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GEVREY ANALYTICITY AND DECAY FOR THE COMPRESSIBLE NAVIER-STOKES SYSTEM WITH CAPILLARITY

FRÉDÉRIC CHARVE, RAPHAËL DANCHIN, AND JIANG XU

Abstract. We are concerned with an isothermal model of viscous and capillary compressible fluids derived by J. E. Dunn and J. Serrin (1985), which can be used as a phase transition model. Compared with the classical compressible Navier-Stokes equations, there is a smoothing effect on the density that comes from the capillary terms. First, we prove that the global solutions with critical regularity that have been constructed in [11] by the second author and B. Desjardins (2001), are Gevrey analytic. Second, we extend that result to a more general critical L^p framework. As a consequence, we obtain algebraic time-decay estimates in critical Besov spaces (and even exponential decay for the high frequencies) for any derivatives of the solution.

Our approach is partly inspired by the work of Bae, Biswas & Tadmor [2] dedicated to the classical incompressible Navier-Stokes equations, and requires our establishing new bilinear estimates (of independent interest) involving the Gevrey regularity for the product or composition of functions.

To the best of our knowledge, this is the first work pointing out Gevrey analyticity for a model of *compressible* fluids.

1. Introduction

When considering a two-phases liquid mixture, it is generally assumed, as a consequence of the Young-Laplace theory, that the phases are separated by a hypersurface and that the jump in the pressure across the hypersurface is proportional to the curvature.

In the most common description – the *Sharp Interface* SI model – the interface between phases corresponds to a discontinuity in the state space. In contrast, in the *Diffuse Interface* DI model, the change of phase corresponds to a fast but regular transition zone for the density and velocity.

The DI approach has become popular lately as its mathematical and numerical study only requires one set of equations to be solved in a single spatial domain (typically, with a Van der Waals pressure, the phase changes are read through the density values). In contrast, with the SI model one has to solve one system per phase coupled with a free-boundary problem, since the location of the interface is unknown (see e.g. [9, 23] for more details about the modelling of phase transitions).

The DI model we here aim at considering originates from the works of Van der Waals and, later, Korteweg more than one century ago. The basic idea is to add to the

1991 *Mathematics Subject Classification.* 76N10, 35D05, 35Q05.

Key words and phrases. Time decay rates; Navier-Stokes-Korteweg system; Gevrey regularity; critical Besov spaces; L^p framework.

The first two authors are partially supported by ANR-15-CE40-0011.

The third author is supported by the National Natural Science Foundation of China (11471158) and the Fundamental Research Funds for the Central Universities (NE2015005). He also benefited from a one month invited professor position of UPEC, when this work has been initiated.

classical compressible fluids equations a *capillary term*, that penalizes high variations of the density. In that way, one selects only physically relevant solutions, that is the ones with density corresponding to either a gas or a liquid, and such that the length of the phase interfaces is minimal. Indeed, if capillary is absent then one can find an infinite number of mathematical solutions (most of them being physically wrong although mathematically correct). The full derivation of the corresponding equations that we shall name the *compressible Navier-Stokes-Korteweg system* is due to Dunn and Serrin in [13]. It reads as follows:

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho u) = 0, \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) - \mathcal{A}u + \nabla \Pi = \operatorname{div} \mathcal{K} \end{cases} \tag{1.1}$$

where $\Pi \triangleq P(\varrho)$ is the pressure function, $\mathcal{A}u \triangleq \operatorname{div}\big(2\mu(\rho)D(u)\big) + \nabla\big(\lambda(\rho)\operatorname{div} u\big)$ is the diffusion operator, $D(u) = \frac{1}{2}(\nabla u +^t \nabla u)$ is the symmetric gradient, and the capillarity tensor is given by

$$\mathcal{K} \triangleq \varrho \operatorname{div}(\kappa(\varrho)\nabla\varrho)\, \mathrm{I}_{\mathbb{R}^d} + \frac{1}{2}(\kappa(\varrho) - \varrho\kappa'(\varrho))|\nabla\varrho|^2\, \mathrm{I}_{\mathbb{R}^d} - \kappa(\varrho)\nabla\varrho \otimes \nabla\varrho.$$

The density-dependent capillarity function κ is assumed to be positive. Note that for smooth enough density and κ, we have (see [4])

$$\operatorname{div}\mathcal{K} = \varrho\nabla\Big(\kappa(\varrho)\Delta\varrho + \frac{1}{2}\kappa'(\varrho)|\nabla\varrho|^2\Big). \tag{1.2}$$

The coefficients $\lambda = \lambda(\varrho)$ and $\mu = \mu(\varrho)$ designate the bulk and shear viscosities, respectively, and are assumed to satisfy in the neighborhood of some reference constant density $\bar\varrho > 0$ the conditions

$$\mu > 0 \quad \text{and} \quad \nu \triangleq \lambda + 2\mu > 0. \tag{1.3}$$

Throughout the paper, we shall assume that the functions λ, μ, κ and P, are real analytic in a neighborhood of $\bar\varrho$. Note that this includes the interesting particular case $\kappa(\rho) = \frac{1}{\rho}$ that corresponds to the so-called quantum fluids. The reader may for instance refer to the recent paper by B. Haspot in [18] where this case is considered under the 'shallow water' assumption for the viscosity coefficients: $(\mu(\rho), \lambda(\rho)) = (\ \rho, 0)$.

System (1.1) is supplemented with initial data

$$(\varrho, u)|_{t=0} = (\varrho_0, u_0), \tag{1.4}$$

and we investigate strong solutions in the whole space $\mathbb{R}^d$ with $d \geq 2$, going to a constant equilibrium $(\bar\varrho, 0)$ with $\bar\varrho > 0$, at infinity.

The starting point of our paper is the global existence result for System (1.1) in so-called critical Besov spaces that has been established by the second author and B. Desjardins in [11]. Before stating the result, let us introduce the following functional space:

$$E = \Big\{(a,u)\Big| a \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot B^{d/2-1}_{2,1} \cap \dot B^{d/2}_{2,1}) \cap L^1(\mathbb{R}_+; \dot B^{d/2+1}_{2,1} \cap \dot B^{d/2+2}_{2,1});$$
$$u \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+; \dot B^{d/2-1}_{2,1}) \cap L^1(\mathbb{R}_+; \dot B^{d/2+1}_{2,1})\Big\},$$

the reader being referred to the appendix for the definition of the Besov spaces coming into play in E.

The following result has been established in [11][1]:

Theorem 1.1. *Let $\bar\varrho > 0$ be such that $P'(\bar\varrho) > 0$. Suppose that the initial density fluctuation $\varrho_0 - \bar\varrho$ belongs to $\dot B^{\frac d2}_{2,1} \cap \dot B^{\frac d2-1}_{2,1}$ and that the initial velocity u_0 is in $\dot B^{\frac d2-1}_{2,1}$.*

There exists a constant $\eta > 0$ depending only on $\kappa, \mu, \nu, \bar\varrho, P'(\bar\varrho)$ and d, such that, if

$$\|\varrho_0 - \bar\varrho\|_{\dot B^{\frac d2}_{2,1}\cap \dot B^{\frac d2-1}_{2,1}} + \|u_0\|_{\dot B^{\frac d2-1}_{2,1}} \leq \eta,$$

then System (1.1) *supplemented with* (1.4) *has a unique global solution (ϱ, u) such that $(\varrho - \bar\varrho, u) \in E$.*

Our first result states that the solutions constructed in Theorem 1.1 are, in fact, Gevrey analytic.

Theorem 1.2. *Let the data (ρ_0, u_0) satisfy the conditions of Theorem 1.1 for some $\bar\varrho > 0$ such that $P'(\bar\varrho) > 0$, and that the functions κ, λ, μ and P are analytic. There exist two positive constants c_0 and η only depending on those functions and on d such that if we set*

$$F = \Big\{U \in E \Big| e^{\sqrt{c_0 t}\Lambda_1} U \in E\Big\},$$

where Λ_1 stands for the Fourier multiplier with symbol[2] $|\xi|_1 = \sum_{i=1}^d |\xi_i|$, then for any data (ϱ_0, u_0) satisfying

$$\|\varrho_0 - \bar\varrho\|_{\dot B^{\frac d2}_{2,1}\cap \dot B^{\frac d2-1}_{2,1}} + \|u_0\|_{\dot B^{\frac d2-1}_{2,1}} \leq \eta, \tag{1.5}$$

System (1.1)-(1.4) *admits a unique solution (ϱ, u) with $(\varrho - \bar\varrho, u) \in F$.*

As a by-product, we shall obtain time-decay estimates in the critical Besov spaces, for any derivative of the solution (see Theorem 3.2 below).

The rest of the paper unfolds as follows. The next section is devoted to proving Theorem 1.2. Then, in Section 3, we extend the statement to the critical L^p Besov framework. First, we establish a result in the same spirit as Theorem 1.1, but in a more general functional framework, then we prove that the solutions constructed therein are also Gevrey analytic (see Theorem 3.1) and fulfill decay estimates (see Theorem 3.2).

Before going into the heart of the matter, let us specify some notations. Throughout the paper, C stands for a positive harmless "constant", the meaning of which is clear from the context. Similarly, $f \lesssim g$ means that $f \leq Cg$ and $f \approx g$ means that $f \lesssim g$ and $g \lesssim f$. It will be also understood that $\|(f,g)\|_X \triangleq \|f\|_X + \|g\|_X$ for all $f, g \in X$. Finally, when $f = (f_1, \cdots, f_d)$ with $f_i \in X$ for $i = 1, \cdots, d$, we shall often use, slightly abusively, the notation $f \in X$ instead of $f \in X^d$.

[1] Actually, only the case of constant capillarity and viscosity coefficients has been considered therein. The case of smooth coefficients may be treated along the same lines (see also the work by B. Haspot in [15] concerning the general polytropic case).

[2] Also for technical reasons, as observed before in [21], it is much more convenient to use the $\ell^1(\mathbb{R}^d)$ norm rather than the usual $\ell^2(\mathbb{R}^d)$ norm associated with $\Lambda = (-\Delta)^{1/2}$.

2. The L^2 framework

Proving Theorem 1.2 relies essentially on the classical fixed point theorem in the space F. To establish that all the conditions are fulfilled however, we need to prove a couple of a priori estimates for smooth enough solutions. To this end, we first recast the system into a more user-friendly shape, then establish Gevrey type estimates for the corresponding linearized system about the constant reference state $(\bar\varrho, 0)$, and new nonlinear estimates.

2.1. **Renormalization of System** (1.1).

Throughout the paper, it is convenient to fix some reference viscosity coefficients $\overline{\lambda}$ and $\overline{\mu}$, pressure $\bar p$ and capillarity coefficient $\bar\kappa$, and to rewrite the diffusion, pressure and capillarity terms as follows:

$$\begin{cases} \mathcal{A}u = \overline{\mu}\mathrm{div}\,\big(2\mu(\rho)D(u)\big) + \overline{\lambda}\nabla\big(\lambda(\rho)\mathrm{div}\,u\big), \\ \overline{p}P'(\varrho)\nabla\varrho, \\ \mathrm{div}\,\mathcal{K} = \overline{\kappa}\varrho\nabla\Big(\kappa(\varrho)\Delta\varrho + \frac{1}{2}\kappa'(\varrho)|\nabla\varrho|^2\Big), \end{cases}$$

in such a way that $\mu(\overline{\varrho}) = \lambda(\overline{\varrho}) = \kappa(\overline{\varrho}) = P'(\overline{\varrho}) = 1$.

If we denote $\overline{\nu} = 2\overline{\mu} + \overline{\lambda}$, then performing the rescaling:

$$\widetilde{\varrho}(t,x) = \frac{1}{\overline{\varrho}}\varrho\Big(\frac{\overline{\nu}}{\overline{\varrho p}}\,t, \frac{\overline{\nu}}{\overline{\varrho}\sqrt{\overline{p}}}\,x\Big), \quad \widetilde{u}(t,x) = \frac{1}{\sqrt{\overline{p}}}u\Big(\frac{\overline{\nu}}{\overline{\varrho p}}\,t, \frac{\overline{\nu}}{\overline{\varrho}\sqrt{\overline{p}}}\,x\Big), \tag{2.1}$$

the parameters $(\overline{\varrho}, \overline{\mu}, \overline{\lambda}, \overline{p}, \overline{\kappa})$ are changed into $(1, \frac{\overline{\mu}}{\overline{\nu}}, \frac{\overline{\lambda}}{\overline{\nu}}, 1, \overline{\kappa}\frac{\overline{\varrho}^2}{\overline{\nu}^2})$. We can therefore assume with no loss of generality that

$$\begin{cases} \overline{\varrho} = 1, \quad \overline{\nu} = 2\overline{\mu} + \overline{\lambda} = 1, \quad \overline{p} = 1, \\ \mu(1) = \lambda(1) = \kappa(1) = P'(1) = 1. \end{cases} \tag{2.2}$$

Then, introducing the density fluctuation $a = \varrho - 1$, System (1.1) becomes

$$\begin{cases} \partial_t a + \mathrm{div}\,u = f, \\ \partial_t u - \overline{\mathcal{A}}u + \nabla a - \overline{\kappa}\nabla\Delta a = g, \end{cases} \tag{2.3}$$

with $f = -\mathrm{div}\,(au)$, and $g = \displaystyle\sum_{j=1}^{5} g_j$, where

$$\begin{cases} \overline{\mathcal{A}} = \overline{\mu}\mathrm{div}\,\big(2D(u)\big) + \overline{\lambda}\nabla\big(\mathrm{div}\,u\big) = \overline{\mu}\Delta u + (\overline{\mu}+\overline{\lambda})\nabla\mathrm{div}\,u, \\ g_1 = -u\cdot\nabla u, \\ g_2 = (1-I(a))\left(2\overline{\mu}\mathrm{div}\,\big(\widetilde{\mu}(a)Du\big) + \overline{\lambda}\nabla\big(\widetilde{\lambda}(a)\mathrm{div}\,u\big)\right), \\ g_3 = -I(a)\overline{\mathcal{A}}u, \\ g_4 = J(a)\cdot\nabla a, \\ g_5 = \overline{\kappa}\nabla\left(\widetilde{\kappa}(a)\Delta a + \frac{1}{2}\nabla\widetilde{\kappa}(a)\cdot\nabla a\right), \end{cases} \tag{2.4}$$

and

$$\begin{cases} \widetilde{\mu}(a) = \mu(1+a) - 1, \quad \widetilde{\lambda}(a) = \lambda(1+a) - 1, \quad \widetilde{\kappa}(a) = \kappa(1+a) - 1, \\ I(a) = \frac{a}{1+a}, \quad J(a) = 1 - \frac{P'(1+a)}{1+a}\cdot \end{cases} \tag{2.5}$$

Let us underline that all those functions are analytic near zero, and vanish at zero.

2.2. **The linearized system.** The present subsection is devoted to exhibiting the smoothing properties of (2.3), assuming that f and g are given. In contrast with the linearized equations for the classical compressible Navier-Stokes system, we shall see that here both the density and the velocity are smoothed out instantaneously. The key to that remarkable property is given by the following lemma where, as in all this subsection, we denote by $\widehat{z}$ the Fourier transform *with respect to the space variable* of the function $z \in \mathcal{C}(\mathbb{R}_+; \mathcal{S}(\mathbb{R}^d))$.

Lemma 2.1. *There exist two positive constants c_0 and C depending only on $(\overline{\kappa}, \overline{\mu})$ and $\overline{\kappa}$, respectively, such that the following inequality holds for all $\xi \in \mathbb{R}^d$ and $t \geq 0$:*

$$|(\widehat{a}, |\xi|\widehat{a}, \widehat{u})(t,\xi)| \leq C\biggl(e^{-c_0|\xi|^2 t}|(\widehat{a}, |\xi|\widehat{a}, \widehat{u})(0,\xi)| + \int_0^t e^{-c_0|\xi|^2(t-\tau)}|(\widehat{f}, |\xi|\widehat{f}, \widehat{g})(\tau,\xi)|\,d\tau \biggr).$$

Proof. It is mainly a matter of adapting to System (2.3) the energy argument of Godunov [14] for partially dissipative first-order symmetric systems (further developed by Kawashima in e.g. [20]).

