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Neighborhood preferences for minimal dominating sets enumeration

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Abstract. We investigate two different approaches to enumerate minimal dominating sets of a graph using structural properties based on neighborhood inclusion. In the first approach, we define a *preference relation* on a graph G as a poset on the set of vertices of G . In particular, we consider the poset of closed neighborhood inclusion $P(G)$ and define the notion of *preferred dominating set* as dominating sets that correspond to minimal ideal of $P(G)$. We show that graphs with a unique preferred dominating set are those who are dominated by simplicial vertices and show that there is a polynomial delay algorithm to enumerate minimal dominating sets if there is one to enumerate preferred dominating sets. In the second approach we consider intersections of minimal dominating sets with *redundant vertices*, i.e., vertices that are not minimal in $P(G)$. We show in a generalized class of split graphs that there is a linear delay algorithm to enumerate minimal dominating sets if these intersections form an independent set system. Graphs that share this property include *completed P_7 -free chordal graphs* which improves results from [14] on P_6 -free chordal graphs.

Keywords: Graphs, Dominating sets, Enumeration algorithms, Preferred enumeration

1 Introduction

We consider the problem of enumerating all minimal (inclusion-wise) dominating sets of a given graph, called the DOM-ENUM problem. A *dominating set* in a graph G is a set of vertices D such that every vertex of G is either in D or is adjacent to some vertex of D . It is said to be *minimal* if it does not contain any other dominating set as a subset. We say that an enumeration algorithm is *output-polynomial* if its running time is bounded by a polynomial depending of the sum of the sizes of the input and the output. An algorithm is said to be *incremental polynomial* if the running time between two outputs is bounded by a polynomial depending of the size of the input and already outputted solutions. Finally, we say that an algorithm is *polynomial delay* if the running time between two outputs is bounded by a polynomial depending of the size of the input. Note that the existence of a polynomial delay algorithm gives an incremental polynomial algorithm which gives an output-polynomial algorithm. For further details on enumeration problems complexity, see [3, 12].

To date, the DOM-ENUM problem is still open for general graphs. Recently, it has been proved in [14] that this problem is equivalent to the problem of enumerating all minimal transversals of a hypergraph, called the TRANS-ENUM problem. The best known algorithm is output quasi-polynomial and comes from hypergraph minimal transversals enumeration [9]. However, several classes of graphs are known to admit output-polynomial algorithms. For example, it has been shown that there exist incremental polynomial algorithms for chordal bi-partite graphs or graphs of bounded conformality [7, 11]. Polynomial delay algorithms are known for degenerate, line, chordal and strongly chordal graphs [5, 6, 15, 16]. Linear delay algorithms are known for split, P_6 -free chordal, permutation, interval graphs and graphs with bounded clique width [2, 13, 14].

We investigate two different approaches to enumerate minimal dominating sets of a graph using structural properties based on neighborhood inclusion. In the first approach, we define a *preference relation* on a graph G as a poset on the set of vertices of G . In particular, we consider the poset of closed neighborhood inclusion $P(G)$ and define the notion of *preferred dominating sets* of G as minimal dominating sets that correspond to minimal ideals of $P(G)$. We show that graphs that have a unique preferred dominating set are those who are dominated by their simplicial vertices. We then rewrite a result from [8] on implicational systems to show that there exists a polynomial delay algorithm to enumerate minimal dominating sets if there is one to enumerate preferred dominating sets. In the second approach we consider intersections of minimal dominating sets with *redundant* vertices, i.e., vertices that are not minimal in $P(G)$. We show in a generalized class of split graphs that there is a linear delay algorithm to enumerate minimal dominating sets if these intersections form an independent set system. Graphs that share this property include *completed* P_7 -free chordal graphs which improves results from [14] on P_6 -free chordal graphs.

The rest of the paper is organized as follows. In the next section, we give preliminary notions and introduce new notations. In sections 3 and 4, we focus on the first approach and show the existence of a polynomial delay algorithm to enumerate minimal dominating sets from preferred dominating sets. In sections 5 and 6, we focus on the second approach and give a linear delay algorithm to enumerate minimal dominating sets on a generalized class of split graphs. We conclude the paper by discussing the outlooks of this work.

2 Preliminaries

We refer to [4] for graph terminology not defined below; all graphs considered in this paper are undirected, finite and simple. A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is the set of *vertices* and $E(G) \subseteq V(G) \times V(G)$ is the set of *edges*. A *clique* is a graph in which every two vertices are adjacent. When considering functions that depend on the size of a graph, we usually use n and m for the size of $V(G)$ and $E(G)$. An edge between x and y is denoted by xy (or yx) instead of $\{x, y\}$. The *subgraph* of G induced by $X \subseteq V(G)$, denoted by $G[X]$, is the graph $(X, E(G) \cap (X \times X))$; $G \setminus X$ is the graph $G[V(G) \setminus X]$. We note $N_G(x)$ (or

simply $N(x)$ if there is no ambiguity) the set of *neighbors* of x in G defined by $N_G(x) = \{y \in V(G) \mid xy \in E(G)\}$; $N_G[x]$ is the set of *closed neighbors* defined by $N_G[x] = N_G(x) \cup \{x\}$. We say that a vertex is *simplicial* if its neighborhood form an induced clique. We note $deg(x)$ the number of neighbors of x , i.e., $deg(x) = |N(x)|$. For a given $X \subseteq V(G)$, we respectively write $N[X]$ and $N(X)$ the sets defined by $\bigcup_{x \in X} N[x]$ and $N[X] \setminus X$.

