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Marc Arnaudon[1] and Pierre Del Moral[2]

[1]University of Bordeaux, Bordeaux Mathematics Institute, France
[2]INRIA, Bordeaux Research Center, France; and UNSW Sydney, Australia

Abstract

Continuous time Feynman-Kac measures on path spaces are central in applied probability, partial differential equation theory, as well as in quantum physics. This article presents a new duality formula between normalized Feynman-Kac distribution and their mean field particle interpretations. Among others, this formula allows us to design a reversible particle Gibbs-Glauber sampler for continuous time Feynman-Kac integration on path spaces. This result extends the particle Gibbs samplers introduced by Andrieu-Doucet-Holenstein [2] in the context of discrete generation models to continuous time Feynman-Kac models and their interacting jump particle interpretations. We also provide new propagation of chaos estimates for continuous time genealogical tree based particle models with respect to the time horizon and the size of the systems. These results allow to obtain sharp quantitative estimates of the convergence rate to equilibrium of particle Gibbs-Glauber samplers. To the best of our knowledge these results are the first of this kind for continuous time Feynman-Kac measures.

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1 Introduction

Feynman-Kac measures on path spaces are central in applied probability as well as in quantum physics. They also arise in a variety of application domains such as in estimation and control theory, as well as a rare event analysis. For a detailed review on Feynman-Kac and their application domains we refer to the three books [7, 8, 13], and the references therein. Their mean field type particle interpretations is defined as a system of particles jumping a given rate uniformly onto the population. From the pure numerical viewpoint, this interacting jump transition can be interpreted as an acceptance-rejection scheme with a recycling. Feynman-Kac interacting particle models encapsulate a variety of algorithms such as the diffusion Monte Carlo used to solve Schrödinger ground states, see for instance the series of articles [3, 4, 16, 23, 26, 27] and the references therein.

Their discrete time versions are encapsulate a variety of well known algorithms such as particle filters [9] (a.k.a. sequential Monte Carlo methods in Bayesian literature [5, 7, 8, 13, 17]), the go-with the winner [1], as well as the self-avoidind random walk pruned-enrichment algorithm by Rosenbluth and Rosenbluth [24], and many others. This list is not exhaustive (see also the references therein). The research monographs [7, 8] provide a detailed discussion on these subjects with precise reference pointers.

The seminal article [2] by Andrieu, Doucet and Holenstein introduced a new way to combine Markov chain Monte Carlo methods with discrete generation particle methods. A variant of the

method, where ancestors are resampled in a forward pass, was developed by Lindsten, Schön and Jordan in [28], and Lindsten and Schön [29]. In all of these studies, the validity of the particle conditional sampler is assessed by interpreting the model as a traditional Markov chain Monte Carlo sampler on an extended state space. The central idea is first to design a detailed encoding of the ancestors at each level in terms of random maps on integers, and then to extend the "target" measure on a sophisticated state space incapsulating these iterated random sequences.

In a more recent article [14], we provide an alternative and we believe more natural interpretation of these particle Markov chain Monte Carlo methods in terms of a duality formula extending the well known unbiasedness properties of Feynman-Kac particle measures on many-body particle measures. This article also provides sharp quantitative estimates of the convergence rate to equilibrium of the models with respect to the time horizon and the size of the systems. The analysis of these models, including backward particle Markov chain Monte Carlo samplers has been further developed in [10, 11].

The main objective of the present article is to extend these methodologies to continuous time Feynman-Kac measures on path spaces.

The first difficulty comes from the fact that the discrete time analysis [10, 11, 14] only applies to simple genetic type particle models, or equivalently to branching models with pure multinomial selection schemes. Thus, these results don't apply to discrete time approximation of continuous time models based on geometric type jumps, and any density type argument cannot be applied.

In contrast with their discrete time version, continuous time Feynman-Kac particle models are not described by conditionally independent local transitions, but in terms of interacting jump processes. This class of processes can be interpreted as Moran type interacting particle systems [21, 22]. They can also be seen as Nanbu type interpretation of a particular spatially homogeneous generalized Boltzmann equation [12, 20].

The analysis of continuous time genetic type particle models is not so developed as their discrete time versions. For instance, uniform convergence estimates are available for continuous time Feynman-Kac models with stable processes [13, 15, 16, 23]. Nevertheless, to the best of our knowledge, sharp estimates for path space models and genealogical tree based particle samplers in continuous time have never been discussed in the literature. These questions are central in the study the convergence to equilibrium of particle Gibbs-Glauber sampler on path spaces.

In the present article we provide a duality formula for continuous time Feynman-Kac measures on path-spaces (cf. theorem 1.1). This formula on generalogical tree based particle models that can be seen as an extension of well known unbiasedness properties of Feynman-Kac models to their many body version (defined in section 4). The second main result of the article is to design and to analyze the stability properties of a particle Gibbs-Glauber sampler of path space (cf. theorem 1.2). Our approach combines a perturbation analysis of nonlinear stochastic semigroups with propagation of chaos techniques (cf. section 3). Incidentally these techniques also provide with little efforts new uniform propagation of chaos estimates w.r.t. the time horizon (cf. corollary 3.12).

1.1 Statement of the main results

Let (X_t, V_t) be a continuous time Markov process and a bounded non negative function on some metric space (S, d_S). We let $\mathbb{P}_t$ be the distribution of $(X_s)_{s\leqslant t}$ on the set $D_t(S)$ of of càdlàg paths from $[0,t]$ to S. As a rule in the further development of the article $\widehat{X}_t := (X_s)_{s\leqslant t}$ stands for the historical process of some process X_t. In this notation, we extend V_t to $D_t(S)$ by setting $\widehat{V}_t(\widehat{X}_t) = V_t(X_t)$.

The Feynman-Kac probability measures $\mathbb{Q}_t$ associated with (X_t, V_t) are defined by

$$d\mathbb{Q}_t = \frac{1}{Z_t}\ \exp\left[-\int_0^t V_s(X_s)ds\right]\ d\mathbb{P}_t$$

where Z_t stands for some normalizing constant.

These measures can be computed in terms the occupation measures of the ancestral lines of an interacting jump process. Consider a system of N particles evolving independently as X_t with jump rate $V_t(X_t)$. At each jump time the particle jumps onto a particle uniformly chosen in the pool.

Equivalently, the N ancestral lines $\xi_t = (\xi_t^i)_{1 \leqslant i \leqslant N}$ of length t can also be seen as a system of N path-valued particles evolving independently as the historical process $\widehat{X}_t$, with jump rate $\widehat{V}_t$.

The occupation measure of the genealogical tree is given by the empirical measures

$$m(\xi_t) := \frac{1}{N} \sum_{1 \leqslant i \leqslant N} \delta_{\xi_t^i} \quad \text{and we denote by } \mathbb{X}_t \text{ a random sample from } m(\xi_t)$$

The dual process $\zeta_t = (\zeta_t^i)_{1 \leqslant i \leqslant N}$ is also defined in terms of N the ancestral lines of length t of an interacting jump process. The main difference is that the first line at any time t is frozen and given by $\zeta_t^1 := \widehat{X}_t$. The remaining $(N-1)$ path-valued particles $\zeta_t^- := (\zeta_t^i)_{2 \leqslant i \leqslant N}$ are defined as above with a rescaled jump rate $(1 - 1/N)\widehat{V}_t$, with an additional jump rate $2\widehat{V}_t/N$ at which the path-particle jump onto the first frozen ancestral line.

The first main result of the article is the following duality formula.

Theorem 1.1 (Duality formula). *For any time horizon $t \geqslant 0$ and any bounded measurable function F on $D_t(S)^2 \times D_t(S)^{N-1}$ symmetric on the last $(N-1)$ coordinates we have*

$$\mathbb{E}\left(F(\mathbb{X}_t, \widehat{\xi}_t) \ \exp\left[- \int_0^t \ m(\xi_s)(\widehat{V}_s) ds \right] \right) = \mathbb{E}\left(F(\widehat{X}_t, \widehat{\zeta}_t) \ \exp\left[- \int_0^t \widehat{V}_s(\widehat{X}_s) ds \right] \right) \tag{1.1}$$

We consider the following regularity condition

$$(H_0) \quad \exists h > 0 \quad \text{s.t.} \quad \forall t \geqslant 0 \quad \forall x \in S \quad \rho(h) \ \mu_{t,h}(dy) \ \leqslant P_{t,t+h}(x, dy) \ \leqslant \rho(h)^{-1} \ \mu_{t,h}(dy) \tag{1.2}$$

for some probability $\mu_{t,h}$ on S and some constant $\rho(h) > 0$ whose value doesn't depend on the parameters (x, y). For instance, condition (1.2) is satisfied for jump-type elliptic diffusions on compact manifolds S with a bounded jump rate.

The second main result of the article can be stated basically as follows.

Theorem 1.2 (Particle Gibbs-Glauber dynamics). *For any time horizon $t \geqslant 0$ the measure $\mathbb{Q}_t$ is reversible w.r.t. the Markov transition $\mathbb{K}_t$ on $D_t(S)$ defined for any bounded measurable function f on $D_t(S)$ and any path $x \in D_t(S)$ by the formula*

$$\mathbb{K}_t(f)(x) := \mathbb{E}\left(m(\zeta_t)(f) \mid \widehat{X}_t = x \right)$$

In addition, when (H_0) is satisfied, for any probability measure μ on $D_t(S)$ we have

$$N \ \mathrm{osc}(\mathbb{K}_t(f)) \leqslant c \ t \ \mathrm{osc}(f) \quad \text{and} \quad \forall n \geqslant 1 \quad \|\mu \mathbb{K}_t^n - \mathbb{Q}_t\|_{tv} \leqslant (c \ t/N)^n$$

for some finite constant c whose value doesn't depend on the parameters (f, t, N).

1.2 Basic notation and preliminary results

Let $\mathcal{B}(S)$ be the Banach space of bounded functions f on S equipped with the uniform norm $\|f\| := \sup_{x \in S} |f(x)|$. Also let $\mathrm{Osc}(S) \subset \mathcal{B}(S)$ be the subset of functions f with unit oscillations; that is s.t. $\mathrm{osc}(f) := \sup_{x,y} |f(x) - f(y)| \leqslant 1$.

We also let $\mathcal{M}(S)$ the set of finite signed measures on S, $\mathcal{M}_+(S) \subset \mathcal{M}(S)$ the subset of positive measures and $\mathcal{P}(S) \subset \mathcal{M}_+(S)$ the subset of probability measures. Given a random measure μ on S we write $\mathbb{E}(\mu)$ the first moment measure given by

$$\mathbb{E}(\mu) \ : \ f \in \mathcal{B}(S) \mapsto \mathbb{E}(\mu)(f) = \mathbb{E}(\mu(f)) \quad \text{with} \quad \mu(f) = \int f(x)\ \mu(dx)$$

The total variation norm on the set $\mathcal{M}(S)$ is defined by

$$\|\mu\|_{tv} \ := \ \sup\{|\mu(f)| \ : \ f \in \mathrm{Osc}(S)\} \tag{1.3}$$

1.2.1 Integral operators

For any bounded positive integral operator $Q(x, dy)$ and any $(\mu, f, x) \in (\mathcal{M}(S) \times \mathcal{B}(S) \times x)$ we define by $\mu Q \in \mathcal{M}(S)$ and $Q(f) \in \mathcal{B}(S)$ by the formulae

$$(\mu Q)(dy) := \int \mu(dx) Q(x, dy) \quad \text{and} \quad Q(f)(x) := \int Q(x, dy)\ f(y)$$

By Fubini theorem we have $\mu Q f := \mu(Q(f)) = (\mu Q)(f)$. We also write Q^n the n iterate of Q defined by the recursion $Q^n(f) = Q(Q^{n-1}(f)) = Q^{n-1}(Q(f))$.

