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# OPTIMAL CONVERGENCE RATES FOR NESTEROV ACCELERATION

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**Abstract.** In this paper, we study the behavior of solutions of the ODE associated to Nesterov acceleration. It is well-known since the pioneering work of Nesterov that the rate of convergence  $O(t^2)$  is optimal for the class of convex functions. In this work, we show that better convergence rates can be obtained with some additional geometrical conditions, such as Łojasiewicz property. More precisely, we prove the optimal convergence rates that can be obtained depending on the geometry of the function  $F$  to minimize. The convergence rates are new, and they shed new light on the behavior of Nesterov acceleration schemes. We prove in particular that the classical Nesterov scheme may provide convergence rates that are worse than the classical gradient descent scheme on sharp functions: for instance, the convergence rate for strongly convex functions is not geometric for the classical Nesterov scheme (while it is the case for the gradient descent algorithm). This shows that applying the classical Nesterov acceleration on convex functions without looking more at the geometrical properties of the objective functions may lead to sub-optimal algorithms.

*Key-words.* Lyapunov functions, rate of convergence, ODEs, optimization, Łojasiewicz property.

**1. Introduction.** The motivation of this paper lies in the minimization of a differentiable function  $F$  with at least one minimizer. Inspired by Nesterov pioneering work [23], we study the following ODE

$$(1.1) \quad \ddot{x}(t) + \frac{\alpha}{t} \dot{x}(t) + \nabla F(x(t)) = 0$$

where  $\alpha > 0$ , with  $t_0 > 0$ ,  $x(t_0) = x_0$  and  $\dot{x}(t_0) = v_0$ . This ODE is associated to FISTA [10] or Accelerated Gradient Method [23] :

$$(1.2) \quad x_{n+1} = y_n - h \nabla F(y_n) \text{ and } y_n = x_n + \frac{n}{n + \alpha} (x_n - x_{n-1})$$

with  $h$  and  $\alpha$  positive parameters. This equation, including or not a perturbation term, has been widely studied in the literature [5, 25, 15, 9, 22]. This equation belongs to a set of similar equations with various viscosity terms. It is impossible to mention all works related to the heavy ball equation or other viscosity terms. We refer the reader to the following recent works [11, 19, 22, 16, 2, 24] and the references that can be found in these articles.

It was proved in [6] that if  $F$  is convex with Lipschitz gradient and if  $\alpha > 3$ , the trajectory  $F(x(t))$ , where  $x$  is the solution of (1.1), converges to the minimum  $F^*$  of  $F$ . It is also known that for  $\alpha \geq 3$  and  $F$  convex we have

$$(1.3) \quad F(x(t)) - F^* = O\left(\frac{1}{t^2}\right)$$

Extending to the continuous setting the proof of Chambolle et al. [17] of the convergence of iterates of FISTA, Attouch et al. [6] proved that for  $\alpha > 3$  the trajectory  $x$  weakly converges to a minimizer of  $F$ . Su et al. [25] proposed some new results, proving the integrability of  $t \mapsto t(F(x(t)) - F^*)$  when  $\alpha > 3$ , and they gave more accurate bounds on  $F(x(t)) - F^*$  in the

case of strong convexity. More recently several studies including a perturbation term [6, 7, 8, 26] have been proposed but always in the setting when  $\alpha \geq 3$ .

In this work, we focus on the decay of  $F(x(t)) - F^*$  depending on the local geometry of the function  $F$  around its set of minimizers. Our aim is to show the optimal convergence rates that can be obtained depending on this local geometry. The paper is organized as follows. In Section 2, we introduce the geometrical hypotheses we consider for the function  $F$ , and their relation with Lojasiewicz property. We then recap the state of the art results on the ODE (1.1) in Section 3. We present the contributions of the paper in Section 4: depending on the geometry of the function  $F$  and the value of the damping parameter  $\alpha$ , we give optimal rates of convergence. The proofs of the theorems are given in Section 5. Some technical proofs are postponed to Appendix A.

**2. Local geometry of convex functions.** In this section we introduce two notions describing the geometry of a convex function around its minimizers.

**DEFINITION 2.1.** *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex differentiable function with  $X^* := \operatorname{argmin} F \neq \emptyset$ . Let:  $F^* := \inf F$ .*

- (i) *Let  $\gamma \geq 1$ . The function  $F$  satisfies the hypothesis  $\mathbf{H}_1(\gamma)$  if, for any critical point  $x^* \in X^*$ , there exists  $\eta > 0$  such that:*

$$\forall x \in B(x^*, \eta), \quad 0 \leq F(x) - F^* \leq \frac{1}{\gamma} \langle \nabla F(x), x - x^* \rangle.$$

- (ii) *Let  $r \geq 1$ . The function  $F$  satisfies the growth condition  $\mathbf{H}_2(r)$  if for any critical point  $x^* \in X^*$ , there exists  $K > 0$  and  $\varepsilon > 0$ , such that:*

$$\forall x \in B(x^*, \varepsilon), \quad K d(x, X^*)^r \leq F(x) - F^*.$$

Note that any convex function automatically satisfies  $\mathbf{H}_1(1)$ , and that any convex function satisfying  $\mathbf{H}_1(\gamma)$  for some  $\gamma \geq 1$ , satisfies  $\mathbf{H}_1(\gamma')$  for any  $\gamma' \in [1, \gamma]$ . The same way, any function satisfying  $\mathbf{H}_2(r)$  for some  $r \geq 1$ , satisfies  $\mathbf{H}_2(r')$  for any  $r' \geq r$ .

The hypothesis  $\mathbf{H}_1(\gamma)$  is a mild assumption, requesting slightly more than the convexity of  $F$  in the neighborhood of its minimizers. In particular, we observe that any function  $F$  ensuring that  $(F - F^*)^{\frac{1}{\gamma}}$  is convex for some  $\gamma \geq 1$ , satisfies  $\mathbf{H}_1(\gamma)$ .

The hypothesis  $\mathbf{H}_2(r)$ ,  $r \geq 1$ , is a growth condition on the function  $F$  around any critical point. It is sometimes also called  $r$ -conditioning [18] or Hölderian error bounds [14]. This assumption is motivated by the fact that, when  $F$  is convex,  $\mathbf{H}_2(r)$  is equivalent to the famous Lojasiewicz inequality [20, 21], a key tool in the mathematical analysis of continuous (or discrete) dynamical systems, with exponent  $\theta = 1 - \frac{1}{r}$  [12, 13]:

**DEFINITION 2.2.** *A proper differentiable function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to have the Lojasiewicz property with exponent  $\theta \in [0, 1)$  if, for any critical point  $x^*$ , there exist  $c > 0$  and  $\varepsilon > 0$  such that:*

$$(2.1) \quad \forall x \in B(x^*, \varepsilon), \quad \|\nabla F(x)\| \geq c |F(x) - F^*|^\theta.$$

where:  $0^0 = 0$  when  $\theta = 0$  by convention.