A *dominating set* in a graph G is a set of vertices D such that every vertex of G is either in D or is adjacent to some vertex of D . It is said to be *minimal* if it does not contain any other dominating set as a subset. The set of all minimal dominating sets of G is denoted by $\mathcal{D}(G)$. Let D be a dominating set of G and $x \in D$. We say that x has a *private neighbor* y in G if $y \notin N[D \setminus \{x\}]$. Note that a private neighbor of a vertex $x \in D$ in G is either x itself or a vertex in $V(G) \setminus D$. The set of private neighbors of $x \in D$ in G is denoted by $Priv(D, x)$; D is a minimal dominating set if and only if $Priv(D, x) \neq \emptyset$ for every $x \in D$. In this paper, we restrict ourselves to connected graphs which has no impact on dominating sets enumeration since disconnected subgraphs can be considered separately.

A *partial order* (or *poset*) on a set X is a binary relation \leq on X which is reflexive, anti-symmetric and transitive, denoted by $P = (X, \leq)$. Here, X is called the *ground set* of P . Let x, y be two elements of P , if $x \leq y$ or $y \leq x$, then x and y are said *comparable*, otherwise they are said *incomparable*. A subset of a poset in which every pair of elements is comparable is called a *chain*. A subset of a poset in which no two distinct elements are comparable is called an *antichain*. A set $I \subseteq X$ is said to be an *ideal* if $x \in I$ and $y \leq x$ implies $y \in I$. For an element $x \in X$ we associate the principal ideal $\downarrow x = \{y \in X \mid y \leq x\}$. The *filter* of $x \in X$ is the dual $\uparrow x = \{y \in X \mid x \leq y\}$. The set of all ideals of P is denoted by $\mathcal{I}(P)$. For a subset $S \subseteq P$, we denote by $\text{Max}(S)$ (resp. $\text{Min}(S)$) the maximal (resp. minimal) elements of S with respect to \leq .

2.1 Preference relation

We define a *preference relation* on G as a partially ordered set P on the set of vertices of G , which compares couples of vertices $x, y \in V(G)$, where $x \leq y$ reads x is preferred to y . The most known preference relations are the *lexicographical order*, where any two vertices are comparable, and *indifference order* or *antichain*, where no two vertices are comparable. In this paper, we define $P(G)$ as *the poset of closed neighborhood inclusion* of G , with ground set $V(G)$ and where $x \leq y$ if $N[x] \subseteq N[y]$. We say that two vertices x and y are *twins* if they share the same closed neighborhood, i.e., $N[x] = N[y]$. These twins are *similar* in $P(G)$ and thus ignored by keeping only one. In fact, it is well known that twins play no role in the complexity of minimal dominating sets enumeration: one can identify and remove them in polynomial time preprocessing, and then generate minimal dominating sets and permute twins at each output to obtain all solutions. In the following, all graphs are considered without twins.

It is well known that the powerset $\mathcal{P}(V(G))$ ordered by inclusion is a boolean lattice in which $\mathcal{D}(G)$ forms an antichain (i.e., a *clutter* or *simple hypergraph*).

The motivation behind the use of a preference relation P is to reduce the space of solutions from this boolean lattice to a distributive lattice. In fact, it can be seen that each comparability $x \leq y$ removes from $\mathcal{P}(V(G))$ the boolean lattice given by the interval $[y, V(G) \setminus \{x\}]$. The use of $P(G)$ as preference relation guarantees that minimal dominating sets are antichains of $P(G)$ and thus that they do not belong to such intervals. We give simple but essential properties on such preference, together with an example of a graph G and its preference relation $P(G)$ in Figure 1.

Lemma 1. *If $D \in \mathcal{D}(G)$, then D is an antichain of $P(G)$.*

Proof. Suppose D is not an antichain of $P(G)$. Then there exist $x, y \in D$ such that $x \leq y$, and thus $N[x] \subseteq N[y]$. Then $\text{Priv}(D, x)$ is empty, which is a contradiction with D being a minimal dominating set. \square

Lemma 2. *$\text{Min}(P(G))$ is always a dominating set.*

Proof. Let $y \notin \text{Min}(P(G))$. Then there exists $x \in \text{Min}(P(G))$ such that $x \leq y$ and thus $N[x] \subseteq N[y]$. In particular, $x \in N[y]$ and thus y is connected to $\text{Min}(P(G))$. \square

Fact 1 *For all $x \in V(G)$, there exists a minimal dominating set containing x .*

Proof. Note that $V(G) \setminus N(x)$ is a dominating set. Thus there is a minimal dominating set containing x . \square

Lemma 3. *If x is simplicial, then $x \in \text{Min}(P(G))$ and $x \leq y$ for all $y \in N(x)$.*

Proof. If x is simplicial, then $N[x]$ forms a clique and therefore $N[x] \subseteq N[y]$ for all $y \in N(x)$. Since G has no twins, then $x < y$ for all $y \in N(x)$. \square

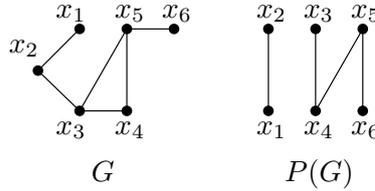


Fig. 1. A graph G and its poset of closed neighborhood inclusion $P(G)$.

2.2 Graph completion

We recall the *graph completion* introduced by Kanté et al. in [14]; this completion adds edges to the graph by keeping its set of minimal dominating sets invariant. Using same notations as in [14], we note $RN(G)$ the set of so-called *redundant vertices* defined by $RN(G) := P(G) \setminus \text{Min}(P(G))$ and $IR(G)$ the set of so-called *irredundant vertices* defined by $IR(G) := \text{Min}(P(G))$.