When $Q(1) > 0$ we let $\overline{Q}$ be the Markov operator

$$\overline{Q} \ : \ f \in \mathcal{B}(S) \mapsto \overline{Q}(f) := Q(f)/Q(1) \in \mathcal{B}(S)$$

We also let ϕ be the mapping from $\mathcal{P}(S)$ into itself defined by

$$\phi(\eta) = \eta Q^\eta \quad \text{with} \quad Q^\eta := \frac{Q}{\eta Q(1)} \Longrightarrow \eta Q^\eta(1) = 1 \quad \text{and} \quad \phi(\delta_x)(f) = \overline{Q}(f)(x) \tag{1.4}$$

Notice that

$$Q^\eta(1) = \ \mu Q^\eta(1)\ Q^\mu(1) \Longrightarrow (\mu Q^\eta(1))^{-1} = \eta Q^\mu(1)$$

1.2.2 Taylor expansions

Observe that for any $\eta, \nu \in \mathcal{P}(S)$ we have the decomposition

$$\phi(\nu) - \phi(\eta) \quad = \quad \eta Q^\nu(1) \ \times \ (\nu - \eta)\partial_\eta \phi$$

with the first order operator

$$\partial_\eta \phi \ : \ f \in \mathcal{B}(S) \mapsto \partial_\eta \phi(f) = Q^\eta\left[f - \phi(\eta)(f)\right] \in \mathcal{B}(S) \Longrightarrow \eta \partial_\eta \phi = 0 = \partial_\eta \phi(1) \tag{1.5}$$

Also observe that

$$\begin{aligned} &\partial_\eta \phi(f)(x) = Q^\eta(1)(x) \ \int \eta(dy)\ Q^\eta(1)(y) \ \left(\overline{Q}(f)(x) - \overline{Q}(f)(y)\right) \\ &\Longrightarrow \|\partial_\eta \phi\| \leqslant \|Q^\eta(1)\|\ \mathrm{osc}(\overline{Q}(f)) \quad \text{and} \quad \|\phi(\nu) - \phi(\eta)\|_{tv} \leqslant [\|Q^\nu(1)\| \wedge \|Q^\eta(1)\|]\ \mathrm{osc}(\overline{Q}(f)) \end{aligned} \tag{1.6}$$

More generally, using the identity

$$\frac{1}{x} = \sum_{0 \leqslant k < n} (1 - x)^k + \frac{(1 - x)^n}{x} \tag{1.7}$$

which is valid for any $x > 0$ and $n \geqslant 1$, we check the Taylor with remainder expansion

$$\phi(\nu) \quad = \quad \phi(\eta) + \sum_{1 \leqslant k \leqslant n} \frac{1}{k!} \, (\nu - \eta)^{\otimes k} \, \partial_\eta^k \phi + \frac{1}{(n+1)!} \, (\nu - \eta)^{\otimes (n+1)} \, \overline{\partial}_{\nu,\eta}^{n+1} \phi \tag{1.8}$$

In the above display, $\partial_\eta^k \phi$ stand for the collection of integral operators

$$\partial_\eta^k \phi(f) := (-1)^{k-1} \, k! \, \left[Q^\eta(1)^{\otimes (k-1)} \otimes \partial_\eta \phi(f) \right] \quad \text{and} \quad \overline{\partial}_{\nu,\eta}^{n+1} \phi := \eta Q^\nu(1) \, \partial_\eta^{n+1} \phi$$

For any $\mu, \eta \in \mathcal{P}(S)$ we have

$$\partial_\eta \phi(f) = Q^\eta[f - \phi(\eta) f] \quad = \quad \mu Q^\eta(1) \, \left(\partial_\mu \phi(f) + Q^\mu(1) \, [\phi(\mu) - \phi(\eta)](f) \right)$$

1.2.3 Carré du champ operators

The carré du champ operator associated with some the generator L acting on an algebra of functions $\mathcal{D}(S) \subset \mathcal{B}(S)$ is defined by the quadratic form

$$(f, g) \in \mathcal{D}(S)^2 \mapsto \Gamma_L(f, g) = L(fg) - f L(g) - g L(f) \in \mathcal{B}(S)$$

When $f = g$ sometimes we write $\Gamma_L(f)$ instead of $\Gamma_L(f, f)$. We also recall the Cauchy-Schwartz inequality

$$|\Gamma_L(f, g)| \leqslant \sqrt{\Gamma_L(f, f) \Gamma_L(g, g)} \quad \text{and} \quad \Gamma_L(cf) = c^2 \Gamma_L(f) \tag{1.9}$$

The above inequality yields the estimate

$$\Gamma_L(f + g) = \Gamma_L(f) + \Gamma_L(g) + 2\Gamma_L(f, g) \leqslant \left[\sqrt{\Gamma_L(f)} + \sqrt{\Gamma_L(g)} \right]^2 \tag{1.10}$$

Let L^d be some bounded jump-type generator of the following form

$$L^d(f)(u) = \lambda(u) \, \int \, (f(v) - f(u)) \, J(u, dv)$$

for some bounded rate function λ and some Markov transition J on S. In this case, we have

$$\Gamma_{L^d}(f, g)(u) = \int \, L^d(u, dv) \, (\delta_v - \delta_u)^{\otimes 2} \, (f \otimes g)$$

We consider the n-th order operators

$$\Gamma_{L^d}^{(n)}(f_1, \ldots, f_n)(u) := \int \, L^d(u, dv) \, (\delta_v - \delta_u)^{\otimes n} \, (f_1 \otimes \ldots \otimes f_n) \tag{1.11}$$

We also have the carré du champ formula

$$(\eta Q^\mu(1))^2 \, \Gamma_L \left(Q^\eta(1), \partial_\eta \phi(f) \right) = \Gamma_L \left(Q^\mu(1), \partial_\mu \phi(f) \right) + \, [\phi(\mu) - \phi(\eta)](f) \, \Gamma_L \left(Q^\mu(1) \right) \tag{1.12}$$

for any $f \in \mathcal{D}(S)$ as soon as $Q^\eta(1), \partial_\eta \phi(f) \in \mathcal{D}(S)$.

1.2.4 Empirical measures

For any $N \geqslant 1$, we let $S_N := S^N/\mathbb{S}_N$ be the N symmetric product of S, where $\mathbb{S}_N$ stands for the symmetric group of order N.

We fix some integer $N \geqslant 2$ and for any $2 \leqslant i < j \leqslant N$ and $x = (x^i)_{1 \leqslant i \leqslant N} \in S_N$ we set

$$\begin{aligned} x^{-i} &= \left(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^N\right) \in S_{N-1} \\ x^{-\{i,j\}} &= \left(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{j-1}, x^{j+1}, \ldots, x^N\right) \in S_{N-2} \end{aligned}$$

For any $2 \leqslant i \leqslant N$ and $x = (x^i)_{1 \leqslant i \leqslant N} \in S_N$ we consider the functions

$$\begin{aligned} \varphi_{x^{-i}} \ : \ u \in S \ &\mapsto \ \varphi_{x^{-i}}(u) = \left(x^1, \ldots, x^{i-1}, u, x^{i+1}, \ldots, x^N\right) \in S^N \\ m \ : \ x \in S_N \ &\mapsto \ m(x) = \frac{1}{N} \sum_{1 \leqslant i \leqslant N} \delta_{x^i} \in \mathcal{P}(S) \end{aligned} \tag{1.13}$$

Let $X = (X^i)_{1 \leqslant i \leqslant N}$ be N independent random samples from some distribution $\eta \in \mathcal{P}(S)$. Using (1.8) we have the first order expansion

$$\phi(m(X)) - \phi(\eta) \ = \ (m(X) - \eta)\partial_\eta \phi - \eta(Q^{m(X)}(1))\ (m(X) - \eta)(Q^\eta(1))\ (m(X) - \eta)\partial_\eta \phi$$

Several estimates can be derived from the above decomposition. For instance using Cauchy-Schwartz inequality we have the bias estimate

$$\log\left(Q(1)(x)/Q(1)(y)\right) \leqslant q \Longrightarrow N\ |\mathbb{E}\left[\phi(m(X))(f)\right] - \phi(\eta)(f)| \leqslant e^q\ \mathrm{osc}(\overline{Q}(f))$$

2 A brief review on Feynman-Kac measures

2.1 Evolution semigroups

Consider the flow of Feynman-Kac measures $(\gamma, \eta) : t \in \mathbb{R}_+ := [0, \infty[\mapsto (\gamma_t, \eta_t) \in (\mathcal{M}_+(S) \times \mathcal{P}(S))$ defined for any $f \in \mathcal{B}(S)$ by the formulae

$$\eta_t(f) = \gamma_t(f)/\gamma_t(1) \quad \text{with} \quad \gamma_t(f) := \mathbb{E}\left(f(X_t) Z_t(X)\right) \tag{2.1}$$

In the above display, $Z_t(X)$ stands for the exponential weight

$$Z_t(X) := \exp\left[-\int_0^t V_s(X_s) ds\right] \Longrightarrow \log \mathbb{E}\left(Z_t(X)\right) = -\int_0^t \eta_s(V_s) ds$$

This shows that

$$\overline{Z}_t(X) := Z_t(X)/\,\mathbb{E}(Z_t(X)) = \exp\left[-\int_0^t \overline{V}_s(X_s) ds\right] \quad \text{with} \quad \overline{V}_t := V_t - \eta_t(V)$$

We also consider the Feynman-Kac semigroup

$$Q_{s,t}(f)(x) = \mathbb{E}\left(f(X_t)\ Z_{s,t}(X) \mid X_s = x\right) \quad \text{with} \quad Z_{s,t}(X) := Z_t(X)/Z_s(X) \tag{2.2}$$

When $V = 0$ the semigroup $Q_{s,t}$ resumes to the Markov semigroup $P_{s,t}$ of the reference process X_t.

The mathematical model defined above is called the Feynman-Kac model associated with the reference process and the potential function (X_t, V_t).

We further assume that the (infinitesimal) generators L_t of X_t are well defined on some common sub-algebra $\mathcal{D}(S) \subset \mathcal{B}(S)$, and for any $s < t$ we have $Q_{s,t}(\mathcal{B}(S)) \subset \mathcal{D}(S)$.

We let $\mathcal{V}_t(f) = V_t f$ the multiplication operator on $\mathcal{B}(S)$. We also let $L_t = L_t^c + L_t^d$ be the decomposition of the generator L_t in terms of a diffusion-type operator L_t^c and a bounded jump-type generator of the following form

$$L_t^d(f)(u) = \lambda_t(u) \int (f(v) - f(u)) \, J_t(u, dv)$$

for some bounded rate function λ_t and some Markov transition J_t on S.

In this notation, for any $f \in \mathcal{D}(S)$ and $s \leqslant t$ we have

$$\partial_t \gamma_t(f) = \gamma_t(L_t^V(f)) \quad \text{with} \quad L_t^V = L_t - \mathcal{V}_t \Longrightarrow \gamma_t = \gamma_s Q_{s,t} \tag{2.3}$$

The semigroup associated with the normalized Feynman-Kac measures η_t is given for any $s \leqslant t$ by the formula

$$\eta_t = \phi_{s,t}(\eta_s) := \frac{\eta_s Q_{s,t}}{\eta_s Q_{s,t}(1)} \Longrightarrow \partial_t \eta_t(f) = \Lambda_t(\eta_t)(f) := \eta_t(L_t^V(f)) + \eta_t(V_t) \; \eta_t(f) \tag{2.4}$$

with the collection of functional linear operators

$$\Lambda_t(\eta) \; : \; f \in \mathcal{D}(S) \mapsto \Lambda_t(\eta)(f) := \eta(L_t^V(f)) + \eta(V_t) \; \eta(f) \in \mathbb{R}$$

Finally we recall that $\eta_t = \mathrm{Law}(\overline{X}_t)$ can be interpreted as the law of a nonlinear Markov process $\overline{X}_t$ associated with the collection of generators L_{t,η_t} defined for any $(\eta, f, x) \in (\mathcal{P}(S), \mathcal{D}(S) \times S)$ by

$$L_{t,\eta}(f)(x) = L_t(f)(x) + V(x) \int (f(y) - f(x)) \; \eta(dy) \quad \Longrightarrow \quad \Lambda_t(\eta) = \eta L_{t,\eta} \tag{2.5}$$

2.2 Path space measures

Consider a Feynman-Kac model $(\gamma'_t, \eta'_t, Q'_{s,t}, \mathbb{Q}'_t, \ldots)$ associated with some auxiliary Markov process X'_t on some metric space $(S', d_{S'})$, and some bounded non negative potential functions V'_t on S'. Also let L'_t be the generator of X'_t defined on some common sub-algebra $\mathcal{D}(L') \subset \mathcal{B}(S')$.

Assume that the process X_t discussed in (2.1) is the historical process

$$X_t := \left(X'_s\right)_{s \leqslant t} \in S := \cup_{t \geqslant 0} D_t(S') \quad \text{and} \quad V_t(X_t) := V'_t(X'_t) \Longrightarrow \eta_t = \mathbb{Q}'_t \tag{2.6}$$

In this situation, the generator L_t and the domain $\mathcal{D}(S)$ of the historical process can be defined in two different ways:

The more conventional approach is to consider cylindrical functions

$$f(X_t) = \varphi(X'_{s_1 \wedge t}, \ldots, X'_{s_n \wedge t})$$

that only depend on a finite collection of time horizons $s_i \leqslant s_{i+1}$, with $1 \leqslant i < n$, and some bounded functions φ from $(S')^n$ into $\mathbb{R}$. The regularity of the "test" function φ depends on the process at hand. For jump-type processes, no additional regularity is required. For diffusion-type processes the function is often required to be compactly supported and twice differentiable.

Another elegant and more powerful approach is to use the functional Itô calculus theory introduced by B. Dupire in an unpublished article [18], and further developed in [6, 19]. This path-dependent stochastic calculus allows to consider more general functions such as running integrals

or running maximum of the process X'_t. It also allows to consider diffusion-type processes with a drift and a diffusion term that depends on the history of the process.

The path space S is equipped with a time-space metric d_S so that (S, d_S) is a complete and separable metric space (cf. for instance proposition 1.1.13 and theorem 1.1.15 in [25]). The smoothness properties of continuous function f on S are defined in terms of time and space functional derivatives. Thus, for diffusion-type historical processes X_t, the generator L_t is defined on functions $f \in \mathcal{D}(L)$ which a differentiable w.r.t. the time parameter and, as before twice differentiable with compactly supported derivatives (cf. for instance theorem 1.3.1 in [25]).

It is clearly not the scope of the article to describe in full details the above functional Itô calculus. We refer the reader to the article [19] and the Ph.D thesis of Saporito [25].

In the further development of the article we shall use these ideas back and forth. We already mention that the mean field particle interpretation of the Feynman-Kac measures associated with an historical process coincides with the genealogical tree-based particle evolution of the marginal model.

