + \frac{1}{N} \sum_{1 \leq i \neq j \leq N} V_t(x^i) F_{x^{-i}}(x^j) - (N-1) m(x)(V_t) F(x) \end{aligned}$$

This implies that

$$\mathcal{G}_t(F)(x) - m(x)(V_t)F(x) = \mathcal{L}_t^V(F)(x) + \frac{1}{N} \sum_{1 \leq i \neq j \leq N} V_t(x^i) F_{x^{-i}}(x^j)$$

On the other hand, we have

$$\begin{aligned} & \frac{1}{N} \sum_{1 \leq i \neq j \leq N} V_t(x^i) F_{x^{-i}}(x^j) \\ &= \frac{1}{N} \sum_{2 \leq i \neq j \leq N} V_t(x^i) F_{x^{-i}}(x^j) + \frac{1}{N} \sum_{2 \leq j \leq N} [V_t(x^1) F_{x^{-1}}(x^j) + V_t(x^j) F_{x^{-j}}(x^1)] \end{aligned}$$

This implies that

$$\begin{aligned} \partial_t \overline{\mathbb{Q}}_t^\xi(F) &= \mathbb{E} \left(\left[\mathcal{L}_t^V(F)(\xi_t) + \sum_{2 \leq i \leq N} V_t(\xi_t^i) \left(1 - \frac{2}{N}\right) \frac{1}{N-2} \sum_{j \notin \{1, i\}} F_{\xi_t^{-i}}(\xi_t^j) \right. \right. \\ &\quad \left. \left. + \left(1 - \frac{1}{N}\right) [V_t(\xi_t^1) F_{\xi_t^{-1}}(\xi_t^2) + V_t(\xi_t^2) F_{\xi_t^{-i}}(\xi_t^1)] \right] Z_t(\xi) \right) \end{aligned}$$

By symmetry arguments, we check that

$$\begin{aligned} \partial_t \overline{\mathbb{Q}}_t^\xi(F) &= \mathbb{E} \left(\left[L_t^V(F_{\xi_t^{-1}})(\xi_t^1) + \sum_{2 \leq i \leq N} L_t(F_{\xi_t^{-i}})(\xi_t^i) - \sum_{2 \leq i \leq N} V_t(\xi_t^i) F(\xi_t) \right. \right. \\ &\quad \left. \left. + \sum_{2 \leq i \leq N} V_t(\xi_t^i) \int F_{\xi_t^{-i}}(u) \left(1 - \frac{2}{N}\right) m_{\xi_t^{-\{1, i\}}}(\xi_t^i)(du) + \sum_{2 \leq i \leq N} V_t(\xi_t^i) \frac{2}{N} F_{\xi_t^{-i}}(\xi_t^1) \right] Z_t(\xi) \right) \end{aligned}$$

We conclude that

$$\begin{aligned} \partial_t \overline{\mathbb{Q}}_t^\xi(F) &= \mathbb{E} \left(\left[L_t(F_{\xi_t^{-1}})(\xi_t^1) - V(\xi_t^1) F(\xi_t) \right. \right. \\ &\quad \left. \left. + \sum_{2 \leq i \leq N} \left(L_t(F_{\xi_t^{-i}})(\xi_t^i) + V_t(\xi_t^i) \int (F_{\xi_t^{-i}}(u) - F(\xi_t)) m_{\xi_t^{-\{1, i\}}}(\xi_t^i)(du) \right) \right] Z_t(\xi) \right) \end{aligned}$$

By symmetry arguments we have

$$\partial_t \overline{\mathbb{Q}}_t^\xi(F) = \mathbb{E}(\mathcal{K}(F)(\xi_t) Z_t(\xi)) = \overline{\mathbb{Q}}_t^\xi(\mathcal{K}(F)) \quad \text{with} \quad \mathcal{K}(F)(x) = \mathcal{H}_t(F)(x) - V(x^1) F(x)$$

In much the same way, we have

$$\partial_t \overline{\mathbb{Q}}_t^\zeta(F) = \mathbb{E}(\mathcal{K}(F)(\zeta_t) Z_t(\zeta^1)) = \overline{\mathbb{Q}}_t^\zeta(\mathcal{K}(F))$$

This ends the proof of the l.h.s. assertion in (4.2). Thus, choosing $F(x) = m(x)(f)$ we have

$$\begin{aligned} \eta_t(f) &= \mathbb{E}(m(\xi_t)(f) \overline{Z}_t(\xi)) \\ &= \mathbb{E}(m(\zeta_t)(f) \overline{Z}_t(\zeta^1)) = \frac{1}{N} \eta_t(f) + \left(1 - \frac{1}{N}\right) \mathbb{E}(f(\zeta_t^2) \overline{Z}_t(\zeta^1)) \end{aligned}$$

This ends the proof of the r.h.s. assertion in (4.2). The proof of the theorem is completed. \blacksquare

We let \mathbb{X}_t be a random sample from $m(\xi_t)$. Next corollary extend the duality formula presented in [31] to continuous time Feynman-Kac models.

Corollary 4.3. *For any $F \in \mathcal{B}(S \times D_t(S \times S_{N-1}))$ we have the duality formula*

$$\Pi_t(F) := \mathbb{E} \left(F(\mathbb{X}_t, \hat{\xi}_t) \bar{Z}_t(\xi) \right) = \mathbb{E} \left(F(\zeta_t^1, \hat{\zeta}_t) \bar{Z}_t(\zeta^1) \right)$$

The above corollary is valid for any Feynman-Kac model \mathbb{Q}_t associated with some process X_t and some potential function V_t on some state space S . Thus it applies to the Feynman-Kac model $\hat{\mathbb{Q}}_t$ defined as in (1.1) by replacing (X_t, V_t) by the path-space model (\hat{X}_t, \hat{V}_t) on the path space \hat{S} . In this notation the t -time marginal $\hat{\eta}_t$ of $\hat{\mathbb{Q}}_t$ coincides with the measure \mathbb{Q}_t defined in (1.1).