Note that taking advantage of the Duhamel formula reduces the proof to the case where $f \equiv 0$ and $g \equiv 0$. Now, applying to the second equation of (2.3) the Leray projector on divergence free vector fields yields

$$\partial_t \mathcal{P}u - \overline{\mu}\Delta \mathcal{P}u = 0, \tag{2.6}$$

from which we readily get, after taking the (space) Fourier transform,

$$|\widehat{\mathcal{P}u}(t)| \leq e^{-\overline{\mu}|\xi|^2 t}|\widehat{\mathcal{P}u}(0)|. \tag{2.7}$$

In order to prove the desired inequality for a and the gradient part of the velocity, it is convenient to introduce the function $v \triangleq \Lambda^{-1}\mathrm{div}\, u$ (with $\Lambda^s z \triangleq \mathcal{F}^{-1}\left(|\xi|^s \mathcal{F}z\right)$ for $s \in \mathbb{R}$) . Then, we discover that (a, v) satisfies (recall that $2\overline{\mu} + \overline{\lambda} = 1$)

$$\begin{cases} \partial_t a + \Lambda v = 0, \\ \partial_t v - \Delta v - \Lambda a - \overline{\kappa}\Lambda^3 a = 0. \end{cases} \tag{2.8}$$

Hence, taking the Fourier transform of both sides of (2.8) gives

$$\begin{cases} \frac{d}{dt}\widehat{a} + |\xi|\widehat{v} = 0, \\ \frac{d}{dt}\widehat{v} + |\xi|^2\widehat{v} - |\xi|(1 + \overline{\kappa}|\xi|^2)\widehat{a} = 0. \end{cases} \tag{2.9}$$

Multiplying the first equation in (2.9) by the conjugate $\overline{\widehat{a}}$ of $\widehat{a}$, and the second one by $\overline{\widehat{v}}$, we get

$$\frac{1}{2}\frac{d}{dt}|\widehat{a}|^2 + |\xi|\mathrm{Re}(\widehat{a}\,\overline{\widehat{v}}) = 0 \tag{2.10}$$

and, because $\mathrm{Re}(\widehat{a}\,\overline{\widehat{v}}) = \mathrm{Re}(\overline{\widehat{a}}\,\widehat{v})$,

$$\frac{1}{2}\frac{d}{dt}|\widehat{v}|^2 + |\xi|^2|\widehat{v}|^2 - |\xi|(1 + \overline{\kappa}|\xi|^2)\mathrm{Re}(\widehat{a}\,\overline{\widehat{v}}) = 0. \tag{2.11}$$

Multiplying (2.10) by $(1 + \overline{\kappa}|\xi|^2)$, and adding up to (2.11) yields

$$\frac{1}{2}\frac{d}{dt}\bigl((1 + \overline{\kappa}|\xi|^2)|\widehat{a}|^2 + |\widehat{v}|^2\bigr) + |\xi|^2|\widehat{v}|^2 = 0. \tag{2.12}$$

In order to track the dissipation arising for a, let us multiply the first and second equations of (2.9) by $-|\xi|\overline{\widehat{v}}$ and $-|\xi|\overline{\widehat{a}}$, respectively. Adding them, we get:

$$\frac{d}{dt}\big(-|\xi|\mathrm{Re}(\widehat{a}\,\overline{\widehat{v}})\big) - |\xi|^3\mathrm{Re}(\overline{\widehat{a}}\widehat{v}) + |\xi|^2(1+\overline{\kappa}|\xi|^2)|\widehat{a}|^2 - |\xi|^2|\widehat{v}|^2 = 0. \tag{2.13}$$

Adding to this $|\xi|^2$(2.10) yields

$$\frac{1}{2}\frac{d}{dt}\big(|\xi|^2|\widehat{a}|^2 - 2|\xi|\mathrm{Re}(\widehat{a}\,\overline{\widehat{v}})\big) + |\xi|^2(1+\overline{\kappa}|\xi|^2)|\widehat{a}|^2 - |\xi|^2|\widehat{v}|^2 = 0. \tag{2.14}$$

Therefore, by multiplying (2.14) by a small enough constant $\beta > 0$ (to be determined later) and adding it to (2.12), we get

$$\frac{1}{2}\frac{d}{dt}\mathcal{L}^2_{|\xi|}(t) + \beta|\xi|^2(1+\overline{\kappa}|\xi|^2)|\widehat{a}|^2 + (1-\beta)|\xi|^2|\widehat{v}|^2 = 0,$$
$$\text{with}\quad \mathcal{L}^2_{|\xi|}(t) \triangleq (1+\overline{\kappa}|\xi|^2)|\widehat{a}|^2 + |\widehat{v}|^2 + \beta\big(|\xi|^2|\widehat{a}|^2 - 2|\xi|\mathrm{Re}(\widehat{a}\,\overline{\widehat{v}})\big).$$

Choosing $\beta = \frac{1}{2}$ we have $\mathcal{L}^2_{|\xi|} \approx |(\widehat{a}, |\xi|\widehat{a}, \widehat{v})|^2$ and using the Cauchy-Schwarz inequality, we deduce that there exists a positive constant c_1 such that on $\mathbb{R}_+$, we have

$$\frac{d}{dt}\mathcal{L}^2_{|\xi|} + c_1|\xi|^2\mathcal{L}^2_{|\xi|} \leq 0,$$

which leads, after time integration, to[3]

$$|(\widehat{a}, |\xi|\widehat{a}, \widehat{v})(t)| \leq Ce^{-c_1|\xi|^2 t}|(\widehat{a}, |\xi|\widehat{a}, \widehat{v})(0)|. \tag{2.15}$$

Putting together with (2.7) completes the proof of the lemma in the case $f \equiv 0$ and $g \equiv 0$. The general case readily stems from Duhamel formula. □

We shall also need the following two results that have been proved in [2].

Lemma 2.2. *The kernel of operator $M_1 := e^{-[\sqrt{t-\tau}+\sqrt{\tau}-\sqrt{t}]\Lambda_1}$ with $0 < \tau < t$ is integrable, and has a L^1 norm that may be bounded independently of τ and t.*

Lemma 2.3. *The operator $M_2 := e^{\frac{1}{2}a\Delta+\sqrt{a}\Lambda_1}$ is a Fourier multiplier which maps boundedly L^p to L^p for all $1 < p < \infty$. Furthermore, its operator norm is uniformly bounded with respect to $a \geq 0$.*

Proving the Gevrey regularity of our solutions will be based on continuity results for the family $(B_t)_{t\geq 0}$ of bilinear operators defined by

$$B_t(f,g)(x) = \Big(e^{\sqrt{c_0 t}\Lambda_1}(e^{-\sqrt{c_0 t}\Lambda_1}f \cdot e^{-\sqrt{c_0 t}\Lambda_1}g)\Big)(x)$$
$$= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot(\xi+\eta)}e^{\sqrt{c_0 t}(|\xi+\eta|_1 - |\xi|_1 - |\eta|_1)}\widehat{f}(\xi)\widehat{g}(\eta)\,d\xi\,d\eta.$$

Following [2] and [22], we introduce the following operators acting on functions depending on one real variable:

$$K_1 f \triangleq \frac{1}{2\pi}\int_0^\infty e^{\mathrm{i}x\xi}\widehat{f}(\xi)\,d\xi \quad\text{and}\quad K_{-1}f \triangleq \frac{1}{2\pi}\int_{-\infty}^0 e^{\mathrm{i}x\xi}\widehat{f}(\xi)\,d\xi,$$

and define $L_{a,1}$ and $L_{a,-1}$ as follows:

$$L_{a,1}f \triangleq f \quad\text{and}\quad L_{a,-1}f \triangleq \frac{1}{2\pi}\int_{\mathbb{R}} e^{\mathrm{i}x\xi}e^{-2a|\xi|}\widehat{f}(\xi)\,d\xi.$$

[3] If one tracks the constants then we get $c_1 = \frac{1}{2}\min(1,\overline{\kappa})$ and $C = \frac{\max(\frac{3}{2},\overline{\kappa}+1)}{\min(\frac{1}{2},\overline{\kappa})}$.

For $t \geq 0$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_d) \in \{-1, 1\}^d$, we set

$$Z_{t,\alpha,\beta} \triangleq K_{\beta_1} L_{\sqrt{c_0 t},\alpha_1\beta_1} \otimes \cdots \otimes K_{\beta_d} L_{\sqrt{c_0 t},\alpha_d\beta_d} \quad \text{and} \quad K_\alpha \triangleq K_{\alpha_1} \otimes \cdots \otimes K_{\alpha_d}.$$

Then we see that

$$B_t(f,g) = \sum_{(\alpha,\beta,\gamma)\in(\{-1,1\}^d)^3} K_\alpha(Z_{t,\alpha,\beta} f Z_{t,\alpha,\gamma} g). \tag{2.16}$$

Since operators K_α and $Z_{t,\alpha,\beta}$ are linear combinations of smooth homogeneous of degree zero Fourier multipliers, they are bounded on L^p for any $1 < p < \infty$ (but they need not be bounded in L^1 and L^∞). Furthermore, they commute with all Fourier multipliers and thus in particular with Λ_1 and with the Littlewood-Paley cut-off operators $\dot{\Delta}_j$. We also have the following fundamental result:

Lemma 2.4. *For any $1 < p, p_1, p_2 < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we have for some constant C independent of $t \geq 0$,*

$$\|B_t(f,g)\|_{L^p} \leq C \|f\|_{L^{p_1}} \|g\|_{L^{p_2}}.$$

2.3. **Results of continuity for the paraproduct, remainder and composition.** The aim of this section is to establish the nonlinear estimates involving Besov Gevrey regularity that will be needed to bound the right-hand side of (2.3). We shall actually prove more general estimates both because they are of independent interest and since they will be used in the next section, when we shall generalize the statement of Theorem 1.2 to L^p related Besov spaces.

The first part of this subsection will be devoted to product estimates, and will require our using Bony's decomposition and to prove new continuity results for the paraproduct and remainder operators.

Recall that, at the formal level, the product of two tempered distributions f and g may be decomposed into

$$fg = T_f g + T_g f + R(f,g) \tag{2.17}$$

with

$$T_f g = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} f \, \dot{\Delta}_j g \quad \text{and} \quad R(f,g) = \sum_{j\in\mathbb{Z}} \sum_{|j'-j|\leq 1} \dot{\Delta}_j f \, \dot{\Delta}_{j'} g.$$

The above operators T and R are called "paraproduct" and "remainder," respectively. The decomposition (2.17) has been first introduced by J.-M. Bony in [5]. The paraproduct and remainder operators possess a lot of continuity properties in Besov spaces (see Chap. 2 in [3]), which motivates their introduction here.

From now on and for notational simplicity, we agree that $F(t) \triangleq e^{\sqrt{c_0 t}\Lambda_1} f$ for $t \geq 0$ (and dependence on t will be often omitted).

Let us start with paraproduct and remainder estimates in the case where all the Lebesgue indices lie in the range $]1, \infty[$.

Proposition 2.1. *Let $s \in \mathbb{R}$ and $1 < p < \infty$, $1 \leq p_1, p_2, r, r_1, r_2 \leq \infty$ with $1/p = 1/p_1 + 1/p_2$ and $1/r = 1/r_1 + 1/r_2$. If $1 < p, p_1, p_2 < \infty$, then there exists a constant C such that for any f, g and $\sigma > 0$ (or $\sigma \geq 0$ if $r_1 = 1$),*

$$\|e^{\sqrt{c_0 t}\Lambda_1} T_f g\|_{\dot{B}^{s-\sigma}_{p,r}} \leq C \|F\|_{\dot{B}^{-\sigma}_{p_1,r_1}} \|G\|_{\dot{B}^{s}_{p_2,r_2}}, \tag{2.18}$$

and for any $s_1, s_2 \in \mathbb{R}$ with $s_1 + s_2 > 0$,

$$\|e^{\sqrt{c_0 t}\Lambda_1} R(f,g)\|_{\dot{B}^{s_1+s_2}_{p,r}} \leq C \|F\|_{\dot{B}^{s_1}_{p_1,r_1}} \|G\|_{\dot{B}^{s_2}_{p_2,r_2}}. \tag{2.19}$$

In order to prove our main results for the Korteweg system, we will need sometimes the estimates corresponding to the case $p_2 = p$ that are contained in the following statement.

Proposition 2.2. *Assume that $1 < p, q < \infty$ and that $1 \leq r, r_1, r_2$ fulfill $1/r = 1/r_1 + 1/r_2$. There exists a constant C such that for any f, g and $\sigma > 0$ (or $\sigma \geq 0$ if $r_1 = 1$),*

$$\|e^{\sqrt{c_0}t\Lambda_1} T_f g\|_{\dot{B}^{s-\sigma}_{p,r}} \leq C \|F\|_{\dot{B}^{\frac{d}{q}-\sigma}_{q,r_1}} \|G\|_{\dot{B}^{s}_{p,r_2}}, \tag{2.20}$$

and for any $s_1, s_2 \in \mathbb{R}$ with $s_1 + s_2 > 0$,

$$\|e^{\sqrt{c_0}t\Lambda_1} R(f,g)\|_{\dot{B}^{s_1+s_2}_{p,r}} \leq C \|F\|_{\dot{B}^{s_1+\frac{d}{q}}_{q,r_1}} \|G\|_{\dot{B}^{s_2}_{p,r_2}}. \tag{2.21}$$

Proof of Proposition 2.1. By the definition of the paraproduct and of B_t, we have

$$e^{\sqrt{c_0}t\Lambda_1} T_f g = \sum_{j \in \mathbb{Z}} W_j \quad \text{with} \quad W_j \triangleq B_t(\dot{S}_{j-1}F, \dot{\Delta}_j G). \tag{2.22}$$

As no Lebesgue index reaches the endpoints, thanks to Lemma 2.4, we obtain

$$\begin{aligned} \|W_j\|_{L^p} &\lesssim \|\dot{S}_{j-1}F\|_{L^{p_1}} \|\dot{\Delta}_j G\|_{L^{p_2}} \\ &\lesssim \Big(\sum_{j' \leq j-2} \|\dot{\Delta}_{j'} F\|_{L^{p_1}} \Big) \|\dot{\Delta}_j G\|_{L^{p_2}}. \end{aligned}$$

Therefore, it holds that

$$2^{j(s-\sigma)} \|W_j\|_{L^p} \lesssim 2^{js} \|\dot{\Delta}_j G\|_{L^{p_2}} \sum_{j' \leq j-2} 2^{\sigma(j'-j)} 2^{-\sigma j'} \|\dot{\Delta}_{j'} F\|_{L^{p_1}}.$$

As $\sigma > 0$, Hölder and Young inequalities for series enable us to obtain

$$\Big(2^{j(s-\sigma)} \|W_j\|_{L^p} \Big)_{\ell^r} \lesssim \|F\|_{\dot{B}^{-\sigma}_{p_1,r_1}} \|G\|_{\dot{B}^{s}_{p_2,r_2}},$$

and one may conclude to (2.18) by using Proposition A.1.

In the case $\sigma = 0$, one just has to use the fact that

$$\|\dot{S}_{j-1}F\|_{L^{p_1}} \lesssim \|F\|_{L^{p_1}} \lesssim \|F\|_{\dot{B}^{0}_{p_1,1}}.$$

Let us now turn to the remainder: we have for all $k \in \mathbb{Z}$,

$$\dot{\Delta}_k e^{\sqrt{c_0}t\Lambda_1} R(f,g) = \sum_{j \geq k-2} \sum_{|j-j'| \leq 1} \dot{\Delta}_k B_t(\dot{\Delta}_j F, \dot{\Delta}_{j'} G).$$

Taking the L^p norm with respect to the spatial variable, we deduce by Lemma 2.4 that

$$\|\dot{\Delta}_k e^{\sqrt{c_0}t\Lambda_1} R(f,g)\|_{L^p} \lesssim \sum_{j \geq k-2} \sum_{|j-j'| \leq 1} \|\dot{\Delta}_j F\|_{L^{p_1}} \|\dot{\Delta}_{j'} G\|_{L^{p_2}}.$$

Then everything now works as for estimating classical Besov norms:

$$\begin{aligned} &2^{k(s_1+s_2)} \|\dot{\Delta}_k e^{\sqrt{c_0}t\Lambda_1} R(f,g)\|_{L^p} \\ &\qquad \lesssim \sum_{j \geq k-2} \sum_{|j-j'| \leq 1} 2^{(k-j)(s_1+s_2)} 2^{js_1} \|\dot{\Delta}_j F\|_{L^{p_1}} 2^{(j-j')s_2} 2^{j's_2} \|\dot{\Delta}_{j'} G\|_{L^{p_2}}, \end{aligned} \tag{2.23}$$

and Young's and Hölder inequalities for series allow to get (2.19) as $s_1 + s_2 > 0$. □

Proof of Proposition 2.2. We argue as in the previous proof, except that one intermediate step is needed for bounding the general term of the paraproduct or remainder. The key point of course is to bound in L^p the general term of B_t in (2.16), while the Lebesgue exponents do not fulfill the conditions of Lemma 2.4.