Definition 1 ([14]). *The completion graph of a graph G is the graph G_{co} with vertex set $V(G)$ and edge set $E(G) \cup \{xy \mid x, y \in RN(G), x \neq y\}$, i.e., G_{co} is obtained from G by adding precisely those edges to G that make $RN(G)$ into a clique.*

Proposition 1 ([14]). *For any graph G , $\mathcal{D}(G) = \mathcal{D}(G_{co})$.*

Proposition 2. *For any graph G , $IR(G) = IR(G_{co})$.*

Proof. By Def. 1, $N_G[x] = N_{G_{co}}[x]$ for every $x \in IR(G)$. Thus, if $y \in RN(G)$ then there exists $x \in IR(G)$ such that $N_G[x] = N_{G_{co}}[x] \subseteq N_G[y] \subseteq N_{G_{co}}[y]$ and therefore $y \in RN(G_{co})$. Now, if $x \in IR(G)$ then $N_G[y] \not\subseteq N_G[x] = N_{G_{co}}[x]$ for every $y \in V(G)$ and thus $x \in IR(G_{co})$. \square

By Lemma 3 vertices that are simplicial in G are simplicial in G_{co} . Note however that the converse is not true. An example is given in Figure 2 where x_3 is simplicial in G_{co} but not in G . Note that a vertex $x \in V(G)$ is simplicial in G_{co} if $y \in N_G(x) \cap IR(G)$ implies that $N_G[x] \subseteq N_G[y]$.

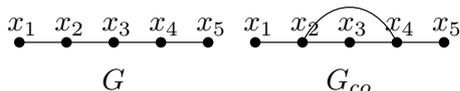


Fig. 2. A graph G and its completion graph G_{co} .

2.3 Preferred dominating sets

It is well known that for a poset P , the set of ideals of P is in bijection with the set of antichains of P . From Lemma 1, we deduce that for $P(G)$ and for any minimal dominating set D there is a unique corresponding ideal $\downarrow D$ with $D = \text{Max}(\downarrow D)$. But the converse is not true, since the empty set is always an ideal. We say that an ideal $I \in \mathcal{I}(P(G))$ is a *dominating ideal* if $\text{Max}(I)$ is a dominating set of G , and that I is a *minimal dominating ideal* if $\text{Max}(I)$ is a minimal dominating set of G . A dominating set D is called *preferred* whenever $J \subseteq \downarrow D$ and J dominates G implies that $J = \downarrow D$. It is clear that a preferred dominating set is a minimal dominating set. Moreover, for every minimal dominating set D , there exists a preferred dominating set D' such that $\downarrow D' \subseteq \downarrow D$. Now, note that if any subset $S \subseteq V(G)$ dominates G , then $\text{Max}(S)$ dominates G . In particular, every ideal $I \in \mathcal{I}(P(G))$ that dominates G is a dominating ideal. This is an important property of our neighborhood preference.

Given a graph G , we denote by $\mathcal{J}(G) \subseteq \mathcal{I}(P(G))$ the set of minimal dominating ideals of G , ordered by inclusion, and note $\mathcal{J}_{\leq}(G) = \text{Min}(\mathcal{J}(G))$ the set of *preferred ideals* of G . We denote by $\mathcal{D}_{\leq}(G)$ the set of preferred dominating sets defined by $\mathcal{D}_{\leq}(G) = \{D \in \mathcal{D}(G) \mid \downarrow D \in \mathcal{J}_{\leq}(G)\}$. Figure 3 shows the bijection between minimal dominating sets and ideals given by Lemma 1, using the same graph as in Figure 1.

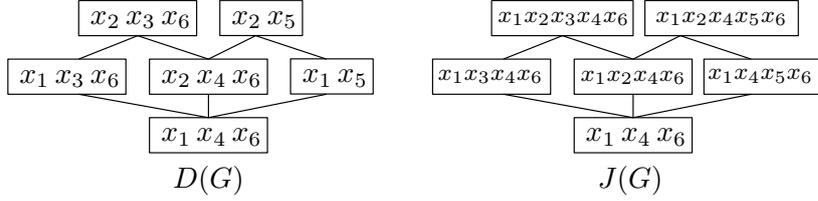


Fig. 3. Bijection between $\mathcal{J}(G)$ and the set of minimal dominating sets $\mathcal{D}(G)$ ordered by inclusion of their ideals in $P(G)$, i.e., $\mathcal{D}(G) = \{\text{Max}(I) : I \in \mathcal{J}(G)\}$. On this graph, $I = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ is a dominating ideal but not a minimal dominating ideal. $J = \{x_1, x_2, x_4, x_5, x_6\}$ is a minimal dominating ideal included in I . The preferred dominating set is unique in this case: $\mathcal{D}_{\leq}(G) = \{\{x_1, x_4, x_6\}\}$.

3 Graphs with a unique preferred dominating set

In this section, we are interested in graphs with a unique preferred dominating set. We show that if a graph has a unique preferred dominating set, then it is directly obtained by minimal elements of P . We then give a simplicial characterization of such graphs and show that they include P_5 -free chordal graphs.