In the same reverse angle, let \mathbb{Q}'_t be the Feynman-Kac model associated with some process X'_t and some potential function V'_t on some state space S' . Also let $\mathbb{Q}_t := \widehat{\mathbb{Q}}'_t$ be the Feynman-Kac model associated with the historical process

$$X_t := \widehat{X}'_t = (X'_s)_{s \leq t} \in S := \hat{S}' = \cup_{s \geq 0} D_s(S')$$

and the potential function V_t on S defined by

$$V_t(X_t) := \widehat{V}'_t(\widehat{X}'_t) = V'_t(X'_t)$$

In this situation, the t -time marginal η_t of \mathbb{Q}_t coincides with the path space measure \mathbb{Q}'_t . In addition, $\hat{\xi}_s = (\xi_s)_{s \leq t}$ is the historical process of N path-valued particles $\xi_s = (\xi_s^i)_{1 \leq i \leq N} \in S_N$. In addition, each particle $\xi_s^i = (\xi_{u,s}^i)_{u \leq s} \in S$ represents the ancestral line of the particle $\xi_s^i := \xi_{s,s}^i \in S'$. For any $u \leq s$, $\xi_{u,s}^i$ stands for the ancestor at level $u \leq s$. Last but not least, the N particle model ξ'_s coincides with the N -particle model associated with the Feynman-Kac model \mathbb{Q}'_t . Therefore the duality theorem 1.1 stated in the introduction is a direct consequence of the above corollary.

Now, we come to the proof of the corollary.

Proof of corollary 4.3:

We associate with a given $F \in \mathcal{B}(S \times D_t(S \times S_{N-1}))$ the function $\bar{F} \in \mathcal{B}(D_t(S \times S_{N-1}))$ defined for any

$$x = (x^1(s), (x^2(s), \dots, x^N(s)))_{s \leq t} \in D_t(S \times S_{N-1}) \quad \text{and} \quad x(t) := (x^1(t), \dots, x^N(t))$$

by the integral formula

$$\bar{F}(x) := \int m(x(t))(du) F(u, x)$$

Using (4.2) we have

$$\mathbb{E} \left(\bar{F}((\xi_s)_{s \leq t}) \bar{Z}_t(\xi) \right) = \mathbb{E} \left(\bar{F}((\zeta_s)_{s \leq t}) \bar{Z}_t(\zeta^1) \right)$$

On the other hand, for any $1 \leq i \leq N$ we have

$$\mathbb{E} \left(\bar{F}((\xi_s)_{s \leq t}) \bar{Z}_t(\xi) \right) = \mathbb{E} \left(F(\xi_t^i, (\xi_s)_{s \leq t}) \bar{Z}_t(\xi) \right) = \mathbb{E} \left(F(\xi_t^1, (\xi_s)_{s \leq t}) \bar{Z}_t(\xi) \right) = \mathbb{E} \left(\underline{F}((\xi_s)_{s \leq t}) \bar{Z}_t(\xi) \right)$$

with the function $\underline{F} \in D_t(S \times S_{N-1})$ given by

$$\underline{F}((x^1(s), \dots, x^N(s))_{s \leq t}) = F(x^1(t), (x^1(s), \dots, x^N(s))_{s \leq t})$$

Using (4.2) we also have

$$\mathbb{E} \left(\underline{F}((\xi_s)_{s \leq t}) \bar{Z}_t(\xi) \right) = \mathbb{E} \left(\underline{F}((\zeta_s)_{s \leq t}) \bar{Z}_t(\zeta^1) \right) = \mathbb{E} \left(F(\zeta_t^1, (\zeta_s)_{s \leq t}) \bar{Z}_t(\zeta^1) \right)$$

This ends the proof of the corollary. \blacksquare

4.3 Particle Gibbs samplers

We further assume that reference process $X_t = (X'_s)_{s \leq t} \in D_t(S')$ in the Feynman-Kac measure (2.1) is the historical of some auxiliary process X'_t taking values in some metric space $(S', d_{S'})$. In this case, X_t is a Markov process taking values in $S = \cup_{s \geq 0} D_s(S')$. Also assume that the potential function V_t is chosen so that $V_t(X_t) = V'_t(X'_t)$. In this situation, the mean field particle model ξ_t coincide with the genealogical tree evolutions of the mean field particle interpretation of the Feynman-Kac measures associated with (X'_t, V'_t) .

In the same vein, the particle model ζ_t is path space genealogical tree based particle model. For instance $\zeta_t^1 = (\zeta_s^1)_{s \leq t}$ is itself the historical process of the path-space process $\zeta_s^1 \in D_s(S')$; so that the jumps onto ζ_t^1 have to be interpreted as a jump of an ancestral line onto ζ_t^1

In this situation, for any given time horizon $t \geq 0$, we have

$$\Pi_t(d(z_1, z_2)) \in \mathcal{P}(E_1 \times E_2) \quad \text{with} \quad E_1 = D_t(S') \quad \text{and} \quad E_2 := D_t(S \times S_{N-1})$$

Observe that for any $z_2 := (z_2(s))_{s \leq t} \in E_2 := D_t(S \times S_{N-1})$ and any $s \leq t$ we have

$$z_2(s) := (z_1^1(s), (z_2^2(s), \dots, z_2^N(s))) \in D_s(S') \times D_s(S')_{N-1}$$

In this notation, we have desintegration formulae

$$\Pi_t(d(z_1, z_2)) = \eta_t(dz_1) \mathbb{M}_t(z_1, dz_2) \quad \text{and} \quad \Pi_t(d(z_1, z_2)) = \mathbb{Q}_t^\xi(dz_2) \mathbb{A}_t(z_2, dz_1)$$

In the above display \mathbb{M}_t stands for the Markov transition from E_1 into E_2 defined by

$$\mathbb{M}_t(z_1, dz_2) := \mathbb{P}(\widehat{\zeta}_t \in dz_2 \mid \zeta_t^1 = z_1)$$

and \mathbb{A}_t the Markov transition from E_2 into E_1 defined by

$$\mathbb{A}_t(z_2, dz_1) := m(z_2(t))(dz_1)$$

The transition of the conventional Gibbs-sampler with target measure Π_t on $E := (E_1 \times E_2)$ is defined by

$$\mathbb{G}_t((z_1, z_2), d(\bar{z}_1, \bar{z}_2)) := \mathbb{M}_t(z_1, d\bar{z}_2) \mathbb{A}_t(\bar{z}_2, d\bar{z}_1) \quad (4.3)$$

This transition is summarized in the following synthetic diagram

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longrightarrow \begin{pmatrix} z_1 \\ \bar{z}_2 \sim (\widehat{\zeta}_t \mid \zeta_t^1 = z_1) \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{z}_1 \sim m(\bar{z}_2(t)) \\ \bar{z}_2 \end{pmatrix}$$