As an example, let us prove Inequality (2.20) for $\sigma = 0$. We write, combining Hölder and Bernstein inequality (A.7), and the properties of continuity of operators K_α and $Z_{t,\alpha,\beta}$,

$$\begin{aligned}\|K_\alpha(Z_{t,\alpha,\beta}\dot{S}_{j-1}F\cdot Z_{t,\alpha,\beta}\dot{\Delta}_j G)\|_{L^p} &\lesssim \|Z_{t,\alpha,\beta}\dot{S}_{j-1}F\cdot Z_{t,\alpha,\beta}\dot{\Delta}_j G\|_{L^p}\\ &\lesssim \sum_{j'\leq j-2}\|\dot{\Delta}_{j'}Z_{t,\alpha,\beta}F\|_{L^\infty}\|Z_{t,\alpha,\beta}\dot{\Delta}_j G\|_{L^p}\\ &\lesssim \sum_{j'\leq j-2}2^{j'\frac{d}{q}}\|\dot{\Delta}_{j'}Z_{t,\alpha,\beta}F\|_{L^q}\|Z_{t,\alpha,\beta}\dot{\Delta}_j G\|_{L^p}\\ &\lesssim \sum_{j'\leq j-2}2^{j'\frac{d}{q}}\|\dot{\Delta}_{j'}F\|_{L^q}\|\dot{\Delta}_j G\|_{L^p}.\end{aligned}$$

From this, we get

$$2^{js}\|W_j\|_{L^p}\lesssim \|F\|_{\dot{B}^{\frac{d}{q}}_{q,1}}\,2^{js}\|\dot{\Delta}_j G\|_{L^p}.$$

We then obtain (2.20) for $\sigma = 0$ thanks to Proposition A.1. □

Combining the above propositions with functional embeddings and Bony's decomposition, one may deduce the following Gevrey product estimates in Besov spaces that will be of extensive use in what follows:

Proposition 2.3. *Let $1 < p < \infty$, $s_1, s_2 \leq d/p$ with $s_1 + s_2 > d\max(0, -1 + 2/p)$. There exists a constant C such that the following estimate holds true:*

$$\|e^{\sqrt{c_0 t}\Lambda_1}(fg)\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,1}} \leq C\|F\|_{\dot{B}^{s_1}_{p,1}}\|G\|_{\dot{B}^{s_2}_{p,1}}. \tag{2.24}$$

Proof. In light of decomposition (2.17), we have

$$e^{\sqrt{c_0 t}\Lambda_1}(fg) = e^{\sqrt{c_0 t}\Lambda_1}T_f g + e^{\sqrt{c_0 t}\Lambda_1}T_g f + e^{\sqrt{c_0 t}\Lambda_1}R(f,g). \tag{2.25}$$

Then (2.20) and standard embedding imply that

$$\begin{cases}\|e^{\sqrt{c_0 t}\Lambda_1}T_f g\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,1}} &\lesssim \|F\|_{\dot{B}^{\frac{d}{p}+(s_1-\frac{d}{p})}_{p,1}}\|G\|_{\dot{B}^{s_2}_{p,1}}\\ \|e^{\sqrt{c_0 t}\Lambda_1}T_g f\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,1}} &\lesssim \|G\|_{\dot{B}^{\frac{d}{p}+(s_2-\frac{d}{p})}_{p,1}}\|F\|_{\dot{B}^{s_1}_{p,1}}.\end{cases}$$

It is easy to deal with the remainder if $p \geq 2$: thanks to embeddings and (2.19), we have

$$\|e^{\sqrt{c_0 t}\Lambda_1}R(f,g)\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,1}} \lesssim \|e^{\sqrt{c_0 t}\Lambda_1}R(f,g)\|_{\dot{B}^{s_1+s_2}_{p/2,1}} \lesssim \|F\|_{\dot{B}^{s_1}_{p,1}}\|G\|_{\dot{B}^{s_2}_{p,1}}.$$

If $1<p<2$, then we use instead that $\dot{B}^{\sigma+d(\frac{1}{p_0}-\frac{1}{p})}_{p_0,1}\hookrightarrow\dot{B}^{\sigma}_{p,1}$ for all $1<p_0<p$, and Inequality (2.19) thus implies that

$$\begin{aligned}\|e^{\sqrt{c_0t}\Lambda_1}R(f,g)\|_{\dot{B}^{s_1+s_2-\frac{d}{p}}_{p,1}}&\lesssim\|e^{\sqrt{c_0t}\Lambda_1}R(f,g)\|_{\dot{B}^{s_1+s_2-2\frac{d}{p}+\frac{d}{p_0}}_{p_0,1}}\\&\lesssim\|F\|_{\dot{B}^{s_1-\frac{2d}{p}+\frac{d}{p_0}}_{p_2,1}}\|G\|_{\dot{B}^{s_2}_{p,1}}\\&\lesssim\|F\|_{\dot{B}^{s_1-\frac{2d}{p}+\frac{d}{p_0}+d(\frac{1}{p}-\frac{1}{p_2})}_{p,1}}\|G\|_{\dot{B}^{s_2}_{p,1}}=\|F\|_{\dot{B}^{s_1}_{p,1}}\|G\|_{\dot{B}^{s_2}_{p,1}},\end{aligned}$$

whenever $1/p+1/p_2=1/p_0$, and $p_2\geq p$. Since $p<2$, it is clear that those two conditions may be satisfied if taking p_0 close enough to 1. □

Remark 2.1. *Proposition 2.3 ensures that the space* $\{f\in\dot{B}^{\frac{d}{p}}_{p,1}\ e^{\sqrt{t}\Lambda_1}f\in\dot{B}^{\frac{d}{p}}_{p,1}\}$ *is an algebra whenever* $1<p<\infty$.

The previous estimates can be adapted to the Chemin-Lerner's spaces $\widetilde{L}^q_T(\dot{B}^{\sigma}_{p,r})$. For example, we have the following result.

Proposition 2.4. *Let* $1<p<\infty$ *and* $1\leq q,q_1,q_2\leq\infty$ *such that* $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}\cdot$ *If* $\sigma_1,\sigma_2\leq d/p$ *and* $\sigma_1+\sigma_2>d\max(0,-1+2/p)$ *then there exists a constant* $C>0$ *such that for all* $T\geq0$,

$$\|e^{\sqrt{c_0t}\Lambda_1}(fg)\|_{\widetilde{L}^q_T(\dot{B}^{\sigma}_{p,1})}\leq C\|F\|_{\widetilde{L}^{q_1}_T(\dot{B}^{\sigma_1}_{p,1})}\|G\|_{\widetilde{L}^{q_2}_T(\dot{B}^{\sigma_2}_{p,1})}.\tag{2.26}$$

In order to prove Theorem 1.2, we need not only bilinear estimates involving Gevrey-Besov regularity, but also composition estimates by real analytic functions.

Lemma 2.5. *Let* F *be a real analytic function in a neighborhood of* 0, *such that* $F(0)=0$. *Let* $1<p<\infty$ *and* $-\min\bigl(\frac{d}{p},\frac{d}{p'}\bigr)<s\leq\frac{d}{p}$ *with* $\frac{1}{p'}=1-\frac{1}{p}\cdot$ *There exist two constants* R_0 *and* D *depending only on* p, d *and* F *such that if for some* $T>0$,

$$\|e^{\sqrt{c_0t}\Lambda_1}z\|_{\widetilde{L}^{\infty}_T(\dot{B}^{\frac{d}{p}}_{p,1})}\leq R_0\tag{2.27}$$

then we have

$$\|e^{\sqrt{c_0t}\Lambda_1}F(z)\|_{\widetilde{L}^{\infty}_T(\dot{B}^{s}_{p,1})}\leq D\|e^{\sqrt{c_0t}\Lambda_1}z\|_{\widetilde{L}^{\infty}_T(\dot{B}^{s}_{p,1})}.\tag{2.28}$$

Remark 2.2. *For proving our main results, we shall use the above lemma with* $s=\frac{d}{p}$ *or* $s=\frac{d}{p}-1$. *Note that the former case requires that* $d\geq2$ *and* $1<p<2d$.

Proof. Let us write

$$F(z)=\sum_{n=1}^{+\infty}a_nz^n$$

and denote by $R_F>0$ the convergence radius of the series. For all $t\geq0$ (as usual $Z=e^{\sqrt{c_0t}\Lambda_1}z$) we have

$$e^{\sqrt{c_0t}\Lambda_1}F(z)=\sum_{n=1}^{+\infty}a_ne^{\sqrt{c_0t}\Lambda_1}z^n=\sum_{n=1}^{+\infty}a_ne^{\sqrt{c_0t}\Lambda_1}(e^{-\sqrt{c_0t}\Lambda_1}Z)^n,\tag{2.29}$$

which implies from (2.26), by induction and thanks to the condition on s, that

$$\begin{aligned}\|e^{\sqrt{c_0t}\Lambda_1}F(z)\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})} &\leq C\sum_{n=1}^{+\infty}|a_n|\Big(C\|e^{\sqrt{c_0t}\Lambda_1}z\|_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\Big)^{n-1}\|e^{\sqrt{c_0t}\Lambda_1}z\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})}\\ &\leq \bar{F}(C\|e^{\sqrt{c_0t}\Lambda_1}z\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})})\|e^{\sqrt{c_0t}\Lambda_1}z\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})},\end{aligned}$$

where we define $\bar{F}(z)=\sum_{n=1}^{+\infty}|a_n|z^{n-1}$. So when $\|e^{\sqrt{c_0t}\Lambda_1}z\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\leq\frac{R_F}{2C}\triangleq R_0$ we have (2.28) with $D=\sup_{z\in\bar{B}(0,\frac{R_F}{2})}|\bar{F}(z)|$. □

Let us end this section with a variant of the previous result:

Lemma 2.6. *Let F be a real analytic function in a neighborhood of 0. Let $1<p<\infty$ and $-\min\big(\frac{d}{p},\frac{d}{p'}\big)<s\leq\frac{d}{p}$. There exist two constants R_0 and D depending only on p, d and F such that if for some $T>0$,*

$$\max_{i=1,2}\|e^{\sqrt{c_0t}\Lambda_1}z_i\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\leq R_0, \tag{2.30}$$

then we have

$$\|e^{\sqrt{c_0t}\Lambda_1}\big(F(z_2)-F(z_1)\big)\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})}\leq D\|e^{\sqrt{c_0t}\Lambda_1}(z_2-z_1)\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})}. \tag{2.31}$$

Proof. With the same notations as before, Proposition 2.4 with $(\sigma_1,\sigma_2)=(s,\frac{d}{p})$ yields:

$$\begin{aligned}\|e^{\sqrt{c_0t}\Lambda_1}\big(F(z_2)-F(z_1)\big)\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})} &\leq \sum_{n=1}^{+\infty}|a_n|\|e^{\sqrt{c_0t}\Lambda_1}(z_2^n-z_1^n)\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})}\\ &\leq \sum_{n=1}^{+\infty}|a_n|\|e^{\sqrt{c_0t}\Lambda_1}\Big((z_2-z_1)\sum_{k=0}^{n-1}z_1^kz_2^{n-1-k}\Big)\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})}\\ &\leq C\sum_{n=1}^{+\infty}|a_n|\|e^{\sqrt{c_0t}\Lambda_1}(z_2-z_1)\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})}\sum_{k=0}^{n-1}\|e^{\sqrt{c_0t}\Lambda_1}(z_1^kz_2^{n-1-k})\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}.\end{aligned}$$

By induction, we get (using $n\leq 2^{n-1}$)

$$\begin{aligned}&\|e^{\sqrt{c_0t}\Lambda_1}\big(F(z_2)-F(z_1)\big)\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})}\\ &\leq C\|e^{\sqrt{c_0t}\Lambda_1}(z_2-z_1)\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})}\sum_{n=1}^{+\infty}|a_n|\sum_{k=0}^{n-1}C^{n-1}\|e^{\sqrt{c_0t}\Lambda_1}z_1\|^k_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\|e^{\sqrt{c_0t}\Lambda_1}z_2\|^{n-1-k}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\ &\leq C\|e^{\sqrt{c_0t}\Lambda_1}(z_2-z_1)\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})}\sum_{n=1}^{+\infty}|a_n|\Big(2C\max_{i=1,2}\|e^{\sqrt{c_0t}\Lambda_1}z_i\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\Big)^{n-1}\\ &\leq C\|e^{\sqrt{c_0t}\Lambda_1}(z_2-z_1)\|_{\widetilde{L}^\infty_T(\dot{B}^s_{p,1})}\bar{F}\big(2C\max_{i=1,2}\|e^{\sqrt{c_0t}\Lambda_1}z_i\|_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\big).\end{aligned}$$

We conclude as before. □

2.4. **The proof of Theorem 1.2.** One can now come back to the proof of Theorem 1.2. Recall the following estimate that has been shown in [11].

Lemma 2.7. *Let (a,u) be a solution in E of System (2.3). There exists a constant $R_0>0$ such that if*

$$\|a\|_{L^\infty(\dot B^{\frac d2}_{2,1})}\leq R_0$$

then one has the following a priori estimate:

$$\|(a,u)\|_E\lesssim\|a_0\|_{\dot B^{\frac d2-1}_{2,1}\cap\dot B^{\frac d2}_{2,1}}+\|u_0\|_{\dot B^{\frac d2-1}_{2,1}}+\bigl(1+\|(a,u)\|_E\bigr)\|(a,u)\|_E^2. \tag{2.32}$$

We want to generalize it in the Gevrey regularity setting, getting the following result:

Lemma 2.8. *Let (a,u) be the global solution constructed in Theorem 1.1. Denote $A\triangleq e^{\sqrt{c_0t}\Lambda_1}a$ and $U\triangleq e^{\sqrt{c_0t}\Lambda_1}u$ where c_0 is the constant of Lemma 2.1. There exists a constant $R_0>0$ such that if*

$$\|A\|_{\widetilde L^\infty(\dot B^{\frac d2}_{2,1})}\leq R_0, \tag{2.33}$$

then we have

$$\|(A,U)\|_E\lesssim\|a_0\|_{\dot B^{\frac d2-1}_{2,1}\cap\dot B^{\frac d2}_{2,1}}+\|u_0\|_{\dot B^{\frac d2-1}_{2,1}}+\bigl(1+\|(A,U)\|_E\bigr)\|(A,U)\|_E^2. \tag{2.34}$$

Proof. Apply $\dot\Delta_q$ to (2.6)-(2.8) and repeat the procedure leading to Lemma 2.1. Multiplying by the factor $e^{\sqrt{c_0t}|\xi|_1}$ we end up with

$$\begin{aligned}|(\widehat{\dot\Delta_qA},\widehat{\dot\Delta_q\nabla A},\widehat{\dot\Delta_qU})(t,\xi)|\leq C\Bigl(&e^{\sqrt{c_0t}|\xi|_1}e^{-c_0|\xi|^2t}|(\widehat{\dot\Delta_qa_0},\widehat{\dot\Delta_q\nabla a_0},\widehat{\dot\Delta_qu_0})|\\&+e^{\sqrt{c_0t}|\xi|_1}\int_0^t e^{-c_0|\xi|^2(t-\tau)}|(\widehat{\dot\Delta_qf},\widehat{\dot\Delta_q\nabla f},\widehat{\dot\Delta_qg})|(\tau,\xi)\,d\tau\Bigr)\cdot\end{aligned}$$