Lemma 4. *If a graph G has a unique preferred minimal dominating set D , then $D = \text{Min}(P)$. Moreover, D is a maximal independent set of G .*

Proof. Suppose $\mathcal{D}_{\leq}(G) = \{D\}$ and $D \neq \text{Min}(P)$. By Lemma 2, we have $\text{Min}(P) \not\subseteq D$ since $\text{Min}(P)$ is a dominating set. Then there exists $d \in \text{Min}(P) \setminus D$ and thus $D' = V(G) \setminus N(d)$ is a dominating set. Moreover, $\downarrow D'$ contains at least one preferred dominating set that contains d since $D' \cap N(d) = \emptyset$. Thus this contradicts that $|\mathcal{D}_{\leq}(G)| = 1$. Suppose that D is not an independent set. Let $x, y \in D$ such that $xy \in E(G)$ and let $x' \in \text{Priv}(D, x)$. Because $D = \text{Min}(P)$, $x' \notin \text{Min}(P)$. Thus $x \leq x'$ since x' is a private neighbor of x . Then $N[x] \subseteq N[x']$ and $x'y \in E(G)$ which contradicts the fact that $x' \in \text{Priv}(D, x)$. We conclude that D is a maximal independent set since D is a minimal dominating set. \square

Note that the converse of Lemma 4 is not true for the P_5 graph given in Figure 2: $\text{Min}(P) = \{x_1, x_3, x_5\}$ is a preferred dominating set but $D = \{x_2, x_4\}$ also is. However, an important consequence of this lemma is that if G has a unique preferred dominating set, then it is directly given by $\text{Min}(P)$. In the following, we give a characterization of graphs and completion graphs that have a unique preferred dominating set.

Lemma 5. *A vertex x is simplicial if and only if $P \setminus \uparrow x$ is not a dominating set.*

Proof. Let us consider any simplicial vertex x and show that $P \setminus \uparrow x$ is not a dominating set. By Lemma 3, $x \leq y$ for all $y \in N(x)$. Hence, $N[x] \subseteq \uparrow x$ and $P \setminus \uparrow x$ does not dominate x . Conversely, let x be a vertex such that $P \setminus \uparrow x$ is not a dominating set. By Lemma 2, $x \in \text{Min}(P)$. Suppose that x is not simplicial, then

there exist $y, z \in N(x)$ such that $yz \notin E(G)$. Thus $N[x] \not\subseteq N[y]$ and $N[x] \not\subseteq N[z]$ and so $y, z \notin \uparrow x$. Moreover, y, z are connected to every $t \in \uparrow x$ since $N[x] \subseteq N[t]$ for all such t and $y, z \in N[x]$. We conclude that $P \setminus \uparrow x$ is a dominating set which is absurd. \square

Theorem 2. *A graph G has a unique preferred dominating set if and only if every vertex in $\text{Min}(P(G))$ is simplicial.*

Proof. Suppose that G has a unique preferred dominating set, i.e., $\mathcal{D}_{\leq}(G) = \{D\}$. By Lemma 4, $D = \text{Min}(P)$. Suppose that there exists $x \in D$ such that x is not simplicial. By Lemma 5, $D' = P \setminus \uparrow x$ is a dominating set and there exists a preferred dominating set D'' such that $\downarrow D'' \subseteq \downarrow D'$. Since $D \not\subseteq \downarrow D'$ we conclude that $|\mathcal{D}_{\leq}(G)| > 1$ which is absurd. Conversely, suppose that all minimal vertices are simplicial. Then by Lemma 2, $D = \text{Min}(P)$ is a dominating set. Moreover, D is minimal and is the only preferred dominating set since by Lemma 5, $D \setminus \uparrow x$ is not a dominating set for all $x \in D$. \square

Corollary 1. *A graph G has a unique preferred dominating set if and only if the set of simplicial vertices is a dominating set.*

The following proposition shows that such graphs include P_5 -free chordal graphs, which strictly include split graphs. However, note that our characterization only relies on simplicial vertices; big induced cycles are not forbidden as long as they are dominated by simplicial vertices.

Proposition 3. *If G is P_5 -free chordal then it has a unique preferred dominating set.*

Proof. Let G be a P_5 -free chordal graph and suppose that there exists at least one minimal vertex u that is not simplicial. Then u has at least two disconnected neighbors x and y . As u is minimal, both x and y have at least one neighbor not in $N[x]$. We note x' and y' such neighbors of x and y . As the graph is chordal, note that $x' \neq y'$ or otherwise it would create an induced C_4 . Therefore, vertices $x'xuyy'$ forms an induced P_5 , which is against the hypothesis. \square

We end this section with a characterization of graphs for which the completion graph has a unique preferred dominating set. Such graphs are those who are completed to a split graph, which were shown in [14] to include P_6 -free chordal graphs.

Theorem 3. *Let G be any graph, the three following statements are equivalent:*

1. G is dominated by vertices that are simplicial in G_{co} ,
2. G_{co} has a unique preferred dominating set,
3. G_{co} is a split graph.

Proof. If G is dominated by vertices that are simplicial in G_{co} then by Corollary of Corollary 1, G_{co} has a unique preferred dominating set. If G_{co} has a unique preferred dominating set then by Theorem 2 every vertex $x \in IR(G)$ is simplicial and thus G_{co} is a split graph. Now, if G_{co} is a split graph, then it is dominated by its set of simplicial vertices and thus G is dominated by vertices that are simplicial in G_{co} . This conclude the proof. \square

4 Enumeration from preferred dominating sets

In this section, we show using results from Ennaoui's thesis [8] on generating *keys of a database* that there exists a polynomial delay algorithm to enumerate $\mathcal{D}(G)$ if there is one to enumerate $\mathcal{D}_{\leq}(G)$. We reformulate and reproved these results in our context for greater clarity. This motivates the study of preferred enumeration.