By construction, we have the duality property

$$\Pi_t(d(z_1, z_2)) \mathbb{G}_t((z_1, z_2), d(\bar{z}_1, \bar{z}_2)) = \Pi_t(d(\bar{z}_1, \bar{z}_2)) \mathbb{G}_t^-((\bar{z}_1, \bar{z}_2), d(z_1, z_2)) \quad (4.4)$$

with the backward transition

$$\mathbb{G}_t^-((\bar{z}_1, \bar{z}_2), d(z_1, z_2)) = \mathbb{A}_t(\bar{z}_2, dz_1) \mathbb{M}_t(z_1, dz_2)$$

Recall that η_t coincide with the marginal Π_t' of Π_t on $E_1 = D_t(S')$. In addition, integrating (4.4) w.r.t. \bar{z}_2 we also have the reversibility property

$$\eta_t(dz_1) \mathbb{K}_t(z_1, d\bar{z}_1) = \eta_t(d\bar{z}_1) \mathbb{K}_t(\bar{z}_1, dz_1)$$

with the Markov transition $\mathbb{K}_t = \mathbb{M}_t \mathbb{A}_t$ from $D_t(S')$ into itself defined by

$$\mathbb{K}_t(f)(z_1) := \int \mathbb{K}_t(z_1, d\bar{z}_1) f(\bar{z}_1) = \mathbb{E} \left(m(\zeta_t)(f) \mid \widehat{\zeta}_t^1 = z_1 \right)$$

We further assume that the Markov transitions of X'_t satisfy condition (H_0) . On this situation, combining corollary 3.3 with corollary 3.12, for any time horizon $t \geq 0$, any function f with unit oscillations and any $\mu \in \mathcal{P}(D_t(S'))$ and $n \geq 1$ we check that

$$\|\mathbb{K}_t(f) - \eta_t(f)\| \leq c(t \vee 1)/N, \text{ which implies } \text{osc}(\mathbb{K}_t(f)) \leq c(t \vee 1)/N,$$

$$\text{and this yields } \|\mu \mathbb{K}_t^n - \eta_t\|_{tv} \leq (c(t \vee 1)/N)^n$$

for some finite constant c whose value doesn't depend on the parameters (f, t, N) . Notice that since we work on historical processes, \mathbb{Q}_t in Theorem 1.2 becomes η_t here.

Appendix

Proof of lemma 3.1

Let $X_{s,t}(x)$, with $t \geq s$, be the stochastic flow associated with the generator L_t starting at $X_{s,s}(x) = x$ at time $t = s$. In this notation, we have the perturbation formula

$$\begin{aligned} & P_{s,t}^\epsilon(f)(x) \\ &= \mathbb{E} \left[f(X_{s,t}(x)) e^{-\epsilon \int_s^t \lambda(X_{s,u}(x)) du} \right] + \int_s^t \mathbb{E} \left[\epsilon \lambda(X_{s,u}(x)) e^{-\epsilon \int_s^u \lambda(X_{s,v}(x)) dv} K_u(P_{u,t}^\epsilon(f))(X_{s,u}(x)) \right] du \end{aligned}$$

For non negative functions f and any $t \geq 0$ and $h > 0$ we have

$$\begin{aligned} P_{t,t+h}^\epsilon(f) &\leq e^{-\epsilon \lambda_1 h} P_{t,t+h}(f) + \epsilon \lambda_2 \varpi_2 \int_t^{t+h} e^{-\epsilon \lambda_1 (u-t)} \kappa_u P_{u,t+h}^\epsilon(f) du \\ &\leq e^{\epsilon(\lambda_2 - \lambda_1)h} \left[e^{-\epsilon \lambda_2 h} P_{t,t+h}(f) + \epsilon \lambda_2 \varpi_2 \int_t^{t+h} e^{-\epsilon \lambda_2 (u-t)} \kappa_u P_{u,t+h}^\epsilon(f) du \right] \end{aligned}$$

In the same vein, we have

$$P_{t,t+h}^\epsilon(f) \geq (\lambda_1/\lambda_2)(\varpi_1/\varpi_2) \left[e^{-\epsilon \lambda_2 h} P_{t,t+h}(f) + \epsilon \lambda_2 \varpi_2 \int_t^{t+h} e^{-\epsilon \lambda_2 (u-t)} \kappa_u P_{u,t+h}^\epsilon(f) du \right]$$

This shows that

$$\rho_\epsilon(h) \leq \frac{d(\delta_x P_{t,t+h}^\epsilon)}{d\mu_{t,h}^\epsilon}(y) \leq \rho_\epsilon(h)^{-1}$$

with the probability measure

$$\mu_{t,h}^\epsilon \propto e^{-\epsilon \lambda_2 h} \mu_{t,h} + \epsilon \lambda_2 \varpi_2 \int_t^{t+h} e^{-\epsilon \lambda_2 (u-t)} \kappa_u P_{u,t+h}^\epsilon(f) du$$

This ends the proof of the lemma. ■

Proof of (3.7)

For any functions $f_i \in \text{Osc}(S)$ and any $l \leq k$ we have

$$\begin{aligned}
& \left| \mathbb{E} \left[\prod_{1 \leq l \leq k} \Delta m(\xi_t)(f_l) \mid \mathcal{F}_{t-} \right] \right| \leq \frac{1}{N^k} \left[\sum_{1 \leq i \leq N} (V_t(\xi_t^i) + \lambda_t(\xi_t^i)) \right] \leq \frac{1}{N^{k-1}} \|\lambda + V\| dt \\
& \implies N^k |\Upsilon_{m(\xi_{s-})}^{k+1} \phi_{s,t}(f)| = \frac{N^k}{(k+1)!} \left| \partial_s \mathbb{E} \left[(\Delta m(\xi_s))^{\otimes(k+1)} \bar{\partial}_{m(\xi_{s-})}^{(k+1)} \phi_{s,t}(f) \mid \mathcal{F}_{s-} \right] \right| \\
& = N^k \left| \partial_s \mathbb{E} \left[\frac{1}{m(\xi_s) Q_{s,t}^{m(\xi_{s-})}(1)} \left(\Delta m(\xi) \left(Q_{s,t}^{m(\xi_{s-})}(1) \right) \right)^k \Delta m(\xi) \partial_{m(\xi_{s-})} \phi_{s,t}(f) \mid \mathcal{F}_{s-} \right] \right| \\
& \leq e^{(2+k)q} \|\lambda + V\|
\end{aligned}$$