Observe that the inequality (2.1) is automatically satisfied at any non-critical point, so that the Lojasiewicz property is in fact a geometrical condition on the function  $F$  around any critical points. Moreover, when the set  $X^*$  of the minimizers is a connected compact set, the Lojasiewicz inequality turns into a geometrical condition on  $F$  around  $X^*$  as stated in [1, Lemma 1] whose proof can be adapted to establish the following result:

LEMMA 2.3. *Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex differentiable function satisfying the growth condition  $\mathbf{H}_2(r)$  for some  $r \geq 1$ . Assume that the set  $X^* = \operatorname{argmin} F$  is compact. Then there exist  $K > 0$  and  $\varepsilon > 0$  such that, for all  $x \in \mathbb{R}^n$ :*

$$d(x, X^*) \leq \varepsilon \Rightarrow Kd(x, X^*)^r \leq F(x) - F^*.$$

Typical examples of functions having the Lojasiewicz property are real-analytic functions and  $C^1$  subanalytic functions [20], or semialgebraic functions [1]. Strongly convex functions satisfy a global Lojasiewicz property with exponent  $\theta = \frac{1}{2}$  [1], or equivalently a global version of the hypothesis  $\mathbf{H}_2(2)$ , namely:

$$\forall x \in \mathbb{R}^n, F(x) - F^* \geq \frac{\mu}{2} d(x, X^*)^2,$$

where  $\mu > 0$  denotes the parameter of strong convexity. By extension, uniformly convex functions of order  $p \geq 2$  satisfy the global version of the hypothesis  $\mathbf{H}_2(p)$  [18].

Let us now present some examples of convex differentiable functions to illustrate situations where the hypothesis  $\mathbf{H}_1$  and  $\mathbf{H}_2$  are satisfied.

We start with the simple example of the function  $F : x \in \mathbb{R} \mapsto |x|^\gamma$  for some  $\gamma > 1$  as illustrated on Figure 1 with  $\gamma = \frac{3}{2}$ . We now introduce the family of straight lines passing through the point  $(x, F(x))$  and defined by:

$$(2.2) \quad T_{\gamma'}(y, x) = F(x) + \frac{1}{\gamma'} \langle \nabla F(x), y - x \rangle.$$

as represented in blue in Figure 1. Note that  $T_1(\cdot, x)$  is exactly the tangent to  $F$  at the point  $(x, F(x))$ . The function  $F : x \mapsto |x|^\gamma$  satisfies the hypothesis  $\mathbf{H}_1(\gamma')$  for some  $\gamma' \geq 1$  if, by definition, at any point  $x$  in a neighborhood of the unique minimizer  $x^* = 0$ , we have:

$$F(x) - F^* \leq \frac{1}{\gamma'} (F(x) - T_1(x^*, x)),$$

or equivalently:

$$F(x) - F^* \leq F(x) - T_{\gamma'}(x^*, x).$$

In other words, as illustrated in Figure 1, the hypothesis  $\mathbf{H}_1(\gamma')$  is satisfied for some  $\gamma' \geq 1$  if the distance between  $F(x)$  and  $F^*$  remains less than a constant fraction of the distance between  $F(x)$  and  $T_1(x^*)$  (in blue in Figure 1), or equivalently if:

$$T_{\gamma'}(x^*, x) \leq F^*$$

in the neighborhood of  $x^* = 0$ . Thus,  $F : x \in \mathbb{R} \mapsto |x|^\gamma$  satisfies  $\mathbf{H}_1(\gamma')$  as long as  $T_{\gamma'}(x^*) \leq F^*$ . Here we easily check that  $F : x \in \mathbb{R} \mapsto |x|^\gamma$  satisfies  $\mathbf{H}_1(\gamma')$  if and only if  $\gamma' \in [1, \gamma]$ .

The growth condition  $\mathbf{H}_2(r)$  applied to  $F$  implies that, in the neighborhood of its minimizer  $x^*$ ,  $F$  should behave at least as  $x \mapsto |x|^r$ . By definition of  $F$ ,  $F$  naturally satisfies  $\mathbf{H}_2(r)$  if and only if  $r \geq \gamma$ .

In the same way, consider now for example the function:

$$(2.3) \quad F(x) = \begin{cases} |x - 1|^\gamma & \text{if } |x| \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

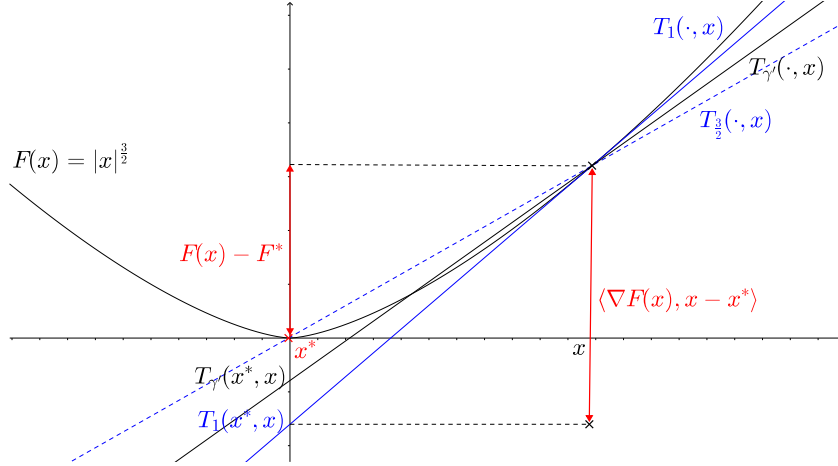


FIG. 1. The function  $F : x \mapsto |x|^{\frac{3}{2}}$  satisfies  $\mathbf{H}_1(\gamma')$  and  $\mathbf{H}_2(r)$  if and only if  $1 \leq \gamma' \leq \gamma = \frac{3}{2}$  and  $r \geq \frac{3}{2}$ .

whose set of minimizers is:  $X^* = [-1, 1]$  as represented in red in Figure 2 with  $\gamma = 3$ . Introducing the family of straight lines parameterized by  $\gamma' \geq 1$  and defined by (2.2), the hypothesis  $\mathbf{H}_1(\gamma')$  can be rewritten as follows:

$$\forall x^* \in X^*, F(x) - F^* \leq \frac{1}{\gamma'}(F(x) - T_1(x^*, x)),$$

or equivalently:

$$\forall x^* \in X^*, T_{\gamma'}(x^*, x) \leq F^*,$$

for any  $x$  in some neighborhood of  $X^*$ . As illustrated on Figure 2, the extremal points ( $x^* = 1$  and  $x^* = -1$  in this example) of  $X^*$  are the most restrictive points  $x^*$  in the previous condition in the sense that  $F$  has the  $\mathbf{H}_1(\gamma')$  property if the associated inequality is satisfied just for  $x^* = \pm 1$ . A straightforward computation shows that, in this example,  $F$  satisfies  $\mathbf{H}_1(\gamma')$  if and only if  $\gamma' \in [1, \bar{\gamma}]$ .

In the same way, the growth condition  $\mathbf{H}_2(r)$  only makes sense in the neighborhood of the set of minimizers  $X^*$  as expressed by Lemma 2.3:  $F$  should behave at least as  $|x|^r$  in the neighborhood of the extremal points  $x^* = -1$  and  $x^* = 1$  of  $X^*$ . Hence the function  $F$  defined by (2.3) satisfies  $\mathbf{H}_2(r)$  if and only if  $r \geq \gamma$ .