Taking the L^2 norm, thanks to the Fourier-Plancherel theorem, we get for all $t\geq0$,

$$\begin{aligned}&\|(\dot\Delta_qA,\dot\Delta_q\nabla A,\dot\Delta_qU)(t)\|_{L^2}\lesssim\|e^{\sqrt{c_0t}\Lambda_1+\frac12c_0t\Delta}e^{\frac12c_0t\Delta}(\dot\Delta_qa_0,\dot\Delta_q\nabla a_0,\dot\Delta_qu_0)\|_{L^2}\\&+\int_0^t\|e^{(\sqrt{c_0(t-\tau)}\Lambda_1+\frac12c_0(t-\tau)\Delta)}e^{-\sqrt{c_0}(\sqrt{t-\tau}+\sqrt\tau-\sqrt t)\Lambda_1}e^{\frac12c_0(t-\tau)\Delta}(\dot\Delta_qF,\dot\Delta_q\nabla F,\dot\Delta_qG)(\tau)\|_{L^2}d\tau\end{aligned}$$

and thanks to Lemmas 2.2 and 2.3, and to the properties of localization of $\dot\Delta_q$, we obtain, denoting $c_1\triangleq\frac{9}{32}c_0$,

$$\begin{aligned}\|(\dot\Delta_qA,\dot\Delta_q\nabla A,\dot\Delta_qU)(t)\|_{L^2}\leq C\Bigl(&e^{-c_1t2^{2q}}\|(\dot\Delta_qa_0,\dot\Delta_q\nabla a_0,\dot\Delta_qu_0)\|_{L^2}\\&+\int_0^t\|e^{-c_1(t-\tau)2^{2q}}(\dot\Delta_qF,\dot\Delta_q\nabla F,\dot\Delta_qG)(\tau)\|_{L^2}\,d\tau\Bigr)\cdot\end{aligned} \tag{2.35}$$

Therefore, multiplying by $2^{q(\frac d2-1)}$ and summing on $q\in\mathbb{Z}$, we obtain that for all $t\geq0$,

$$\begin{aligned}\|(A,U)\|_{E_t}\triangleq\|(A,\nabla A,U)\|_{\widetilde L^\infty_t(\dot B^{\frac d2-1}_{2,1})}+c_0\|(A,\nabla A,U)\|_{L^1_t(\dot B^{\frac d2+1}_{2,1})}\\\leq C\Bigl(\|(a_0,\nabla a_0,u_0)\|_{\dot B^{\frac d2-1}_{2,1}}+\|(F,\nabla F,G)\|_{L^1_t(\dot B^{\frac d2-1}_{2,1})}\Bigr)\cdot\end{aligned}$$

We are left with estimating the external force terms as in the classical Besov case, but using the laws suited to Gevrey regularity.

Regarding F, we have thanks to Proposition 2.3

$$\int_0^t \|F(\tau)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} d\tau \leq \int_0^t \|e^{\sqrt{c_0\tau}\Lambda_1}(au)\|_{\dot{B}^{\frac{d}{2}}_{2,1}} d\tau$$
$$\leq C\int_0^t \|A\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|U\|_{\dot{B}^{\frac{d}{2}}_{2,1}} d\tau \leq \|A\|_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})} \|U\|_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}.$$

Estimating ∇F is also based on Proposition 2.3, after using that $f = -u\cdot\nabla a - \nabla(a\operatorname{div} u)$. Then one may write that

$$\int_0^t \|\nabla F(\tau)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} d\tau \leq \int_0^t \|e^{\sqrt{c_0\tau}\Lambda_1}(u\cdot\nabla a + a\operatorname{div} u)\|_{\dot{B}^{\frac{d}{2}}_{2,1}} d\tau$$
$$\leq C\int_0^t \big(\|U\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|\nabla A\|_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|A\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \|\operatorname{div} U\|_{\dot{B}^{\frac{d}{2}}_{2,1}}\big) d\tau$$
$$\leq C\Big(\|U\|_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})} \|A\|_{L^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|A\|_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} \|U\|_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}\Big).$$

One can now turn to g: using Proposition 2.3 with $(s_1, s_2) = (\frac{d}{2}-1, \frac{d}{2})$ yields

$$\int_0^t \|e^{\sqrt{c_0\tau}\Lambda_1} g_1\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} d\tau = \int_0^t \|e^{\sqrt{c_0\tau}\Lambda_1}(u\cdot\nabla u)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} d\tau \leq C\|U\|^2_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}. \tag{2.36}$$

Using the same product law together with Lemma 2.5, and under the following condition that depends on the convergence radii of the analytic functions appearing in g:

$$\|e^{\sqrt{c_0 t}\Lambda_1} a\|_{\widetilde{L}^\infty(\dot{B}^{\frac{d}{2}}_{2,1})} \leq \frac{1}{2C}\min(R_I, R_{\widetilde{\mu}}, R_{\widetilde{\lambda}}, R_{\widetilde{\kappa}}, R_J), \tag{2.37}$$

we get that

$$\int_0^t \|e^{\sqrt{c_0\tau}\Lambda_1} g_3\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} d\tau \leq C\|A\|_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} \|U\|_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}. \tag{2.38}$$

Similarly, we obtain:

$$\begin{cases} \displaystyle\int_0^t \|e^{\sqrt{c_0\tau}\Lambda_1} g_2\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} d\tau \leq C(1+\|A\|_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})})\|A\|_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} \|U\|_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}, \\ \displaystyle\int_0^t \|e^{\sqrt{c_0\tau}\Lambda_1} g_4\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} d\tau \leq C\|A\|^2_{L^2_t(\dot{B}^{\frac{d}{2}}_{2,1})}, \\ \displaystyle\int_0^t \|e^{\sqrt{c_0\tau}\Lambda_1}\nabla(\widetilde{\kappa}(a)\Delta a)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} d\tau \leq C\|A\|_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})} \|A\|_{L^1_t(\dot{B}^{\frac{d}{2}+2}_{2,1})}. \end{cases} \tag{2.39}$$

We have to be careful with the second part of g_5: as Lemma 2.5 requires the regularity index to be less than $\frac{d}{2}$, we have to rewrite the term into:

$$\int_0^t \|e^{\sqrt{c_0\tau}\Lambda_1}\nabla(\nabla(\widetilde{\kappa}(a))\cdot\nabla a)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}} d\tau \leq \int_0^t \|e^{\sqrt{c_0\tau}\Lambda_1}(\widetilde{\kappa}'(a)\nabla a\cdot\nabla a)\|_{\dot{B}^{\frac{d}{2}}_{2,1}} d\tau$$
$$\leq C(1+\|A\|_{L^\infty_t(\dot{B}^{\frac{d}{2}}_{2,1})})\|A\|^2_{L^2_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}.$$

Putting all the above estimates together, we conclude the proof of Lemma 2.8. □

Now we are able to complete the proof of Theorem 1.2 by means of the fixed point theorem. Let $W(t)$ be the semi-group associated to the left-hand side of (2.3). According to the standard Duhamel formula, one has

$$\left(\begin{array}{c} a(t) \\ u(t) \end{array}\right) = \left(\begin{array}{c} a_L \\ u_L \end{array}\right) + \int_0^t W(t-\tau)\left(\begin{array}{c} f(\tau) \\ g(\tau) \end{array}\right) d\tau \quad \text{with} \quad \left(\begin{array}{c} a_L \\ u_L \end{array}\right) \triangleq W(t)\left(\begin{array}{c} a_0 \\ u_0 \end{array}\right).$$

Define the functional $\Psi_{(a_L,u_L)}$ in a neighborhood of zero in the space F by

$$\Psi_{(a_L,u_L)}(\bar{a},\bar{u}) = \int_0^t W(t-\tau)\left(\begin{array}{c} f(a_L+\bar{a},u_L+\bar{u}) \\ g(a_L+\bar{a},u_L+\bar{u}) \end{array}\right) d\tau. \tag{2.40}$$

To get the existence part of the theorem, it suffices to show that $\Psi_{(a_L,u_L)}$ has a fixed point in F. Our procedure is divided into two steps : stability of some closed ball $\mathcal{B}(0,r)$ of F by $\Psi_{(a_L,u_L)}$, then contraction in that ball. As those two properties have been established for the space E in [11], we shall concentrate on proving suitable bounds for $e^{\sqrt{c_0\tau}\Lambda_1}\Psi_{(a_L,u_L)}(\bar{a},\bar{u})$.

Step 1: Stability of some ball $\mathcal{B}(0,r)$. We prove that the ball $\mathcal{B}(0,r)$ of F is stable under $\Psi_{(a_L,u_L)}$, provided the radius r is small enough. Let $a = a_L + \bar{a}$ and $u = u_L + \bar{u}$. If the data fulfill (1.5), then from Lemmas 2.7, 2.8 and the definition of the space F, we get

$$\|e^{\sqrt{c_0\tau}\Lambda_1}(a_L,u_L)\|_E \leq C(\|a_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1}} + \|u_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}) \leq C\eta, \tag{2.41}$$

and

$$\|e^{\sqrt{c_0\tau}\Lambda_1}\Psi_{(a_L,u_L)}(\bar{a},\bar{u})\|_E \leq C\|e^{\sqrt{c_0\tau}\Lambda_1}(f,\nabla f,g)\|_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}. \tag{2.42}$$

Assuming that r is so small that:

$$\|e^{\sqrt{c_0 t}\Lambda_1}a\|_{L^\infty(\dot{B}^{\frac{d}{2}}_{2,1})} \leq \|(a,u)\|_F \leq r \leq \frac{1}{2C}\min(R_I, R_{\widetilde{\mu}}, R_{\widetilde{\lambda}}, R_{\widetilde{\kappa}}, R_J), \tag{2.43}$$

and also that $2C\eta \leq r$, we get

$$\begin{aligned} \|\Psi_{(a_L,u_L)}(\bar{a},\bar{u})\|_F &\leq C\|(a_L+\bar{a},u_L+\bar{u})\|_F^2\Big(1+\|(a_L+\bar{a},u_L+\bar{u})\|_F\Big) \\ &\leq C(C\eta+r)^2(1+C\eta+r) \leq C\frac{9}{4}r^2\Big(1+\frac{3}{2}r\Big). \end{aligned} \tag{2.44}$$

Finally, choosing (r,η) such that

$$r \leq \min\Big(1, \frac{8}{45C}, \frac{1}{2C}\min(R_I, R_{\widetilde{\mu}}, R_{\widetilde{\lambda}}, R_{\widetilde{\kappa}}, R_J)\Big) \quad \text{and} \quad \eta \leq \frac{r}{2C},$$

assumption (2.43) is satisfied. Hence, it follows from (2.44) that

$$\Psi_{(a_L,u_L)}(\mathcal{B}(0,r)) \subset \mathcal{B}(0,r).$$

Step 2: The contraction property. Let $(\bar{a}_1,\bar{u}_1)$ and $(\bar{a}_2,\bar{u}_2)$ be in $\mathcal{B}(0,r)$. Denote $a_i = a_L + \bar{a}_i$ and $u_i = u_L + \bar{u}_i$ for $i = 1,2$. According to (2.40) and Lemmas 2.7, 2.8, we have (as already explained, we focus on bounds for the Gevrey estimates, the estimates in E are in [11])

$$\begin{aligned} &\|\Psi_{(a_L,u_L)}(\bar{a}_2,\bar{u}_2) - \Psi_{(a_L,u_L)}(\bar{a}_1,\bar{u}_1)\|_F \\ &= \|e^{\sqrt{c_0\tau}\Lambda_1}\Big(f(a_2,u_2)-f(a_1,u_1), \nabla f(a_2,u_2)-\nabla f(a_1,u_1), g(a_2,u_2)-g(a_1,u_1)\Big)\|_{L^1(\dot{B}^{\frac{d}{2}-1}_{2,1})} \end{aligned}$$

where f and g are defined in (2.4). All terms are estimated exactly as in the previous step except that we use in addition Lemma 2.6. Let us for example give details for $g_2 = (1-I(a))\mathrm{div}\,(\widetilde{\mu}(a)\cdot\nabla a))$ (assume that $\widetilde{\lambda}=0$ for conciseness):

$$\begin{aligned}(2.45)\quad \|g_2(a_2,u_2)-g_2(a_1,u_1)\|_{L^1(\dot{B}^{\frac{d}{2}-1}_{2,1})} &\leq \|\big(I(a_2)-I(a_1)\big)\mathrm{div}\,(\widetilde{\mu}(a_2)\cdot\nabla a_2)\|_{L^1(\dot{B}^{\frac{d}{2}-1}_{2,1})}\\ &+\|(1-I(a_1))\mathrm{div}\,\big((\widetilde{\mu}(a_2)-\widetilde{\mu}(a_1))\nabla u_2+\widetilde{\mu}(a_1)\cdot\nabla(u_2-u_1)\big)\|_{L^1(\dot{B}^{\frac{d}{2}-1}_{2,1})}.\end{aligned}$$

Following the previous computations (together with Lemma 2.6 for the first and second terms), we obtain, if r and η are small enough,

$$\begin{aligned}&\|\Psi_{(a_L,u_L)}(\bar{a}_2,\bar{u}_2)-\Psi_{(a_L,u_L)}(\bar{a}_1,\bar{u}_1)\|_F\\ &\quad\leq C\Big(\|(a_1,u_1)\|_F+\|(a_2,u_2)\|_F\Big)\Big(1+\|(a_1,u_1)\|_F+\|(a_2,u_2)\|_F\Big)\|(\bar{a}_2-\bar{a}_1,\bar{u}_2-\bar{u}_1)\|_F\\ &\quad\leq 4C(r+C\eta)(1+r+C\eta)\|(\bar{a}_2-\bar{a}_1,\bar{u}_2-\bar{u}_1)\|_F\\ &\quad\leq \frac{1}{4}\|(\bar{a}_2-\bar{a}_1,\bar{u}_2-\bar{u}_1)\|_F.\end{aligned}$$

Hence, combining the two steps completes the proof of Theorem 1.2. □

3. The L^p framework

Our aim here is to extend Theorem 1.2 to more general critical Besov spaces. Recall that for the classical compressible Navier-Stokes equations, the first two authors [6] and Chen-Miao-Zhang [8] established a global existence result for small data in L^p type critical Besov spaces. The proofs therein are based on the study of the paralinearized system combined with a Lagrangian change of coordinates. A more elementary method has been proposed afterward by B. Haspot in [17]. It relies on the introduction of some suitable *effective velocity* that, somehow, allows to uncouple the velocity equation from the mass equation.

In the present section, by combining Haspot's approach with estimates in the same spirit as the previous section, we shall not only extend the critical regularity result in L^p spaces to the capillary case, but also obtain Gevrey analytic regularity:

Theorem 3.1. *Assume that the functions κ, λ, μ and P are real analytic and that the condition $P'(\overline{\varrho})>0$ is fulfilled. Let $p\in[2,\min(4,2d/(d-2))]$ with, additionally, $p\neq 4$ if $d=2$. There exists an integer $k_0\in\mathbb{N}$ and a real number $\eta>0$ depending only on the functions κ, λ, μ and P, and on p and d, such that if one defines the threshold between low and high frequencies as in* (A.11)*, if $a_0\in\dot{B}^{\frac{d}{p}}_{p,1}$ and $u_0\in\dot{B}^{\frac{d}{p}-1}_{p,1}$ with, besides, (a_0^ℓ,u_0^ℓ) in $\dot{B}^{\frac{d}{2}-1}_{2,1}$ satisfy*

$$(3.1)\qquad \mathcal{X}_{p,0}\triangleq \|(a_0,u_0)\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|a_0\|^h_{\dot{B}^{\frac{d}{p}}_{p,1}}+\|u_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\leq\eta,$$

then (2.3) *has a unique global-in-time solution (a,u) in the space X_p defined by*

$$\begin{aligned}X_p\triangleq\{(a,u)|(a,u)^\ell\in\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{2}+1}_{2,1}),a^h\in\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{\frac{d}{p}}_{p,1})\cap L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{p}+2}_{p,1}),\\ u^h\in\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{\frac{d}{p}-1}_{p,1})\cap L^1(\mathbb{R}_+;\dot{B}^{\frac{d}{p}+1}_{p,1})\}.\end{aligned}$$

Furthermore, there exists a constant c_0 so that (a,u) belongs to the space

$$Y_p\triangleq\{(a,u)\in X_p|e^{\sqrt{c_0}t\Lambda_1}(a,u)\in X_p\}.$$

Remark 3.1. *In the physical dimensions $d = 2, 3$, Condition* (3.1) *allows us to consider the case $p > d$, and the velocity regularity exponent $d/p - 1$ thus becomes negative. Therefore, our result applies to* large *highly oscillating initial velocities (see e.g.* [6] *for more explanations).*

3.1. **Global estimates in X_p for** (2.3)**.** As in Section 2, the proof of Theorem 3.1 is based on the fixed point theorem in complete metric spaces. Another important ingredient is the following endpoint maximal regularity property of the heat equation with *complex* diffusion coefficient.