2.3 Some regularity conditions

This section discusses in some details the two main regularity conditions used in the further development of the article.

Firstly, observe that the semigroup $P_{s,t}$ associated with the historical process $X_t = (X'_s)_{s\leqslant t}$ discussed in (2.6) never satisfies the regularity condition (H_0) stated in (1.2). Nevertheless it may happens that the semigroup $P'_{s,t}$ associated with X'_t satisfies condition (H_0). In this situation, to avoid repetition or unnecessary long discussions we simply say that (H'_0) is met.

We also use the following weaker conditions:

$$(H_1) \qquad \exists \alpha < \infty \quad \exists \beta > 0 \quad \text{s.t.} \quad \forall s \leqslant t \quad \operatorname{osc}(\overline{Q}_{s,t}(f)) \leqslant \alpha\, e^{-\beta(t-s)}\, \operatorname{osc}(f)$$

$$(H_2) \quad \exists q < \infty \quad \text{s.t.} \quad \forall s \leqslant t \quad \forall x, y \in S \qquad \log\left(Q_{s,t}(1)(x)/Q_{s,t}(1)(y)\right) \leqslant q$$

As before when the semigroup $\overline{Q}'_{s,t}$ and $Q'_{s,t}$ of Feynman-Kac model associated with some parameters (X'_t, V'_t) satisfy condition (H_i), to avoid repetition or unnecessary long discussions we simply say that (H'_i) is met. We recall that

$$(H_0) \Longrightarrow (H_1) \Longrightarrow (H_2)$$

The proof of the l.h.s. assertion can be found in [15]. To check the second we observe that

$$\log\left(Q_{s,t}(1)(x)/Q_{s,t}(1)(y)\right) = \int_s^t \left[\phi_{s,u}(\delta_y)(V_u) - \phi_{s,u}(\delta_x)(V_u)\right]\, du \tag{2.7}$$

This implies that

$$(H_1) \Longrightarrow (H_2) \quad \text{with} \quad q = \alpha\beta^{-1}\, \operatorname{osc}(V) \quad \text{with} \quad \operatorname{osc}(V) = \sup_{t\geqslant 0} \operatorname{osc}(V_t)$$

Using (1.6) we also have

$$\begin{aligned} (H_2) \quad &\Longrightarrow \quad \|\partial_\eta \phi_{s,t}(f)\| \leqslant e^q\, \operatorname{osc}(f) \quad \left(\text{since} \;\; \operatorname{osc}(\overline{Q}_{s,t}(f)) \leqslant \;\operatorname{osc}(f)\right) \\ (H_1) \quad &\Longrightarrow \quad \|\partial_\eta \phi_{s,t}(f)\| \leqslant r\, e^{-\beta(t-s)}\, \operatorname{osc}(f) \quad \text{with} \quad r = \alpha\, e^q \quad \text{and} \quad q = \alpha\beta^{-1}\, \operatorname{osc}(V) \end{aligned} \tag{2.8}$$

We return to the historical process $X_t = (X'_s)_{s\leqslant t}$ discussed in (2.6). In this case, for any $x_s = (x'(u))_{u\leqslant s}$ and $y_s = (y'(u))_{u\leqslant s} \in D_s(S')$ we have

$$Q_{s,t}(f)(x_s) = Q'_{s,t}(f')(x'_s)$$

This implies that

$$(H'_1) \Longrightarrow (H_2) \quad \text{is met with} \quad q = \alpha\beta^{-1}\, \operatorname{osc}(V) \quad \text{and} \quad \|\partial_\eta \phi_{s,t}(f)\| \leqslant e^q\, \operatorname{osc}(f) \tag{2.9}$$

2.4 Forward and backward equations

Proposition 2.1. *For any $s \leqslant t$ and $\eta \in \mathcal{P}(S)$ we have the Gelfand-Pettis forward and backward differential equations*

$$\partial_t \phi_{s,t}(\eta_s) = \Lambda_t(\phi_{s,t}(\eta_s)) \quad \textit{and} \quad \partial_s \phi_{s,t}(\eta) = -\Lambda_s(\eta)\partial_\eta \phi_{s,t} \tag{2.10}$$

In addition, for any mapping ϕ of the form (1.4) we also have

$$\partial_t \phi\left(\phi_{s,t}(\eta)\right) = \Lambda_t(\eta)\partial_{\phi_{s,t}(\eta)}\phi \quad \textit{and} \quad \partial_s \phi\left(\phi_{s,t}(\eta)\right) = -\Lambda_s(\eta)\partial_{\phi_{s,t}(\eta)}\phi \tag{2.11}$$

Proof. The l.h.s. assertion in (2.10) is a direct consequence of (2.4). Applying these decompositions to $\phi_{s,t}$, for any $s + h \leqslant t$ we find that

$$\begin{aligned}
&\phi_{s+h,t}\left(\eta + [\phi_{s,s+h}(\eta) - \eta]\right) \\
&= \phi_{s+h,t}(\eta) + [\phi_{s,s+h}(\eta) - \eta] \circ \partial_\eta \phi_{s+h,t} + \frac{1}{\phi_{s,s+h}(\eta) Q^\eta(1)} \ \frac{1}{2} \ [\phi_{s,s+h}(\eta) - \eta]^{\otimes 2} \circ \partial_\eta^2 \phi_{s+h,t}
\end{aligned}$$

On the other hand we have

$$\phi_{s,s+h}(\eta) = \eta + \Lambda_s(\eta) \ h + \mathrm{O}(h^2) \quad \text{and} \quad \phi_{s,s+h}(\eta) Q^\eta(1) = 1 + \mathrm{O}(h)$$

This yields the backward evolution formula

$$h^{-1}\left[\phi_{s+h,t}\left(\eta\right) - \phi_{s,t}(\eta)\right] \longrightarrow_{h \to 0} \partial_s \phi_{s,t}(\eta) = -\Lambda_s(\eta)\partial_\eta \phi_{s,t}$$

For any mapping ϕ of the form (1.4) we also have

$$\begin{aligned}
&\phi\left(\phi_{s+h,t}(\eta)\right) - \phi\left(\phi_{s,t}(\eta)\right) = \left(\phi_{s+h,t}(\eta) - \phi_{s,t}(\eta)\right)\partial_{\phi_{s,t}(\eta)}\phi \\
&+ \frac{1}{2} \ \left(\phi_{s+h,t}(\eta) - \phi_{s,t}(\eta)\right)^{\otimes 2} \partial^2_{\phi_{s,t}(\eta)}\phi + \frac{1}{\phi_{s+h,t}(\eta) \ Q^{\phi_{s,t}(\eta)}(1)} \ \frac{1}{3} \ \left(\phi_{s+h,t}(\eta) - \phi_{s,t}(\eta)\right)^{\otimes 3} \circ \partial^3_{\phi_{s,t}(\eta)}\phi
\end{aligned}$$

Arguing as above we check (2.11). This ends the proof of the proposition. ∎

2.5 Mean field particle systems

Let $\mathcal{B}(S_N) \subset \mathcal{B}(S^N)$ be the subset of symmetric functions on S^N, and $\mathcal{B}(S \times S_{N-1}) \subset \mathcal{B}(S^N)$ be the set of functions F on S^N symmetric with respect to the last $(N-1)$ arguments. Also let $\mathcal{D}(S^N) \subset \mathcal{B}(S^N)$ be the set of functions $F \in \mathcal{B}(S^N)$ s.t. for any $x \in S_N$ we have

$$F_{x^{-i}} := F \circ \varphi_{x^{-i}} \in \mathcal{D}(S)$$

with the functions $\varphi_{x^{-i}}$ and the set $\mathcal{D}(S)$ introduced in (1.13) and (2.3).

Also let $\mathcal{D}(S_N) \subset \mathcal{B}(S_N)$, resp. $\mathcal{D}(S \times S_{N-1}) \subset \mathcal{B}(S \times S_{N-1})$ the trace of $\mathcal{D}(S^N)$ on $\mathcal{B}(S_N)$, resp. $\mathcal{B}(S \times S_{N-1})$.

Definition 2.2. *The N-mean field particle interpretation of the nonlinear process discussed in (2.5) is defined by the Markov process $\xi_t = \left(\xi^i_t\right)_{1 \leqslant i \leqslant N} \in S_N$ with generators $\mathcal{G}_t$ given for any $F \in \mathcal{D}(S_N)$ and any $x = (x^i)_{1 \leqslant i \leqslant N} \in S_N$ by*

$$\mathcal{G}_t(F)(x) = \sum_{1 \leqslant i \leqslant N} L_{t,m(x)}(F_{x^{-i}})(x^i) \tag{2.12}$$

We let $\mathcal{F} := (\mathcal{F}_t)_{t\geqslant 0}$, with $\mathcal{F}_t = \sigma(\xi_u \ : \ u \leqslant s)$ be the filtration generated by the mean field particle model defined in (2.12).

We let $D([0,t], S_N)$ be the set of function $F \ : \ (s,x) \in ([0,t] \times S_N) \mapsto F_s(x) \in \mathbb{R}$ with a bounded derivative w.r.t. the first argument and s.t. $F_t \in D(S_N)$. For any $F \in D([0,T], S_N)$, and any $T \geqslant 0$, we have

$$dF_t(\xi_t) = \left[\partial_t F_t + \mathcal{G}_t(F_t)\right](\xi_t)\ dt + d\mathcal{M}_t(F)$$

In the above display $\mathcal{M}_t$ stands for a martingale random field on $D([0,T], S_N)$ with angle bracket defined for any functions $F, G \in D([0,T], S_N)$ and any time horizon $t \in [0,T]$ by the formula

$$\partial_t \langle \mathcal{M}(F), \mathcal{M}(G) \rangle_t = \Gamma_{\mathcal{G}_t}(F_t, G_t)(\xi_t)$$

Choosing functions of the form

$$F_t(x) = m(x)(f_t) \quad \text{and} \quad G_t(x) = m(x)(g_t) \Longrightarrow \Gamma_{\mathcal{G}_t}(F_t, G_t)(\xi_t) = m(\xi_t)\Gamma_{L_{t,m(\xi_t)}}(f_t, g_t) \tag{2.13}$$

we also check that the occupation measure $m(\xi_t) \in \mathcal{P}(S)$ satisfies the stochastic equation

$$dm(\xi_t)(f_t) = \left[m(\xi_t)(\partial_t f_t) + \Lambda_t(m(\xi_t))(f_t)\right]\ dt + \frac{1}{\sqrt{N}}\ dM_t(f) \tag{2.14}$$

with a martingale random field M_t on $D([0,T], S)$ with angle brackets by the formula

$$\partial_t \langle M(f), M(g) \rangle_t$$

$$= m(\xi_t)(\Gamma_{L_t}(f_t, g_t)) + \int\ m(\xi_t)(du)\ m(\xi_t)(dv)\ V_t(u)\ (f_t(v) - f_t(u))(g_t(v) - g_t(u))$$

With a slight abuse of notation we also write M_t the extension of the random field M_t to $\mathcal{F}$-predictable functions $D([0,T], S)$.

In the further development of the article we write $(M_t^c, \mathcal{M}_t^c)$ and $(M_t^d, \mathcal{M}_t^d)$ the continuous and the discontinuous part of the martingales $(M_t, \mathcal{M}_t)$; as well as

$$L_{t,\eta} = L_{t,\eta}^c + L_{t,\eta}^d \quad \text{with} \quad L_{t,\eta}^c = L_t^c$$

The angle bracket of $\mathcal{M}_t^d$ is given for any functions $F, G \in D([0,T], S_N)$ and any time horizon $t \in [0,T]$ by the formula

$$\partial_t \langle \mathcal{M}^d(F), \mathcal{M}^d(G) \rangle_t$$

$$= \sum_{1 \leqslant i \leqslant N} \int\ \left[F_{t,\xi_t^{-i}}(v) - F_t(\xi_t)\right]\left[G_{t,\xi_t^{-i}}(v) - G_t(\xi_t)\right]\ \left[V_t(\xi_t^i) m(\xi_t)(dv) + \lambda_t(\xi_t^i) J_t(\xi_t^i, dv)\right]$$

Definition 2.3. *Let $\zeta_t = \left(\zeta_t^i\right)_{1 \leqslant i \leqslant N} \in (S \times S_{N-1})$ be the Markov process with initial condition $\zeta_0 = \xi_0$ and generators $\mathcal{H}_t$ defined for any $F \in \mathcal{D}(S \times S_{N-1})$ and $x = (x^i)_{1 \leqslant i \leqslant N} \in (S \times S_{N-1})$ by*

$$\mathcal{H}_t(F)(x)$$

$$= L_t(F_{x^{-1}})(x^1) + \sum_{2 \leqslant i \leqslant N}\ \left(L_t(F_{x^{-i}})(x^i) + V_t(x^i)\ \int\ (F_{x^{-i}}(u) - F(x))\, m_{x^1}(x^{-\{1,i\}})(du)\right)$$

with the empirical probability measures

$$m_{x^1}(x^{-\{1,i\}}) := \left(1 - \frac{2}{N}\right)\ m(x^{-\{1,i\}}) + \frac{2}{N}\ \delta_{x^1}$$