Finally, consider a convex differentiable function  $F$  with a  $L$ -Lipschitz continuous gradient for some  $L > 0$ . Such a function automatically satisfies the hypothesis  $\mathbf{H}_1(\gamma)$  with  $\gamma \in ]1, 2]$  provided that it has the Łojasiewicz property with exponent  $\theta = \frac{1}{2}$ , or equivalently satisfies the growth condition  $\mathbf{H}_2(2)$ . Indeed, since  $F$  is convex with Lipschitz continuous gradient, we have:

$$\forall (x, y) \in \mathbb{R}^n, F(y) - F(x) - \langle \nabla F(x), y - x \rangle \geq \frac{1}{2L} \|\nabla F(y) - \nabla F(x)\|^2,$$

hence:

$$\forall x \in \mathbb{R}^n, F(x) - F^* \leq \langle \nabla F(x), x - x^* \rangle - \frac{1}{2L} \|\nabla F(x)\|^2.$$

Assuming in addition that  $F$  has the Łojasiewicz property with exponent  $\theta = \frac{1}{2}$ , or equivalently that  $F$  satisfies  $\mathbf{H}_2(2)$ , we get:

$$\left(1 + \frac{c^2}{2L}\right) |F(x) - F^*| \leq \langle \nabla F(x), x - x^* \rangle,$$

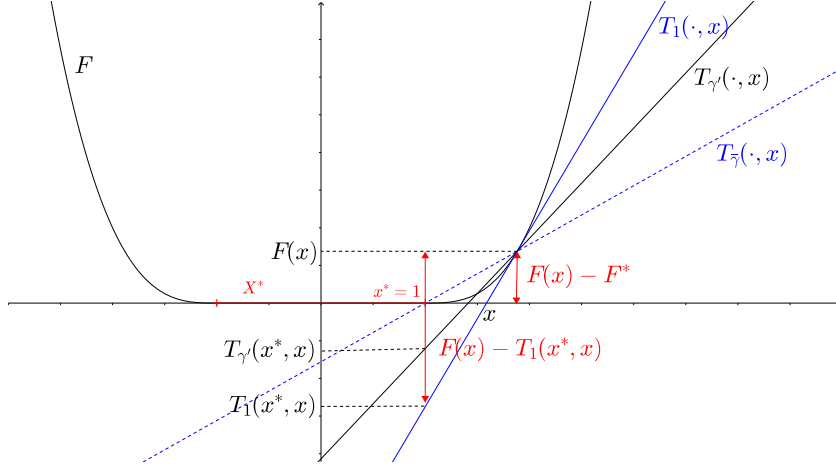


FIG. 2. The function  $F$  defined by (2.3) with  $\gamma = 3$  satisfies  $\mathbf{H}_1(\gamma')$  and  $\mathbf{H}_2(r)$  iff  $\gamma' \in [1, \gamma]$  and  $r \geq 3$ .

in the neighborhood of its minimizers. In other words,  $F$  satisfies the hypothesis  $\mathbf{H}_1(\gamma)$  with  $\gamma = 1 + \frac{c^2}{2L} > 1$ .

REMARK 1. If a convex differentiable function  $F$  has both a  $L$ -Lipschitz continuous gradient and the Lojasiewicz property with exponent  $\theta$  and constant  $c > 0$ , then necessarily:  $\theta \in [\frac{1}{2}, 1]$ . In addition, if  $\theta = \frac{1}{2}$ , then:  $2L \geq c^2$ .

**3. Related results.** In this section, we recall some classical state of the art results related to the ODE (1.1).

Let us first recall that as soon as  $\alpha > 0$ , then  $F(x(t))$  converges to  $F^*$  [7, 4]. As recalled in Section 1, a larger value of  $\alpha$  is required to show the convergence of the trajectory  $x(t)$ . Nevertheless, the convergence result on  $F(x(t))$  will prove to be useful in several proofs of this paper.

The  $\mathbf{H}_1(\gamma)$  hypothesis was first introduced in [7]. If  $F$  satisfies  $\mathbf{H}_1(\gamma)$  and  $\alpha > 1 + \frac{2}{\gamma}$  then

$$(3.1) \quad F(x(t)) - F^* = o\left(\frac{1}{t^2}\right),$$

and the trajectory  $x(t)$  (weakly in an infinite dimensional space) converges to a minimizer  $x^*$  of  $F$  [25, 7, 4]. This means that thanks to the additional hypothesis  $\mathbf{H}_1(\gamma)$ , the damping parameter  $\alpha$  can be chosen smaller.

Now, if  $F$  satisfies  $\mathbf{H}_1(\gamma)$  and  $\alpha \leq 1 + \frac{2}{\gamma}$ , then we can no longer prove the convergence of the trajectory  $x(t)$  but we still have the following convergence rate for  $F(x(t))$ :

$$(3.2) \quad F(x(t)) - F^* = O\left(\frac{1}{t^{\frac{2\gamma\alpha}{2+\gamma}}}\right).$$

Moreover, this decay is optimal and reached for  $F(x) = |x|^\gamma$  if  $\gamma \geq 1$  (see [4] for the case  $\gamma = 1$  in  $\mathbb{R}$  and [7] for the general case and the optimality of the rate). Notice that for convex functions, the decay is  $O\left(\frac{1}{t^{\frac{2\gamma}{3}}}\right)$ . We can also notice that the bound hidden in the big  $O$  is explicit and available also for  $\gamma < 1$ , that is for non convex functions (for example for functions whose square is convex).

If  $F$  is the function  $F(x) = |x|^\gamma$ , where  $x \in \mathbb{R}$ , with  $\gamma > 2$  and  $\alpha \geq \frac{\gamma+2}{\gamma-2}$ , then the ODE (1.1) admits an explicit solution of the form  $x(t) = \frac{K}{t^{\frac{2}{\gamma-2}}}$  [4]. This simple calculation will show the optimality of the convergence rate we obtain in Theorem 4.3.

Moreover, if  $F$  is strongly convex, then for any  $\alpha > 0$  we have [25]

$$(3.3) \quad F(x(t)) - F^* = O\left(\frac{1}{t^{\frac{2\alpha}{3}}}\right).$$

This shows that by assuming more on the geometry of the function  $F$  (in this case, strong convexity), better rates of convergence can be achieved.

Eventually several results about the convergence rate of the solutions of ODE associated to the classical gradient descent :

$$(3.4) \quad \dot{x}(t) + \nabla F(x(t)) = 0$$

or the ODE associated to the heavy ball method

$$(3.5) \quad \ddot{x} + \alpha \dot{x}(t) + \nabla F(x(t)) = 0$$

under geometrical conditions such that Łojasiewicz properties have been proposed, see for example Polyak et al. [24]. The authors prove that if the function  $F$  satisfies  $\mathbf{H}_2(2)$  and some other conditions, the decay of  $F(x(t)) - F^*$  is exponential for the solutions of both previous equations. These rates are the continuous counterparts of the exponential decay rate of the classical gradient descent algorithm and the heavyball method algorithm for strongly convex functions.

In the next section we will prove that this geometric rate is not true for solutions of (1.1) even for quadratic functions, and we will prove that from an optimization point of view, the classical Nesterov acceleration may be less efficient than the classical gradient descent.