Lemma 3.1. *Let $T > 0$, $s \in \mathbb{R}$ and $1 \leq \rho_2, p, r \leq \infty$. Let u satisfy*

$$\begin{cases} \partial_t u - \beta \Delta u = f, \\ u|_{t=0} = u_0(x), \end{cases} \tag{3.2}$$

where $\beta \in \mathbb{C}$ is a constant parameter with $\operatorname{Re}\beta > 0$. Then, there exists a constant C depending only on d and such that for all $\rho_1 \in [\rho_2, \infty]$, one has

$$(\operatorname{Re}\beta)^{\frac{1}{\rho_1}} \|u\|_{\widetilde{L}^{\rho_1}_T(\dot{B}^{s+\frac{2}{\rho_1}}_{p,r})} \leq C\Bigl(\|u_0\|_{\dot{B}^s_{p,r}} + (\operatorname{Re}\beta)^{\frac{1}{\rho_2}-1}\|f\|_{\widetilde{L}^{\rho_2}_T(\dot{B}^{s-2+\frac{2}{\rho_2}}_{p,r})}\Bigr)\cdot \tag{3.3}$$

Proof. We claim that there exists some absolute constants c and C such that

$$\|\dot{\Delta}_j e^{\beta t\Delta} z\|_{L^p} \leq C e^{-c\operatorname{Re}\beta\, t 2^{2j}} \|\dot{\Delta}_j z\|_{L^p}, \qquad t \geq 0, \quad j \in \mathbb{Z}. \tag{3.4}$$

Indeed, using a suitable rescaling, it suffices to prove (3.4) for $j = 0$. Now, if we fix some smooth function $\widetilde{\varphi}$ compactly supported away from 0 and with value 1 on φ, then we may write

$$\begin{aligned} e^{\beta t\Delta}\dot{\Delta}_0 z &= \mathcal{F}^{-1}\Bigl(\widetilde{\varphi} e^{-\beta t|\cdot|^2}\widehat{\dot{\Delta}_0 z}\Bigr) \\ &= g_{\beta t} \star \dot{\Delta}_0 z \quad \text{with} \quad g_{\beta t}(x) \triangleq (2\pi)^{-d}\int e^{ix\cdot\xi}\widetilde{\varphi}(\xi) e^{-\beta t|\xi|^2} d\xi. \end{aligned}$$

Then, integrating by parts, we discover that for all $x \in \mathbb{R}^d$, we have

$$g_{\beta t}(x) = (1 + |x|^2)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}(\mathrm{Id} - \Delta_\xi)^d\Bigl(\widetilde{\varphi}(\xi) e^{-\beta t|\xi|^2}\Bigr) d\xi.$$

Expanding the last term and using the fact that integration may be performed on some annulus, we get for some positive constants c, C and C',

$$\|g_{\beta t}\|_{L^1} \leq C\|(1 + |\cdot|^2)^d g_{\beta t}\|_{L^\infty} \leq C' e^{-ct\operatorname{Re}\beta}.$$

Then, using the convolution inequality $L^1 \star L^p \to L^p$ yields (3.4). From it, we get

$$\|\dot{\Delta}_j u(t)\|_{L^p} \leq C\Bigl(e^{-c\operatorname{Re}\beta\, 2^{2j}t}\|\dot{\Delta}_j u_0\|_{L^p} + \int_0^t e^{-c\operatorname{Re}\beta\, 2^{2j}(t-\tau)}\|\dot{\Delta}_j f(\tau)\|_{L^p}\, d\tau\Bigr)\cdot \tag{3.5}$$

Then, (3.3) follows from exactly the same calculations as in [3]. □

Combining Lemma 3.1 with the low frequency estimates of the previous section and introducing some suitable effective velocity will enable us to get the following result.

Lemma 3.2. *There exists some constant C such that for all $t \geq 0$,*

$$\mathcal{X}_p(t) \leq C(\mathcal{X}_{p,0} + \mathcal{X}_p^2(t) + \mathcal{X}_p^3(t)), \tag{3.6}$$

where

$$\mathcal{X}_p(t) \triangleq \|(a,u)\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|(a,u)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} + \|a\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})\cap L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})} + \|u\|^{h}_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})\cap L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}. \tag{3.7}$$

Proof. We start from the linearized system (2.3):

$$\begin{cases} \partial_t a + \operatorname{div} u = f, \\ \partial_t u - \overline{\mathcal{A}} u + \nabla a - \overline{\kappa}\nabla\Delta a = g. \end{cases}$$

The incompressible part of the velocity fulfills the heat equation

$$\partial_t \mathcal{P}u - \overline{\mu}\Delta\mathcal{P}u = \mathcal{P}g. \tag{3.8}$$

Hence, using the notation $z_j := \dot{\Delta}_j z$ for z in $\mathcal{S}'$, we see that there exists a constant $c>0$ such that we have for all $j\in\mathbb{Z}$,

$$\|\mathcal{P}u_j(t)\|_{L^p} \le Ce^{-c2^{2j}t}\Big(\|\mathcal{P}u_j(0)\|_{L^p} + \int_0^t e^{c2^{2j}\tau}\|\mathcal{P}g_j\|_{L^p}\,d\tau\Big), \tag{3.9}$$

which leads for all $T>0$, after summation on $j\ge k_0$, to

$$\|\mathcal{P}u\|^{h}_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|\mathcal{P}u\|^{h}_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})} \lesssim \|\mathcal{P}u_0\|^{h}_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|\mathcal{P}g\|^{h}_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}. \tag{3.10}$$

To estimate a and $\mathcal{Q}u$, following Haspot in [17], we introduce the modified velocity

$$v \triangleq \mathcal{Q}u + (-\Delta)^{-1}\nabla a$$

so that $\operatorname{div} v = \operatorname{div} u - a$, and discover that, since $\bar{\lambda}+2\bar{\mu}=1$,

$$\begin{cases} \partial_t\nabla a + \nabla a + \Delta v = \nabla f, \\ \partial_t v - \Delta v - \overline{\kappa}\Delta\nabla a = \mathcal{Q}g + (-\Delta)^{-1}\nabla f + v - (-\Delta)^{-1}\nabla a. \end{cases}$$

In the Fourier space, the eigenvalues of the associated matrix read (with the convention that $\sqrt{r} := i\sqrt{|r|}$ if $r<0$):

$$\lambda^{\pm}(\xi) = \frac{1}{2}\Big(1+|\xi|^2 \pm \sqrt{(1-4\overline{\kappa})|\xi|^4 - 2|\xi|^2 + 1}\Big).$$

Therefore, in the high frequency regime, we expect that for any $\overline{\kappa}>0$, the system has a parabolic behavior. This may be easily justified by considering suitable linear combinations of v and ∇a. Indeed, for all $\alpha\in\mathbb{C}$, we have

$$\partial_t(v+\alpha\nabla a) - (1-\alpha)\Delta v - \overline{\kappa}\Delta\nabla a + \alpha\nabla a = \alpha\nabla f + \mathcal{Q}g + (-\Delta)^{-1}\nabla f + v - (-\Delta)^{-1}\nabla a.$$

Therefore, if we set

$$w \triangleq v + \alpha\nabla a \quad \text{with} \quad \alpha \ \text{ satisfying } \ \alpha = \frac{\overline{\kappa}}{1-\alpha},$$

then we have

$$\partial_t w - (1-\alpha)\Delta w = -\alpha\nabla a + \alpha\nabla f + \mathcal{Q}g + (-\Delta)^{-1}\nabla f + v - (-\Delta)^{-1}\nabla a.$$

A possible choice is

$$\alpha = \frac{1}{2}\big(1+\sqrt{1-4\overline{\kappa}}\big) \quad \text{so that} \quad 1-\alpha = \frac{1}{2}\big(1-\sqrt{1-4\overline{\kappa}}\big).$$

Obviously, the real part of $1-\alpha$ is positive for any value of $\overline{\kappa}$. Hence one can take advantage of (3.3) and get

$$\begin{aligned}(3.11)\quad \|w\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|w\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}&\lesssim\|w_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\\&+\|\alpha\nabla f+\mathcal{Q}g+(-\Delta)^{-1}\nabla f\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|v-\alpha\nabla a-(-\Delta)^{-1}\nabla a\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}.\end{aligned}$$

Because $\nabla(-\Delta)^{-1}$ is an homogeneous Fourier multiplier of degree -1, we have

$$\begin{aligned}\|\alpha\nabla f+\mathcal{Q}g+(-\Delta)^{-1}\nabla f\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}&\lesssim\|f\|^h_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}+\|f-\mathrm{div}g\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-2}_{p,1})}\\&\lesssim\|f\|^h_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}+\|g\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}.\end{aligned}$$

Next, let us observe that, owing to the high frequency cut-off, we have for some universal constant C,

$$\|\alpha\nabla a\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}\leq C2^{-2k_0}\|a\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+2}_{p,1})},\qquad\|v\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}\leq C2^{-2k_0}\|v\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}$$
$$\text{and}\quad\|(-\Delta)^{-1}\nabla a\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}\leq C2^{-4k_0}\|a\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+2}_{p,1})}.$$

Consequently, it follows that

$$\begin{aligned}(3.12)\quad \|w\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|w\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}&\lesssim\|w_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|f\|^h_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}+\|g\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}\\&+2^{-2k_0}\|a\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+2}_{p,1})}+2^{-2k_0}\|v\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}+2^{-4k_0}\|a\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+2}_{p,1})}.\end{aligned}$$

Now, in order to estimate v, we use the fact that

$$(3.13)\qquad\nabla a=\frac{w-v}{\alpha}$$

so that the equation for v rewrites

$$\partial_t v-\frac{\alpha-\overline{\kappa}}{\alpha}\,\Delta v=\frac{\overline{\kappa}}{\alpha}\Delta w+\nabla(-\Delta)^{-1}(f-\mathrm{div}g)+v-(-\Delta)^{-1}\nabla a.$$

The important observation is that

$$\frac{\alpha-\overline{\kappa}}{\alpha}=\frac{\overline{\kappa}}{1-\alpha}\cdot$$

Hence one can again take advantage of (3.3), and get

$$\begin{aligned}\|v\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|v\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}&\lesssim\|v_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|\nabla(-\Delta)^{-1}(f-\mathrm{div}g)\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}\\&+\Big\|\frac{\overline{\kappa}}{\alpha}\Delta w+v-(-\Delta)^{-1}\nabla a\Big\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})},\end{aligned}$$

whence

$$\begin{aligned}(3.14)\quad \|v\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|v\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}&\lesssim\|v_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|g\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|f\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-2}_{p,1})}\\&+\|w\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}+2^{-2k_0}\|v\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}+2^{-4k_0}\|a\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+2}_{p,1})}.\end{aligned}$$

Plugging (3.12) in (3.14) and taking k_0 large enough, we arrive at

$$\|v\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|v\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim\|v_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|f\|^h_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}+\|g\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}+2^{-2k_0}\|a\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+2}_{p,1})}.$$

Then, inserting that latter inequality in (3.12) and using (3.13), we get

$$\|(\nabla a,v)\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|(\nabla a,v)\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim\|(\nabla a_0,v_0)\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|f\|^h_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}+\|g\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}.$$

Finally, keeping in mind that $u=v-(-\Delta)^{-1}\nabla a+\mathcal{P}u$, we conclude that

$$\begin{aligned}\|(\nabla a,u)\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|(\nabla a,u)\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}&\\ \lesssim\|(\nabla a_0,u_0)\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|f\|^h_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}+\|g\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}&.\end{aligned}\tag{3.15}$$

Let us next turn to estimates for the nonlinear terms. For the high frequencies of f, we just write that

$$\begin{aligned}\|f\|^h_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}&\lesssim\|a\|_{L^\infty_T(L^\infty)}\|u\|_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|u\|_{L^2_T(L^\infty)}\|a\|_{L^2_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}\\&\lesssim\|a\|_{L^\infty_T(\dot{B}^{\frac{d}{p}}_{p,1})}\|u\|_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|u\|_{L^2_T(\dot{B}^{\frac{d}{p}}_{p,1})}\|a\|_{L^2_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}\\&\lesssim\mathcal{X}^2_p(T).\end{aligned}\tag{3.16}$$

All terms in g, but $\nabla(\widetilde{\kappa}(a)\Delta a)$ and $\nabla(\widetilde{\kappa}'(a)|\nabla a|^2)$ have been treated in e.g. [17] for the classical compressible Navier-Stokes equations; they are bounded by the right-hand side of (3.6). Now, regarding the high frequencies of these two capillary terms, one can just use the fact that the space $\dot{B}^{\frac{d}{p}}_{p,1}$ is stable by product and composition, and get

$$\|\nabla(\widetilde{\kappa}(a)\Delta a)\|^h_{L^1(\dot{B}^{\frac{d}{p}-1}_{p,1})}\lesssim\|\widetilde{\kappa}(a)\Delta a\|_{L^1(\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim\|a\|_{L^\infty(\dot{B}^{\frac{d}{p}}_{p,1})}\|\Delta a\|_{L^1(\dot{B}^{\frac{d}{p}}_{p,1})}.$$

Similarly,

$$\begin{aligned}\|\nabla(\widetilde{\kappa}'(a)|\nabla a|^2)\|^h_{L^1(\dot{B}^{\frac{d}{p}-1}_{p,1})}&\lesssim\|\widetilde{\kappa}'(1)+(\widetilde{\kappa}'(a)-\widetilde{\kappa}'(1))|\nabla a|^2\|_{L^1(\dot{B}^{\frac{d}{p}}_{p,1})}\\&\lesssim(1+\|a\|_{L^\infty(\dot{B}^{\frac{d}{p}}_{p,1})})\|\nabla a\|^2_{L^2(\dot{B}^{\frac{d}{p}}_{p,1})}.\end{aligned}\tag{3.17}$$

To handle the low frequencies, one can use the fact that, owing to Lemma 2.1,

$$\|(a,u)\|^\ell_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})}\lesssim\|(a_0,u_0)\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|(f,g)\|^\ell_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}.\tag{3.18}$$

Again, taking advantage of prior works on the compressible Navier-Stokes equations, we just have to check that the capillary terms satisfy (3.6). Now, we have

$$\|\nabla(\widetilde{\kappa}(a)\Delta a)\|^\ell_{L^1(\dot{B}^{\frac{d}{2}-1}_{2,1})}\lesssim\|\widetilde{\kappa}(a)\Delta a\|^\ell_{L^1(\dot{B}^{\frac{d}{2}}_{2,1})}.$$

In order to estimate the r.h.s., we use the following Bony decomposition:

$$\widetilde{\kappa}(a)\Delta a=T_{\widetilde{\kappa}(a)}\Delta a+T_{\Delta a}\widetilde{\kappa}(a)+R(\widetilde{\kappa}(a),\Delta a).$$

Recall that $T:\dot{B}^{\frac{d}{p}-1}_{p,1}\times\dot{B}^{\frac{d}{p}}_{p,1}\to\dot{B}^{\frac{d}{2}-1}_{2,1}$ for $2\leq p\leq\min(4,\frac{2d}{d-2})$. Hence we have

$$\begin{aligned}\|T_{\widetilde{\kappa}(a)}\Delta a + T_{\Delta a}\widetilde{\kappa}(a)\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}} &\lesssim \|T_{\widetilde{\kappa}(a)}\Delta a + T_{\Delta a}\widetilde{\kappa}(a)\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\ &\lesssim \|\widetilde{\kappa}(a)\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|\Delta a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|\Delta a\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|\widetilde{\kappa}(a)\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\\ &\lesssim \|a\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|\Delta a\|_{\dot{B}^{\frac{d}{p}}_{p,1}}.\end{aligned}$$