Theorem 2.4. *Given the historical process $\widehat{\zeta}_t^1$ the process $(\zeta_s^-)_{s\leqslant t}$ coincides with the $(N-1)$-mean field interpretation (2.12) of the Feynman-Kac model $(\eta_s^-)_{s\leqslant t}$ defined as in (2.1) and (2.12) by replacing (L_s, V_s) by (L_s^-, V_s^-), with the jump generator*

$$L_s^-(f)(u) = L_s(f)(u) + \frac{2}{N}\ V_s(u)\ \left(f(\zeta_s^1) - f(u)\right) \quad \textit{and} \quad V_s^- := \left(1 - \frac{1}{N}\right)\ V_s$$

Proof. By construction, the generators $\mathcal{G}_s^-$ of the process $(\zeta_s^-)_{s\leqslant t}$ given $\widehat{\zeta}_t^1$ are defined for any $s \leqslant t$, any $F \in \mathcal{D}(S_{N-1})$ and any $x = (x^i)_{1\leqslant i<N} \in S_{N-1}$ by the formula

$$\mathcal{G}_s^-(F)(x) = \sum_{1\leqslant i<N} L_s(F_{x^{-i}})(x^i) + \sum_{1\leqslant i<N} V_s(x^i)\ \int\ (F_{x^{-i}}(u) - F(x))\, m_{\zeta_s^1}(x^{-i})(du)$$

Observe that for any $x = (x^i)_{1\leqslant i<N} \in S_{N-1}$ and $y \in S$ we have

$$\int\ (F_{x^{-i}}(u) - F(x))\, m_y(x^{-i})(du)$$

$$= \left(1 - \frac{1}{N}\right)\ \int\ (F_{x^{-i}}(u) - F(x))\, m(x)(du) + \frac{2}{N}\ (F_{x^{-i}}(y) - F(x))$$

This implies that

$$\sum_{1\leqslant i<N} V_s(x^i) \int\ (F_{x^{-i}}(u) - F(x))\, m_y(x^{-\{i\}})(du)$$

$$= \sum_{1\leqslant i<N} V_s^-(x^i)\ \int\ (F_{x^{-i}}(u) - F(x))\, m(x)(du) + \frac{2}{N} \sum_{1\leqslant i<N} V_s(x^i)\ (F_{x^{-i}}(y) - F(x))$$

We conclude that

$$\mathcal{G}_s^-(F)(x) = \sum_{1\leqslant i<N} \left[L_s^-(F_{x^{-i}})(x^i) + V_s^-(x^i)\ \int\ (F_{x^{-i}}(u) - F(x))\, m(x)(du)\right]$$

This ends the proof of the theorem. ∎

3 Perturbation analysis

3.1 Semigroup estimates

We consider a collection of generators L_t^ϵ and potential functions V_t^δ of the form

$$L_t^\epsilon = L_t + \epsilon\ \overline{L}_t \quad \text{and} \quad V_t^\delta = V_t + \delta \overline{V}_t \quad \text{with} \quad |\epsilon|, |\delta| \in [0, 1]$$

In the above display, $\overline{V}_t$ stands for some uniformly bounded function and $\overline{L}_t$ a bounded generator of an auxiliary jump type Markov process of the form

$$\overline{L}_t(f)(x) = \lambda\ \int\ (f(y) - f(x))\ K_t(x, dy)$$

for some $\lambda \geqslant 0$ and some Markov transition K_t.

We let $P^\epsilon_{s,t}$ be the transition semigroup of the process with generator L^ϵ_t. In this notation, we also have the perturbation formula

$$P^\epsilon_{s,t} = e^{-\epsilon\lambda(t-s)} P_{s,t} + \epsilon\lambda \int_s^t e^{-\epsilon\lambda(u-s)}\ P_{s,u} K_u P^\epsilon_{u,t}\ du$$

This shows that

$$(H_0) \quad \Longrightarrow \quad \rho(h)\ \leqslant \frac{d(\delta_x P^\epsilon_{t,t+h})}{d\mu^\epsilon_{t,h}}(y) \leqslant \rho(h)^{-1} \tag{3.1}$$

with the probability measure

$$\mu^\epsilon_{t,h} := e^{-\epsilon\lambda h}\ \mu + \epsilon\lambda \int_t^{t+h} e^{-\epsilon\lambda(u-t)}\ \mu K_u P^\epsilon_{u,t+h}\ du$$

We consider the Feynman-Kac semigroup $Q^{\delta,\epsilon}_{s,t}$ be defined as $Q_{s,t}$ by replacing V_t by V^δ_t and X_t by a Markov process with generator L^ϵ_t.

Also let $\phi^{(\delta,\epsilon)}_{s,t}$ be defined as $\phi_{s,t}$ by replacing $Q_{s,t}$ by $Q^{\delta,\epsilon}_{s,t}$, and set

$$L^{\delta,\epsilon}_t = \epsilon \overline{L}_t - \delta\ \overline{\mathcal{V}}_t \quad \text{and} \quad L^{\delta,\epsilon}_{t,\eta} = \epsilon \overline{L}_t - \delta(\overline{\mathcal{V}}_t - \eta(\overline{\mathcal{V}}_t))$$

Theorem 3.1. *For any $|\epsilon|, |\delta| \in [0,1]$ and any $s \leqslant t$ we have the semigroup perturbation formulae*

$$Q^{\delta,\epsilon}_{s,t} - Q_{s,t} = \int_s^t Q^{\delta,\epsilon}_{s,u}\ L^{\delta,\epsilon}_u\ Q_{u,t}\ du = \int_s^t Q_{s,u}\ L^{\delta,\epsilon}_u\ Q^{\delta,\epsilon}_{u,t}\ du \tag{3.2}$$

In addition, for any $\eta \in \mathcal{P}(S)$ we have

$$\phi^{\delta,\epsilon}_{s,t}(\eta) - \phi_{s,t}(\eta) = \int_s^t \phi^{\delta,\epsilon}_{s,u}(\eta)\ L^{\delta,\epsilon}_{u,\phi^{\delta,\epsilon}_{s,u}(\eta)}\ \partial_{\phi^{\delta,\epsilon}_{s,u}(\eta)}\phi_{u,t}\ du = \int_s^t \phi_{s,u}(\eta)\ L^{\delta,\epsilon}_{u,\phi_{s,u}(\eta)}\ \partial_{\phi_{s,u}(\eta)}\phi^{\delta,\epsilon}_{u,t}\ du$$

Proof. We check (3.2) the fact that

$$\partial_u(Q^{\delta,\epsilon}_{s,u} Q_{u,t}) = Q^{\delta,\epsilon}_{s,u}\ (L^\epsilon_u - L_u - \delta\ \overline{\mathcal{V}}_u)\ Q_{u,t} = \epsilon\ Q^{\delta,\epsilon}_{s,u}\ \overline{L}_u\ Q_{u,t} - \delta\ Q^{\delta,\epsilon}_{s,u}\ \overline{\mathcal{V}}_u\ Q_{u,t}$$

and

$$\partial_u(Q_{s,u} Q^{\delta,\epsilon}_{u,t}) = -\epsilon\ Q_{s,u}\ \overline{L}_u\ Q^{\delta,\epsilon}_{u,t} + \delta\ Q_{s,u}\ \overline{\mathcal{V}}_u\ Q^{\delta,\epsilon}_{u,t}$$

The perturbation analysis of the normalized semigroups $\phi^{\delta,\epsilon}_{s,t}$ is slightly more involved.

Let $\Lambda^{\delta,\epsilon}_t$ be defined as Λ_t by replacing (L_t, V_t) by $(L^\epsilon_t, V^\delta_t)$. Notice that

$$h^{-1}\left[\phi^{\delta,\epsilon}_{t,t+h}(\eta) - \eta\right] = \Lambda^{\delta,\epsilon}_t(\eta) + \mathrm{O}(h)$$

For any given $s \leqslant t$, we consider the interpolating maps $u \in [s,t] \mapsto \Delta^{\delta,\epsilon}_{s,u,t}$ defined by

$$\Delta^{\delta,\epsilon}_{s,u,t} := \phi_{u,t} \circ \phi^{\delta,\epsilon}_{s,u}$$

On the other hand, for any $s \leqslant u \leqslant u+h \leqslant t$ we have the decomposition

$$\begin{aligned}
&\Delta^{\delta,\epsilon}_{s,u+h,t}(\eta) - \Delta^{\delta,\epsilon}_{s,u,t}(\eta) \\
&= \phi_{u+h,t}\left(\phi^{\delta,\epsilon}_{s,u+h}(\eta)\right) - \phi_{u,t}\left(\phi^{\delta,\epsilon}_{s,u+h}(\eta)\right) + \phi_{u,t}\left(\phi^{\delta,\epsilon}_{s,u+h}(\eta)\right) - \phi_{u,t}\left(\phi^{\delta,\epsilon}_{s,u}(\eta)\right) \\
&= -\Lambda_u(\phi^{\delta,\epsilon}_{s,u}(\eta))\left(\partial_{\phi^{\delta,\epsilon}_{s,u}(\eta)}\phi_{u,t}\right)h \\
&\qquad + \phi_{u,t}\left(\phi^{\delta,\epsilon}_{s,u}(\eta) + \left[\phi^{\delta,\epsilon}_{s,u+h}(\eta) - \phi^{\delta,\epsilon}_{s,u}(\eta)\right]\right) - \phi_{u,t}\left(\phi^{\delta,\epsilon}_{s,u}(\eta)\right) + \mathrm{O}(h^2)
\end{aligned}$$

This implies that

$$\begin{aligned}
&h^{-1}\left[\Delta^{\delta,\epsilon}_{s,u+h,t}(\eta) - \Delta^{\delta,\epsilon}_{s,u,t}(\eta)\right] \\
&= -\Lambda_u(\phi^{\delta,\epsilon}_{s,u}(\eta))\partial_{\phi^{\delta,\epsilon}_{s,u}(\eta)}\phi_{u,t} + h^{-1}\left[\phi^{\delta,\epsilon}_{s,u+h}(\eta) - \phi^{\delta,\epsilon}_{s,u}(\eta)\right]\partial_{\phi^{\delta,\epsilon}_{s,u}(\eta)}\phi_{u,t} + \mathrm{O}(h)
\end{aligned}$$

We conclude that

$$\partial_u \Delta^{\delta,\epsilon}_{s,u,t}(\eta) = \left[\Lambda^{\delta,\epsilon}_u(\phi^{\delta,\epsilon}_{s,u}(\eta)) - \Lambda_u(\phi^{\delta,\epsilon}_{s,u}(\eta))\right]\partial_{\phi^{\delta,\epsilon}_{s,u}(\eta)}\phi_{u,t}$$

On the other hand, we have

$$\left[\Lambda^{\delta,\epsilon}_t(\eta) - \Lambda_t(\eta)\right](f) = \epsilon\ \eta(\overline{L}_t(f)) - \delta\ \eta(f\ (\overline{V}_t - \eta(\overline{V}_t)))$$

By symmetry arguments, this ends the proof of the theorem. ∎

Corollary 3.2. *For any $s \leqslant t$ and any $\eta \in \mathcal{P}(S)$ we have the estimates*

$$\begin{aligned}
(H_1) \quad &\Longrightarrow \quad \|\phi^{\delta,\epsilon}_{s,t}(\eta) - \phi_{s,t}(\eta)\|_{tv} \leqslant c\ (\epsilon + \delta) \\
(H_2) \quad &\Longrightarrow \quad \|\phi^{\delta,\epsilon}_{s,t}(\eta) - \phi_{s,t}(\eta)\|_{tv} \leqslant c\ (\epsilon + \delta)\ (t-s)
\end{aligned}$$

for some finite constant c whose value doesn't depend on the parameters (s,t,η), nor on (ϵ,δ).

