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**LEVEL SETS OF CERTAIN NEUMANN  
EIGENFUNCTIONS UNDER DEFORMATION OF  
LIPSCHITZ DOMAINS  
APPLICATION TO THE EXTENDED COURANT  
PROPERTY**

PIERRE BÉRARD AND BERNARD HELFFER

ABSTRACT. In this paper, we prove that the Extended Courant Property fails to be true for certain  $C^\infty$  domains with Neumann boundary condition: there exists a linear combination of a second and a first Neumann eigenfunctions, with three nodal domains. For the proof, we revisit a deformation argument of Jerison and Nadirashvili (J. Amer. Math. Soc. 2000, vol. 13). This argument being interesting in itself, we give full details. In particular, we carefully control the dependence of the constants on the geometry of our Lipschitz domains along the deformations.

1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain (open and connected), with  $n \geq 2$ . We assume that  $\Omega$  is smooth enough, and we consider the eigenvalue problem

$$(1.1) \quad \begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ B(u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where the boundary condition  $B(u)$  is either the Dirichlet boundary condition  $u|_{\partial\Omega} = 0$ , or the Neumann boundary condition  $\frac{\partial u}{\partial n_e}|_{\partial\Omega} = 0$  (where  $n_e$  denotes the exterior unit normal).

We write the eigenvalues of (1.1) in nondecreasing order, with multiplicities, starting with the index 1,

$$(1.2) \quad \mu_1(\Omega, \mathbf{a}) < \mu_2(\Omega, \mathbf{a}) \leq \mu_3(\Omega, \mathbf{a}) \leq \dots,$$

where  $\mathbf{a} \in \{\mathfrak{D}, \mathfrak{N}\}$  denotes the boundary condition.

Given an eigenvalue  $\mu(\Omega, \mathbf{a})$  of (1.1), we denote by  $\mathcal{E}(\mu(\Omega, \mathbf{a}))$  the corresponding eigenspace. Given an eigenfunction  $u \in \mathcal{E}(\mu(\Omega, \mathbf{a}))$ , we denote by

$$(1.3) \quad \mathcal{Z}(u) = \overline{\{x \in \Omega \mid u(x) = 0\}}$$

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the *nodal set* of  $u$ , and by  $\beta_0(u)$  the number of *nodal domains* (the connected components of  $\Omega \setminus \mathcal{Z}(u)$ ) of the function  $u$ .

Given an eigenvalue  $\mu = \mu(\Omega, \mathbf{a})$  of (1.1), we denote by  $\kappa(\mu)$  the *least index* of  $\mu$ ,

$$(1.4) \quad \kappa(\mu) = \min\{k \mid \mu_k(\Omega, \mathbf{a}) = \mu\}.$$

The following classical theorem was proved by R. Courant in 1923, see for example [17, § VI.6].

**Theorem 1.1** (Courant's nodal domain theorem). *Let  $\mu$  be an eigenvalue of (1.1), and  $u \in \mathcal{E}(\mu)$  a corresponding eigenfunction. Then,*

$$(1.5) \quad \beta_0(u) \leq \kappa(\mu).$$

When  $d = 1$ , given a finite interval  $] \alpha, \beta[$ , and a smooth real function  $q$  on  $[\alpha, \beta]$ , instead of the eigenvalue problem for the Laplacian, we consider the Sturm-Liouville eigenvalue problem,

$$(1.6) \quad \begin{cases} -y'' + qy = \mu y \text{ in } ] \alpha, \beta[, \\ B(y) = 0 \text{ at } \{\alpha, \beta\}. \end{cases}$$

There are striking differences between the eigenvalue problems (1.6) ( $d = 1$ ) and (1.1) ( $d \geq 2$ ).

*First difference.*

When  $d = 1$ , a classical theorem of C. Sturm [44] states that the eigenvalues of (1.6) are all simple, and that an eigenfunction of (1.6), associated with the  $n$ th eigenvalue, has exactly  $n$  nodal domains.