For the remainder term, one can use that $R : \dot{B}^{\frac{d}{p}}_{p,1} \times \dot{B}^{\frac{d}{p}}_{p,1} \to \dot{B}^{\frac{d}{2}}_{2,1}$ if $2 \leq p \leq 4$. Hence we eventually get, if $p \leq \min(4, \frac{2d}{d-2})$,

$$\|\nabla(\widetilde{\kappa}(a)\Delta a)\|^{\ell}_{L^1(\dot{B}^{\frac{d}{2}-1}_{2,1})} \lesssim \|a\|_{L^\infty(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}-1}_{p,1})}\|\Delta a\|_{L^1(\dot{B}^{\frac{d}{p}-1}_{p,1})}.$$

In order to estimate the other capillary term, we simply use that $\widetilde{\kappa}'(a)\nabla a = \nabla(\widetilde{\kappa}(a))$, with $\widetilde{\kappa}(0) = 0$. Now, thanks to Bony's decomposition:

$$\nabla a \cdot \nabla(\widetilde{\kappa}(a)) = T_{\nabla a}\nabla(\widetilde{\kappa}(a)) + T_{\nabla(\widetilde{\kappa}(a))}\nabla a + R(\nabla(\widetilde{\kappa}(a)), \nabla a).$$

and to

$$\begin{aligned}\|T_{\nabla a}\nabla(\widetilde{\kappa}(a)) + T_{\nabla(\widetilde{\kappa}(a))}\nabla a\|^{\ell}_{\dot{B}^{\frac{d}{2}}_{2,1}} &\lesssim \|\nabla a\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|\nabla(\widetilde{\kappa}(a))\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|\nabla(\widetilde{\kappa}(a))\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|\nabla a\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\\ &\lesssim \|a\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|\nabla a\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\end{aligned}$$

and

$$\|R(\nabla(\widetilde{\kappa}(a)), \nabla a)\|_{\dot{B}^{\frac{d}{2}}_{2,1}} \lesssim \|\nabla(\widetilde{\kappa}(a))\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|\nabla a\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \lesssim \|a\|_{\dot{B}^{\frac{d}{p}+1}_{p,1}}\|\nabla a\|_{\dot{B}^{\frac{d}{p}}_{p,1}},$$

we end up with

$$\|\nabla(\kappa'(a)|\nabla a|^2)\|^{\ell}_{L^1(\dot{B}^{\frac{d}{2}-1}_{2,1})} \lesssim \|a\|_{L^2(\dot{B}^{\frac{d}{p}}_{p,1}\cap\dot{B}^{\frac{d}{p}+1}_{p,1})}\|\nabla a\|_{L^2(\dot{B}^{\frac{d}{p}}_{p,1})}.$$

Combining with the already proved estimates for the other nonlinear terms (see [10]), we conclude that (3.6) is fulfilled. From this, it is not difficult to work out a fixed point argument as in the previous section, and to prove the first part of Theorem 3.1. □

3.2. **More paraproduct, remainder and product estimates.** In order to investigate the Gevrey regularity of solutions in the L^p framework, resorting only to Propositions 2.1-2.2 does not allow to get suitable bounds for the low frequency part of some nonlinear terms. The goal of this short subsection is to establish more estimates for the paraproduct, remainder operators in L^2 based Besov spaces, when the two functions under consideration belong to some L^p type Besov space.

Proposition 3.1. *Assume that* $2 \leq p \leq \min(4, \frac{2d}{d-2})$ *and* $s \in \mathbb{R}$. *There exists a constant* $C > 0$ *such that*

$$\|e^{\sqrt{c_0 t}\Lambda_1}T_f g\|_{\dot{B}^s_{2,1}} \leq C\|F\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|G\|_{\dot{B}^{s+1-\frac{d}{2}+\frac{d}{p}}_{p,1}} \quad \text{with} \quad F \triangleq e^{\sqrt{t}\Lambda_1}f \quad \text{and} \quad G \triangleq e^{\sqrt{t}\Lambda_1}g.$$

Proof. If $p > 2$ then we define p^* by the relation $\frac{1}{2} = \frac{1}{p} + \frac{1}{p^*}$. Then applying inequality (2.18) with the exponents $(s, \sigma, p, p_1, p_2, r, r_1, r_2) = (s + 1 - \frac{d}{p^*}, 1 - \frac{d}{p^*}, 2, p^*, p, 1, 1, \infty)$

which is possible since $p^* \geq p$ (that is $2 \leq p \leq 4$) and $-\sigma \triangleq \frac{d}{p^*} - 1 \leq 0$ (or, equivalently, $p \leq \frac{2d}{d-2}$), we get

$$\|e^{\sqrt{c_0}t\Lambda_1} T_f g\|_{\dot{B}^s_{2,1}} \leq C\|F\|_{\dot{B}^{\frac{d}{p^*}-1}_{p^*,1}} \|G\|_{\dot{B}^{s+1-\frac{d}{p^*}}_{p,\infty}}.$$

Then using the embedding $\dot{B}^{\frac{d}{p}}_{p,1} \hookrightarrow \dot{B}^{\frac{d}{p^*}}_{p^*,1}$ (note that $p^* \geq p$) and $\dot{B}^{s+1-\frac{d}{2}+\frac{d}{p}}_{p,1} \hookrightarrow \dot{B}^{\frac{d}{p^*}}_{p^*,1}$ gives the desired inequality.

The endpoint case $p = 2$ stems from (2.20) with the exponents $(s, \sigma, p, q, r, r_1, r_2) = (s+1, 1, 2, 2, 1, 1, \infty)$. □

As a consequence of Proposition 2.1 and of the embedding $\dot{B}^{\sigma+d(\frac{2}{p}-\frac{1}{2})}_{p/2,1} \hookrightarrow \dot{B}^{\sigma}_{2,1}$ for any $2 \leq p \leq 4$ and $\sigma \in \mathbb{R}$, we readily get:

Proposition 3.2. *Let $d \geq 2$ and $2 \leq p \leq 4$. If $s_1 + s_2 > d(\frac{1}{2} - \frac{2}{p})$ then there exists a constant $C > 0$ such that*

$$\|e^{\sqrt{c_0}t\Lambda_1} R(f,g)\|_{\dot{B}^{s_1+s_2}_{2,1}} \leq C\|F\|_{\dot{B}^{s_1+d\left(\frac{2}{p}-\frac{1}{2}\right)}_{p,1}} \|G\|_{\dot{B}^{s_2}_{p,1}}. \tag{3.19}$$

Proposition 3.3. *Assume that $2 \leq p \leq \min(4, \frac{2d}{d-2})$ and $p < 2d$. There exists a constant $C > 0$ such that:*

$$\begin{cases} \|e^{\sqrt{c_0\tau}\Lambda_1}(fg)\|^\ell_{\dot{B}^{\frac{d}{2}}_{2,1}} \lesssim \|F\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \|G\|_{\dot{B}^{\frac{d}{p}+1}_{p,1}} + \|F\|_{\dot{B}^{\frac{d}{p}+1}_{p,1}} \|G\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}, \\ \|e^{\sqrt{c_0\tau}\Lambda_1}(fg)\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \lesssim \|F\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \|G\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|F\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \|G\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}, \\ \|e^{\sqrt{c_0\tau}\Lambda_1}(fg)\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \lesssim \|F\|_{\dot{B}^{\frac{d}{p}-1}_{p,1} \cap \dot{B}^{\frac{d}{p}}_{p,1}} \|G\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}. \end{cases} \tag{3.20}$$

Proof. From Bony's decomposition, we have

$$\|e^{\sqrt{c_0\tau}\Lambda_1}(fg)\|^\ell_{\dot{B}^{\frac{d}{2}}_{2,1}} = \|e^{\sqrt{c_0\tau}\Lambda_1}(T_f g + T_g f + R(f,g))\|^\ell_{\dot{B}^{\frac{d}{2}}_{2,1}}.$$

Thanks to Propositions 3.1 and 3.2 (with $(s, s_1, s_2) = (\frac{d}{2}, \frac{d}{2} - \frac{d}{p} - 1, \frac{d}{p} + 1)$), we get that:

$$\begin{cases} \|e^{\sqrt{c_0\tau}\Lambda_1}(T_f g)\|^\ell_{\dot{B}^{\frac{d}{2}}_{2,1}} + \|e^{\sqrt{c_0\tau}\Lambda_1}(R(f,g))\|^\ell_{\dot{B}^{\frac{d}{2}}_{2,1}} \lesssim \|F\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \|G\|_{\dot{B}^{\frac{d}{p}+1}_{p,1}}, \\ \|e^{\sqrt{c_0\tau}\Lambda_1}(T_g f)\|^\ell_{\dot{B}^{\frac{d}{2}}_{2,1}} \lesssim \|G\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \|F\|_{\dot{B}^{\frac{d}{p}+1}_{p,1}}. \end{cases}$$

The second estimate is proved the same way but with $(s, s_1, s_2) = (\frac{d}{2} - 1, \frac{d}{2} - \frac{d}{p} - 1, \frac{d}{p})$. For the last estimate, we write that, taking advantage of the low frequency cut-off,

$$\|e^{\sqrt{c_0\tau}\Lambda_1}(fg)\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}} \lesssim \|e^{\sqrt{c_0\tau}\Lambda_1}(T_f g + T_g f)\|^\ell_{\dot{B}^{\frac{d}{2}-2}_{2,1}} + \|e^{\sqrt{c_0\tau}\Lambda_1} R(f,g))\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}}.$$

The last term may be bounded as before, and for the first two terms, we apply Proposition 3.1 with $s = \frac{d}{2} - 2$. □

3.3. **A priori estimates for Gevrey regularity.** That paragraph is devoted to proving estimates for Gevrey regularity in the L^p Besov framework. This will be based on the following lemma.

Lemma 3.3. *If (a,u) satisfies (2.3), then the following a priori estimate holds true:*

$$\|(a,u)\|_{Y_p} \leq C\big(\mathcal{X}_{p,0} + \|(a,u)\|_{Y_p}^2 + \|(a,u)\|_{Y_p}^3\big)\cdot \tag{3.21}$$

Proof. Summing up inequality (2.35) for $j \leq k_0$, we get for all $t \geq 0$,

$$\begin{aligned}\|(A,U)\|^\ell_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \|(A,U)\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})} &\\ \lesssim \|(a_0,u_0)\|^\ell_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \|F\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} &+ \|G\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\cdot\end{aligned} \tag{3.22}$$

Regarding the high frequency estimates, we plan to repeat the computations of the previous section after introducing $e^{\sqrt{c_0}t\Lambda_1}$ everywhere. Now, using again the auxiliary functions

$$v \triangleq \mathcal{Q}u + (-\Delta)^{-1}\nabla a \quad\text{and}\quad w \triangleq v + \alpha\nabla a \quad\text{with}\quad \alpha = \frac{1}{2}\big(1+\sqrt{1-4\overline{\kappa}}\big),$$

and setting $\widetilde{\alpha} \triangleq 1-\alpha$ and $\widetilde{g} \triangleq \mathcal{Q}g + (-\Delta)^{-1}\nabla f + v - (-\Delta)^{-1}\nabla a$, we discover that

$$w(t) = e^{\widetilde{\alpha}t\Delta}w_0 + \int_0^t e^{\widetilde{\alpha}(t-\tau)\Delta}(-\alpha\nabla a + \alpha\nabla f + \widetilde{g})(\tau)\,d\tau.$$

Hence $W(t) \triangleq e^{\sqrt{c_0}t\Lambda_1}w(t)$ fulfills (with obvious notation):

$$W(t) = e^{\sqrt{t}\Lambda_1 + \widetilde{\alpha}t\Delta}w_0 + \int_0^t e^{[(\sqrt{t}-\sqrt{\tau})\Lambda_1 + \widetilde{\alpha}(t-\tau)\Delta]}(-\alpha\nabla A + \alpha\nabla F + \widetilde{G})(\tau)\,d\tau.$$

It follows from Lemmas 2.2-2.3 that for the same threshold k_0 as in (3.11) and (3.12), we have

$$\begin{aligned}\|W\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|W\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})} &\lesssim \|w_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|A\|^h_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})} + \|F\|^h_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})} + \|\widetilde{G}\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}\\ &\lesssim \|w_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + 2^{-2k_0}\|A\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+2}_{p,1})} + 2^{-2k_0}\|V\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}\\ &\quad + \|F\|^h_{L^1_T(\dot{B}^{\frac{d}{p}}_{p,1})} + \|G\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}\cdot\end{aligned}$$

Then one can revert to v as in (3.13), applying $e^{\sqrt{c_0}t\Lambda_1}$ to:

$$\partial_t v - \frac{\overline{\kappa}}{1-\alpha}\Delta v = \frac{\overline{\kappa}}{\alpha}\Delta w + \widetilde{g}.$$

Denoting $V \triangleq e^{\sqrt{c_0}t\Lambda_1}v$ and following the procedure leading to (3.14), one gets

$$\begin{aligned}\|V\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})} + \|V\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})} &\lesssim \|W\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})} + \|\widetilde{G}\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}\\ &\lesssim \|v_0\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}} + \|W\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})} + 2^{-2k_0}\|V\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}\\ &\quad + 2^{-4k_0}\|A\|^h_{L^1_T(\dot{B}^{\frac{d}{p}+2}_{p,1})} + \|F\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-2}_{p,1})} + \|G\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}\cdot\end{aligned}$$

For the incompressible part of velocity, applying $e^{\sqrt{c_0}t\Lambda_1}$ to (3.8) yields

$$\|\mathcal{P}U\|^h_{\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|\mathcal{P}U\|^h_{\widetilde{L}^1_T(\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim\|\mathcal{P}u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|G\|^h_{L^1_T(\dot{B}^{\frac{d}{p}-1}_{p,1})}. \tag{3.23}$$

Therefore, taking the same large enough k_0 as in the previous section, and using (3.13), we deduce that

$$\begin{aligned}\|(\nabla A,U)\|^h_{\widetilde{L}^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}+\|(\nabla A,U)\|^h_{\widetilde{L}^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}&\\ \lesssim\|(\nabla a_0,u_0)\|^h_{\dot{B}^{\frac{d}{p}-1}_{p,1}}+\|F\|^h_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|G\|^h_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}.&\end{aligned} \tag{3.24}$$

Putting together with (3.22), we end up with

$$\begin{aligned}\|(A,U)\|_{X_p(t)}\lesssim X_{p,0}+\|F\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}}_{2,1})}+\|G\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}&\\ +\|F\|^h_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|G\|^h_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}.&\end{aligned} \tag{3.25}$$

All that remains to do is to bound F and G, which will be strongly based on Proposition 3.3 as regards the low frequencies.