3.2 Particle stochastic flows

For any $t \geqslant 0$, we let $\Delta m(\xi_t)$ be the random jump occupation measure

$$\Delta m(\xi_t) := m(\xi_t) - m(\xi_{t-}) = \Delta M_t = M_t - M_{t-}$$

with the martingale random field M_t defined in (2.14). In this notation, we have

$$N^{n-1}\ \partial_t \mathbb{E}\left[(\Delta m(\xi_t))^{\otimes n}(f^{(1)}_t \otimes \ldots \otimes f^{(n)}_t) \mid \mathcal{F}_{t-}\right] = m(\xi_{t-})\Gamma^{(n)}_{L^d_{m(\xi_{t-})}}(f^{(1)}_t, \ldots, f^{(n)}_t) \tag{3.3}$$

with the operators $\Gamma^{(n)}_{L^d_{m(\xi_{t-})}}$ defined in (1.11). When $n = 2$ the above formula resumes to

$$\begin{aligned}
&\partial_t \mathbb{E}\left[\Delta m(\xi_t)(f_t)\ \Delta m(\xi_t)(g_t) \mid \mathcal{F}_{t-}\right] = \frac{1}{N}\ m(\xi_{t-})\Gamma_{L^d_{m(\xi_{t-})}}(f_t, g_t) \\
&= \partial_t \langle M^d_t(f), M^d(g)\rangle_t = \partial_t \langle \mathcal{M}^d_t(F), \mathcal{M}^d(G)\rangle_t \quad \text{with } (F,G) \text{ defined in (2.13)}
\end{aligned}$$

Definition 3.3. *For any $t \geqslant s$ and $n \geqslant 1$, we consider the integral random operators*

$$\Delta^n \phi_{s,t}(m(\xi_s)) \quad := \quad N^{n-1} \ \frac{1}{n!} \ (\Delta m(\xi_s))^{\otimes n} \ \overline{\partial}^n_{m(\xi_{s-})+\Delta m(\xi_s), m(\xi_{s-})} \phi_{s,t}$$

and their first variational measure

$$\Upsilon^n_{m(\xi_{s-})} \phi_{s,t} := \partial_s \mathbb{E}\left[\Delta^n \phi_{s,t}(m(\xi_s)) \mid \mathcal{F}_{s-}\right]$$

Choosing $n = 1$ we have

$$\Delta \phi_{s,t}(m(\xi_s)) \quad := \quad \Delta^1 \phi_{s,t}(m(\xi_s)) = \phi_{s,t}(m(\xi_s)) - \phi_{s,t}(m(\xi_{s-}))$$

Arguing as in the proof of (2.8) and using (3.3), for any collection of functions $f^{(n)} \in \mathrm{Osc}(S)$ we have the estimate

$$N^{n-1} \ \partial_s \, \mathbb{E}\left[\Delta \phi_{s,t}(m(\xi_s))^{\otimes n} (f^{(1)}_t \otimes \ldots \otimes f^{(n)}_t) \mid \mathcal{F}_{s-}\right] \leqslant e^{nq} \ \|\lambda + V\| \tag{3.4}$$

Proposition 3.4. *For any $t \geqslant s$ and $n \geqslant 1$, we have*

$$\Delta^n \phi_{s,t}(m(\xi_s)) = \frac{N^{n-1}}{n!} \ (\Delta m(\xi_s))^{\otimes n} \partial^n_{m(\xi_{s-})} \phi_{s,t} + \frac{1}{N} \ \Delta^{n+1} \phi_{s,t}(m(\xi_s)) \tag{3.5}$$

In addition, for any $f \in \mathcal{B}(S)$ we have

$$\begin{aligned} &\Upsilon^n_{m(\xi_{s-})} \phi_{s,t}(f) \\ &= (-1)^{n-1} \ m(\xi_{s-}) \Gamma^{(n)}_{L^d_{m(\xi_{s-})}} \left(Q^{m(\xi_{s-1})}_{s,t}(1), \ldots, Q^{m(\xi_{s-1})}_{s,t}(1), \partial_{m(\xi_{s-})} \phi_{s,t}(f)\right) + \frac{1}{N} \ \Upsilon^{(n+1)}_{m(\xi_{s-})} \phi_{s,t}(f) \end{aligned} \tag{3.6}$$

Proof. We have

$$\begin{aligned} \Delta^{n+1} \phi_{s,t}(m(\xi_s)) &= N^n \left[\Delta \phi_{s,t}(m(\xi_s)) - \sum_{1 \leqslant k \leqslant n} \frac{1}{k!} \ (\Delta m(\xi_s))^{\otimes k} \partial^k_{m(\xi_{s-})} \phi_{s,t}\right] \\ &= N \ \Delta^n \phi_{s,t}(m(\xi_s)) - \frac{N^n}{n!} \ (\Delta m(\xi_s))^{\otimes n} \partial^n_{m(\xi_{s-})} \phi_{s,t} \Longleftrightarrow (3.5) \end{aligned}$$

This implies that

$$\begin{aligned} &\partial_s \mathbb{E}\left[\Delta^n \phi_{s,t}(m(\xi_s)) \mid \mathcal{F}_{s-}\right] := \Upsilon^n_{m(\xi_{s-})} \phi_{s,t} \\ &= \frac{N^{n-1}}{n!} \ \partial_s \mathbb{E}\left[(\Delta m(\xi_s))^{\otimes n} \partial^n_{m(\xi_{s-})} \phi_{s,t} \mid \mathcal{F}_{s-}\right] + \frac{1}{N} \ \partial_s \mathbb{E}\left[\Delta^{n+1} \phi_{s,t}(m(\xi_s)) \mid \mathcal{F}_{s-}\right] \end{aligned}$$

This ends the proof of the proposition. ∎

Lemma 3.5. *For any $n \geqslant 1$ and $s \leqslant t$ we have the almost sure uniform estimates*

$$(H_2) \Longrightarrow \|\Upsilon^n_{m(\xi_{s-})} \phi_{s,t}\|_{tv} \leqslant 2^{n-1} e^{(n+1)q} \ \|\lambda + V\| \tag{3.7}$$

The detailed proof of the above estimate is provided in the appendix, on page 22.

In the further development of this section, for any given time horizon t and any $f \in \mathcal{B}(S)$ we let

$$s \in [0,t] \mapsto \mathcal{M}_s^d(\phi_{.,t}(m(.))(f))$$

be the martingale $s \in [0,t] \mapsto \mathcal{M}_s^d(F)$ associated with the function

$$(s,x) \in [0,t] \times S_N \mapsto F(s,x) = \phi_{s,t}(m(x))(f)$$

We also denote by

$$s \in [0,t] \mapsto M_s^c\left(\partial_{m(\xi_.)}\phi_{.,t}(f)\right) \ , \quad \text{resp.} \quad M_s^c\left(Q_{.,t}^{m(\xi_.)}(1)\right)$$

the martingale $M_s^c(f)$ associated with the $\mathcal{F}$-predictable bounded function

$$(s,x) \in [0,t] \times S \mapsto f_s(x) = \partial_{m(\xi_{s-})}\phi_{s,t}(f)(x) \ , \quad \text{resp.} \quad f_s(x) = Q_{s,t}^{m(\xi_{s-})}(1)(x)$$

We are now in position to state and to prove the main result of this section.

Theorem 3.6. *For any time horizon $t \geqslant 0$ and any $f \in \mathcal{B}(S)$ the interpolating function*

$$s \in [0,t] \mapsto \phi_{s,t}(m(\xi_s))(f) \in \mathbb{R}$$

satisfies the stochastic differential equation

$$\begin{aligned} d\phi_{s,t}(m(\xi_s))(f) &= \frac{1}{\sqrt{N}}\ dM_s^c\left(\partial_{m(\xi_.)}\phi_{.,t}(f)\right) + d\mathcal{M}_s^d\left(\phi_{.,t}(m(.))(f)\right) \\ &\quad + \frac{1}{N}\ \Upsilon^2_{m(\xi_{s-})}\phi_{s,t}(f)\ ds - \frac{1}{N}\ m(\xi_s)\Gamma_{L_s^c}\left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)}\phi_{s,t}(f)\right)\ ds \end{aligned}$$

Proof. Observe that

$$dm(\xi_s) = \Lambda_s(m(\xi_s))\ ds + \frac{1}{\sqrt{N}}\ dM_s^c + \Delta m(\xi_s) - \underbrace{\mathbb{E}(\Delta m(\xi_s) \mid \mathcal{F}_{s-})}_{\ll ds}$$

Using Itô formula and the backward formula (2.10) we have

$$\begin{aligned} d\ \phi_{s,t}(m(\xi_s))(f) &= -\Lambda_s(m(\xi_s))\left(\partial_{m(\xi_s)}\phi_{s,t}(f)\right)\ ds + \left[\phi_{s,t}(m(\xi_{s-}) + dm(\xi_s)) - \phi_{s,t}(m(\xi_{s-}))\right](f) \\ &= \frac{1}{\sqrt{N}}\ dM_s^c\left(\partial_{m(\xi_.)}\phi_{.,t}(f)\right) + d\mathcal{M}_s^d(\phi_{.,t}(m(.))(f)) \\ &\quad + \frac{1}{2N}\ (dM_s^c \otimes dM_s^c)\ \partial^2_{m(\xi_s)}\phi_{s,t}(f) + \partial_s\mathbb{E}\left[\Delta\phi_{s,t}(m(\xi_s))(f) - \Delta m(\xi_s)\partial_{m(\xi_{s-})}\phi_{s,t}(f) \mid \mathcal{F}_{s-}\right]\ ds \end{aligned}$$

This ends the proof of the theorem. ∎

Next corollary is a direct consequence of the recursion (3.6).

Corollary 3.7. *For any $t \geqslant 0$ and any $f \in \mathcal{B}(S)$ we have the almost sure formula*

$$\begin{aligned}
&\phi_{s,t}(m(\xi_s))(f) - \phi_{0,t}(m(\xi_0))(f) \\
&= \frac{1}{\sqrt{N}}\ M_s^c\left(\partial_{m(\xi_\bullet)}\phi_{\bullet,t}(f)\right) + \mathcal{M}_s^d(\phi_{\bullet,t}(m(\bullet))(f)) \\
&\qquad -\frac{1}{N}\ \int_0^s\ m(\xi_u)\Gamma_{L_{u,m(\xi_u)}}\left(Q_{u,t}^{m(\xi_u)}(1), \partial_{m(\xi_u)}\phi_{u,t}(f)\right)\ du + \frac{1}{N^2}\ \int_0^s\ \Upsilon^3_{m(\xi_{u-})}\phi_{u,t}(f)\frac{1}{2}du
\end{aligned} \tag{3.8}$$

Choosing $s = t$ and taking the expectation in (3.8) we obtain the following result.

Corollary 3.8. *For any $t \geqslant 0$ and $f \in \mathcal{D}(S)$ we have the formula*

$$\begin{aligned}
&\mathbb{E}(m(\xi_t)(f)) - \mathbb{E}(\phi_{0,t}(m(\xi_0))(f)) \\
&= -\frac{1}{N}\ \int_0^t\ \mathbb{E}\left[m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)}\phi_{s,t}(f)\right)\right]\ ds + \frac{1}{N^2}\int_0^t\ \mathbb{E}\left[\Upsilon^3_{m(\xi_{s-})}\phi_{s,t}(f)\right]\frac{1}{2}\ ds
\end{aligned}$$

3.3 Some non asymptotic estimates

Theorem 3.9. *For any time horizon $t \geqslant 0$ and any function $f \in \mathrm{Osc}(S)$ we have*

$$\begin{aligned}
(H_1) \quad &\Longrightarrow \quad |\mathbb{E}(m(\xi_t)(f)) - \eta_t(f)| \leqslant c/N \\
(H_2) \quad &\Longrightarrow \quad |\mathbb{E}(m(\xi_t)(f)) - \eta_t(f)| \leqslant c\ t/N
\end{aligned}$$

for some finite constant c whose value doesn't depend on the parameters (t, N).

The proof of the above theorem is mainly based on the decomposition presented in corollary 3.8. The estimates rely on elementary but rather technical carré du champ inequalities, and semigroup techniques. Thus, the detail of the proof is housed in the appendix, on page 22.

The first estimate stated in the above corollary extend the bias estimate obtained in [23] to time varying Feynman-Kac models. The central difference between homogeneous and time varying models lies on the fact that we cannot use h-process techniques. The latter allows to interpret the Feynman-Kac semigroups in terms of more conventional Markov semigroups.

We end this section with a some more or less direct consequences of the above estimates in the analysis of the measures discussed in theorem 2.4.

By corollary 3.2, for any $N > 1$ we have

$$(H_1) \Longrightarrow \|\eta_t^- - \eta_t\|_{tv} \leqslant c/N \quad \text{and} \quad (H_2) \Longrightarrow \|\eta_t^- - \eta_t\|_{tv} \leqslant c\ t/N$$

By (3.1), when (H_0) is satisfied, the Feynman-Kac model defined in terms of (L_s^-, V^-) satisfy conditions the stability property (H_1). Thus, using theorem 3.9 we readily deduce the following estimates.

Corollary 3.10. *We have almost sure and uniform estimates*

$$(H_0) \Longrightarrow |\mathbb{E}\left(m(\zeta_t^-)(f)\ |\ \widehat{\zeta}_t^1\right) - \eta_t(f)| \leqslant c/N$$

We further assume that the Feynman-Kac model is associated with the historical process $X_t = (X'_s)_{s\leqslant t}$ discussed in (2.6). Also assume that the transition semigroup $P'_{s,t}$ of the auxiliary process X'_t satisfies condition (H_0); that is (H'_0) is met. In this situation, using (2.9) we check that the Feynman-Kac model associated with the historical process X_t satisfies (H_2). Thus, using corollary 3.9 we also deduce the following estimates.

Corollary 3.11. *Assume that the Feynman-Kac model is associated with the historical process $X_t = (X'_s)_{s\leqslant t}$ of the auxiliary process X'_t. In this situation, for any $N > 1$ we have almost sure and uniform estimates*

$$(H'_0) \Longrightarrow |\mathbb{E}\left(m(\zeta_t^-)(f) \mid \widehat{\zeta}_t^1\right) - \eta_t(f)| \leqslant c\ t/N$$

The above results give some information on the bias of the occupation measures. We end this section with some propagation of chaos estimate. Using (3.8), for any functions $f_i \in \mathrm{Osc}(S)$ we have

$$\begin{aligned}
&\mathbb{E}\left(m(\xi_t)(f_1)\ m(\xi_t)(f_2)\right) - \mathbb{E}\left(\phi_{0,t}(m(\xi_0))(f_1)\ \phi_{0,t}(m(\xi_0))(f_2)\right) \\
&= -\frac{1}{N} \sum_{(k,l)\in\{(1,2),(2,1)\}} \int_0^t \mathbb{E}\left[\phi_{s,t}(m(\xi_s))(f_k)\ m(\xi_u)\Gamma_{L_{u,m(\xi_u)}}\left(Q_{u,t}^{m(\xi_u)}(1), \partial_{m(\xi_u)}\phi_{u,t}(f_l)\right)\ \right] du \\
&\quad + \frac{1}{N}\int_0^t \mathbb{E}\left[m(\xi_u)\Gamma_{L_{u,m(\xi_u)}}\left(\partial_{m(\xi_u)}\phi_{u,t}(f_1), \partial_{m(\xi_u)}\phi_{u,t}(f_2)\right)\ \right] \\
&\quad + \int_0^t \partial_s\, \mathbb{E}\left[\Delta\phi_{s,t}(m(\xi_s))(f_1)\ \frac{1}{2}\Delta\phi_{s,t}(m(\xi_s))(f_2)\right]\ ds \\
&\quad + \frac{1}{N^2} \sum_{(k,l)\in\{(1,2),(2,1)\}} \int_0^t \mathbb{E}\left[\phi_{s,t}(m(\xi_s))(f_k)\ \Upsilon^3_{m(\xi_{s-})}\phi_{s,t}(f_l)\frac{1}{2}\right]\ ds
\end{aligned}$$

By (3.4) and using the same lines of arguments as in the proof of theorem 3.9 we check the following estimates.