**4. Contributions.** In this section, we state the optimal convergence rates that can be achieved when  $F$  satisfies hypotheses such as  $\mathbf{H}_1(\gamma)$  and/or  $\mathbf{H}_2(r)$ . The first result gives optimal control for functions whose geometry is sharp :

**THEOREM 4.1.** *Let  $\gamma \geq 1$  and  $\alpha > 0$ .*

1. *If  $F$  satisfies the hypothesis  $\mathbf{H}_1(\gamma)$  and if  $\alpha \leq 1 + \frac{2}{\gamma}$ , then*

$$(4.1) \quad F(x(t)) - F^* = O\left(\frac{1}{t^{\frac{2\gamma\alpha}{\gamma+2}}}\right)$$

2. *If  $F$  satisfies the hypotheses  $\mathbf{H}_1(\gamma)$  and  $\mathbf{H}_2(2)$  and if  $F$  has a unique minimizer, if  $\alpha > 1 + \frac{2}{\gamma}$  then*

$$(4.2) \quad F(x(t)) - F^* = O\left(\frac{1}{t^{\frac{2\gamma\alpha}{\gamma+2}}}\right)$$

Moreover this decay is optimal in the sense that for any  $\gamma \in (1, 2]$  this rate is achieved for the function  $F(x) = |x|^\gamma$ .

Note that the first point of Theorem 4.1 is already proven in [7]. The optimality of convergence rate result is precisely stated in the next Proposition:

**PROPOSITION 4.2.** *Let  $\gamma \in (1, 2]$ . Let us assume that  $\alpha > 1 + \frac{2}{\gamma}$ . Let  $x$  be a solution of (1.1) with  $F(x) = |x|^\gamma$  with  $t_0 > \sqrt{|\frac{\xi}{2}|}$ ,  $|x(t_0)| < 1$  and  $v_0 = 0$ . There exists  $K > 0$  such that for any  $T > 0$ , there exists  $t \geq T$  such that*

$$(4.3) \quad F(x(t)) - F^* \geq \frac{K}{t^{\frac{2\gamma\alpha}{\gamma+2}}}.$$

Let us make several observations: first, to apply the second point of Theorem 4.1, more conditions are needed than for the first point: the hypothesis  $\mathbf{H}_2(2)$  and the uniqueness of the minimizer are needed (only for the second point) to prove a decay faster than  $O(\frac{1}{t^2})$ , which is the uniform rate than can be achieved with  $\alpha \geq 3$  for convex functions [25]. The uniqueness of the minimizer in the second point is crucial in the proof, but it is still an open problem to know if this uniqueness is a necessary condition.

Note that the second point of Theorem 4.1 mostly applies for  $\gamma \leq 2$ , since we did not find any function satisfying both conditions  $\mathbf{H}_1(\gamma)$  with  $\gamma > 2$  and  $\mathbf{H}_2(2)$ .

We can remark that if  $F$  is a quadratic function in the neighborhood of  $x^*$ , Theorem 4.1 applies with  $\gamma = 2$  and thus

$$(4.4) \quad F(x(t)) - F^* = O\left(\frac{1}{t^\alpha}\right)$$

which is better than  $O\left(\frac{1}{t^{\frac{2}{3}}}\right)$  that is proposed for strongly convex functions by Su et al. [25] and Attouch et al. [6].

Finally, we can remark that the decay for quadratic or strongly convex functions is not geometric, while it is the case for the classical gradient descent scheme (see e.g. [18]). This shows that applying the classical Nesterov acceleration on convex functions without looking more at the geometrical properties of the objective functions may lead to sub-optimal algorithms.

The second result deals with geometries associated to  $\gamma > 2$ .

**THEOREM 4.3.** *Let  $\gamma_1 > 2$  and  $\gamma_2 > 2$ , if  $F$  is coercive and satisfies  $\mathbf{H}_1(\gamma_1)$  and  $\mathbf{H}_2(\gamma_2)$ :*

1. *If  $\gamma_1 \leq \gamma_2$  and  $\alpha \geq \frac{\gamma_1+2}{\gamma_1-2}$  then we have*

$$(4.5) \quad F(x(t)) - F^* = O\left(\frac{1}{t^{\frac{2\gamma_2}{\gamma_2-2}}}\right).$$

2. *If  $\gamma_2 \leq \gamma_1$  then for any  $\gamma \in [\gamma_2, \gamma_1]$ , if  $\alpha \geq \frac{\gamma+2}{\gamma-2}$  then we have*

$$(4.6) \quad F(x(t)) - F^* = O\left(\frac{1}{t^{\frac{2\gamma}{\gamma-2}}}\right).$$

One can notice that in Theorem 4.3 the uniqueness of the minimizer is not needed anymore.

In the case when  $\gamma_1 = \gamma_2$ , we have furthermore the convergence of the trajectory:

**COROLLARY 4.4.** *Let  $\gamma > 2$ , if  $F$  is coercive and satisfies  $\mathbf{H}_1(\gamma)$  and  $\mathbf{H}_2(\gamma)$  and if  $\alpha \geq \frac{\gamma+2}{\gamma-2}$  then we have*

$$(4.7) \quad F(x(t)) - F^* = O\left(\frac{1}{t^{\frac{2\gamma}{\gamma-2}}}\right)$$

and

$$(4.8) \quad \|\dot{x}(t)\| = O\left(\frac{1}{t^{\frac{\gamma}{\gamma-2}}}\right).$$

Moreover the trajectory  $x(t)$  is finite and it converges to a minimizer  $x^*$  of  $F$ .

This decay is optimal since Attouch et al. proved that it is achieved for the function  $F(x) = |x|^\gamma$  in [3].



*Comments on the hypotheses  $\mathbf{H}_1$  and  $\mathbf{H}_2$ .* From these two theorems we can make the following comments:

1. If the function  $F$  behaves like  $\|x - x^*\|^\gamma$  in the neighborhood of its unique minimizer  $x^*$ , then the decay of  $F(x(t)) - F^*$  depends directly on  $\alpha$  if  $\gamma \leq 2$ , but it does not depend on  $\alpha$ , for large  $\alpha$  if  $\gamma > 2$ .
2. For such functions the best decay rate of  $F(x(t)) - F^*$  is achieved for  $\gamma = 2$ , that is for quadratic like functions around the minimizer, which is  $O\left(\frac{1}{t^\alpha}\right)$ . If  $\gamma < 2$ , it seems that the oscillations of the solution  $x(t)$  prevent us to get an optimal decay rate. The inertia seems to be too large for such functions. If  $\gamma > 2$ , for large  $\alpha$ , the decay is not as fast because the gradient of the functions decays too fast in the neighborhood of the minimizer. For these functions a larger inertia could be more efficient.
3. In the second point of Theorem 4.1 and in Theorem 4.3 both conditions  $\mathbf{H}_1$  and  $\mathbf{H}_2$  are used to get a decay rate. It turns out that these two conditions are important. Condition  $\mathbf{H}_1(\gamma)$  ensures that the function is not too sharp and it may prevent from bad oscillations of the solution, while condition  $\mathbf{H}_2(\gamma)$  ensures that the magnitude of the gradient of the function is not too low in the neighborhood of the minimizers.
4. With the sole condition  $\mathbf{H}_1(\gamma)$  on  $F$  for any  $\gamma \geq 1$ , it is impossible to get a bound on the decay rate like  $O\left(\frac{1}{t^\delta}\right)$  with  $\delta > 2$ . Indeed, for any  $\eta > 2$  and for a large friction parameter  $\alpha$ , the solution  $x$  of the ODE associated to  $F(x) = |x|^\eta$  satisfies  $F(x(t)) - F^* = Kt^{-\frac{2\eta}{\eta-2}}$  and the power  $\frac{2\eta}{\eta-2}$  can be chosen arbitrary close to 2.
5. With the sole hypothesis  $\mathbf{H}_2(\gamma)$ , it seems difficult to establish optimal rate. Indeed the function  $F(x) = |x|^3$  satisfies  $\mathbf{H}_2(3)$ . Applying Theorem 4.3 with  $\gamma_1 = \gamma_2 = 3$ , we know that for this function with  $\alpha = \frac{\gamma_1+2}{\gamma_1-2} = 5$ , we have  $F(x(t)) - F^* = O\left(\frac{1}{t^6}\right)$ . Nevertheless, with the sole hypothesis  $\mathbf{H}_2(3)$ , such a decay cannot be achieved. Indeed, the function  $F(x) = |x|^2$  satisfies  $\mathbf{H}_2(3)$ , but from the optimality part of Theorem 4.1 we know that we cannot achieve a decay better than  $\frac{1}{t^{\frac{2\alpha\gamma}{\gamma+2}}} = \frac{1}{t^5}$  for  $\alpha = 5$ .
6. It seems that the use of both hypotheses is a simple way to provide optimal decay rates.

**5. Proofs.** Both proofs rely on Lyapunov functions  $\mathcal{E}$  and  $\mathcal{H}$  introduced by Su et al. [25] Attouch et al. [6] and Aujol et al. [7] :

$$(5.1) \quad \mathcal{E}_{\lambda,\xi}(t) = t^2(F(x(t)) - F^*) + \frac{1}{2} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2 + \frac{\xi}{2} \|x(t) - x^*\|^2.$$

where  $x^*$  is a minimizer of  $F$  and  $\lambda$  and  $\xi$  are two real numbers. The function  $\mathcal{H}$  is defined from  $\mathcal{E}$  and it depends on another real parameter  $p$  :

$$(5.2) \quad \mathcal{H}(t) = t^p \mathcal{E}(t)$$

Using the following notations

$$(5.3) \quad a(t) = t(F(x(t)) - F^*)$$

$$(5.4) \quad b(t) = \frac{1}{2t} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2$$

$$(5.5) \quad c(t) = \frac{1}{2t} \|x(t) - x^*\|^2$$

we have

$$(5.6) \quad \mathcal{E}(t) = t(a(t) + b(t) + \xi c(t))$$

From now on we will choose

$$(5.7) \quad \xi = \lambda(\lambda + 1 - \alpha)$$

and we will use the following Lemma whose proof is postponed to Appendix A:

LEMMA 5.1. *If  $F$  satisfies the hypothesis  $\mathbf{H}_1(\gamma)$ , for any  $\gamma > 0$  and if  $\xi = \lambda(\lambda - \alpha + 1)$  then*

$$(5.8) \quad \mathcal{H}'(t) \leq t^p ((2 - \gamma\lambda + p)a(t) + (2\lambda + 2 - 2\alpha + p)b(t) + \lambda(\lambda + 1 - \alpha)(-2\lambda + p)c(t))$$

One can remark that this inequality is actually an equality for the specific choice  $F(x) = |x|^\gamma$ .

**5.1. Proof of Theorem 4.1.** In this section we prove the second point of the Theorem and we refer to [7] for a complete proof of the first point, including the optimality of the rate. The proof of this first point is actually similar to the following one but simpler. The choice of  $p$  and  $\lambda$  are the same, but due to the value of  $\alpha$ , the function  $\mathcal{H}$  is non increasing and sum of non negative terms, which simplifies the analysis and necessitates less hypotheses to conclude.

*Proof.* We choose here  $p = \frac{2\gamma\alpha}{\gamma+2} - 2$  and  $\lambda = \frac{2\alpha}{\gamma+2}$  and thus

$$(5.9) \quad \xi = \frac{2\alpha}{(\gamma+2)^2}(2 + \gamma(1 - \alpha))$$

Since we consider the case when  $\alpha > 1 + \frac{2}{\gamma}$ , we have  $\xi < 0$  and thus  $\mathcal{H}$  is not defined as a sum of positive functions.

Moreover, from Lemma 5.1 it appears that

$$(5.10) \quad \mathcal{H}'(t) \leq K_1 t^p c(t) \quad \text{with } K_1 = \lambda(\lambda + 1 - \alpha)(-2\lambda + p).$$

Let us compute explicitly the value of  $K_1$ . We have  $p = \frac{2\gamma\alpha}{\gamma+2} - 2$  and  $\lambda = \frac{2\alpha}{\gamma+2}$ , so that:

$$\begin{aligned} K_1 &= \frac{2\alpha}{\gamma+2} \left( \frac{2\alpha}{\gamma+2} + 1 - \alpha \right) \left( -2\frac{2\alpha}{\gamma+2} + \frac{2\gamma\alpha}{\gamma+2} - 2 \right) \\ &= \frac{4\alpha}{(\gamma+2)^3} (2\alpha + \gamma + 2 - \alpha\gamma - 2\alpha) (-2\alpha + \gamma\alpha - \gamma - 2) \\ &= \frac{4\alpha}{(\gamma+2)^3} (\gamma + 2 - \alpha\gamma) (\alpha(-2 + \gamma) - \gamma - 2) \end{aligned}$$

Hence

$$(5.11) \quad K_1 = \frac{4\alpha(\gamma(1 - \alpha) + 2)(\alpha(\gamma - 2) - (\gamma + 2))}{(\gamma + 2)^3}$$

Since  $\alpha > 1 + \frac{2}{\gamma}$ , we have  $\gamma(1 - \alpha) + 2 < 0$ . Hence the sign of  $K_1$  is the opposit of the sign of  $\alpha(\gamma - 2) - (\gamma + 2)$ . More precisely, if  $\gamma \leq 2$ , then  $\alpha(\gamma - 2) - (\gamma + 2) < 0$  and thus  $K_1 > 0$ . If  $\gamma > 2$ , then  $\alpha(\gamma - 2) - (\gamma + 2) > 0$  iff  $\alpha > \frac{\gamma+2}{\gamma-2}$ . Hence  $K_1 > 0$  iff  $\alpha < \frac{\gamma+2}{\gamma-2}$ .