Proof of theorem 3.10

We use (1.12) to check that

$$\begin{aligned}
& m(\xi_s) \Gamma_{L_s, m(\xi_s)} \left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)} \phi_{s,t}(f) \right) \\
& = (\eta_s Q_{s,t}^{m(\xi_s)}(1))^2 m(\xi_s) \Gamma_{L_s, m(\xi_s)} \left(Q_{s,t}^{\eta_s}(1), \partial_{\eta_s} \phi_{s,t}(f) \right) \\
& \quad + (\eta_s Q_{s,t}^{m(\xi_s)}(1))^2 [\phi_{s,t}(\eta_s) - \phi_{s,t}(m(\xi_s))](f) m(\xi_s) \Gamma_{L_s, m(\xi_s)} \left(Q_{s,t}^{\eta_s}(1) \right)
\end{aligned}$$

Using (1.6) we also have the estimate

$$\begin{aligned}
& \left| m(\xi_s) \Gamma_{L_s, m(\xi_s)} \left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)} \phi_{s,t}(f) \right) \right| \leq e^{3q} \text{osc}(\bar{Q}_{s,t}(f)) m(\xi_s) \Gamma_{L_s, m(\xi_s)} \left(Q_{s,t}^{\eta_s}(1) \right) \\
& \quad + e^{2q} \sqrt{m(\xi_s) \Gamma_{L_s, m(\xi_s)} \left(Q_{s,t}^{\eta_s}(1) \right)} \sqrt{m(\xi_s) \Gamma_{L_s, m(\xi_s)} \left(\partial_{\eta_s} \phi_{s,t}(f) \right)}
\end{aligned} \tag{4.5}$$

On the other hand, we have

$$\begin{aligned}
& \partial_{\eta_s} \phi_{s,t}(f) = Q_{s,t}^{\eta_s} [f - \eta_t(f)] \quad \text{and} \quad Q_{s,t}^{\eta_s}(f)(x) = \mathbb{E} \left(f(X_t) e^{-\int_s^t \bar{V}_u(X_u) du} \mid X_s = x \right) \\
& \implies \partial_s Q_{s,t}^{\eta_s}(f) = -L_s(Q_{s,t}^{\eta_s}(f)) + \bar{V}_s Q_{s,t}^{\eta_s}(f) \\
& \implies \partial_s (Q_{s,t}^{\eta_s}(f) Q_{s,t}^{\eta_s}(g)) = -Q_{s,t}^{\eta_s}(f) L_s(Q_{s,t}^{\eta_s}(g)) - Q_{s,t}^{\eta_s}(g) L_s(Q_{s,t}^{\eta_s}(f)) + 2\bar{V}_s Q_{s,t}^{\eta_s}(f) Q_{s,t}^{\eta_s}(g)
\end{aligned}$$

We also have

$$L_{t,\mu}(f) = L_t(f) + V_t [\mu(f) - f] \iff L_t(f) - L_{t,\mu}(f) = V_t [f - \mu(f)]$$

This yields the formula

$$\begin{aligned}
\eta \Gamma_{L_t, \eta}(f, g) - \eta \Gamma_{L_t}(f, g) & = \int \eta(dx) \eta(dy) V_t(y) [f(y) - f(x)][g(y) - g(x)] \\
& = \eta(V_t(fg)) + \eta(V_t) \eta(fg) - \eta(fV_t) \eta(g) - \eta(gV_t) \eta(f)
\end{aligned}$$

For any given time horizon t and $s \in [0, t]$ we have

$$\begin{aligned}
& dm(\xi_s)(Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g)) - \frac{1}{\sqrt{N}} dM_s(Q_{\cdot,t}^{\eta_s}(f)Q_{\cdot,t}^{\eta_s}(g)) \\
&= m(\xi_s) \left[L_{s,m(\xi_s)}(Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g)) - Q_{s,t}^{\eta_s}(f) L_s(Q_{s,t}^{\eta_s}(g)) - Q_{s,t}^{\eta_s}(g) L_s(Q_{s,t}^{\eta_s}(f)) + 2\bar{V}_s Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g) \right] ds \\
&= m(\xi_s) \left[\Gamma_{s,L_{s,m(\xi_s)}}(Q_{s,t}^{\eta_s}(f), Q_{s,t}^{\eta_s}(g)) + V_s Q_{s,t}^{\eta_s}(f) (m(\xi_s)Q_{s,t}^{\eta_s}(g) - Q_{s,t}^{\eta_s}(g)) \right. \\
&\quad \left. + V_t Q_{s,t}^{\eta_s}(g) (m(\xi_s)Q_{s,t}^{\eta_s}(f) - Q_{s,t}^{\eta_s}(f)) + 2\bar{V}_s Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g) \right] ds
\end{aligned}$$

This implies that

$$\begin{aligned}
& m(\xi_t)(fg) - m(\xi_0)(Q_{0,t}^{\eta_0}(f)Q_{0,t}^{\eta_0}(g)) - \frac{1}{\sqrt{N}} M_t(Q_{\cdot,t}^{\eta_s}(f)Q_{\cdot,t}^{\eta_s}(g)) \\
&= \int_0^t m(\xi_s) \Gamma_{s,L_{s,m(\xi_s)}}(Q_{s,t}^{\eta_s}(f), Q_{s,t}^{\eta_s}(g)) ds \\
&+ \int_0^t m(\xi_s) \left[V_s Q_{s,t}^{\eta_s}(f) (m(\xi_s)Q_{s,t}^{\eta_s}(g) - Q_{s,t}^{\eta_s}(g)) \right. \\
&\quad \left. + V_s Q_{s,t}^{\eta_s}(g) (m(\xi_s)Q_{s,t}^{\eta_s}(f) - Q_{s,t}^{\eta_s}(f)) + 2(V_s - \eta_s(V_s)) Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g) \right] ds
\end{aligned}$$

After some simplifications we check that

$$\begin{aligned}
& \int_0^t m(\xi_s) \Gamma_{s,L_{s,m(\xi_s)}}(Q_{s,t}^{\eta_s}(f), Q_{s,t}^{\eta_s}(g)) ds \\
&= m(\xi_t)(fg) - m(\xi_0)(Q_{0,t}^{\eta_0}(f)Q_{0,t}^{\eta_0}(g)) - \frac{1}{\sqrt{N}} M_t(Q_{\cdot,t}^{\eta_s}(f)Q_{\cdot,t}^{\eta_s}(g)) \\
&+ \int_0^t \left[2\eta_s(V_s) m(\xi_s)(Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g)) \right. \\
&\quad \left. - m(\xi_s)(V_s Q_{s,t}^{\eta_s}(f)) m(\xi_s)Q_{s,t}^{\eta_s}(g) - m(\xi_s)(V_s Q_{s,t}^{\eta_s}(g)) m(\xi_s)Q_{s,t}^{\eta_s}(f) \right] ds
\end{aligned}$$