Corollary 3.12. *For any time horizon $t \geqslant 0$ and any $f, g \in \mathrm{Osc}(S)$ we have*

$$\begin{aligned}
(H_1) &\Longrightarrow |\mathbb{E}\left(f(\xi_t^1)\ g(\xi_t^2)\right) - \eta_t(f)\ \eta_t(g)| \leqslant c/N \\
(H_2) &\Longrightarrow |\mathbb{E}\left(f(\xi_t^1)\ g(\xi_t^2)\right) - \eta_t(f)\ \eta_t(g)| \leqslant c\ t/N
\end{aligned}$$

In the settings of corollary 3.11 we also check the almost sure estimate

$$(H'_0) \Longrightarrow |\mathbb{E}\left(f(\zeta_t^2)\ g(\zeta_t^3) \mid \widehat{\zeta}_t^1\right) - \eta_t(f)\ \eta_t(g)| \leqslant c\ t/N$$

We can extend the above arguments to any finite block of particles.

4 Many-body Feynman-Kac measures

4.1 Description of the models

We let $\mathbb{P}_t^\xi$ and $\mathbb{P}_t^\zeta$ be the distribution of the historical process

$$\widehat{\xi}_t := (\xi_s)_{s\leqslant t} := \left(\xi_s^1, \ldots, \xi_s^N\right)_{s\leqslant t} \quad \text{and} \quad \widehat{\zeta}_t := (\zeta_s)_{s\leqslant t} := \left(\zeta_s^1, \ldots, \zeta_s^N\right)_{s\leqslant t} \in D_t(S \times S_{N-1})$$

We set

$$Z_t(\xi) := \exp\left[-\int_0^t m(\xi_s)(V_s)ds\right] \quad \text{and} \quad \overline{Z}_t(\xi) := \exp\left[-\int_0^t m(\xi_s)(\overline{V}_s)ds\right]$$

We recall for any $f \in \mathcal{B}(S)$ the unbiased property

$$\mathbb{E}\left(m(\xi_t)(f)\ \overline{Z}_t(\xi)\right) = \eta_t(f)$$

For any $1 \leqslant i \leqslant N$ we also consider the historical process

$$\widehat{\zeta}_t^i := (\zeta_s^i)_{s\leqslant t} \quad \text{and the reduced particle system} \quad \zeta_t^- = \left(\zeta_t^2, \ldots, \zeta_t^N\right) \in S_{N-1}$$

Definition 4.1. *For any time horizon $t \in \mathbb{R}_+$, the N-many-body Feynman-Kac measures $\mathbb{Q}_t^\xi \in \mathcal{P}(D_t(S \times S_{N-1}))$ and $\mathbb{Q}_t^\zeta \in \mathcal{P}(D_t(S \times S_{N-1}))$ are defined by Radon-Nikodym the formulae*

$$d\mathbb{Q}_t^\xi/d\mathbb{P}_t^\xi := \overline{Z}_t(\xi) \qquad \text{and} \qquad d\mathbb{Q}_t^\zeta/d\mathbb{P}_t^\zeta := \overline{Z}_t(\zeta^1) \tag{4.1}$$

4.2 A duality formula

In contrast with conventional changes of probability measures the exponential terms $\overline{Z}_t(\xi)$ and $\overline{Z}_t(\zeta^1)$ have unit mean but they are not martingales w.r.t. the laws $\mathbb{P}_t^\xi$ and $\mathbb{P}_t^\zeta$. We let $\mathbb{Q}_t^{\zeta^i}$ be the $\widehat{\zeta}_t^i$-marginal of $\mathbb{Q}_t^\zeta$, with $1 \leqslant i \leqslant N$.

Theorem 4.2. *For any $1 \leqslant i \leqslant N$ and any time horizon $t \geqslant 0$ we have*

$$\mathbb{Q}_t^\xi = \mathbb{Q}_t^\zeta \quad \text{and} \quad \mathbb{Q}_t^{\zeta^i} = \mathbb{Q}_t \tag{4.2}$$

Proof. Observe that $\widehat{\xi}_t := (\xi_s)_{s\leqslant t}$ and $\widehat{\zeta}_t := (\zeta_s)_{s\leqslant t}$ coincide with the historical processes of processes ξ_s and ζ_s. In addition, for any $x = (x_s)_{s\leqslant t} \in D_t(S)$ we have

$$\widehat{V}_t(x) := V_t(x_t) \Longrightarrow m(\widehat{\xi}_s)(\widehat{V}_s) = m(\xi_s)(V_s)$$

In this case, $\mathbb{Q}_t^\xi$ and $\mathbb{Q}_t^\zeta$ coincide with the t-time marginal of the measures of the measures $\widehat{\mathbb{Q}}_t^\xi$ and $\widehat{\mathbb{Q}}_t^\zeta$ defined as above by replacing (ξ_t, ζ_t, V_t) by $(\widehat{\xi}_t, \widehat{\zeta}_t, \widehat{V}_t)$. In this situation the state space S is replaced by the space of paths

$$\widehat{S} = \cup_{t\geqslant 0}\ D_t(S)$$

In addition, the generators $(\mathcal{G}_t, \mathcal{H}_t, \mathcal{G}_t^-)$ are replaced by the generators $(\widehat{\mathcal{G}}_t, \widehat{\mathcal{H}}_t, \widehat{\mathcal{J}}_t)$ of the historical processes $(\widehat{\xi}_t, \widehat{\zeta}_t, \widehat{\zeta}_t^{-1})$. These generators are defined as above by replacing (S, L_t, V_t) by $(\widehat{S}, \widehat{L}_t, \widehat{V}_t)$ where $\widehat{L}_t$ stands for the generator of the historical process $\widehat{X}_t := (X_s)_{s\leqslant t}$. Thus, there is no loss of generality to prove (4.2) for the t-marginal probability measures $(\overline{\mathbb{Q}}_t^\xi, \overline{\mathbb{Q}}_t^\zeta)$ of $(\mathbb{Q}_t^\xi, \mathbb{Q}_t^\zeta)$.

For any $(F, x) \in (\mathcal{D}(S \times S_{N-1}) \times S_N)$ we set

$$\begin{aligned}
\mathcal{L}_t(F)(x) &:= \sum_{1\leqslant i\leqslant N} L_t(F_{x^{-i}})(x^i) \\
\mathcal{L}_t^V(F)(x) &:= \sum_{1\leqslant i\leqslant N} L_t^V(F_{x^{-i}})(x^i) = \mathcal{L}_t(F)(x) - N\ m(x)(V_t)\ F(x)
\end{aligned}$$

Observe that

$$\begin{aligned}\mathcal{G}_t(F)(x) &= \mathcal{L}_t(F)(x) + \sum_{1\leqslant i\leqslant N} V_t(x^i)\ \int\ (F_{x^{-i}}(u) - F(x))\ m(x)(du) \\ &= \mathcal{L}_t(F)(x) + \frac{1}{N}\sum_{1\leqslant i\neq j\leqslant N} V_t(x^i)\ \left[F_{x^{-i}}(x^j) - F_{x^{-i}}(x^i)\right] \\ &= \mathcal{L}_t(F)(x) + \frac{1}{N}\sum_{1\leqslant i\neq j\leqslant N} V_t(x^i)\ F_{x^{-i}}(x^j) - (N-1)\ m(x)(V_t)F(x)\end{aligned}$$

This implies that

$$\mathcal{G}_t(F)(x) - m(x)(V_t)F(x) = \mathcal{L}_t^V(F)(x) + \frac{1}{N}\sum_{1\leqslant i\neq j\leqslant N} V_t(x^i)\ F_{x^{-i}}(x^j)$$

On the other hand, we have

$$\frac{1}{N}\sum_{1\leqslant i\neq j\leqslant N} V_t(x^i)\ F_{x^{-i}}(x^j)$$

$$= \frac{1}{N}\sum_{2\leqslant i\neq j\leqslant N} V_t(x^i)\ F_{x^{-i}}(x^j) + \frac{1}{N}\sum_{2\leqslant j\leqslant N}\ \left[V_t(x^1)\ F_{x^{-1}}(x^j) + V_t(x^j)\ F_{x^{-j}}(x^1)\right]$$

This implies that

$$\begin{aligned}\partial_t\overline{\mathbb{Q}}_t^{\xi}(F) = \mathbb{E}\Bigg(\Bigg[&\mathcal{L}_t^V(F)(\xi_t) + (N-1)\ \left(1-\frac{2}{N}\right)\ V_t(\xi_t^2)\ F_{\xi_t^{-2}}(\xi_t^3) \\ &+ \left(1-\frac{1}{N}\right)\ \left[V_t(\xi_t^1)\ F_{\xi_t^{-1}}(\xi_t^2) + V_t(\xi_t^2)\ F_{\xi_t^{-2}}(\xi_t^1)\right]\Bigg]\ Z_t(\xi)\Bigg)\end{aligned}$$

By symmetry arguments we check that

$$\begin{aligned}\partial_t\overline{\mathbb{Q}}_t^{\xi}(F) = \mathbb{E}\Bigg(\Bigg[&L_t^V(F_{\xi_t^{-1}})(\xi_t^1) + \sum_{2\leqslant i\leqslant N}\ L_t^V(F_{\xi_t^{-i}})(\xi_t^i) + (N-1)\ V_t(\xi_t^2)\ \left(1-\frac{2}{N}\right)\ F_{\xi_t^{-2}}(\xi_t^3) \\ &+ (N-1)\ V_t(\xi_t^2)\ \frac{2}{N}\ F_{\xi_t^{-2}}(\xi_t^1)\Bigg]\ Z_t(\xi)\Bigg)\end{aligned}$$

This implies that

$$\begin{aligned}\partial_t\overline{\mathbb{Q}}_t^{\xi}(F) = \mathbb{E}\Bigg(\Bigg[&L_t^V(F_{\xi_t^{-1}})(\xi_t^1) + \sum_{2\leqslant i\leqslant N}\ L_t(F_{\xi_t^{-i}})(\xi_t^i) - \frac{1}{N-2}\sum_{2\leqslant i\neq\leqslant N} V_t(\xi_t^i)\ F_{\xi_t^{-i}}(\xi_t^i) \\ &+\frac{1}{N-2}\sum_{2\leqslant i\neq j\leqslant N} V_t(\xi_t^i)\ \left(1-\frac{2}{N}\right)\ F_{\xi_t^{-i}}(\xi_t^j) + \frac{1}{N-2}\sum_{2\leqslant i\neq j\leqslant N} V_t(\xi_t^i)\ \frac{2}{N}\ F_{\xi_t^{-i}}(\xi_t^1)\Bigg]\ Z_t(\xi)\Bigg)\end{aligned}$$

We conclude that

$$\begin{aligned}\partial_t\overline{\mathbb{Q}}_t^{\xi}(F) = \mathbb{E}\Bigg(\Bigg[&L_t(F_{\xi_t^{-1}})(\xi_t^1) - V(\xi_t^1)\ F(\xi_t) \\ &+ \sum_{2\leqslant i\leqslant N}\ \left(L_t(F_{\xi_t^{-i}})(\xi_t^i) + V_t(\xi_t^i)\ \int\ \left(F_{\xi_t^{-i}}(u) - F(\xi_t)\right) m_N(\xi_t^{-\{1,i\}})(du)\right)\Bigg]\ Z_t(\xi)\Bigg)\end{aligned}$$

By symmetry arguments we have

$$\partial_t \overline{\mathbb{Q}}_t^{\xi}(F) = \mathbb{E}\left(\mathcal{K}(F)(\xi_t)\ Z_t(\xi)\right) = \overline{\mathbb{Q}}_t^{\xi}(\mathcal{J}(F)) \quad \text{with} \quad \mathcal{K}(F)(x) = \mathcal{H}_t(F)(x) - V(x^1)\ F(x)$$

In much the same way, we have

$$\partial_t \overline{\mathbb{Q}}_t^{\zeta}(F) = \mathbb{E}\left(\mathcal{K}(F)(\zeta_t)\ Z_t(\zeta^1)\right) = \overline{\mathbb{Q}}_t^{\zeta}(\mathcal{K}(F))$$

This ends the proof of the l.h.s. assertion in (4.2). Thus, choosing $F(x) = m(x)(f)$ we have

$$\begin{aligned}
\eta_t(f) &= \mathbb{E}\left(m(\xi_t)(f)\ \overline{Z}_t(\xi)\right) \\
&= \mathbb{E}\left(m(\zeta_t)(f)\ \overline{Z}_t(\zeta^1)\right) = \frac{1}{N}\ \eta_t(f) + \left(1 - \frac{1}{N}\right) \mathbb{E}\left(f(\zeta_t^2)\ \overline{Z}_t(\zeta^1)\right)
\end{aligned}$$

This ends the proof of the r.h.s. assertion in (4.2). The proof of the theorem is completed. ∎

We let $\mathbb{X}_t$ be a random sample from $m(\xi_t)$. Next corollary extend the duality formula presented in [14] to continuous time Feynman-Kac models.