Let us start with F. Then, thanks to $(3.20)_1$ and Besov injections (as $p\geq 2$), we get

$$\begin{aligned}\|F\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}&\lesssim\|A\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|U\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|U\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|A\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\\ &\lesssim\Big(\|A\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}+\|A\|^h_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\Big)\Big(\|U\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}+\|U\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\Big)\\ &\quad+\Big(\|A\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}+\|A\|^h_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\Big)\Big(\|U\|^\ell_{L^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}+\|U\|^h_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\Big)\\ &\lesssim\|(a,u)\|^2_{Y_p}.\end{aligned}$$

Next, we bound the norm $\|G\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}$. Using $(3.20)_2$ we obtain

$$\begin{aligned}\|G_1\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}&=\|e^{\sqrt{c_0\tau}\Lambda_1}(u\cdot\nabla u)\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\\ &\lesssim\|U\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|U\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|U\|^2_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\lesssim\|(a,u)\|^2_{Y_p}.\end{aligned}$$

Let us now turn to $G_3=-e^{\sqrt{c_0\tau}\Lambda_1}(I(a)\overline{\mathcal{A}}u)$. Thanks to $(3.20)_3$ and Proposition 2.5:

$$\begin{aligned}\|G_3\|^\ell_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}&\lesssim\int_0^t\|e^{\sqrt{c_0\tau}\Lambda_1}I(a)\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1}}\|U\|_{\dot{B}^{\frac{d}{p}+1}_{p,1}}d\tau\\ &\lesssim\|A\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1}\cap\dot{B}^{\frac{d}{p}}_{p,1})}\|U\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\lesssim\|(a,u)\|^2_{Y_p}.\end{aligned}$$

Similarly, we estimate $G_4 \triangleq e^{\sqrt{c_0\tau}\Lambda_1}(J(a)\nabla a)$ using $(3.20)_2$ and Proposition 2.5:

$$\begin{aligned}\|G_4\|^\ell_{L^1_t(\dot B^{\frac{d}{2}-1}_{2,1})} &\lesssim \int_0^t \Big(\|e^{\sqrt{c_0\tau}\Lambda_1}J(a)\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\|\nabla A\|_{\dot B^{\frac{d}{p}}_{p,1}} + \|e^{\sqrt{c_0\tau}\Lambda_1}J(a)\|_{\dot B^{\frac{d}{p}}_{p,1}}\|\nabla A\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\Big)d\tau\\ &\lesssim \int_0^t \Big(\|A\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\|\nabla A\|_{\dot B^{\frac{d}{p}}_{p,1}} + \|A\|_{\dot B^{\frac{d}{p}}_{p,1}}\|\nabla A\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\Big)d\tau\\ &\lesssim \|(a,u)\|^2_{Y_p}.\end{aligned}$$

In order to bound the term corresponding to g_3, it suffices to consider $\widetilde G_3 \triangleq e^{\sqrt{c_0\tau}\Lambda_1}(1-I(a))\nabla(\widetilde\mu(a)\nabla u)$, the other term being similar. Now, we have:

$$\|G_2\|^\ell_{L^1_t(\dot B^{\frac{d}{2}-1}_{2,1})} \le \underbrace{\|e^{\sqrt{c_0\tau}\Lambda_1}\big(\widetilde\mu(a)\nabla u\big)\|^\ell_{L^1_t(\dot B^{\frac{d}{2}}_{2,1})}}_{I} + \underbrace{\|e^{\sqrt{c_0\tau}\Lambda_1}\Big(I(a)\nabla\big(\widetilde\mu(a)\nabla u\big)\Big)\|^\ell_{L^1_t(\dot B^{\frac{d}{2}-1}_{2,1})}}_{II}.$$

The first term may be bounded (taking once again advantage of the low frequencies cut-off) according to $(3.20)_2$ and Proposition 2.5 as follows:

$$\begin{aligned}I &\lesssim \|e^{\sqrt{c_0\tau}\Lambda_1}\big(\widetilde\mu(a)\nabla u\big)\|^\ell_{L^1_t(\dot B^{\frac{d}{2}-1}_{2,1})}\\ &\lesssim \int_0^t \Big(\|e^{\sqrt{c_0\tau}\Lambda_1}\big(\widetilde\mu(a)\big)\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\|U\|_{\dot B^{\frac{d}{p}+1}_{p,1}} + \|e^{\sqrt{c_0\tau}\Lambda_1}\big(\widetilde\mu(a)\big)\|_{\dot B^{\frac{d}{p}}_{p,1}}\|U\|_{\dot B^{\frac{d}{p}}_{p,1}}\Big)d\tau.\end{aligned}$$

The second term is bounded using $(3.20)_3$ and Propositions 2.5 and 2.3:

$$\begin{aligned}II &\lesssim \int_0^t \|e^{\sqrt{c_0\tau}\Lambda_1}I(a)\|_{\dot B^{\frac{d}{p}-1}_{p,1}\cap\dot B^{\frac{d}{p}}_{p,1}}\|e^{\sqrt{c_0\tau}\Lambda_1}\nabla\big(\widetilde\mu(a)\nabla u\big)\|_{\dot B^{\frac{d}{p}-1}_{p,1}}d\tau\\ &\lesssim \int_0^t \|A\|_{\dot B^{\frac{d}{p}-1}_{p,1}\cap\dot B^{\frac{d}{p}}_{p,1}}\|e^{\sqrt{c_0\tau}\Lambda_1}\widetilde\mu(a)\|_{\dot B^{\frac{d}{p}}_{p,1}}\|U\|_{\dot B^{\frac{d}{p}+1}_{p,1}}\,d\tau.\end{aligned}$$

We finally obtain that

$$\|G_2\|^\ell_{L^1_t(\dot B^{\frac{d}{2}-1}_{2,1})} \lesssim (1+\|(a,u)\|_{Y_p})\|(a,u)\|^2_{Y_p}.$$

To bound the capillary terms, we use $(3.20)_2$, writing that

$$\begin{aligned}\|e^{\sqrt{c_0\tau}\Lambda_1}\nabla(\widetilde\kappa(a)\Delta a)\|^\ell_{L^1_t(\dot B^{\frac{d}{2}-1}_{2,1})} &\lesssim \|e^{\sqrt{c_0\tau}\Lambda_1}(\widetilde\kappa(a)\Delta a)\|^\ell_{L^1_t(\dot B^{\frac{d}{2}-1}_{2,1})}\\ &\lesssim \|e^{\sqrt{c_0\tau}\Lambda_1}\widetilde\kappa(a)\|_{L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}\|\Delta A\|_{L^1_t(\dot B^{\frac{d}{p}}_{p,1})}\\ &\qquad + \|\Delta A\|_{L^1_t(\dot B^{\frac{d}{p}-1}_{p,1})}\|e^{\sqrt{c_0\tau}\Lambda_1}\widetilde\kappa(a)\|_{L^\infty_t(\dot B^{\frac{d}{p}}_{p,1})}\\ &\lesssim \|A\|_{L^\infty_t(\dot B^{\frac{d}{p}-1}_{p,1})}\|A\|_{L^1_t(\dot B^{\frac{d}{p}+2}_{p,1})} + \|A\|_{L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})}\|A\|_{L^\infty_t(\dot B^{\frac{d}{p}}_{p,1})}.\end{aligned}$$

As we just have to bound the low frequencies, one gets thanks to $(3.20)_2$,

$$\begin{aligned}\Big\|e^{\sqrt{c_0\tau}\Lambda_1}\nabla\Big(\frac{1}{2}\nabla\widetilde{\kappa}(a)\cdot\nabla a\Big)\Big\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})} &\lesssim \|e^{\sqrt{c_0\tau}\Lambda_1}(\nabla\widetilde{\kappa}(a)\cdot\nabla a)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\\ &\lesssim \int_0^t\Big(\|e^{\sqrt{c_0\tau}\Lambda_1}(\nabla\widetilde{\kappa}(a))\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\|\nabla A\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\\ &\qquad+\|e^{\sqrt{c_0\tau}\Lambda_1}(\nabla\widetilde{\kappa}(a))\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|\nabla A\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\Big)d\tau.\end{aligned}$$

Thanks to Proposition 2.5 we see that the first term is bounded by:

$$\begin{aligned}\|e^{\sqrt{c_0\tau}\Lambda_1}\widetilde{\kappa}(a)\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|A\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})} &\lesssim \|A\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|A\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}\\ &\lesssim \Big(\|A\|^{\ell}_{L^\infty_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}+\|A\|^{h}_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\Big)\Big(\|A\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}+1}_{2,1})}+\|A\|^{h}_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\Big).\end{aligned}$$

We have to be careful for the last term as $\frac{d}{p}+1$ is not in the range of Proposition 2.5. However, we have $\nabla\widetilde{\kappa}(a)=\widetilde{\kappa}'(a)\nabla a$ and thus,

$$\begin{aligned}&\int_0^t\|e^{\sqrt{c_0\tau}\Lambda_1}(\widetilde{\kappa}'(a)\nabla a)\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|A\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\,d\tau\\ &\qquad\lesssim\int_0^t\Big(\|e^{\sqrt{c_0\tau}\Lambda_1}\big(\widetilde{\kappa}'(a)-\widetilde{\kappa}'(0)\big)\|_{\dot{B}^{\frac{d}{p}}_{p,1}}+|\widetilde{\kappa}'(0)|\Big)\|\nabla A\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|A\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\,d\tau\\ &\qquad\lesssim(1+\|A\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})})\|A\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|A\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})},\end{aligned}$$

which enables us to obtain:

$$\|e^{\sqrt{c_0\tau}\Lambda_1}g_5(\tau)\|^{\ell}_{L^1_t(\dot{B}^{\frac{d}{2}-1}_{2,1})}\lesssim(1+\|(a,u)\|_{Y_p})\|(a,u)\|^2_{Y_p}.$$

To complete the proof, we need to bound the high frequencies of F and G. This turns out to be rather straightforward, as we only need Proposition 2.4 and Lemma 2.5. More precisely, we get

$$\begin{aligned}\|F\|^{h}_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} &\lesssim \|e^{\sqrt{c_0\tau}\Lambda_1}(a\operatorname{div}u+u\cdot\nabla a)\|^{h}_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\ &\lesssim \|A\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|\operatorname{div}U\|_{L^1_t(\dot{B}^{\frac{d}{p}}_{p,1})}+\|U\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|\nabla A\|_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\end{aligned}$$

and

$$\begin{aligned}\|G\|^{h}_{L^1_t(\dot{B}^{\frac{d}{p}-1}_{p,1})} &\lesssim \|U\|_{L^\infty_t(\dot{B}^{\frac{d}{p}-1}_{p,1})}\|U\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|A\|^2_{L^2_t(\dot{B}^{\frac{d}{p}}_{p,1})}\\ &\quad+\|A\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|U\|_{L^1_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}+\|A\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})}\|A\|_{L^1_t(\dot{B}^{\frac{d}{p}+2}_{p,1})}\\ &\quad+(1+\|A\|_{L^\infty_t(\dot{B}^{\frac{d}{p}}_{p,1})})\|A\|^2_{L^2_t(\dot{B}^{\frac{d}{p}+1}_{p,1})}.\end{aligned}$$

Putting all the previous estimates together ends the proof of Lemma 3.3. □

Finally, as in the previous section, using a suitable contracting mapping argument enables us to complete the proof of Theorem 3.1. The details are left to the reader. As for uniqueness, it stems from [11, Thm. 5].

3.4. **The time-decay of solutions in Besov spaces.** The aim of this part is to exhibit the time-decay properties of the solutions that have been constructed in Theorems 1.2 and 3.1. Those properties will come up as a consequence of the following lemma.

Lemma 3.4. *There exists a universal constant $c>0$ such that for all $s\in\mathbb{R}$, there exists a constant C_s such that for any tempered distribution u, real number $\alpha>0$ and integer $j\in\mathbb{Z}$, the following inequality holds true:*

$$\|\Lambda^s e^{-\alpha\Lambda_1}\dot\Delta_j u\|_{L^p}\leq C_s 2^{js}e^{-c\alpha 2^j}\|\dot\Delta_j u\|_{L^p}. \tag{3.1}$$

Proof. The starting point is the fact that, by definition of operator $e^{-\alpha\Lambda_1}$, we have for all $v\in\mathcal{S}'(\mathbb{R}^d)$,

$$e^{-\alpha\Lambda_1}v=h_\alpha\star v\quad\text{with}\quad h_\alpha=\mathcal{F}^{-1}(e^{-\alpha|\cdot|_1}).$$

Now, we notice that h_α is nonnegative, since

$$\int_{\mathbb{R}}e^{-|\eta|}e^{ix\eta}\,d\eta=\frac{2}{1+x^2}$$

and, owing to the definition of $|\xi|_1$, we have

$$\mathcal{F}^{-1}(e^{-\alpha|\cdot|_1})(x)=\frac{1}{(2\pi)^d}\prod_{j=1}^d\biggl(\int_{\mathbb{R}}e^{-\alpha|\xi_j|}e^{ix_j\xi_j}\,d\xi_j\biggr).$$

Therefore

$$\|h_\alpha\|_{L^1}=\int_{\mathbb{R}^d}h_\alpha(x)\,dx=\mathcal{F}(\mathcal{F}^{-1}(e^{-\sqrt{\alpha}|\cdot|_1}))(0)=1.$$

From this, we deduce by Young inequality that for all $\alpha\geq 0$,

$$\|e^{-\alpha\Lambda_1}v\|_{L^p}\leq\|v\|_{L^p}. \tag{3.2}$$

In order to get (3.1) for $s=0$, one has to refine the argument. First, performing a suitable rescaling reduces the proof to the case $j=0$. Then we introduce a family $(\phi_k)_{1\leq k\leq d}$ of smooth functions on $\mathbb{R}^d$ such that

(1) $\operatorname{Supp}\phi_k\subset\bigl\{\xi\in\mathbb{R}^d\,,\ \frac34\leq|\xi|\leq\frac83\quad\text{and}\quad\frac{3}{4\sqrt{d}}\leq|\xi_k|\bigr\}$;
(2) $\sum_{k=1}^d\phi_k\equiv 1$ on $\operatorname{Supp}\varphi$, where φ is the function used in the definition of the Littlewood-Paley decomposition.

As we obviously have

$$e^{-\alpha|\xi|_1}\mathcal{F}(\dot\Delta_0 u)(\xi)=\sum_{k=1}^d(e^{-\alpha|\xi|_1}\phi_k(\xi))\,\mathcal{F}(\dot\Delta_0 u)(\xi),$$

one may write

$$e^{-\alpha\Lambda_1}\dot\Delta_0 u=\sum_{k=1}^d h_k\star\dot\Delta_0 u\quad\text{with}\quad h_k\triangleq\mathcal{F}^{-1}(e^{-\alpha|\cdot|_1}\phi_k).$$

If we prove that for some $c>0$ and $C>0$ independent of α, we have

$$\|h_k\|_{L^1}\leq C\biggl(\frac{1+\alpha}{\alpha}\biggr)^d e^{-c\alpha}, \tag{3.3}$$

then, combining with (3.2) will complete the proof of the lemma for $s=0$.

Let us prove (3.3) for $k=1$ (the other cases being similar). Then we introduce the notation $\xi=(\xi_1,\xi')$ and $x=(x_1,x')$. Since $(\alpha^2+x_1^2)e^{ix\cdot\xi}=(\alpha^2-\partial^2_{\xi_1\xi_1})(e^{ix\cdot\xi})$, integrating by parts with respect to the variable ξ_1 in the integral defining h_1 yields:

$$(\alpha^2+x_1^2)h_1(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{ix\cdot\xi}e^{-\alpha|\xi'|_1}(\alpha^2-\partial_{11}^2)(\phi_1(\xi)e^{-\alpha|\xi_1|})\,d\xi.$$

Now, let us observe that

$$(e^{-\alpha|r|})'=-\alpha e^{-\alpha|r|}\operatorname{sgn}r\quad\text{and}\quad\alpha^2e^{-\alpha|r|}-(e^{-\alpha|r|})''=2\alpha\delta_0.$$

Therefore,

$$(\alpha^2-\partial_{11}^2)(\phi_1(\xi)e^{-\alpha|\xi_1|})=2\alpha\phi_1(0,\xi')\delta_{\xi_1=0}+e^{-\alpha|\xi_1|}\bigl(2\alpha\operatorname{sgn}(\xi_1)\partial_1\phi_1(\xi)-\partial_{11}^2\phi_1(\xi)\bigr),$$

and thus (taking advantage of the fact that $\phi_1(0,\xi')=0$)

$$(\alpha^2+x_1^2)h_1(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{ix\cdot\xi}e^{-\alpha|\xi|_1}(2\alpha\operatorname{sgn}(\xi_1)\partial_1\phi_1(\xi)-\partial_{11}^2\phi_1(\xi))\,d\xi\cdot$$

Multiplying by $(\alpha^2+x_2^2)$, the same arguments lead to (denoting $\xi_2'=(\xi_1,0,\xi_2,...,\xi_d)$ and $\phi_1^2(\xi)=2\alpha\operatorname{sgn}(\xi_1)\partial_1\phi_1(\xi)-\partial_{11}^2\phi_1(\xi)$)

$$\begin{aligned}(\alpha^2+x_1^2)(\alpha^2+x_2^2)h_1(x)=\frac{1}{(2\pi)^d}\biggl(&2\alpha\int_{\mathbb{R}^{d-1}}e^{ix_2'\cdot\xi_2'}e^{-\alpha|\xi_2'|_1}\phi_1^2(\xi_2')\,d\xi_2'\\&+\int_{\mathbb{R}^d}e^{ix\cdot\xi}e^{-\alpha|\xi|_1}(2\alpha\operatorname{sgn}(\xi_2)\partial_2\phi_1^2(\xi)-\partial_{22}^2)\phi_1^2(\xi)\,d\xi\biggr)\cdot\end{aligned}$$

Multiplying the above equality by $(\alpha^2+x_3^2)\cdots(\alpha^2+x_d^2)$, repeating the above computation, and using the fact that,

$$\forall\xi\in\operatorname{Supp}\phi_1,\ e^{-\alpha|\xi|_1}\leq e^{-\alpha|\xi_1|}\leq e^{-\frac{3\alpha}{4\sqrt{d}}},$$

we end up with

$$\prod_{\ell=1}^d(\alpha^2+x_\ell^2)h_1(x)\leq C(\alpha+1)^de^{-\frac{3\alpha}{4\sqrt{d}}},$$

which implies (3.3), and thus the lemma for $s=0$.