When  $d \geq 2$ , the eigenvalues of (1.1) may have multiplicities (this is for example the case for a square with either Dirichlet or Neumann condition on the boundary). By Courant's nodal domain theorem, an eigenfunction of (1.1), associated with the  $n$ th-eigenvalue has at most  $n$  nodal domains. However,

- (1) For the round sphere  $\mathbb{S}^2$ , and for the square with Dirichlet boundary condition, examples of A. Stern [7, 8] show that there is no general lower bound on  $\beta_0(u)$  for higher energy eigenvalues, except the trivial bound  $\beta_0(u) \geq 2$ . Note that the example of the square suggests that such a statement might not be true for the Neumann boundary condition, see the paragraph before Proposition 10.2 in [24].
- (2) A theorem of Å. Pleijel [39] shows that the upper bound  $\beta_0(u) \leq \kappa(\mu)$  is sharp for finitely many eigenvalues  $\mu$  only.

*Second difference.*

Another, not so well-known, theorem of C. Sturm [45] states that, for  $n \geq m \geq 1$ , a linear combination  $\sum_{k=m}^n a_k V_k$  of eigenfunctions of (1.6), in the range  $k \in \{m, \dots, n\}$ , has at least  $(m - 1)$ , and at most  $(n - 1)$  zeros in the interval  $] \alpha, \beta[$ . We refer to [10] for a more precise

statement of Sturm’s theorem, and to [19], in particular Theorem 1 in Section IV.3, for a different point of view.

In dimension  $d \geq 2$ , a similar statement (for the upper-bound) appears in Footnote 1, page 454 of [17, Chap. VI.6], namely:

*Any linear combination of the first  $n$  eigenfunctions divides the domain, by means of its nodes, into no more than  $n$  subdomains. See the Göttingen dissertation of H. Herrmann, *Beiträge zur Theorie der Eigenwerte und Eigenfunktionen*, 1932.*

This statement is sometimes referred to as the “Courant-Herrmann theorem” [22, § 9.2], or the “Courant-Herrmann conjecture” [20]. We shall call this statement the “Extended Courant Property”, and refer to it as the  $\text{ECP}(\Omega, \mathbf{a})$ , when applied to the boundary value problem (1.1), with the boundary condition  $\mathbf{a}$ .

In [6], see also [5, 33], V. Arnold points out that the  $\text{ECP}(\mathbb{S}^2, g_0)$  is true for the round metric  $g_0$ , and that the  $\text{ECP}(\mathbb{S}^3, g_0)$  is false, with counterexamples constructed by O. Viro [47]. Arnold also claims that  $\text{ECP}(\mathbb{S}^2, g)$  is false for a generic metric  $g$ . As far as we understand, the only known proof that the assertion “the  $\text{ECP}(\mathbb{S}^2, g_0)$  is true”, is a real algebraic geometry proof. Such a proof can be found in [36] (Theorem 1, and second remark on page 305). To our knowledge, no proof of the second claim has been published.

Little seems to be known on the ECP. In [11, 12], we gave some examples of domains such that  $\text{ECP}(\Omega, \mathbf{a})$  is false, with either the Dirichlet or the Neumann boundary condition. However, all these examples are singular (domains or surfaces with cracks), or have a nonsmooth boundary (polygonal domains). A natural question is whether one can construct counterexamples to the ECP with a  $C^\infty$  boundary. Numerical simulation for the equilateral triangle with rounded corners (the corners of the triangle are replaced with circular caps tangent to the sides) suggest that this should be true. Note however that a triangle with rounded corners is  $C^1$ , not  $C^2$ .

The pictures in the first row of Figure 1.1 display the level sets and nodal domains of a second Neumann eigenfunction  $\phi$  of the equilateral triangle with rounded corners, as calculated by MATLAB. The function is almost symmetric<sup>1</sup> with respect to one of the axes of symmetry of the triangle. The pictures in the second row display the nodal sets of the function  $a + \phi$  for two values of  $a$ . They provide a numerical evidence that ECP is not true for the equilateral triangle with rounded corners, and Neumann boundary condition.

In this paper, we prove,

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<sup>1</sup>Generally speaking, numerical softwares do not necessarily produce the symmetric eigenfunctions when an eigenvalue is not simple.