Using Hypothesis  $\mathbf{H}_2(2)$  and the uniqueness of the minimizer, there exists  $K > 0$  such that

$$(5.12) \quad Kt \|x(t) - x^*\|^2 \leq t(F(x(t)) - F^*) = a(t),$$

and thus

$$(5.13) \quad c(t) \leq \frac{1}{2Kt^2} a(t).$$

Since  $\xi < 0$  with our choice of parameters, we get:

$$\mathcal{H}(t) \geq t^{p+1}(a(t) + \xi c(t)) \geq t^{p+1}\left(1 + \frac{\xi}{2Kt^2}\right)a(t).$$

It follows that it exists  $t_1$  such that for all  $t \geq t_1$ ,  $\mathcal{H}(t) \geq 0$  and

$$(5.14) \quad \mathcal{H}(t) \geq \frac{1}{2}t^{p+1}a(t).$$

From (5.10), (5.13) and (5.14), we get

$$(5.15) \quad \mathcal{H}'(t) \leq \frac{K_1}{K} \frac{\mathcal{H}(t)}{t^3}$$

From the Gronwall Lemma it exists  $A > 0$  such that  $\forall t \geq t_1$   $\mathcal{H}(t) \leq A$ . According to (5.14), we then conclude that  $t^{p+2}(F(x(t)) - F^*) = t^{p+1}a(t)$  is bounded which concludes the proof of the point 2. of Theorem 4.1.

□

**5.2. Proof of Proposition 4.2 (Optimality of the convergence rates).** Before proving the optimality of the convergence rate stated in Proposition 4.2, we need the following technical lemma:

LEMMA 5.2. *Let  $y$  a continuously differentiable function with values in  $\mathbb{R}$ . Let  $T > 0$  and  $\epsilon > 0$ . If  $y$  is bounded, then there exists  $t_1 > T$  such that:*

$$(5.16) \quad |\dot{y}(t_1)| \leq \frac{\epsilon}{t_1}$$

*Proof.* We split the proof into two cases.

1. There exists  $t_1 > T$  such that  $\dot{y}(t_1) = 0$ .
2.  $\dot{y}(t)$  is of constant sign for  $t > T$ . For instance we assume  $\dot{y}(t) > 0$ . By contradiction, let us assume that  $\dot{y}(t) > \frac{\epsilon}{t} \forall t > T$ . Then  $y(t)$  cannot be a bounded function as assumed.

□

Let us now prove the Proposition 4.2: the idea of the proof is the following: we first show that  $\mathcal{H}$  is bounded from below. Since  $\mathcal{H}$  is a sum of 3 terms including the term  $F - F^*$ , we then show that given  $t_1 \geq t_0$ , there always exists a time  $t \geq t_1$  such that the value of  $\mathcal{H}$  is concentrated on the term  $F - F^*$ .

We start the proof by using the fact that, for the function  $F(x) = |x|^\gamma$ , the inequality of Lemma 5.1 is actually an equality, and using the values  $p = \frac{2\gamma\alpha}{\gamma+2} - 2$  and  $\lambda = \frac{2\alpha}{\gamma+2}$  of the previous Theorem, we have a closed form for the derivative of function  $\mathcal{H}$ :

$$(5.17) \quad \mathcal{H}'(t) = K_1 t^p c(t) = \frac{K_1}{2} t^{p-1} |x(t)|^2$$

where  $K_1$  is a positive constant. This implies in particular that  $\mathcal{H}$  is non-decreasing.

From the previous Theorem,  $\mathcal{H}$  is bounded above. Observing that under the hypothesis of the Proposition 4.2, we have:  $\mathcal{H}(t_0) > 0$ , it follows that it exists  $\ell > 0$  such that  $\mathcal{H}(t) \rightarrow \ell$  when  $t \rightarrow +\infty$ . In particular, we have that for  $t$  large enough,

$$(5.18) \quad \mathcal{H}(t) \geq \frac{\ell}{2}$$

i.e.

$$(5.19) \quad a(t) + b(t) + \xi c(t) \geq \frac{\ell}{2t^{p+1}}$$

Moreover, since  $\xi < 0$ , we then have:

$$(5.20) \quad a(t) + b(t) \geq \frac{\ell}{2t^{p+1}}$$

Let  $T > 0$  and  $\epsilon > 0$ . We set:

$$(5.21) \quad y(t) := t^\lambda x(t)$$

where:  $\lambda = \frac{2\alpha}{\gamma+2}$ . From the previous Theorem, we know that  $y(t)$  is bounded. Hence, from Lemma 5.2, there exists  $t_1 > T$  such that

$$(5.22) \quad |\dot{y}(t_1)| \leq \frac{\epsilon}{t_1}$$

But

$$(5.23) \quad \dot{y}(t) = t^{\lambda-1} (\lambda x(t) + t\dot{x}(t))$$

Hence using (5.22):

$$(5.24) \quad t_1^\lambda |\lambda x(t_1) + t_1 \dot{x}(t_1)| \leq \epsilon$$

We remind the reader that  $b(t) = \frac{1}{2t} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2$ . We thus have:

$$(5.25) \quad b(t_1) \leq \frac{\epsilon^2}{2t_1^{2\lambda+1}}$$

Since  $\gamma \leq 2$ ,  $\lambda = \frac{2\alpha}{\gamma+2}$  and  $p = \frac{2\gamma\alpha}{\gamma+2} - 2$ , we have  $2\lambda + 1 \geq p + 1$ , and thus

$$(5.26) \quad b(t_1) \leq \frac{\epsilon^2}{2t_1^{p+1}}.$$

For  $\epsilon = \sqrt{\frac{\ell}{2}}$  for example, there exists thus some  $t_1 > T$  such that  $b(t_1) \leq \frac{\ell}{4t_1^{p+1}}$ . Then  $a(t_1) \geq \frac{\ell}{4t_1^{p+1}}$ , i.e.  $F(x(t_1)) - F^* \geq \frac{\ell}{4t_1^{p+2}}$ . Since  $p + 2 = \frac{2\gamma\alpha}{\gamma+2}$ , this concludes the proof.

**5.3. Proof of Theorem 4.3.** We detail here the proof of Theorem 4.3. We split the proof into two parts, depending on the sign of  $(\gamma_1 - \gamma_2)$ .

*Case  $\gamma_1 \leq \gamma_2$ .* Let us consider  $\gamma_1 > 2$ ,  $\gamma_2 > 2$ , and  $\alpha \geq \frac{\gamma_1+2}{\gamma_1-2}$ .

We consider here functions  $\mathcal{H}$  for all  $x^*$  in the set  $X^*$  of minimizers of  $F$  and prove that these functions are uniformly bounded. More precisely for any  $x^* \in X^*$  we define  $\mathcal{H}(t)$  with  $p = \frac{4}{\gamma_1-2}$  and  $\lambda = \frac{2}{\gamma_1-2}$ . With this choice of  $\lambda$  and  $p$ , using Hypothesis  $\mathbf{H}_1(\gamma_1)$  we have from Lemma 5.1:

$$(5.27) \quad \mathcal{H}'(t) \leq 2t^{\frac{4}{\gamma_1-2}} \left( \frac{\gamma_1+2}{\gamma_1-2} - \alpha \right) b(t).$$

which is non positive when  $\alpha \geq \frac{\gamma_1+2}{\gamma_1-2}$ , which implies that the function  $\mathcal{H}$  is bounded above. Hence for any choice of  $x^*$  in the set of minimizers  $X^*$ , the function  $\mathcal{H}$  is bounded above and since the set of minimizers is bounded ( $F$  is coercive), there exists  $A > 0$  and  $t_0$  such that for all choices of  $x^*$  in  $X^*$ ,

$$(5.28) \quad \mathcal{H}(t_0) \leq A,$$

which implies that for all  $x^* \in X^*$  and for all  $t \geq t_0$

$$(5.29) \quad \mathcal{H}(t) \leq A.$$

Hence for all  $t \geq t_0$  and for all  $x^* \in X^*$

$$(5.30) \quad t^{\frac{4}{\gamma_1-2}} t^2 (F(x(t)) - F^*) \leq \frac{|\xi|}{2} t^{\frac{4}{\gamma_1-2}} \|x(t) - x^*\|^2 + A,$$

which implies that

$$(5.31) \quad t^{\frac{4}{\gamma_1-2}} t^2 (F(x(t)) - F^*) \leq \frac{|\xi|}{2} t^{\frac{4}{\gamma_1-2}} d(x(t), X^*)^2 + A.$$