Choosing $f = g = 1$ and taking the expectations we find that

$$\begin{aligned}
& \int_0^t \mathbb{E} \left[m(\xi_s) \Gamma_{s,L_{s,m(\xi_s)}}(Q_{s,t}^{\eta_s}(1)) \right] ds \\
&= 1 - \eta_0(Q_{0,t}^{\eta_0}(1)^2) + 2 \int_0^t \mathbb{E} \left[\eta_s(V_s) m(\xi_s)(Q_{s,t}^{\eta_s}(1)^2) - m(\xi_s)(V_s Q_{s,t}^{\eta_s}(1)) m(\xi_s)Q_{s,t}^{\eta_s}(1) \right] ds \\
&\leq 1 + 2e^{2q} \|V\| t
\end{aligned}$$

Choosing $f = g = h - \eta_t(h)$, with $h \in \text{Osc}(S)$ and taking the expectations we find that

$$\begin{aligned}
& \int_0^t \mathbb{E} \left[m(\xi_s) \Gamma_{s, L_{s, m(\xi_s)}} (\partial_{\eta_s} \phi_{s,t}(h)) \right] ds \\
&= \mathbb{E} \left[m(\xi_t) ([h - \eta_t(h)]^2) \right] - \eta_0([\partial_{\eta_0} \phi_{0,t}(h)]^2) \\
&+ 2 \int_0^t \mathbb{E} \left[\eta_s(V_s) m(\xi_s) ([\partial_{\eta_s} \phi_{s,t}(h)]^2) - m(\xi_s) (V_s \partial_{\eta_s} \phi_{s,t}(h)) m(\xi_s) (\partial_{\eta_s} \phi_{s,t}(h)) \right] ds \\
&\leq 1 + 4e^{2q} \|V\| t
\end{aligned}$$

For any $f \in \text{Osc}(S)$ combining (4.5) with Cauchy-Schwartz inequality we find that that

$$\int_0^t |m(\xi_s) \Gamma_{L_{s, m(\xi_s)}} \left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)} \phi_{s,t}(f) \right)| ds \leq 2e^{3q} [1 + 4e^{2q} \|V\|] t$$

Combining the above estimate with (3.7) and corollary 3.9 we conclude that

$$(H_2) \implies N |\mathbb{E}(m(\xi_t)(f)) - \mathbb{E}(\phi_{0,t}(m(\xi_0))(f))| \leq 2e^{3q} \left(1 + 4e^{2q} \|V\| + \frac{1}{N^2} 2e^q \|\lambda + V\| \right) t$$

We further assume that (H_1) is satisfied. In this case, using (4.5) we also have

$$\begin{aligned}
|m(\xi_s) \Gamma_{L_{s, m(\xi_s)}} \left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)} \phi_{s,t}(f) \right)| &\leq e^{3q} \alpha e^{-\beta(t-s)} m(\xi_s) \Gamma_{L_{s, m(\xi_s)}} \left(Q_{s,t}^{\eta_s}(1) \right) \\
&+ e^{2q} \sqrt{m(\xi_s) \Gamma_{L_{s, m(\xi_s)}} \left(Q_{s,t}^{\eta_s}(1) \right)} \sqrt{m(\xi_s) \Gamma_{L_{s, m(\xi_s)}} \left(\partial_{\eta_s} \phi_{s,t}(f) \right)}
\end{aligned}$$

For any $\tilde{\beta} \in \mathbb{R}$ we set

$$\begin{aligned}
\tilde{Q}_{s,t}^{\eta_s}(f)(x) &:= e^{\tilde{\beta}(t-s)} Q_{s,t}^{\eta_s}(f)(x) \\
&= \mathbb{E} \left(f(X_t) e^{-\int_s^t \tilde{V}_u(X_u) du} \mid X_s = x \right) \quad \text{with} \quad \tilde{V}_t(x) = \bar{V}_t(x) - \tilde{\beta}
\end{aligned}$$

Arguing as above, we have

$$\partial_s(\tilde{Q}_{s,t}^{\eta_s}(f) \tilde{Q}_{s,t}^{\eta_s}(g)) = -\tilde{Q}_{s,t}^{\eta_s}(f) L_s(\tilde{Q}_{s,t}^{\eta_s}(g)) - \tilde{Q}_{s,t}^{\eta_s}(g) L_s(\tilde{Q}_{s,t}^{\eta_s}(f)) + 2 \tilde{V}_s \tilde{Q}_{s,t}^{\eta_s}(f) \tilde{Q}_{s,t}^{\eta_s}(g)$$

and

$$\begin{aligned}
& dm(\xi_s) (\tilde{Q}_{s,t}^{\eta_s}(f) \tilde{Q}_{s,t}^{\eta_s}(g)) - \frac{1}{\sqrt{N}} dM_s(\tilde{Q}_{s,t}^{\eta_s}(f) \tilde{Q}_{s,t}^{\eta_s}(g)) \\
&= m(\xi_s) \left[\Gamma_{s, L_{s, m(\xi_s)}} (\tilde{Q}_{s,t}^{\eta_s}(f), \tilde{Q}_{s,t}^{\eta_s}(g)) + V_s \tilde{Q}_{s,t}^{\eta_s}(f) (m(\xi_s) \tilde{Q}_{s,t}^{\eta_s}(g) - \tilde{Q}_{s,t}^{\eta_s}(g)) \right. \\
&\quad \left. + V_t \tilde{Q}_{s,t}^{\eta_s}(g) (m(\xi_s) \tilde{Q}_{s,t}^{\eta_s}(f) - \tilde{Q}_{s,t}^{\eta_s}(f)) + 2 \tilde{V}_s \tilde{Q}_{s,t}^{\eta_s}(f) \tilde{Q}_{s,t}^{\eta_s}(g) \right] ds
\end{aligned}$$