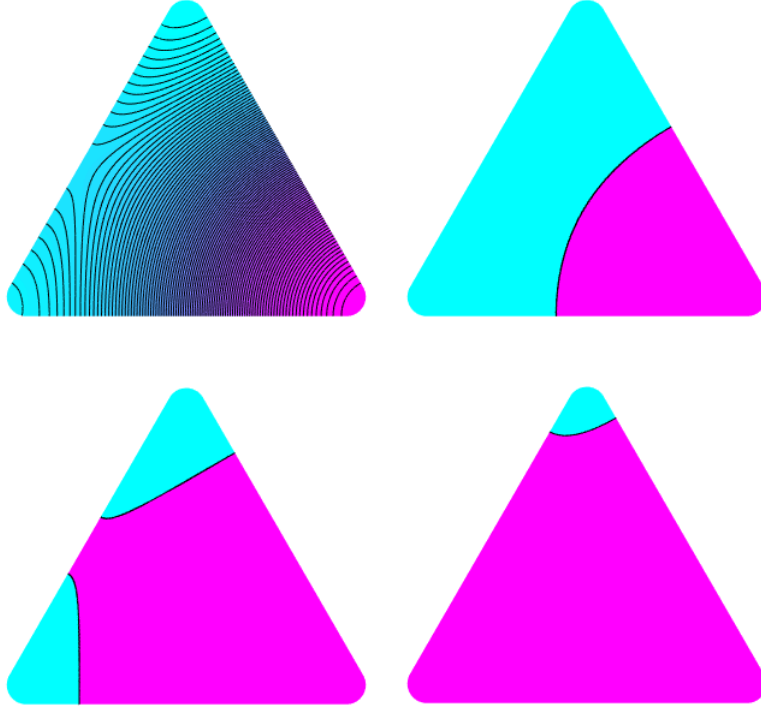


FIGURE 1.1. Level sets of one of the second Neumann eigenfunctions of the equilateral triangle with rounded corners

**Theorem 1.2.** *There exists a one parameter family of  $C^\infty$  domains  $\{\Omega_t, 0 < t < t_0\}$  in  $\mathbb{R}^2$ , with the symmetry of the equilateral triangle  $\mathcal{T}_e$ , such that:*

- (1) *The family is strictly increasing, and  $\Omega_t$  tends to  $\mathcal{T}_e$ , in the sense of the Hausdorff distance, as  $t$  tends to 0.*
- (2) *For any  $t \in ]0, t_0[$ , the  $\text{ECP}(\Omega_t)$  is false. More precisely, for each  $t$ , there exists a linear combination of a symmetric 2nd Neumann eigenfunction and a 1st Neumann eigenfunction of  $\Omega_t$ , with precisely three nodal domains.*

As we have shown in [11], for the equilateral triangle  $\mathcal{T}_e$ , the  $\text{ECP}(\mathcal{T}_e, \mathbf{a})$  is false for both the Dirichlet, and the Neumann boundary conditions. The idea of the proof of Theorem 1.2 is to show that one can find a deformation of  $\mathcal{T}_e$  such that the symmetric second Neumann eigenfunction deforms nicely. For this purpose, we revisit a deformation argument given by Jerison and Nadirashvili [28] in the framework of the “hot spots” conjecture. This argument being interesting in itself, we give full details, and extend its applications.

**Organization of the paper.** In Section 2, we develop the deformation argument. In Section 3, we show how to construct smooth domains

with the symmetry of the equilateral triangle, and we establish some properties of these domains, to be used in the following section. In Section 4, we apply the deformation argument to prove Theorem 1.2. The appendices provide some complements.

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## 2. A DEFORMATION ARGUMENT

In this section, we revisit a deformation argument of Jerison and Nadirashvili [28, Section 2]. Note that our framework is different: they are interested in antisymmetric eigenfunctions in domains with two orthogonal lines of mirror symmetry; we are interested in symmetric eigenfunctions in domains with the symmetries of an equilateral triangle. We also aim at controlling the constants which appear in the analytic inequalities, and at making sure that they are uniform in a large class of domains. This aspect is not always taken care of clearly in the literature.