Proving the general case $s\geq0$ follows from the case $s=0$: indeed, Inequality (A.8) ensures that

$$\|\Lambda^se^{-\alpha\Lambda_1}\dot\Delta_ju\|_{L^p}\leq C_s2^{js}\|e^{-\alpha\Lambda_1}\dot\Delta_ju\|_{L^p},$$

and bounding the right-hand side according to (3.1) thus yields the desired inequality. □

One can now state our main decay estimates.

Theorem 3.2. *Let (ϱ,u) be the solution constructed in Theorem 3.1. Then for any $s\in[0,\infty[$, there exists a constant C_s such that for all $t>0$, it holds that*

$$\|\varrho(t)-\bar\varrho\|^\ell_{\dot B^{\frac{d}{2}-1+s}_{2,1}}\leq C_sX_{p,0}t^{-\frac{s}{2}},\qquad\|u(t)\|^\ell_{\dot B^{\frac{d}{2}-1+s}_{2,1}}\leq C_sX_{p,0}t^{-\frac{s}{2}},$$

$$\|\varrho(t)-\bar\varrho\|^h_{\dot B^{\frac{d}{p}+s}_{p,1}}\leq C_sX_{p,0}t^{-\frac{s}{2}}e^{-c\sqrt{t}},\qquad\|u(t)\|^h_{\dot B^{\frac{d}{p}-1+s}_{p,1}}\leq C_sX_{p,0}t^{-\frac{s}{2}}e^{-c\sqrt{t}}.$$

Proof. Recall that the solution constructed in Theorem 3.1 fulfills

$$\|(\varrho-\bar{\varrho},u)\|_{Y_p}\leq CX_{p,0}.$$

Now, Inequality (A.8) implies that

$$\|u(t)\|^{\ell}_{\dot{B}^{\frac{d}{2}-1+s}_{2,1}}\leq C_s\|\Lambda^s u(t)\|^{\ell}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}.$$

Then we write, denoting $U=e^{\sqrt{c_0t}\Lambda_1}u$ and using the previous lemma, that

$$\begin{aligned}
t^{\frac{s}{2}}\|\Lambda^s u\|^{\ell}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}&=\sum_{j\leq k_0}t^{\frac{s}{2}}2^{j(\frac{d}{2}-1)}\|\Lambda^s e^{-\sqrt{c_0t}\Lambda_1}\dot{\Delta}_jU(t)\|_{L^2}\\
&\leq C_s\sum_{j\leq k_0}(\sqrt{t}2^j)^s e^{-c\sqrt{c_0t}2^j}2^{j(\frac{d}{2}-1)}\|\dot{\Delta}_jU(t)\|_{L^2}\\
&\leq C_s\|U(t)\|^{\ell}_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\\
&\leq C_sX_{p,0}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
t^{\frac{s}{2}}\|u(t)\|^{h}_{\dot{B}^{\frac{d}{p}-1+s}_{p,1}}&\leq C_s\|\Lambda^s u(t)\|^{h}_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\\
&\leq C_s\sum_{j\geq k_0}2^{j(\frac{d}{p}-1)}t^{\frac{s}{2}}\|e^{-\sqrt{c_0t}\Lambda_1}\Lambda^s\dot{\Delta}_jU(t)\|_{L^p}\\
&\leq C_s\sum_{j\geq k_0}e^{-\frac{c}{2}\sqrt{c_0t}2^j}2^{j(\frac{d}{p}-1)}(\sqrt{t}2^j)^s e^{-\frac{c}{2}\sqrt{c_0t}2^j}\|\dot{\Delta}_jU(t)\|_{L^p}\\
&\leq C_se^{-\frac{c}{2}\sqrt{c_0t}2^{k_0}}\|U(t)\|^{h}_{\dot{B}^{\frac{d}{p}-1}_{p,1}}\\
&\leq C_se^{-\frac{c}{2}\sqrt{c_0t}2^{k_0}}X_{p,0}.
\end{aligned}$$

Proving the inequalities for ϱ is totally similar. □

Remark 3.2. *The decay estimate pointed out in Theorem 3.2 is much better than that of the usual compressible Navier-Stokes (see for example* [12]*). This reflects the parabolicity of the compressible Navier-Stokes-Korteweg system.*

Appendix A. Littlewood-Paley Decomposition and Besov Spaces

Here we recall a few basic results concerning the Littlewood-Paley decomposition and Besov spaces. More details may be found in e.g. [3, Chap. 2].

To build the Littlewood-Paley decomposition, one need a smooth radial function χ supported in the ball $\mathcal{B}(0,\frac{4}{3})$ and with value 1 on $\mathcal{B}(0,\frac{3}{4})$. Let $\varphi(\xi)\triangleq\chi(\xi/2)-\chi(\xi)$. Then, φ is compactly supported in the annulus $\{\xi\in\mathbb{R}^d,\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$ and fulfills

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\cdot)=1\ \text{ in }\ \mathbb{R}^d\setminus\{0\}\cdot$$

Define the dyadic blocks $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ by $\dot{\Delta}_j=\varphi(2^{-j}D)$ (that is, $\widehat{\dot{\Delta}_jf}:=\varphi(2^{-j}\xi)\widehat{f}(\xi)$ for all tempered distribution f). The (formal) homogeneous Littlewood-Paley decomposition of f reads

$$f=\sum_{j\in\mathbb{Z}}\dot{\Delta}_jf.$$

That equality holds true in the set $\mathcal{S}'$ of tempered distributions whenever f belongs to

$$\mathcal{S}'_h \triangleq \big\{ f \in \mathcal{S}', \ | \ \lim_{j\to-\infty} \|\dot{S}_j f\|_{L^\infty} = 0 \big\},$$

where $\dot{S}_j$ stands for the low frequency cut-off defined by $\dot{S}_j = \chi(2^{-j}D)$.

Definition A.1. *For $\sigma \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we set*

$$\|f\|_{\dot{B}^\sigma_{p,r}} = \Big\| 2^{j\sigma} \|\dot{\Delta}_j f\|_{L^p(\mathbb{R}^d)} \Big\|_{\ell^r(\mathbb{Z})}.$$

We then define the homogeneous Besov space $\dot{B}^\sigma_{p,r}$ to be the subset of distributions $f \in \mathcal{S}'_h$ such that $\|f\|_{\dot{B}^\sigma_{p,r}} < \infty$.

Homogeneous Besov spaces on $\mathbb{R}^d$ possess the following scaling invariance for any $\sigma \in \mathbb{R}$ and $(p,r) \in [1,+\infty]^2$:

$$C^{-1}\lambda^{\sigma-\frac{d}{p}} \|f\|_{\dot{B}^\sigma_{p,r}} \leq \|f(\lambda\cdot)\|_{\dot{B}^\sigma_{p,r}} \leq C\lambda^{\sigma-\frac{d}{p}} \|f\|_{\dot{B}^\sigma_{p,r}}, \qquad \lambda > 0, \tag{A.1}$$

where the constant C depends only on σ, p and on the dimension d.

The following properties have been used repeatedly in the paper:

- The space $\dot{B}^s_{p,r}$ is complete whenever $s < d/p$, or $s \leq d/p$ and $r = 1$.
- For any $p \in [1,\infty]$, we have the continuous embedding $\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty}$.
- If $\sigma \in \mathbb{R}$, $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$, then $\dot{B}^\sigma_{p_1,r_1} \hookrightarrow \dot{B}^{\sigma-d(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$.
- The space $\dot{B}^{\frac{d}{p}}_{p,1}$ is continuously embedded in the set of bounded continuous functions (going to 0 at infinity if $p < \infty$).
- If K is a smooth homogeneous of degree m function on $\mathbb{R}^d \setminus \{0\}$ that maps $\mathcal{S}'_h$ to itself, then

$$K(D) : \dot{B}^\sigma_{p,r} \to \dot{B}^{\sigma-m}_{p,r}. \tag{A.2}$$

  In particular, the gradient operator maps $\dot{B}^\sigma_{p,r}$ to $\dot{B}^{\sigma-1}_{p,r}$.

Let us also mention the following interpolation inequality that is satisfied whenever $1 \leq p, r_1, r_2, r \leq \infty$, $\sigma_1 \neq \sigma_2$ and $\theta \in (0,1)$:

$$\|f\|_{\dot{B}^{\theta\sigma_2+(1-\theta)\sigma_1}_{p,r}} \lesssim \|f\|^{1-\theta}_{\dot{B}^{\sigma_1}_{p,r_1}} \|f\|^{\theta}_{\dot{B}^{\sigma_2}_{p,r_2}}. \tag{A.3}$$

The following proposition has been used in this paper.

Proposition A.1. *Let $\sigma \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. Let $(f_j)_{j\in\mathbb{Z}}$ be a sequence of L^p functions such that $\sum_{j\in\mathbb{Z}} f_j$ converges to some distribution f in $\mathcal{S}'_h$ and*

$$\Big\| 2^{j\sigma} \|f_j\|_{L^p(\mathbb{R}^d)} \Big\|_{\ell^r(\mathbb{Z})} < \infty.$$

If $\mathrm{Supp}\,\widehat{f_j} \subset \mathcal{C}(0, 2^j R_1, 2^j R_2)$ for some $0 < R_1 < R_2$, then f belongs to $\dot{B}^\sigma_{p,r}$ and there exists a constant C such that

$$\|f\|_{\dot{B}^\sigma_{p,r}} \leq C \Big\| 2^{j\sigma} \|f_j\|_{L^p(\mathbb{R}^d)} \Big\|_{\ell^r(\mathbb{Z})}. \tag{A.4}$$

The following result was used to bound the terms of System (1.1) involving compositions of functions:

Proposition A.2. *Let $F:\mathbb{R}\to\mathbb{R}$ be smooth with $F(0)=0$. For all $1\leq p,r\leq\infty$ and $\sigma>0$ we have $F(f)\in\dot B^\sigma_{p,r}\cap L^\infty$ for $f\in\dot B^\sigma_{p,r}\cap L^\infty$, and*

$$\|F(f)\|_{\dot B^\sigma_{p,r}}\leq C\|f\|_{\dot B^\sigma_{p,r}} \tag{A.5}$$

with C depending only on $\|f\|_{L^\infty}$, F' (and higher derivatives), σ, p and d.

If $\sigma>-\min(\frac{d}{p},\frac{d}{p'})$, then $f\in\dot B^\sigma_{p,r}\cap\dot B^{\frac{d}{p}}_{p,1}$ implies that $F(f)\in\dot B^\sigma_{p,r}\cap\dot B^{\frac{d}{p}}_{p,1}$, and

$$\|F(f)\|_{\dot B^\sigma_{p,r}}\leq C(1+\|f\|_{\dot B^{\frac{d}{p}}_{p,1}})\|f\|_{\dot B^\sigma_{p,r}}. \tag{A.6}$$

Let us finally recall the following classical *Bernstein inequality*:

$$\|D^kf\|_{L^b}\leq C^{1+k}\lambda^{k+d(\frac{1}{a}-\frac{1}{b})}\|f\|_{L^a} \tag{A.7}$$

that holds for all function f such that $\operatorname{Supp}\mathcal{F}f\subset\{\xi\in\mathbb{R}^d:|\xi|\leq R\lambda\}$ for some $R>0$ and $\lambda>0$, if $k\in\mathbb{N}$ and $1\leq a\leq b\leq\infty$.

Let us also recall that, as a consequence of [3, Lemma 2.2], we have for all $s\in\mathbb{R}$ if $\operatorname{Supp}\mathcal{F}f\subset\{\xi\in\mathbb{R}^d:r\lambda\leq|\xi|\leq R\lambda\}$ for some $0<r<R$,

$$\|\Lambda^sf\|_{L^b}\approx\lambda^s\|f\|_{L^b}\quad\text{with}\quad\Lambda^s\triangleq(-\Delta)^{\frac{s}{2}}. \tag{A.8}$$

When localizing PDE's by means of Littlewood-Paley decomposition, one ends up with bounds for each dyadic block in spaces of type $L^q_T(L^p)\triangleq L^q(0,T;L^p(\mathbb{R}^d))$. To get a Besov type information, we then have to perform a summation on $\ell^r(\mathbb{Z})$, which motivates the following definition that has been first introduced by J.-Y. Chemin in [7] for $0\leq T\leq+\infty$, $\sigma\in\mathbb{R}$ and $1\leq p,q,r\leq\infty$:

$$\|f\|_{\widetilde{L}^q_T(\dot B^\sigma_{p,r})}\triangleq\Big\|\big(2^{j\sigma}\|\dot\Delta_jf\|_{L^q_T(L^p)}\big)\Big\|_{\ell^r(\mathbb{Z})}.$$

For notational simplicity, index T is omitted if $T=+\infty$.

We also used the following functional space:

$$\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot B^\sigma_{p,r})\triangleq\big\{f\in\mathcal{C}(\mathbb{R}_+;\dot B^\sigma_{p,r})\ s.t.\ \|f\|_{\widetilde{L}^\infty(\dot B^\sigma_{p,r})}<\infty\big\}. \tag{A.9}$$

The above norms may be compared with those of the more standard Lebesgue-Besov spaces $L^q_T(\dot B^\sigma_{p,r})$ via Minkowski's inequality:

$$\|f\|_{\widetilde{L}^q_T(\dot B^\sigma_{p,r})}\leq\|f\|_{L^q_T(\dot B^\sigma_{p,r})}\ \text{ if }\ r\geq q,\qquad\|f\|_{\widetilde{L}^q_T(\dot B^\sigma_{p,r})}\geq\|f\|_{L^\rho_T(\dot B^\sigma_{p,r})}\ \text{ if }\ r\leq q. \tag{A.10}$$

Restricting the above norms to the low or high frequencies parts of distributions is fundamental in our approach. For some fixed integer k_0 (the value of which follows from the proof of the main theorem), we put $z^\ell\triangleq\dot S_{k_0}z$ and $z^h\triangleq z-z^\ell$, and[4]

$$\|z\|^\ell_{\dot B^s_{p,1}}\triangleq\sum_{k\leq k_0}2^{ks}\|\dot\Delta_kz\|_{L^p},\qquad\|z\|^h_{\dot B^s_{p,1}}\triangleq\sum_{k\geq k_0-1}2^{ks}\|\dot\Delta_kz\|_{L^p}, \tag{A.11}$$

$$\|z\|^\ell_{\widetilde{L}^\infty_T(\dot B^s_{p,1})}\triangleq\sum_{k\leq k_0}2^{ks}\|\dot\Delta_kz\|_{L^\infty_T(L^p)}\quad\text{and}\quad\|z\|^h_{\widetilde{L}^\infty_T(\dot B^s_{p,1})}\triangleq\sum_{k\geq k_0-1}2^{ks}\|\dot\Delta_kz\|_{L^\infty_T(L^p)}.$$

[4]For technical reasons, we need a small overlap between low and high frequencies.

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Université Paris-Est, LAMA (UMR 8050), UPEMLV, UPEC, CNRS, 61 avenue du Général de Gaulle, 94010 Créteil Cedex 10
E-mail address: frederic.charve@u-pec.fr

Université Paris-Est, LAMA (UMR 8050), UPEMLV, UPEC, CNRS, 61 avenue du Général de Gaulle, 94010 Créteil Cedex 10
E-mail address: danchin@univ-paris12.fr

Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, P.R.China,
E-mail address: jiangxu_79math@yahoo.com