Lemma 6 ([8]). *Let $I \in \mathcal{J}(G) \setminus \mathcal{J}_{\leq}(G)$. There exists $a \in \text{Max}(I)$ and $I' \in \mathcal{J}(G)$ such that $I = I' \cup \downarrow a$.*

Proof. Since I is not preferred, there exists $a \in \text{Max}(I)$ such that $I \setminus \{a\}$ is still a dominating ideal. Since $\text{Max}(I)$ is a minimal dominating set, every $b \in \text{Max}(I) \setminus \{a\}$ has a private t such that $N[t] \cap \downarrow a = \emptyset$. Then, there exists a minimal dominating D set such that $\text{Max}(I) \setminus \{a\} \subseteq D$ and $D \subseteq \text{Max}(I \setminus \{a\})$. In other words, D is equal to $\text{Max}(I) \setminus \{a\}$ plus a subset of predecessors of a . Therefore $I = \downarrow D \cup \downarrow a$ and $I' = \downarrow D$. \square

A consequence of Lemma 6 is that every minimal dominating set D that is not preferred is *accessible* from a minimal dominating set D' by adding one vertex to D' and by removing its predecessors in P . It thus gives the existence of a *transition graph* where nodes are elements of $\mathcal{J}(G)$ and where there is an edge from $I' \in \mathcal{J}(G)$ to $I \in \mathcal{J}(G)$ if $I = I' \cup \downarrow a$ for some a . The idea behind this transition graph notion, as stated in [10] by Gely et al., is to define a rooted spanning tree on this transition graph such that any algorithm that search the graph using this tree will enumerate all $\mathcal{J}(G)$. This method was also known as reverse search in [1]. Note that in the case of several preferred dominating sets we are interested in a spanning forest. Lemma 7 defines such spanning forest with $\mathcal{J}_{\leq}(G)$ for roots nodes and Theorem 4 shows how it can be used to get a polynomial delay algorithm for minimal dominating ideals enumeration.

Lemma 7 ([8]). *For all $I \in \mathcal{J}(G) \setminus \mathcal{J}_{\leq}(G)$, there exists a unique $I' \in \mathcal{J}(G)$, denoted by $I' = \text{Parent}(I)$, satisfying the following:*

1. $I = I' \cup \downarrow a$ where a is the lexicographically largest element in $\text{Max}(I)$ such that $I \setminus \{a\}$ is a dominating set.
2. $\text{Max}(I')$ is the lexicographically largest minimal dominating set contained $\text{Max}(I \setminus \{a\})$.

Proof. Let $I \in \mathcal{J}(G) \setminus \mathcal{J}_{\leq}(G)$, $S = \{a \in \text{Max}(I) \mid \text{Max}(I \setminus \{a\}) \text{ is a dominating set}\}$ and $\mathcal{I}_a = \{I' \in \mathcal{J}(G) \mid I = I' \cup \downarrow a\}$ for some $a \in S$. Since $I \notin \mathcal{J}_{\leq}(G)$, S is not empty and by Lemma 6 there exist $a \in \text{Max}(I)$ and $I' \in \mathcal{J}(G)$ such that $I = I' \cup \downarrow a$ and thus \mathcal{I}_a is not empty. Then, for a given $I \in \mathcal{J}(G) \setminus \mathcal{J}_{\leq}(G)$, we consider the lexicographically largest $a \in S$ such that $\mathcal{I}_a \neq \emptyset$, and I' the lexicographically largest ideal in \mathcal{I}_a . This defines the *Parent*(I) relation. \square

Theorem 4 ([8]). *There is polynomial delay and space algorithm to enumerate $\mathcal{J}_{\leq}(G)$ if there is one to enumerate $\mathcal{J}_{\leq}(G)$.*

Proof. Suppose that there is a polynomial delay and space algorithm to generate $\mathcal{J}_{\leq}(G)$ and let I_1, I_2, \dots, I_m be the list of preferred ideals generated by the algorithm. Let us consider the k -th step, when I_k is outputted. Using Lemma 6, we search the rooted tree at I_k using a depth-first algorithm in polynomial delay and space, before continuing to I_{k+1} . The *parent* relation can be computed in polynomial time and ensures that rooted trees are disjoint and solutions only generated once. \square

As a consequence, knowing how to generate $\mathcal{D}_{\leq}(G)$ in polynomial delay and space is sufficient to generate $\mathcal{D}(G)$ in polynomial delay and space. Thus, this gives a polynomial delay algorithm to enumerate $\mathcal{D}(G)$ whenever G or its completion graph has a unique preferred dominating set, since $\mathcal{D}_{\leq}(G)$ is directly obtained by $\text{Min}(P)$ on these graphs. However, we show using a second approach for generating minimal dominating sets using neighborhoods that there exists a linear delay algorithm for a generalized class of split graphs. Such graphs include split graphs and thus (completion) graphs that have a unique preferred dominating set.

5 Bipartitions of minimal dominating sets

In this section, we investigate an approach for enumerating minimal dominating sets from their intersection with redundant vertices. For any dominating set D , we note $D_{RN} = D \cap RN(G)$ its subset of redundant vertices and $D_{IR} = D \cap IR(G)$ its subset of irredundant vertices. Then D_{RN} and D_{IR} form a bipartition of D . We note $\mathcal{D}_{RN}(G)$ the set defined by $\mathcal{D}_{RN}(G) = \{D_{RN} \mid D \in \mathcal{D}(G)\}$.

We call a graph G *quasi-split* if the vertex set $V(G)$ can be partitioned into one clique $C_0 = RN(G)$ and a set of k disconnected cliques $C_1, \dots, C_l \subseteq IR(G)$ such that every vertex of C_i has a neighbor in C_0 and none in C_j , for all $i \neq j \neq 0$. Note that for a given graph, C_0 is uniquely determined. Clearly, G is a split graph if $|C_i| = 1$ for all $i \neq 0$. Note that quasi-split graphs contain co-bipartite incidence graphs that were used to polynomially reduce minimal dominating sets enumeration to minimal transversals in hypergraphs in [14]. Thus, enumerating minimal dominating sets of a quasi-split graph is harder than minimal transversals. We now recall some results from Mary's thesis [17] on a generalization of graph twins for minimal dominating sets.