This implies that

$$\begin{aligned}
& \int_0^t e^{2\tilde{\beta}(t-s)} m(\xi_s) \Gamma_{s, L_s, m(\xi_s)}(Q_{s,t}^{\eta_s}(f), Q_{s,t}^{\eta_s}(g)) ds \\
&= m(\xi_t)(fg) - e^{\tilde{\beta}t} m(\xi_0)(Q_{0,t}^{\eta_0}(f)Q_{0,t}^{\eta_0}(g)) - \frac{1}{\sqrt{N}} M_t(\tilde{Q}_{\cdot,t}^{\eta_\cdot}(f)\tilde{Q}_{\cdot,t}^{\eta_\cdot}(g)) \\
&+ \int_0^t e^{2\tilde{\beta}(t-s)} \left[2(\eta_s(V_s) + \tilde{\beta}) m(\xi_s)(Q_{s,t}^{\eta_s}(f)Q_{s,t}^{\eta_s}(g)) - m(\xi_s)(V_s Q_{s,t}^{\eta_s}(f)) m(\xi_s)Q_{s,t}^{\eta_s}(g) \right. \\
&\quad \left. - m(\xi_s)(V_s Q_{s,t}^{\eta_s}(g)) m(\xi_s)Q_{s,t}^{\eta_s}(f) \right] ds
\end{aligned}$$

Choosing $f = g = 1$ and $\tilde{\beta} < 0$ we have

$$\begin{aligned}
& \int_0^t e^{2\tilde{\beta}(t-s)} \mathbb{E} \left[m(\xi_s) \Gamma_{s, L_s, m(\xi_s)}(Q_{s,t}^{\eta_s}(1)) \right] ds \\
&= 1 - e^{\tilde{\beta}t} \eta_0(Q_{0,t}^{\eta_0}(1)^2) \\
&+ 2 \int_0^t e^{2\tilde{\beta}(t-s)} \mathbb{E} \left[(\eta_s(V_s) + \tilde{\beta}) m(\xi_s)(Q_{s,t}^{\eta_s}(1)^2) - m(\xi_s)(V_s Q_{s,t}^{\eta_s}(1)) m(\xi_s)Q_{s,t}^{\eta_s}(1) \right] ds \\
&\leq 1 + e^{2a} \left(1 + 2|\tilde{\beta}|^{-1}\|V\| \right) = 1 + e^{2a} \left(1 + 4\beta^{-1}\|V\| \right) \quad \text{when } \tilde{\beta} = -\beta/2
\end{aligned}$$

Choosing $f = g = [h - \eta_t(h)]$, with $h \in \text{Osc}(S)$ and $0 < \tilde{\beta} < \beta$ we have

$$\begin{aligned}
& \int_0^t e^{2\tilde{\beta}(t-s)} \mathbb{E} \left[m(\xi_s) \Gamma_{s, L_s, m(\xi_s)}(\partial_{\eta_s} \phi_{s,t}(h)) \right] ds \\
&\leq \mathbb{E} \left[m(\xi_t)([h - \eta_t(h)]^2) \right] \\
&+ 2 \int_0^t e^{2\tilde{\beta}(t-s)} \left[(\eta_s(V_s) + \tilde{\beta}) m(\xi_s) \left([\partial_{\eta_s} \phi_{s,t}(h)]^2 \right) - m(\xi_s)(V_s \partial_{\eta_s} \phi_{s,t}(h)) m(\xi_s) \partial_{\eta_s} \phi_{s,t}(h) \right] ds \\
&\leq 1 + 2r^2(2\|V\| + \tilde{\beta}) \int_0^t e^{-2(\beta-\tilde{\beta})(t-s)} ds \\
&\leq 1 + r^2(2\|V\| + \tilde{\beta}) (\beta - \tilde{\beta})^{-1} = 1 + r^2(4\|V\|\beta^{-1} + 1) \quad \text{when } \tilde{\beta} = \beta/2
\end{aligned}$$

We end the proof of the theorem using the fact that

$$\begin{aligned}
& |m(\xi_s) \Gamma_{L_s, m(\xi_s)} \left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)} \phi_{s,t}(f) \right) | \\
&\leq e^{3q}(1 + \alpha) e^{-\beta(t-s)/2} m(\xi_s) \Gamma_{L_s, m(\xi_s)} \left(Q_{s,t}^{\eta_s}(1) \right) + e^{2q} e^{\beta(t-s)/2} m(\xi_s) \Gamma_{L_s, m(\xi_s)} \left(\partial_{\eta_s} \phi_{s,t}(f) \right)
\end{aligned}$$

In the last assertion we have used the fact that the estimate $\sqrt{ab} \leq ca + b/c$, for all $a, b, c > 0$. This ends the proof of the theorem. \blacksquare

References

- [1] D. Aldous, U. Vazirani. Go with the winners algorithms. In *Foundations of Computer Science. Proceedings., 35th Annual Symposium on* (pp. 492–501). IEEE (1994).
- [2] C. Andrieu, A. Doucet, R. Holenstein. Particle Markov chain Monte Carlo for efficient numerical simulation. In L’Ecuyer, P. and Owen, A. B., editors, *Monte Carlo and Quasi-Monte Carlo Methods 2008*, pp. 45–60. Springer-Verlag Berlin Heidelberg (2009).
- [3] M. Arnaudon, P. Del Moral. A variational approach to nonlinear and interacting diffusions. ArXiv preprint:1812.04269 (2018).
- [4] D. Benedetto, E. Caglioti, M. Pulvirenti. A kinetic equation for granular media. *RAIRO Modél. Math. Anal. Numér.* vol. 31, no. 5, pp. 615–641 (1997).
- [5] D. Benedetto, E. Caglioti, E., Carrillo, M. Pulvirenti. A non-Maxwellian steady distribution for one-dimensional granular media. *J. Statist. Phys.* vol. 91, pp. 979–990 (1998).
- [6] F. Bolley, I. Gentil, A. Guillin. Uniform convergence to equilibrium for granular media. *Archive for Rational Mechanics and Analysis*, vol. 208, no. 2, pp. 429–445 (2013).
- [7] A. Beskos, G.O. Roberts. Exact simulation of diffusions. *The Annals of Applied Probability*, vol. 15, no. 4, pp. 2422–2444 (2005).
- [8] A. Beskos, O. Papaspiliopoulos, G.O. Roberts. Retrospective exact simulation of diffusion sample paths with applications. *Bernoulli*, vol. 12, no. 6, pp. 1077–1098. (2006).
- [9] A. Beskos, O. Papaspiliopoulos, G.O. Roberts. A factorisation of diffusion measure and finite sample path constructions. *Methodol. Comput. Appl. Probab.*, vol. 10, pp. 85–104 (2008).
- [10] A. Beskos, O. Papaspiliopoulos, G.O. Roberts. P. Fearnhead. Exact and computationally efficient likelihood-based estimation for discretely observed diffusion processes. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, vol. 68, pp. 333–382 (2006).
- [11] M. Caffarel, R. Assaraf. A pedagogical introduction to quantum Monte Carlo. In *Mathematical models and methods for ab initio Quantum Chemistry in Lecture Notes in Chemistry*, eds. M. Defranceschi and C. Le Bris, Springer p.45 (2000).
- [12] B. Casella, G.O. Roberts. Exact simulation of jump-diffusion processes with Monte Carlo applications. *Methodology and Computing in Applied Probability*, vol. 13, no. 3, pp. 449–473 (2011).
- [13] E. Cancès, B. Jourdain, T. Lelièvre. Quantum Monte Carlo simulations of fermions. A mathematical analysis of the fixed-node approximation, *Mathematical Models and Methods in Applied Sciences*, 16(9), 1403-1440, (2006).
- [14] O. Cappé, E. Moulines, and T. Rydèn. *Inference in Hidden Markov Models*, Springer-Verlag (2005).
- [15] J. A. Carrillo, R. J. McCann, and C. Villani. Contractions in the 2-Wasserstein length space and thermalization of granular media. *Arch. Rational Mech. Anal.*, vol. 179, pp. 217–263, (2006).
- [16] P. Cattiaux, A. Guillin, and F. Malrieu. Probabilistic approach for granular media equations in the non uniformly convex case. *Prob. Theor. Rel. Fields*, vol. 140, no. 1-2, pp. 19–40 (2008).