**2.1. Geometric framework: the class  $\mathcal{L}_M$ .** Let  $M$  be a positive constant.

**Definition 2.1.** *The class  $\mathcal{L}_M$  comprises the sets  $\Omega \subset \mathbb{R}^2$  which satisfy the following conditions.*

$$(2.1) \quad \Omega \text{ is convex and open, with } 0 \in \Omega.$$

$$(2.2) \quad \overline{B}(M^{-1}) \subset \Omega \subset \overline{\Omega} \subset B(M),$$

where  $B(R)$  denotes the open ball centered at 0, with radius  $R$ , and  $\overline{B}(R)$  denotes the corresponding closed ball.

$$(2.3) \quad \Omega \text{ is symmetric with respect to the line } D := \{u = 0\}.$$

$$(2.4) \quad \partial\Omega \text{ is regular at } D \cap \partial\Omega,$$

i.e. in a neighborhood of  $m \in D \cap \partial\Omega$ , the boundary  $\partial\Omega$  is piecewise  $C^1$ , and  $\partial\Omega \setminus \{m\}$  is  $C^1$ .

The domain  $\Omega$  can be described by a polar equation,

$$(2.5) \quad \Omega = \{(r, \theta) \mid 0 \leq r < \rho(\theta)\},$$

where the function  $\rho$  is a  $2\pi$ -periodic, Lipschitz function, with Lipschitz constant bounded from above by  $M$ .

We define the domain,

$$(2.6) \quad \Omega_+ := \Omega \cap \{u > 0\}.$$

We decompose its boundary  $\partial\Omega_+$  as

$$(2.7) \quad \partial\Omega_+ = \overline{\Gamma} \sqcup \overline{\Gamma_D},$$

with  $\Gamma = \partial\Omega_+ \cap \{u > 0\}$ , and  $\Gamma_D = D \cap \Omega_+$ .

**Notation.** In the sequel, we denote by  $D$  both the line, and the mirror symmetry with respect to the line  $D$ . We denote by  $D^*$  the action of the symmetry  $D$  on functions,  $D^*\phi = \phi \circ D$ .

**Remarks 2.2.** *We note the following properties for later reference.*

- (1) *According to Proposition 2.4.4 in [26], domains satisfying conditions (2.1) and (2.2) satisfy a uniform (i.e. depending only on  $M$ ) cone property. It follows from Theorem 2.4.7, and Remark 2.4.8 in [26] that such domains are uniformly Lipschitz domains (i.e., the boundary is locally the graph of a Lipschitz function, ibidem Definition 2.4.5).*
- (2) *With the definitions of Appendix A, for such domains, the inclusion  $H^1(\Omega) \hookrightarrow L_2(\Omega)$  is compact, and we can define eigenvalues using the variational approach.*
- (3) *The fact that a domain  $\Omega$ , defined in polar coordinates as in (2.5), is a Lipschitz domain also follows from [48, Theorem 7.1].*
- (4) *Let  $\Omega$  be a domain defined by a polar equation, as in (2.5). Define the function  $r(\theta)$  by  $r(\theta) = 1/\rho(\theta)$ . If  $\Omega$  is convex, then the second derivative of  $r$ , in the sense of distributions, is a measure such that  $r''(\theta) + r(\theta) \geq 0$ , see Appendix B.*

We consider the Neumann eigenvalue problem for  $-\Delta$  in  $\Omega$ . We denote the Neumann eigenvalues by  $\nu_i(\Omega)$ , and arrange them in nondecreasing order, starting with the index 1. We also consider the eigenvalue problems for  $-\Delta$  in  $\Omega_+$ , with either the Neumann boundary condition on  $\partial\Omega_+$ , or the mixed boundary conditions, Neumann on  $\Gamma$  and Dirichlet on  $\Gamma_D$ . We denote these eigenvalues respectively by  $\mu_i(\Omega_+, \mathbf{nn})$ , and  $\mu_i(\Omega_+, \mathbf{nd})$ , and arrange them in nondecreasing order, starting with the index 1.