Definition 2 ([17]). *Two connected vertices x and y are called d -equivalent if $z \in N[x] \Delta N[y]$ implies that $z \in RN(G)$. A class X of d -equivalent vertices is a set such that x, y are d -equivalent for all $x, y \in X$; then $G[X]$ is a clique.*

Proposition 4 ([17]). *Two vertices x and y are d -equivalent if and only if $\mathcal{D}_x(G) = \{(D \setminus \{y\}) \cup \{x\} \mid D \in \mathcal{D}_y(G)\}$, where $\mathcal{D}_x(G)$ is the set of minimal dominating sets containing x .*

Note that if G is quasi-split then every clique $C_i, i \neq 0$, is a set of d-equivalent vertices since $N(C_i) \subseteq RN(G)$ for all $i \neq 0$. We show for such graphs that $\mathcal{D}_{RN}(G)$ is equal to the set of every subset $A \subseteq RN(G)$ for which all $x \in A$ have a private neighbor in some $C_i \subseteq N(A)$.

Lemma 8. *Let G be a quasi-split graph and $A \subseteq RN(G)$. If every $x \in A$ has a private neighbor in some $C_i \subseteq N(A), i \neq 0$, then $A \cup D^A \in \mathcal{D}(G)$ for every $D^A \in \{\{c_1, \dots, c_l\} \mid (c_1, \dots, c_l) \in \prod_{C_i \not\subseteq N[A]} C_i\}$.*

Proof. Let $A \subseteq RN(G)$ and $D^A \in \{\{c_1, \dots, c_l\} \mid (c_1, \dots, c_l) \in \prod_{C_i \not\subseteq N[A]} C_i\}$. Note that $D = A \cup D^A$ is a dominating set. Indeed, if $A = \emptyset$, then by Lemma 2, $IR(G) = \cup_{i \neq 0} C_i$ is a dominating set of G . Since $C_i, i \neq 0$, are sets of d-equivalent vertices, D^A is a dominating set of G and so is D . If $A \neq \emptyset$, then $C_0 \subseteq N[A]$ and by construction of D^A , $C_i \subseteq N[D^A]$ for all $i \neq 0$ such that $C_i \not\subseteq N[A]$. Thus, D is a dominating set of G . We now show that every $x \in D$ has a private neighbor. If $x \in D^A$, then $x \in C_i$ for some $C_i \not\subseteq N[A], i \neq 0$, and $|D \cap C_i| = 1$; thus x has a private neighbor. Now, since every $x \in A$ has a private neighbor y in some $C_i \subseteq N(A), i \neq 0$, then y is not connected to any vertex of D^A and thus $D \in \mathcal{D}(G)$. \square

Lemma 9. *Let G be a quasi-split graph and $D \in \mathcal{D}(G)$. Then every $x \in D_{RN}$ has a private neighbor in some $C_i \subseteq N(D_{RN}), i \neq 0$, and $D_{IR} \in \{\{c_1, \dots, c_l\} \mid (c_1, \dots, c_l) \in \prod_{C_i \not\subseteq N[D_{RN}]} C_i\}$.*

Proof. Let $D \in \mathcal{D}(G)$, we first show that every vertex in D_{RN} has a private neighbor in some $C_i \subseteq N(D_{RN}), i \neq 0$. Suppose that there exists $x \in D_{RN}$ such that every $y \in Priv(D, x)$ verifies $y \in C_i$ for some $C_i \not\subseteq N(D_{RN}), i \neq 0$. Since D dominates C_i and $C_i \not\subseteq N(D_{RN}), C_i \cap D_{IR} \neq \emptyset$ and thus $Priv(D, x) = \emptyset$ which is absurd. We now show that $D_{IR} \in \{\{c_1, \dots, c_l\} \mid (c_1, \dots, c_l) \in \prod_{C_i \not\subseteq N[D_{RN}]} C_i\}$. Since $D_{IR} \subseteq IR(G)$, D_{IR} intersects at least every clique $C_i, i \neq 0$, such that $C_i \not\subseteq N(D_{RN})$. Moreover, D_{IR} does not intersect any clique $C_i, i \neq 0$, such that $C_i \subseteq N(D_{RN})$ since no vertex in $D_{IR} \cap C_i$ can have private neighbors as $N(C_i) \subseteq RN(G)$ and thus $N[C_i] \subseteq N[D_{RN}]$. At last, $|D_{IR} \cap C_i| = 1$ for every $i \neq 0$ since every C_i is a set of d-equivalent vertices. We conclude that $D_{IR} \in \{\{c_1, \dots, c_l\} \mid (c_1, \dots, c_l) \in \prod_{C_i \not\subseteq N[D_{RN}]} C_i\}$. \square

Theorem 5. *If G is quasi-split then $\mathcal{D}_{RN}(G) = \{A \subseteq RN(G) \mid \forall x \in A, x \text{ has a private neighbor in some } C_i \subseteq N(A)\}$.*

Proof. By Lemma 8, if $A \subseteq RN(G)$ verifies that every $x \in A$ has a private neighbor in some $C_i \subseteq N(A)$, then $A = D_{RN}$ for some $D = A \cup D^A \in \mathcal{D}(G)$ and thus $A \in \mathcal{D}_{RN}(G)$. Now, if $A \in \mathcal{D}_{RN}(G)$ then $A = D_{RN}$ for some $D \in \mathcal{D}(G)$ and thus by Lemma 9 every $x \in A$ has a private neighbor in some $C_i \subseteq N(A)$. \square

Corollary 2. *Let G be a quasi-split graph. Then $\mathcal{D}_{RN}(G)$ is an independent set system if and only if $A \in \mathcal{D}_{RN}(G)$ implies that for all $x, y \in A$, x has a private neighbor in some $C_i \subseteq N(A \setminus \{y\})$.*

In the following, we call *IS-quasi-split* a quasi-split graph on which the set $\mathcal{D}_{RN}(G)$ is an independent set system and show that such graphs include completed P_7 -free chordal graphs.