- [17] D. Cordero-Erausquin, W. Gangbo, and C. Houdré. Inequalities for generalized entropy and optimal transportation. In *Recent Advances in the Theory and Applications of Mass Transport*, Contemp. Math. 353. A. M. S., Providence (2004).
- [18] R. Cont, D.A. Fournié. Functional Itô calculus and stochastic integral representation of martingales. *The Annals of Probability*, vol. 41, no. 1, pp. 109–133 (2013).
- [19] A.M. Cox, S.C. Harris, E.L. Horton, A. E. Kyprianou. Multi-species neutron transport equation. *Journal of Statistical Physics*, pp. 1–31, (2019).
- [20] A. S. Dalalyan. Theoretical guarantees for approximate sampling from smooth and log-concave densities. In: *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, vol. 79, no. 3 pp. 651–676 (2017).
- [21] P. Del Moral. **Feynman-Kac formulae**. Genealogical and interacting particle systems with applications. *Probability and its Applications (New York)*. (573p.) Springer-Verlag, New York (2004).
- [22] P. Del Moral. **Mean field simulation for Monte Carlo integration**. Chapman & Hall. *Monographs on Statistics & Applied Probability* (2013).
- [23] P. Del Moral. Nonlinear filtering: interacting particle solution. *Markov Process. & Related Fields*, vol. 2, no. 4, pp. 555–579 (1996).
- [24] P. Del Moral, A. Doucet. Particle motions in absorbing medium with hard and soft obstacles. *Stochastic Analysis and Applications*, vol. 22, no. 5, pp. 1175–1207 (2004).
- [25] P. Del Moral, A. Guionnet. On the stability of interacting processes with applications to filtering and genetic algorithms. *Annales de l’Institut Henri Poincaré (B) Probability and Statistics*, vol. 37, no. 2, pp. 155–194 (2001).
- [26] P. Del Moral, A. Guionnet. On the stability of measure valued processes with applications to filtering. *Comptes Rendus de l’Académie des Sciences-Series I-Mathematics*, vol. 329, no. 5, pp. 429–434 (1999).
- [27] P. Del Moral, A. Jasra. A sharp first order analysis of Feynman-Kac particle models, Part I: Propagation of chaos. *Stochastic Processes and their Applications*, 128(1), pp. 332–353 (2018).
- [28] P. Del Moral, A. Jasra. A sharp first order analysis of Feynman-Kac particle models, Part II: Particle Gibbs samplers. *Stochastic Processes and their Applications*, 128(1), pp. 354–371 (2018).
- [29] P. Del Moral, P. Jacob, A. Lee, L. Murray, G.W. Peters. Feynman-Kac particle integration with geometric interacting jumps, *Stochastic Analysis and Applications*, vol. 31, no. 5, pp. 830–871 (2013).
- [30] P. Del Moral, J. Jacod. The Monte-Carlo method for filtering with discrete-time observations: Central limit theorems. *Numerical Methods and stochastics*, vol. 34, pp. 29–53 (2002).
- [31] P. Del Moral, R. Kohn and F. Patras *On particle Gibbs sampler*. *Ann. Inst. Henri Poincaré Probab. Stat.* vol. 52, no. 4, pp. 1687–1733 (2016).
- [32] P. Del Moral, L. Miclo. A Moran particle system approximation of Feynman-Kac formulae. *Stochastic processes and their applications*, 86(2), pp. 193–216 (2000).