We now set:

$$(5.32) \quad v(t) := t^{\frac{4}{\gamma_2-2}} d(x(t), X^*)^2$$

where  $d$  denotes the euclidean distance. Using (5.31) we have:

$$(5.33) \quad t^{\frac{2\gamma_1}{\gamma_1-2}} (F(x(t)) - F^*) \leq \frac{|\xi|}{2} t^{\frac{4}{\gamma_1-2} - \frac{4}{\gamma_2-2}} v(t) + A.$$

Using the hypothesis  $\mathbf{H}_2(\gamma_2)$  applied under the form given by Lemma 2.3 (since  $X^*$  is compact), there exists  $K > 0$  such that

$$(5.34) \quad K \left( t^{-\frac{4}{\gamma_2-2}} v(t) \right)^{\frac{\gamma_2}{2}} \leq F(x(t)) - F^*,$$

which is equivalent to

$$(5.35) \quad K v(t)^{\frac{\gamma_2}{2}} t^{\frac{-2\gamma_2}{\gamma_2-2}} \leq F(x(t)) - F^*.$$

Hence:

$$(5.36) \quad K t^{\frac{2\gamma_1}{\gamma_1-2} - \frac{-2\gamma_2}{\gamma_2-2}} v(t)^{\frac{\gamma_2}{2}} \leq t^{\frac{2\gamma_1}{\gamma_1-2}} (F(x(t)) - F^*).$$

Using (5.33), we obtain:

$$(5.37) \quad K t^{\frac{2\gamma_1}{\gamma_1-2} - \frac{-2\gamma_2}{\gamma_2-2}} v(t)^{\frac{\gamma_2}{2}} \leq \frac{|\xi|}{2} t^{\frac{4}{\gamma_1-2} - \frac{4}{\gamma_2-2}} v(t) + A,$$

i.e.:

$$(5.38) \quad K v(t)^{\frac{\gamma_2}{2}} \leq \frac{|\xi|}{2} v(t) + A t^{\frac{4}{\gamma_2-2} - \frac{4}{\gamma_1-2}}.$$

Since  $\gamma_1 \leq \gamma_2$ , we deduce that  $v$  is bounded. Hence, using (5.33) there exists some positive constant  $B$  such that:

$$(5.39) \quad F(x(t)) - F^* \leq B t^{\frac{-2\gamma_2}{\gamma_2-2}} + A t^{\frac{-2\gamma_1}{\gamma_1-2}}.$$

Since  $\gamma_1 \leq \gamma_2$ , we have  $\frac{-2\gamma_2}{\gamma_2-2} \geq \frac{-2\gamma_1}{\gamma_1-2}$ . Hence we deduce that  $F(x(t)) - F^* = O\left(t^{\frac{-2\gamma_2}{\gamma_2-2}}\right)$ .

Case  $\gamma_2 \leq \gamma_1$ . Since  $F$  satisfies  $\mathbf{H}_1(\gamma_1)$  and  $\mathbf{H}_2(\gamma_2)$ , with  $\gamma_2 \leq \gamma_1$ , then for any  $\gamma \in [\gamma_2, \gamma_1]$  we also have that  $F$  satisfies  $\mathbf{H}_2(\gamma)$  and  $\mathbf{H}_1(\gamma)$ . We can thus only consider the case when  $\gamma_1 = \gamma_2 = \gamma$ , as done in the previous case.

**5.3.1. Proof of Corollary 4.4.** We are now in position to prove Corollary 4.4.

*Proof.* The first point of Corollary 4.4 is just a particular instance of Theorem 4.3. In the sequel, we prove the second point of Corollary 4.4.

Let  $t \geq t_0$  and  $\tilde{x} \in X^*$  such that

$$(5.40) \quad d(x(t), \tilde{x}) = d(x(t), X^*)$$

We previously proved that it exists  $A > 0$  such that for any  $t \geq t_0$  and any  $x^* \in X^*$

$$(5.41) \quad \mathcal{H}(t) \leq A.$$

For the choice  $x^* = \tilde{x}$  this inequality ensures that

$$(5.42) \quad \frac{t^{\frac{4}{\gamma-2}}}{2} \|\lambda(x(t) - \tilde{x}) + t\dot{x}(t)\|^2 + t^{\frac{4}{\gamma-2}} \frac{\xi}{2} d(x(t), \tilde{x})^2 \leq A$$

which is equivalent to

$$(5.43) \quad \frac{t^{\frac{4}{\gamma-2}}}{2} \|\lambda(x(t) - \tilde{x}) + t\dot{x}(t)\|^2 \leq \frac{|\xi|}{2} v(t) + A$$

where  $v(t)$  is defined in (5.32) with  $\gamma = \gamma_2$ . Using the fact that the function  $v$  is bounded (a consequence of (5.38)) we deduce that it exists a real number  $A_1 > 0$  such that

$$(5.44) \quad \|\lambda(x(t) - \tilde{x}) + t\dot{x}(t)\| \leq \frac{A_1}{t^{\frac{2}{\gamma-2}}}.$$

Thus:

$$(5.45) \quad t \|\dot{x}(t)\| \leq \frac{A_1}{t^{\frac{2}{\gamma-2}}} + |\lambda| d(x(t), \tilde{x}) = \frac{A_1 + |\lambda| \sqrt{v(t)}}{t^{\frac{2}{\gamma-2}}}.$$

Using once again the fact that the function  $v$  is bounded we deduce that it exists a real number  $A_2$  such that

$$(5.46) \quad \|\dot{x}(t)\| \leq \frac{A_2}{t^{\frac{\gamma}{\gamma-2}}}$$

which implies that  $\|\dot{x}(t)\|$  is an integrable function. As a consequence, we deduce that the trajectory  $x(t)$  is finite.

□

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**Appendix A. Proof of Lemma 5.1.** We prove here Lemma 5.1. Notice that the computations are standard (see e.g. [7]).

We will make use of the following results:

LEMMA A.1. Let  $\gamma > 0$ , if  $g(x) = (F(x) - F(x^*))^{\frac{1}{\gamma}}$  is convex then

$$(A.1) \quad \gamma(F(x(t)) - F(x^*)) \leq \langle \nabla F(x(t)), x(t) - x^* \rangle$$

*Proof.* Since  $g$  is convex we have

$$(A.2) \quad g(x(t)) \leq \langle \nabla g(x(t)), x(t) - x^* \rangle$$

and  $\nabla g(x(t)) = \frac{1}{\gamma}(F(x(t)) - F(x^*))^{\frac{1}{\gamma}-1} \nabla F(x(t))$ . Replacing  $g(x(t))$  by  $(F(x(t)) - F(x^*))^{\frac{1}{\gamma}}$  we get the result.