Lemma 10. *If G is a connected P_7 -free chordal graph then every two connected irredundant vertices are d -equivalent.*

Proof. Let $x, y \in IR(G)$ be two connected vertices that are not d -equivalent. By definition there exists $z \in N[x] \Delta N[y]$ such that $z \in IR(G)$. Without loss of generality, let us choose x (rather than y) to have such neighbor z . Note that $N[z] \not\subseteq N[x]$ and $N[y] \not\subseteq N[x]$ since $x \in IR(G)$ and thus there exist z', y' such that $z' \in N(z) \setminus N[x]$ and $y' \in N(y) \setminus N[x]$. Now, since $y, z \in IR(G)$, there exist y'', z'' such that $z'' \in N(z') \setminus N[z]$ and $y'' \in N(y') \setminus N[y]$. Since G is chordal, $z''z'zyy'y''$ forms an induced P_7 which is excluded. \square

Theorem 6. *If G is a connected P_7 -free chordal, then G_{co} is IS-quasi-split.*

Proof. Let $C_0 = RN(G)$ and $C_1, \dots, C_k \subseteq IR(G)$ be the k connected component of $IR(G)$. By Lemma 10, C_i, C_j are disconnected cliques for every $i \neq j \neq 0$. Thus, G_{co} is quasi-split. Let us suppose that G is not IS-quasi-split. Using Corollary 2, there exist $A \in \mathcal{D}_{RN}(G)$ and $x, y \in A$ such that every $z \in P(A, x)$ belongs to a clique $C_i \not\subseteq N(A \setminus \{y\})$. Let $z \in P(A, x)$ be one such vertex and let us note Z the clique $C_i, i \neq 0$, in which z belongs. Then $yz \notin E(G)$ since $z \in P(A, x)$. Moreover, since $Z \not\subseteq N(A \setminus \{y\})$ there exists $z' \in Z \cap N(y)$ such that $z' \notin N(A \setminus \{y\})$. In particular $xz' \notin E(G)$. Note that as $z' \in IR(G)$ then $N[y] \not\subseteq N[z']$ and thus there exists y' such that $y' \in N(y) \setminus N[z']$. Now, since $x \in RN(G)$ there exists $x' \in IR(G)$ such that $N[x'] \subseteq N[x]$. We note X the clique $C_i, i \neq 0$, in which x' belongs. Since $N[X] \subseteq N[x'] \subseteq N[x]$ and $x \in A \setminus \{y\}, X \subseteq N(A \setminus \{y\})$. Then by hypothesis, $P(A, x) \cap X = \emptyset$ and therefore x' is connected to some vertex $u \in A$ which is connected to x as $N[x'] \subseteq N[x]$. Note that since $z \in P(A, x), u$ is not connected to z . At last, since $x, u \in A$ then $N[u] \not\subseteq N[x]$ and thus there exists some $u' \in N(u) \setminus N[x]$. Since G is chordal and since every two vertices at distance two in path $u'uxzz'y'y'$ are disconnected, this leads to an induced P_7 in G which is excluded as G is P_7 -free. \square

One question remains to know if there is a polynomial time algorithm to recognize IS-quasi-split graphs.

6 Enumeration algorithm

We now describe an algorithm which takes a quasi-split graph G , a linear ordering σ of $RN(G)$, and enumerate $\mathcal{D}_{RN}(G)$ in linear delay whenever $\mathcal{D}_{RN}(G)$ is an independent set system. At each step of the algorithm, given $A \in \mathcal{D}_{RN}(G)$, the algorithm computes the largest $a = \text{Max}\{\sigma(y) \mid y \in A\}$ ¹ and check if A can be extended by adding some candidate $c \in RN(G)$ such that $\sigma(c) > a$. The

¹ $\text{Max}\{\sigma(y) \mid y \in A\} = 0$ if $A = \emptyset$.

recursive tree stops when A cannot be extended, that is if no minimal dominating set intersects $RN(G)$ on $A \cup \{c\}$.

Algorithm 1: ENUM(G, σ, A)

Data: An IS-quasi-split graph G , a linear ordering σ of $RN(G)$ and $A \subseteq RN(G)$.

Result: Output $\mathcal{D}_{RN}(G)$.

```

1 output  $A$ 
2  $a \leftarrow \text{Max}\{\sigma(y) \mid y \in A\}^1$ 
3 for all  $c \in RN(G)$  such that  $\sigma(c) > a$  do
4   | if  $\exists C_i$  such that  $C_i \not\subseteq N(A)$  and  $C_i \subseteq N(A \cup \{c\})$ 
5   |   then ENUM( $G, \sigma, A \cup \{c\}$ )
6 end

```

Theorem 7. *Algorithm 1 enumerates $\mathcal{D}_{RN}(G)$ in linear delay and polynomial space on quasi-split graphs whenever $\mathcal{D}_{RN}(G)$ is an independent set system.*

Proof. We prove the completeness of the algorithm using induction on the number of elements in $A \in \mathcal{D}_{RN}(G)$. First, by Lemma 2, $A = \emptyset$ belongs to $\mathcal{D}_{RN}(G)$ since $IR(G)$ is a dominating set and $A \cap IR(G) = \emptyset$. At first call, A is outputted (Line 1). Assume now that every $A \in \mathcal{D}_{RN}(G)$ such that $|A| \leq k$ has been outputted by the algorithm and let $A' \in \mathcal{D}_{RN}(G)$ such that $|A'| = k + 1$. Let $a \leftarrow \text{Max}\{\sigma(y) \mid y \in A'\}$ and $x \in A'$ be the vertex such that $\sigma(x) = a$. Since $\mathcal{D}_{RN}(G)$ is an independent set system, there exists $A \in \mathcal{D}_{RN}(G)$ such that $A = A' \setminus \{x\}$. Now, by inductive hypothesis since $|A| = |A' \setminus \{x\}| = k$, A is outputted by the algorithm. Also x is greater than every vertex in A with respect to σ (Line 3). By Theorem 5, since $A' \in \mathcal{D}_{RN}(G)$, x has a private neighbor in some C_i such that $C_i \subseteq N(A \cup \{x\})$ and thus $C_i \not\subseteq N(A)$ (Line 4). Therefore A' is outputted by the call (Line 5) of ENUM(G, σ, A').