Corollary 4.3. *For any $F \in \mathcal{B}(S \times D_t(S \times S_{N-1}))$ we have the duality formula*

$$\Pi_t(F) := \mathbb{E}\left(F(\mathbb{X}_t, \widehat{\xi}_t)\ \overline{Z}_t(\xi)\right) = \mathbb{E}\left(F(\zeta_t^1, \widehat{\zeta}_t)\ \overline{Z}_t(\zeta^1)\right)$$

Proof. We associate with a given $F \in \mathcal{B}(S \times D_t(S \times S_{N-1}))$ the function $\overline{F} \in \mathcal{B}(D_t(S \times S_{N-1}))$ defined for any

$$x = (x^1(s), (x^2(s), \ldots, x^N(s)))_{s \leqslant t} \in D_t(S \times S_{N-1}) \quad \text{and} \quad x(t) := (x^1(t), \ldots, x^N(t))$$

by the integral formula

$$\overline{F}(x) := \int m(x(t))(du)\ F(u, x)$$

Using (4.2) we have

$$\mathbb{E}\left(\overline{F}((\xi_s)_{s \leqslant t})\ \overline{Z}_t(\xi)\right) = \mathbb{E}\left(\overline{F}((\zeta_s)_{s \leqslant t})\ \overline{Z}_t(\zeta^1)\right)$$

On the other hand, for any $1 \leqslant i \leqslant N$ we have

$$\mathbb{E}\left(\overline{F}((\xi_s)_{s \leqslant t})\ \overline{Z}_t(\xi)\right) = \mathbb{E}\left(F(\xi_t^i, (\xi_s)_{s \leqslant t})\ \overline{Z}_t(\xi)\right) = \mathbb{E}\left(F(\xi_t^1, (\xi_s)_{s \leqslant t})\ \overline{Z}_t(\xi)\right) = \mathbb{E}\left(\underline{F}((\xi_s)_{s \leqslant t})\ \overline{Z}_t(\xi)\right)$$

with the function $\underline{F} \in D_t(S \times S_{N-1})$ given by

$$\underline{F}((x^1(s), \ldots, x^N(s))_{s \leqslant t}) = F(x^1(t), (x^1(s), \ldots, x^N(s))_{s \leqslant t}))$$

Using (4.2) we also have

$$\mathbb{E}\left(\underline{F}((\xi_s)_{s \leqslant t})\ \overline{Z}_t(\xi)\right) = \mathbb{E}\left(\underline{F}((\zeta_s)_{s \leqslant t})\ \overline{Z}_t(\zeta^1)\right) = \mathbb{E}\left(F(\zeta_t^1, (\zeta_s)_{s \leqslant t})\ \overline{Z}_t(\zeta^1)\right)$$

This ends the proof of the corollary. ∎

4.3 Particle Gibbs samplers

We further assume that reference process $X_t = (X'_s)_{s\leqslant t} \in D_t(S')$ in the Feynman-Kac measure (2.1) is the historical of some auxiliary process X'_t taking values in some metric space $(S', d_{S'})$. In this case, X_t is a Markov process taking values in $S = \cup_{s\geqslant 0} D_s(S')$. Also assume that the potential function V_t is chosen so that $V_t(X_t) = V'_t(X'_t)$. In this situation, the mean field particle model ξ_t coincide with the genealogical tree evolutions of the mean field particle interpretation of the Feynman-Kac measures associated with (X'_t, V'_t).

In the same vein, the particle model ζ_t is path space genealogical tree based particle model. For instance $\zeta^1_t = (\zeta^1_s)_{s\leqslant t}$ is itself the historical process of the path-space process $\zeta^1_s \in D_s(S')$; so that the jumps onto ζ^1_t have to be interpreted as a jump of an ancestral line onto ζ^1_t

In this situation, for any given time horizon $t \geqslant 0$, we have

$$\Pi_t(d(z_1, z_2)) \in \mathcal{P}(E_1 \times E_2) \quad \text{with} \quad E_1 = D_t(S') \quad \text{and} \quad E_2 := D_t(S \times S_{N-1})$$

Observe that for any $z_2 := (z_2(s))_{s\leqslant t} \in E_2 := D_t(S \times S_{N-1})$ and any $s \leqslant t$ we have

$$z_2(s) := (z^1_1(s), (z^2_2(s), \ldots, z^N_2(s))) \in D_s(S') \times D_s(S')_{N-1}$$

In this notation, we have desintegration formulae

$$\Pi_t(d(z_1, z_2)) = \eta_t(dz_1)\ \mathbb{M}_t(z_1, dz_2) \quad \text{and} \quad \Pi_t(d(z_1, z_2)) = \mathbb{Q}^{\xi}_t(dz_2)\ \mathbb{A}_t(z_2, dz_1)$$

In the above display $\mathbb{M}_t$ stands for the Markov transition from E_1 into E_2 defined by

$$\mathbb{M}_t(z_1, dz_2) := \mathbb{P}(\widehat{\zeta}_t \in dz_2 \mid \zeta^1_t = z_1)$$

and $\mathbb{A}_t$ the Markov transition from E_2 into E_1 defined by

$$\mathbb{A}_t(z_2, dz_1) := m(z_2(t))(dz_1)$$

The transition of the conventional Gibbs-sampler with target measure Π_t on $E := (E_1 \times E_2)$ is defined by

$$\mathbb{G}_t((z_1, z_2), d(\overline{z}_1, \overline{z}_2)) := \mathbb{M}_t(z_1, d\overline{z}_2)\ \mathbb{A}_t(\overline{z}_2, d\overline{z}_1) \tag{4.3}$$

This transition is summarized in the following synthetic diagram

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longrightarrow \begin{pmatrix} z_1 & \\ \overline{z}_2 & \sim\ (\widehat{\zeta}_t \mid \zeta^1_t = z_1) \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{z}_1 & \sim\ m(\overline{z}_2(t)) \\ \overline{z}_2 & \end{pmatrix}$$

By construction, we have the duality property

$$\Pi_t(d(z_1, z_2))\ \mathbb{G}_t((z_1, z_2), d(\overline{z}_1, \overline{z}_2)) = \Pi_t(d(\overline{z}_1, \overline{z}_2))\ \mathbb{G}^-_t((\overline{z}_1, \overline{z}_2), d(z_1, z_2)) \tag{4.4}$$

with the backward transition

$$\mathbb{G}^-_t((\overline{z}_1, \overline{z}_2), d(z_1, z_2)) = \mathbb{A}_t(\overline{z}_2, dz_1)\ \mathbb{M}_t(z_1, dz_2)$$

Recall that η_t coincide with the marginal Π'_t of Π_t on $E_1 = D_t(S')$. In addition, integrating (4.4) w.r.t. $\overline{z}_2$ we also have the reversibility property

$$\eta_t(dz_1)\ \mathbb{K}_t(z_1, d\overline{z}_1)\ = \eta_t(d\overline{z}_1)\ \mathbb{K}_t(\overline{z}_1, dz_1)$$

with the Markov transition $\mathbb{K}_t = \mathbb{M}_t\mathbb{A}_t$ from $D_t(S')$ into itself defined by

$$\mathbb{K}_t(f)(z_1) := \int \mathbb{K}_t(z_1, d\overline{z}_1)\ f(\overline{z}_1) = \mathbb{E}\left(m(\zeta_t)(f) \mid \widehat{\zeta}_t^1 = z_1\right)$$

We further assume that the Markov transitions of X'_t satisfy condition (H_0). In this situation, combining corollary 3.2 with corollary 3.11, for any time horizon $t \geqslant 0$, any function f with unit oscillations and any $\mu \in \mathcal{P}(D_t(S'))$ and $n \geqslant 1$ we check that

$$\|\mathbb{K}_t(f) - \eta_t(f)\| \leqslant ct/N, \text{ which implies } \mathrm{osc}(\mathbb{K}_t(f)) \leqslant ct/N, \text{ which implies } \|\mu\mathbb{K}_t^n - \eta_t\|_{tv} \leqslant (ct/N)^n$$

for some finite constant c whose value doesn't depend on the parameters (f, t, N).

Appendix

Proof of (3.7)

For any functions $f_i \in \mathrm{Osc}(S)$ and any $l \leqslant k$ we have

$$\left|\mathbb{E}\left[\prod_{1\leqslant l\leqslant k} \Delta m(\xi_t)(f_l) \mid \mathcal{F}_{t-}\right]\right| \leqslant \frac{1}{N^k}\ \left[\sum_{1\leqslant i\leqslant N}\left(V_t(\xi_t^i) + \lambda_t(\xi_t^i)\right)\right] \leqslant \frac{1}{N^{k-1}}\ \|\lambda + V\|dt$$

$$\Longrightarrow N^k\ |\Upsilon^{k+1}_{m(\xi_{s-})}\phi_{s,t}(f)| = \frac{N^k}{(k+1)!}\ \left|\partial_s\mathbb{E}\left[(\Delta m(\xi_s))^{\otimes(k+1)}\ \overline{\partial}^{(k+1)}_{m(\xi_{s-})}\phi_{s,t}(f) \mid \mathcal{F}_{s-}\right]\right|$$

$$= N^k\ \left|\partial_s\mathbb{E}\left[\frac{1}{m(\xi_s)Q_{s,t}^{m(\xi_{s-})}(1)}\ \left(\Delta m(\xi)\left(Q_{s,t}^{m(\xi_{s-})}(1)\right)\right)^k\ \Delta m(\xi)\partial_{m(\xi_{s-})}\phi_{s,t}(f) \mid \mathcal{F}_{s-}\right]\right|$$

$$\leqslant\ e^{(2+k)q}\ \|\lambda + V\|$$

Proof of theorem 3.9

We use (1.12) to check that

$$m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)}\phi_{s,t}(f)\right)$$

$$= (\eta_s Q_{s,t}^{m(\xi_s)}(1))^2\ m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{\eta_s}(1), \partial_{\eta_s}\phi_{s,t}(f)\right)$$

$$+(\eta_s Q_{s,t}^{m(\xi_s)}(1))^2\ \ [\phi_{s,t}(\eta_s) - \phi_{s,t}(m(\xi_s))](f)\ m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{\eta_s}(1)\right)$$

Using (1.6) we also have the estimate

$$\left|m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)}\phi_{s,t}(f)\right)\right| \leqslant e^{3q}\ \ \mathrm{osc}(\overline{Q}_{s,t}(f))\ m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{\eta_s}(1)\right)$$

$$+e^{2q}\ \sqrt{m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{\eta_s}(1)\right)}\ \sqrt{m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(\partial_{\eta_s}\phi_{s,t}(f)\right)} \tag{4.5}$$

On the other hand, we have

$$\partial_{\eta_s}\phi_{s,t}(f) = Q^{\eta_s}_{s,t}[f-\eta_t(f)] \quad \text{and} \quad Q^{\eta_s}_{s,t}(f)(x) = \mathbb{E}\left(f(X_t)\ e^{-\int_s^t \overline{V}_u(X_u)\ du} \mid X_s = x\right)$$

$$\Longrightarrow \partial_s Q^{\eta_s}_{s,t}(f) = -L_s(Q^{\eta_s}_{s,t}(f)) + \overline{V}_s\ Q^{\eta_s}_{s,t}(f)$$

$$\Longrightarrow \partial_s(Q^{\eta_s}_{s,t}(f)Q^{\eta_s}_{s,t}(g)) = -Q^{\eta_s}_{s,t}(f)\ L_s(Q^{\eta_s}_{s,t}(g)) - Q^{\eta_s}_{s,t}(g)\ L_s(Q^{\eta_s}_{s,t}(f)) + 2\overline{V}_s\ Q^{\eta_s}_{s,t}(f)Q^{\eta_s}_{s,t}(g)$$

We also have

$$L_{t,\mu}(f) = L_t(f) + V_t\ [\mu(f)-f] \Longleftrightarrow L_t(f) - L_{t,\mu}(f) = V_t\ [f-\mu(f)]$$

This yields the formula

$$\begin{aligned}\eta\Gamma_{L_{t,\eta}}(f,g) - \eta\Gamma_{L_t}(f,g) &= \int \eta(dx)\ \eta(dy)\ V_t(y)\ [f(y)-f(x)][g(y)-g(x)] \\ &= \eta(V_t(fg)) + \eta(V_t)\ \eta(fg) - \eta(fV_t)\ \eta(g) - \eta(gV_t)\ \eta(f)\end{aligned}$$