- [33] P. Del Moral and L. Miclo. **Branching and interacting particle systems** approximations of Feynman-Kac formulae with applications to non-linear filtering. In *Séminaire de Probabilités, XXXIV*, volume 1729, *Lecture Notes in Math.*, pages 1–145. Springer, Berlin (2000).
- [34] P. Del Moral, L. Miclo. On the Stability of Non Linear Semigroup of Feynman-Kac Type. *Ann. de la Faculté des Sciences de Toulouse*, vol.11, no.2, pp. (2002).
- [35] P. Del Moral, L. Miclo. Particle approximations of Lyapunov exponents connected to Schrödinger operators and Feynman-Kac semigroups. *ESAIM: Probability and Statistics*, no. 7, pp. 171–208 (2003).
- [36] P. Del Moral, S. Penev. *Stochastic Processes: From Applications to Theory*. Chapman and Hall/CRC (2017).
- [37] P. Del Moral, f. Patras, S. Rubenthaler. Convergence of U-statistics for interacting particle systems. *Journal of Theoretical Probability*, vol. 24, no. 4:1002 (2011).
- [38] P. Del Moral, D. Villemonais. Exponential mixing properties for time inhomogeneous diffusion processes with killing. *Bernoulli*, vol. 24, no. 2, pp. 1010–1032 (2018).
- [39] B. Delyon, F. Cérou, A. Guyader, M. Rousset. A central limit theorem for Fleming-Viot particle systems with hard killing. *arXiv preprint arXiv:1709.06771* (2017).
- [40] A. Doucet, N. De Freitas, N. Gordon, Editors. *An introduction to sequential Monte Carlo methods*. In *Sequential Monte Carlo methods in practice*. Springer, New York, NY (2001).
- [41] B. Dupire. *Functional itô calculus*. (2009).
- [42] A. Durmus, E. Moulines. Nonasymptotic convergence analysis for the unadjusted Langevin algorithm. In: *Ann. Appl. Probab.*, vol. 27, no. 3 , pp. 1551–1587 (2017).
- [43] P. Fearnhead, K. Latuszynski, G. O. Roberts, G. Sermaidis. Continuous-time importance sampling: Monte Carlo methods which avoid time-discretization error. *arXiv preprint arXiv:1712.06201* (2017).
- [44] D. T. Gillespie. Exact stochastic simulation of coupled chemical reactions. *J. Phys. Chem.*, vol. 81, no. 25, pp. 2340–2361 (1977).
- [45] S. Jazaerli, Y.F. Saporito. Functional Itô calculus, path-dependence and the computation of Greeks. *Stochastic Processes and their Applications*, vol. 127, no.12, pp. 3997–4028 (2017).
- [46] C. Jarzynski. Equilibrium free energy differences from non equilibrium measurements: a master equation approach, *Phys. Rev. E*, vol. 56, no. 5, pp. 5018–5035 (1997).
- [47] C. Jarzynski. Nonequilibrium equality for free energy differences, *Phys. Rev. Lett.*, vol. 78, no. 14, pp. 2690–2693 (1997).
- [48] P.E. Kloeden, E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer, Berlin (1992).
- [49] R. Kubo. The fluctuation-dissipation theorem. *Reports on progress in physics*, vol. 29, no. 1, p. 255 (1966).

- [50] A.E. Kyprianou, S. Palau. Extinction properties of multi-type continuous-state branching processes. *Stochastic Processes and their Applications*, vol. 128, no.10, pp. 3466–3489 (2018).
- [51] R. Lande, S. Engen, B.E. Saether. *Stochastic population dynamics in ecology and conservation*. Oxford University Press (2003).
- [52] P. Langevin. Sur la théorie du mouvement brownien, *Comptes-Rendus de l'Académie des Sciences*, vol. 146, pp. 530–532 (1908).
- [53] T. Lelièvre, M. Rousset and G. Stoltz. *Free energy computations: A mathematical perspective*. Imperial College Press (2010).
- [54] T. Lelièvre, M. Rousset and G. Stoltz. Computation of free energy differences through non-equilibrium stochastic dynamics: the reaction coordinate case, *Journal of Computational Physics*, 222(2), 624-643, (2007).
- [55] F. Lindsten, T. Schön, M. I. Jordan. Ancestor Sampling for Particle Gibbs. *Conference in Advances in Neural Information Processing Systems*, vol. 25, pp. 2600–2608 (2012).
- [56] F. Lindsten, T. Schön. On the use of backward simulation in the particle Gibbs sampler. *Acoustics, Speech and Signal Processing (ICASSP), 2012 IEEE International Conference on*. IEEE (2012).
- [57] F. Malrieu. Logarithmic Sobolev inequalities for some nonlinear PDE's. *Stochastic Process. Appl.* vol. 95, no. 1, 109–132 (2001).
- [58] F. Malrieu. Convergence to equilibrium for granular media equations and their Euler schemes. *Ann. Appl. Probab.*, vol. 13, no. 2, pp. 540–560 (2003).
- [59] H. P. McKean. A class of Markov processes associated with nonlinear parabolic equations. *Proc. Nat. Acad. Sci. U.S.A.*, vol. 56, pp. 1907–1911 (1966).
- [60] H. P. McKean. Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations, Lecture Series in Differential Equations, Session 7*, Catholic Univ. (1967), pp. 41–57. Air Force Office Sci. Res., Arlington, Va., (1967).
- [61] S. Méléard. Asymptotic behaviour of some interacting particle systems, McKean-Vlasov and Boltzmann models. In: Talay, D., Tubaro, L. (Eds.), *Probabilistic Models for Nonlinear Partial Differential Equations*, Montecatini Terme, 1995, *Lecture Notes in Mathematics*, Vol. 1627. Springer, Berlin (1996).
- [62] P.A.P. Moran. *The Statistical Processes of Evolutionary Theory*. Oxford: Clarendon Press (1962).
- [63] P.A.P. Moran. Random processes in genetics. *Mathematical Proceedings of the Cambridge Philosophical Society*. 54 (1): pp. 60–71 (1958).
- [64] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, vol. 26, pp. 101–174 (2001).
- [65] F. Otto, V. Villani. Generalization of an inequality by Talagrand, and links with the logarithmic Sobolev inequality. *J. Funct. Anal.* vol. 173, pp. 361–400 (2000).

- [66] G.O. Roberts, J.S. Rosenthal. Optimal scaling for various Metropolis-Hastings algorithms. *Statistical science*, vol. 16, no. 4, pp. 351–367 (2001). .
- [67] M. Rousset. On the control of an interacting particle approximation of Schrödinger ground states. *SIAM J. Math. Anal.*, vol. 38, no. 3, pp. 824–844 (2006).
- [68] M.Rousset and G.Stoltz. Equilibrium sampling from non equilibrium dynamics. *J. Stat. Phys.*, vol. 123, no. 6, pp. 1251–1272 (2006).
- [69] M.N. Rosenbluth, A.W. Rosenbluth. Monte Carlo calculation of the average extension of molecular chains. *The Journal of Chemical Physics*, 23(2), pp. 356–359 (1955).
- [70] Y.F. Saporito. Topics on Functional Itô Calculus and Multiscale Stochastic Volatility Modeling. PhD. thesis Santa Barbara (2014).
- [71] R.A. Singer. Estimating optimal tracking filter performance for manned maneuvering targets. *IEEE Transactions on Aerospace and Electronic Systems*, vol. 4, pp. 473–483 (1970).
- [72] Y. Tamura. On asymptotic behaviors of the solution of a nonlinear diffusion equation. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, vol.31, no. 1, pp. 195–221 (1984).
- [73] Y. Tamura. Free energy and the convergence of distributions of diffusion processes of McKean type. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, vol. 34, no. 2, pp. 443–484, (1987).
- [74] D. Villemonais. General approximation method for the distribution of Markov processes conditioned not to be killed. *ESAIM Probab. Stat.*, vol. 18, pp. 441–467 (2014).