Now, if a set A is outputted by the algorithm, then either $A = \emptyset$ (Line 1) or every vertex $x \in A$ has a private neighbor in some clique C_i such that $C_i \subseteq N(A)$ (Line 4). By Theorem 5, $A \in \mathcal{D}_{RN}(G)$.

We now analyze the complexity of the algorithm. Note that computing $a = \text{Max}\{\sigma(y) \mid y \in A\}$ (Line 2) and $c \in RN(G)$ such that $\sigma(c) > a$ (Line 3) takes $\mathcal{O}(n)$ operations. We show using arrays that testing in the loop if there exists C_i such that $C_i \not\subseteq N(A)$ and $C_i \subseteq N(A \cup \{c\})$ (Line 4) is bounded by $\mathcal{O}(\text{deg}(c))$. Before calling the algorithm, compute an array T_1 of size k such that $T_1[i] = |C_i|$ for every $i \neq 0$. This array will be used to know the number of remaining vertices to dominate in each clique C_1, \dots, C_k . Also, compute an array T_2 of size n such that $T_2[x] = i$ if $x \in C_i$. Using these two arrays, one can access in constant time to the number of remaining vertices to dominate in the unique clique C_i in which x belongs by calling $T_1[T_2[x]]$. Finally, consider an array W of size n initialized to zero. This array will be used to know if a vertex x is dominated by A , by setting $W[x] = \sigma(z)$ if x is connected to some $z \in A$ and $W[x] = 0$ if $x \notin N[A]$. Note that this preprocessing takes at most $\mathcal{O}(n+m)$ steps.

Now, at each iteration of the loop (Line 3), when considering a new candidate $c \in RN(G)$, do the following : for each $x \in N(c)$, update $W[x] := \sigma(x)$ and $T_1[T_2[x]] := T_1[T_2[x]] - 1$ whenever $W[x] = 0$. Note that $T_1[T_2[x]]$ is decreased to zero if and only if c verifies $C_i \not\subseteq N(A)$ and $C_i \subseteq N(A \cup \{c\})$ for $i = T_2[x]$. This testing takes at most $\mathcal{O}(deg(c))$ steps. When backtracking c , undo the changes by setting $W[x] = 0$ and $T_1[T_2[x]] := T_1[T_2[x]] + 1$ for every $x \in N(c)$ such that $W[x] := \sigma(x)$. This also takes at most $\mathcal{O}(deg(c))$. Since $\sum_{x \in V(G)} deg(x)$ is bounded by $\mathcal{O}(n+m)$, the whole loop (Line 3) takes at most $\mathcal{O}(n+m)$ steps. \square

As a consequence, there exists a linear delay algorithm to enumerate minimal dominating sets in P_7 -free chordal graphs after polynomial preprocessing to compute $RN(G)$, $IR(G)$ and thus the partition C_0, C_1, \dots, C_k . Indeed, at each output of Algorithm 1, one just has to extend A to $A \cup D^A$ for every set D^A in the cartesian product of all $C_i \not\subseteq N[A]$ that can be computed in constant delay.

Also, note that Algorithm 1 can be extended to other set system structures of $\mathcal{D}_{RN}(G)$.

An *accessible set system* is a family of set in which every nonempty set X contains an element x such that $X \setminus \{x\}$ belongs to the family. We obtain the same result as in Theorem 5 for accessible set systems, i.e., $\mathcal{D}_{RN}(G)$ is an accessible set system if $A \in \mathcal{D}_{RN}(G)$ implies that there exists $y \in A$ such that for all $x \in A$, x has a private neighbor in some $C_i \subseteq N(A \setminus \{y\})$. For such graphs, we can enumerate $\mathcal{D}_{RN}(G)$ in polynomial delay by testing every candidate at each step to extend A using same techniques as in Section 4.

For future research, we are interested in structural properties of $\mathcal{D}_{RN}(G)$ and in an extension of the clique restriction on quasi-split to other simple class of graphs such as co-graphs, split graphs, etc.

7 Conclusion

In this paper, we investigated two different approaches to enumerate minimal dominating sets of a graph using structural properties based on neighborhood inclusion. We introduced preferred enumeration and showed that there is a polynomial delay algorithm to enumerate minimal dominating sets if there is one to enumerate preferred dominating sets. We then studied intersections of minimal dominating sets with redundant vertices and gave a linear delay algorithm to enumerate minimal dominating sets in a generalized class of split graphs whenever such intersections form an independent set system. We showed that such graphs include completed P_7 -free chordal graphs which improves results from [14] on P_6 -free chordal graphs.

We highlight two directions for future fundamental research in enumeration problems: (1) preferred enumeration may be of interest in some applications like data mining where the size of the output is usually exponential, (2) preferred enumeration may reveal structural properties of graphs and discrete structures to obtain more efficient algorithms. Indeed, more structural properties can be captured by the preference relation as shown in this paper for graphs with unique preferred dominating set.

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