□

LEMMA A.2.

$$(A.3) \quad \mathcal{E}'_{\lambda,\xi}(t) = 2a(t) + t(\lambda + 1 - \alpha) \|\dot{x}(t)\|^2$$

$$(A.4) \quad + \lambda t \langle -\nabla F(x(t)), x(t) - x^* \rangle + (\lambda(\lambda + 1) - \alpha\lambda + \xi) \langle \dot{x}(t), x(t) - x^* \rangle$$

*Proof.*

We differentiate :

$$(A.5) \quad \mathcal{E}'_{\lambda,\xi}(t) = 2a(t) + t^2 \langle \nabla F(x(t)), \dot{x}(t) \rangle$$

$$(A.6) \quad + \langle \lambda \dot{x}(t) + t\ddot{x}(t) + \dot{x}(t), \lambda(x(t) - x^*) + t\dot{x}(t) \rangle + \xi \langle \dot{x}(t), x(t) - x^* \rangle$$

$$(A.7) \quad \mathcal{E}'_{\lambda,\xi}(t) = 2a(t) + t^2 \langle \nabla F(x(t)) + \ddot{x}(t), \dot{x}(t) \rangle$$

$$(A.8) \quad + (\lambda + 1)t \|\dot{x}(t)\|^2 + \lambda t \langle \ddot{x}(t), x(t) - x^* \rangle + (\lambda(\lambda + 1) + \xi) \langle \dot{x}(t), x(t) - x^* \rangle$$

$$(A.9) \quad \mathcal{E}'_{\lambda,\xi}(t) = 2a(t) + t^2 \langle -\frac{\alpha}{t} \dot{x}(t), \dot{x}(t) \rangle$$

$$(A.10) \quad + (\lambda + 1)t \|\dot{x}(t)\|^2 + \lambda t \langle \ddot{x}(t), x(t) - x^* \rangle + (\lambda(\lambda + 1) + \xi) \langle \dot{x}(t), x(t) - x^* \rangle$$

$$(A.11) \quad \mathcal{E}'_{\lambda,\xi}(t) = 2a(t) + t(\lambda + 1 - \alpha) \|\dot{x}(t)\|^2$$

$$(A.12) \quad + \lambda t \langle \ddot{x}(t), x(t) - x^* \rangle + (\lambda(\lambda + 1) + \xi) \langle \dot{x}(t), x(t) - x^* \rangle$$

And now, using (1.1), we get:

$$(A.13) \quad \mathcal{E}'_{\lambda,\xi}(t) = 2a(t) + t(\lambda + 1 - \alpha) \|\dot{x}(t)\|^2$$

$$(A.14) \quad + \lambda t \langle -\nabla F(x(t)) - \frac{\alpha}{t} \dot{x}(t), x(t) - x^* \rangle + (\lambda(\lambda + 1) + \xi) \langle \dot{x}(t), x(t) - x^* \rangle$$

$$(A.15) \quad \mathcal{E}'_{\lambda,\xi}(t) = 2a(t) + t(\lambda + 1 - \alpha) \|\dot{x}(t)\|^2$$

$$(A.16) \quad + \lambda t \langle -\nabla F(x(t)), x(t) - x^* \rangle + (\lambda(\lambda + 1) - \alpha\lambda + \xi) \langle \dot{x}(t), x(t) - x^* \rangle$$

□

LEMMA A.3.

$$(A.17) \quad \mathcal{E}'_{\lambda,\xi}(t) = 2a(t) + \lambda t \langle -\nabla F(x(t)), x(t) - x^* \rangle$$

$$(A.18) \quad + (\xi - \lambda(\lambda + 1 - \alpha)) \langle \dot{x}(t), x(t) - x^* \rangle$$

$$(A.19) \quad + 2(\lambda + 1 - \alpha)b(t) - 2\lambda^2(\lambda + 1 - \alpha)c(t)$$

*Proof.*

We start from the result of Lemma A.2. Observing that

$$(A.20) \quad \frac{1}{t} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2 = t \|\dot{x}(t)\|^2 + 2\lambda \langle \dot{x}(t), x(t) - x^* \rangle + \frac{\lambda^2}{t} \|x(t) - x^*\|^2$$

we can write

$$(A.21) \quad \mathcal{E}'_{\lambda,\xi}(t) = 2a(t) + \lambda t \langle -\nabla F(x(t)), x(t) - x^* \rangle$$

$$(A.22) \quad + (\xi - \lambda(\lambda + 1 - \alpha)) \langle \dot{x}(t), x(t) - x^* \rangle$$

$$(A.23) \quad + (\lambda + 1 - \alpha) \frac{1}{t} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2 - \frac{\lambda^2(\lambda + 1 - \alpha)}{t} \|x(t) - x^*\|^2$$

□

LEMMA A.4. *If  $\xi = \lambda(\lambda + 1 - \alpha)$ , then*

$$(A.24) \quad \mathcal{E}'_{\lambda,\xi}(t) = 2a(t) + \lambda t \langle -\nabla F(x(t)), x(t) - x^* \rangle + 2(\lambda + 1 - \alpha)b(t) - 2\lambda^2(\lambda + 1 - \alpha)c(t)$$

LEMMA A.5.

*If  $F$  satisfies the hypothesis  $\mathbf{H}_1(\gamma)$  and  $\xi = \lambda(\lambda + 1 - \alpha)$ , then:*

$$(A.25) \quad \mathcal{E}'_{\lambda,\xi}(t) \leq (2 - \gamma\lambda)a(t) + 2(\lambda + 1 - \alpha)b(t) - 2\lambda^2(\lambda + 1 - \alpha)c(t)$$

*Proof.* To prove Lemma A.5, we only apply the inequality of Lemma A.1 in the equality of Lemma A.4.

□

One can notice that if  $F(x) = |x|^\gamma$  the inequality of Lemma A.4 is actually an equality, which ensures that for this specific function  $F$  the inequality in Lemma 5.1 is an equality.

LEMMA A.6. *If  $\xi = \lambda(\lambda + 1 - \alpha)$ , then*

$$(A.26) \quad \mathcal{H}'(t) = t^p ((2 + p)a(t) + \lambda t \langle -\nabla F(x(t)), x(t) - x^* \rangle + (2\lambda + 2 - 2\alpha + p)b(t)$$

$$(A.27) \quad + \lambda(\lambda + 1 - \alpha)(-2\lambda + p)c(t))$$

*Proof.*

We have  $\mathcal{H}(t) = t^p \mathcal{E}(t)$ .

Hence  $\mathcal{H}'(t) = t^p \mathcal{E}'(t) + p t^{p-1} \mathcal{E}(t) = t^{p-1} (t \mathcal{E}'(t) + p \mathcal{E}(t))$ . We conclude by using Lemma A.4.

□

*Proof.* [Proof of Lemma 5.1] We only apply the inequality of Lemma A.1 in the equality of Lemma A.6.

□

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