For any given time horizon t and $s \in [0,t]$ we have

$$dm(\xi_s)(Q^{\eta_s}_{s,t}(f)Q^{\eta_s}_{s,t}(g)) - \tfrac{1}{\sqrt{N}}\ dM_s(Q^{\eta_\cdot}_{\cdot,t}(f)Q^{\eta_\cdot}_{\cdot,t}(g))$$

$$= m(\xi_s)\left[L_{s,m(\xi_s)}(Q^{\eta_s}_{s,t}(f)Q^{\eta_s}_{s,t}(g)) - Q^{\eta_s}_{s,t}(f)\ L_s(Q^{\eta_s}_{s,t}(g)) - Q^{\eta_s}_{s,t}(g)\ L_s(Q^{\eta_s}_{s,t}(f)) + 2\overline{V}_s\ Q^{\eta_s}_{s,t}(f)Q^{\eta_s}_{s,t}(g)\right]\ ds$$

$$= m(\xi_s)\Big[\Gamma_{s,L_{s,m(\xi_s)}}(Q^{\eta_s}_{s,t}(f), Q^{\eta_s}_{s,t}(g)) + V_s\ Q^{\eta_s}_{s,t}(f)\ \ (m(\xi_s)Q^{\eta_s}_{s,t}(g) - Q^{\eta_s}_{s,t}(g))$$

$$+V_t\ Q^{\eta_s}_{s,t}(g)\ \ (m(\xi_s)Q^{\eta_s}_{s,t}(f) - Q^{\eta_s}_{s,t}(f)) + 2\overline{V}_s\ Q^{\eta_s}_{s,t}(f)Q^{\eta_s}_{s,t}(g)\Big]\ ds$$

This implies that

$$m(\xi_t)(fg) - m(\xi_0)(Q^{\eta_0}_{0,t}(f)Q^{\eta_0}_{0,t}(g)) - \frac{1}{\sqrt{N}}\ M_t(Q^{\eta_\cdot}_{\cdot,t}(f)Q^{\eta_\cdot}_{\cdot,t}(g))$$

$$= \int_0^t m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(Q^{\eta_s}_{s,t}(f), Q^{\eta_s}_{s,t}(g))\ ds$$

$$+ \int_0^t m(\xi_s)\Big[V_s\ Q^{\eta_s}_{s,t}(f)\ \ (m(\xi_s)Q^{\eta_s}_{s,t}(g) - Q^{\eta_s}_{s,t}(g))$$

$$+V_s\ Q^{\eta_s}_{s,t}(g)\ \ (m(\xi_s)Q^{\eta_s}_{s,t}(f) - Q^{\eta_s}_{s,t}(f)) + 2(V_s - \eta_s(V_s))\ Q^{\eta_s}_{s,t}(f)Q^{\eta_s}_{s,t}(g)\Big]\ ds$$

After some simplifications we check that

$$\int_0^t m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(Q^{\eta_s}_{s,t}(f), Q^{\eta_s}_{s,t}(g))\ ds$$

$$= m(\xi_t)(fg) - m(\xi_0)(Q^{\eta_0}_{0,t}(f)Q^{\eta_0}_{0,t}(g)) - \frac{1}{\sqrt{N}}\ M_t(Q^{\eta_\cdot}_{\cdot,t}(f)Q^{\eta_\cdot}_{\cdot,t}(g))$$

$$+ \int_0^t \Big[2\eta_s(V_s)\ m(\xi_s)(Q^{\eta_s}_{s,t}(f)Q^{\eta_s}_{s,t}(g))$$

$$-m(\xi_s)(V_s\ Q^{\eta_s}_{s,t}(f))\ \ m(\xi_s)Q^{\eta_s}_{s,t}(g) - m(\xi_s)(V_s\ Q^{\eta_s}_{s,t}(g))\ \ m(\xi_s)Q^{\eta_s}_{s,t}(f)\Big]\ ds$$

Choosing $f = g = 1$ and taking the expectations we find that

$$\int_0^t \mathbb{E}\left[m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(Q_{s,t}^{\eta_s}(1))\right]\ ds$$

$$= 1 - \eta_0(Q_{0,t}^{\eta_0}(1)^2) + 2\int_0^t \mathbb{E}\left[\eta_s(V_s)\ m(\xi_s)(Q_{s,t}^{\eta_s}(1)^2) - m(\xi_s)(V_s\ Q_{s,t}^{\eta_s}(1))\ \ m(\xi_s)Q_{s,t}^{\eta_s}(1)\right]\ ds$$

$$\leqslant 1 + 2e^{2q}\ \|V\|\ t$$

Choosing $f = g = h - \eta_t(h)$, with $h \in \mathrm{Osc}(S)$ and taking the expectations we find that

$$\int_0^t \mathbb{E}\left[m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(\partial_{\eta_s}\phi_{s,t}(h))\right]\ ds$$

$$= \mathbb{E}\left[m(\xi_t)([h-\eta_t(h)]^2)\right] - \eta_0([\partial_{\eta_0}\phi_{0,t}(h)]^2)$$

$$+2\int_0^t \mathbb{E}\left[\eta_s(V_s)\ m(\xi_s)([\partial_{\eta_s}\phi_{s,t}(h)]^2) - m(\xi_s)(V_s\ \partial_{\eta_s}\phi_{s,t}(h))\ \ m(\xi_s)(\partial_{\eta_s}\phi_{s,t}(h))\right]\ ds$$

$$\leqslant 1 + 4e^{2q}\ \|V\|\ t$$

For any $f \in \mathrm{Osc}(S)$ combining (4.5) with Cauchy-Schwartz inequality we find that that

$$\int_0^t |m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)}\phi_{s,t}(f)\right)|\ ds \leqslant 2e^{3q}\ \left[1 + 4e^{2q}\ \|V\|\right]\ t$$

Combining the above estimate with (3.7) and corollary 3.8 we conclude that

$$(H_2) \implies N\ |\mathbb{E}(m(\xi_t)(f)) - \mathbb{E}(\phi_{0,t}(m(\xi_0))(f))| \leqslant 2e^{3q}\left(1 + 4e^{2q}\ \|V\| + \frac{1}{N^2}\ 2e^q\ \|\lambda + V\|\right)\ t$$

We further assume that (H_1) is satisfied. In this case, using (4.5) we also have

$$|m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{m(\xi_s)}(1), \partial_{m(\xi_s)}\phi_{s,t}(f)\right)| \leqslant e^{3q}\ \alpha\ e^{-\beta(t-s)}\ m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{\eta_s}(1)\right)$$

$$+e^{2q}\ \sqrt{m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q_{s,t}^{\eta_s}(1)\right)}\ \sqrt{m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(\partial_{\eta_s}\phi_{s,t}(f)\right)}$$

For any $\widetilde{\beta} \in \mathbb{R}$ we set

$$\begin{aligned}\widetilde{Q}_{s,t}^{\eta_s}(f)(x) &:= e^{\widetilde{\beta}(t-s)}\ Q_{s,t}^{\eta_s}(f)(x)\\ &= \mathbb{E}\left(f(X_t)\ e^{-\int_s^t \widetilde{V}_u(X_u)\ du}\ |\ X_s = x\right) \quad \text{with} \quad \widetilde{V}_t(x) = \overline{V}_t(x) - \widetilde{\beta}\end{aligned}$$

Arguing as above, we have

$$\partial_s(\widetilde{Q}_{s,t}^{\eta_s}(f)\widetilde{Q}_{s,t}^{\eta_s}(g)) = -\widetilde{Q}_{s,t}^{\eta_s}(f)\ L_s(\widetilde{Q}_{s,t}^{\eta_s}(g)) - \widetilde{Q}_{s,t}^{\eta_s}(g)\ L_s(\widetilde{Q}_{s,t}^{\eta_s}(f)) + 2\ \widetilde{V}_s\ \widetilde{Q}_{s,t}^{\eta_s}(f)\ \widetilde{Q}_{s,t}^{\eta_s}(g)$$

and

$$dm(\xi_s)(\widetilde{Q}_{s,t}^{\eta_s}(f)\widetilde{Q}_{s,t}^{\eta_s}(g)) - \frac{1}{\sqrt{N}}\ dM_s(\widetilde{Q}_{\cdot,t}^{\eta_\cdot}(f)\widetilde{Q}_{\cdot,t}^{\eta_\cdot}(g))$$

$$= m(\xi_s)\left[\Gamma_{s,L_{s,m(\xi_s)}}(\widetilde{Q}_{s,t}^{\eta_s}(f), \widetilde{Q}_{s,t}^{\eta_s}(g)) + V_s\ \widetilde{Q}_{s,t}^{\eta_s}(f)\ \ (m(\xi_s)\widetilde{Q}_{s,t}^{\eta_s}(g) - \widetilde{Q}_{s,t}^{\eta_s}(g))\right.$$

$$\left.+V_t\ \widetilde{Q}_{s,t}^{\eta_s}(g)\ \ (m(\xi_s)\widetilde{Q}_{s,t}^{\eta_s}(f) - \widetilde{Q}_{s,t}^{\eta_s}(f)) + 2\widetilde{V}_s\ \widetilde{Q}_{s,t}^{\eta_s}(f)\ \widetilde{Q}_{s,t}^{\eta_s}(g)\right]\ ds$$

This implies that

$$\int_0^t e^{2\widetilde{\beta}(t-s)}\ m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(Q^{\eta_s}_{s,t}(f),Q^{\eta_s}_{s,t}(g))\ ds$$

$$= m(\xi_t)(fg) - e^{\widetilde{\beta}t}\ m(\xi_0)(Q^{\eta_0}_{0,t}(f)Q^{\eta_0}_{0,t}(g)) - \frac{1}{\sqrt{N}}\ M_t(\widetilde{Q}^{\eta_\cdot}_{\cdot,t}(f)\widetilde{Q}^{\eta_\cdot}_{\cdot,t}(g))$$

$$+\int_0^t e^{2\widetilde{\beta}(t-s)}\ \Big[2(\eta_s(V_s)+\widetilde{\beta})\ m(\xi_s)(Q^{\eta_s}_{s,t}(f)Q^{\eta_s}_{s,t}(g)) - m(\xi_s)(V_s\ Q^{\eta_s}_{s,t}(f))\ \ m(\xi_s)Q^{\eta_s}_{s,t}(g)$$

$$-m(\xi_s)(V_s\ Q^{\eta_s}_{s,t}(g))\ \ m(\xi_s)Q^{\eta_s}_{s,t}(f)\Big]\ ds$$

Choosing $f=g=1$ and $\widetilde{\beta}<0$ we have

$$\int_0^t e^{2\widetilde{\beta}(t-s)}\ \mathbb{E}\left[m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(Q^{\eta_s}_{s,t}(1))\right]\ ds$$

$$= 1 - e^{\widetilde{\beta}t}\ \eta_0(Q^{\eta_0}_{0,t}(1)^2)$$

$$+2\ \int_0^t e^{2\widetilde{\beta}(t-s)}\ \mathbb{E}\left[(\eta_s(V_s)+\widetilde{\beta})\ m(\xi_s)(Q^{\eta_s}_{s,t}(1)^2) - m(\xi_s)(V_s\ Q^{\eta_s}_{s,t}(1))\ \ m(\xi_s)Q^{\eta_s}_{s,t}(1)\right]\ ds$$

$$\leqslant 1 + e^{2q}\ \left(1+2|\widetilde{\beta}|^{-1}\|V\|\right) = 1+e^{2q}\ \left(1+4\beta^{-1}\|V\|\right) \quad \text{when} \quad \widetilde{\beta}=-\beta/2$$

Choosing $f=g=[h-\eta_t(h)]$, with $h\in \mathrm{Osc}(S)$ and $0<\widetilde{\beta}<\beta$ we have

$$\int_0^t e^{2\widetilde{\beta}(t-s)}\ \mathbb{E}\left[m(\xi_s)\Gamma_{s,L_{s,m(\xi_s)}}(\partial_{\eta_s}\phi_{s,t}(h)\right]\ ds$$

$$\leqslant \mathbb{E}\left[m(\xi_t)([h-\eta_t(h)]^2)\right]$$

$$+2\int_0^t e^{2\widetilde{\beta}(t-s)}\ \left[(\eta_s(V_s)+\widetilde{\beta})\ m(\xi_s)\left([\partial_{\eta_s}\phi_{s,t}(h)]^2\right) - m(\xi_s)(V_s\ \partial_{\eta_s}\phi_{s,t}(h))\ \ m(\xi_s)\partial_{\eta_s}\phi_{s,t}(h)\right]\ ds$$

$$\leqslant 1 + 2r^2(2\|V\|+\widetilde{\beta})\int_0^t e^{-2(\beta-\widetilde{\beta})(t-s)}\ ds$$

$$\leqslant 1 + r^2(2\|V\|+\widetilde{\beta})\ (\beta-\widetilde{\beta})^{-1} = 1+r^2(4\|V\|\beta^{-1}+1) \quad \text{when} \quad \widetilde{\beta}=\beta/2$$

We end the proof of the theorem using the fact that

$$|m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q^{m(\xi_s)}_{s,t}(1),\partial_{m(\xi_s)}\phi_{s,t}(f)\right)|$$

$$\leqslant e^{3q}(1+\alpha)\ e^{-\beta(t-s)/2}m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(Q^{\eta_s}_{s,t}(1)\right) + e^{2q}\ e^{\beta(t-s)/2}m(\xi_s)\Gamma_{L_{s,m(\xi_s)}}\left(\partial_{\eta_s}\phi_{s,t}(f)\right)$$

In the last assertion we have used the fact that the estimate $\sqrt{ab}\leqslant ca+b/c$, for all $a,b,c>0$. This ends the proof of the theorem. ∎

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