We are interested in the *least positive* eigenvalues associated with the symmetry  $D$ . More precisely, we introduce

$$(2.8) \quad \nu^-(\Omega) := \inf\{\nu_i(\Omega) \mid i \geq 2, \exists \varphi, -\Delta\varphi = \nu_i(\Omega)\varphi, D^*\varphi = -\varphi\},$$

and

$$(2.9) \quad \nu^+(\Omega) := \inf\{\nu_i(\Omega) \mid i \geq 2, \exists \varphi, -\Delta\varphi = \nu_i(\Omega)\varphi, D^*\varphi = \varphi\},$$

where the equations  $-\Delta\varphi = \nu_i(\Omega)\varphi$  are to be understood in  $\Omega$ .

It is easy to see that

$$(2.10) \quad \begin{cases} \nu^-(\Omega) &= \mu_1(\Omega_+, \mathbf{nd}), \\ \nu^+(\Omega) &= \mu_2(\Omega_+, \mathbf{nn}), \\ \nu_2(\Omega) &= \min\{\nu^-(\Omega), \nu^+(\Omega)\}. \end{cases}$$

**Remarks 2.3.** *About the eigenvalues  $\nu^-(\Omega)$  and  $\nu^+(\Omega)$ .*

- (1) Because  $\mu_1(\Omega_+, \mathbf{n}\mathfrak{d})$  is simple, there is, up to scaling<sup>2</sup>, only one anti-symmetric eigenfunction of  $-\Delta$  in  $\Omega$ , associated with the eigenvalue  $\nu^-(\Omega)$ .
- (2) If  $\nu_2(\Omega)$  is a simple eigenvalue, then either  $\nu_2(\Omega) = \nu^+(\Omega) < \nu^-(\Omega)$  or  $\nu_2(\Omega) = \nu^-(\Omega) < \nu^+(\Omega)$ , and the corresponding eigenfunction is either invariant, or anti-invariant under  $D$ . If  $\dim \mathcal{E}(\nu_2(\Omega)) \geq 2$ , then

$$\mathcal{E}(\nu_2) = (\mathcal{E}(\nu_2) \cap \mathcal{S}_+) \oplus (\mathcal{E}(\nu_2) \cap \mathcal{S}_-),$$

with  $\dim \mathcal{E}(\nu_2) \cap \mathcal{S}_+ \leq 1$ .

Recall that  $\dim \mathcal{E}(\nu_2) \leq 3$ , at least if  $\Omega$  is regular, see [27].

- (3) Let  $\mathcal{T}_i(\alpha)$  be an isosceles triangle with aperture  $\alpha \in ]0, \pi[$ . According to [34, § 10],

$$\nu_2(\mathcal{T}_i(\alpha)) = \nu^+(\mathcal{T}_i(\alpha)) < \nu^-(\mathcal{T}_i(\alpha)) \text{ when } 0 < \alpha < \frac{\pi}{3},$$

$$\nu_2(\mathcal{T}_i(\alpha)) = \nu^-(\mathcal{T}_i(\alpha)) < \nu^+(\mathcal{T}_i(\alpha)) \text{ when } \frac{\pi}{3} < \alpha < \pi.$$

There is a bifurcation at  $\frac{\pi}{3}$ , in which case

$$\nu_2(\mathcal{T}_i(\frac{\pi}{3})) = \nu^-(\mathcal{T}_i(\frac{\pi}{3})) = \nu^+(\mathcal{T}_i(\frac{\pi}{3})).$$

- (4) In Section 3, we consider domains  $\Omega$  which admit the symmetry group  $\mathcal{G}_0$  of the equilateral triangle, see (3.7). For such domains, Proposition 3.6 tells us that

$$\nu^-(\Omega) = \nu^+(\Omega) = \nu_2(\Omega) = \nu_3(\Omega) < \nu_4(\Omega).$$

**Remark 2.4.** Since  $\nu^-(\Omega)$  is a simple eigenvalue, there is a unique corresponding eigenfunction of  $-\Delta$  in  $\Omega$ ,  $\psi_\Omega$ , which is anti-symmetric with respect to  $D$ , and satisfies,

$$(2.11) \quad \int_{\Omega} \psi_\Omega^2 = 1 \text{ and } \psi_\Omega|_{\Omega_+} > 0.$$

**Notation.** In (2.4), and henceforth, we skipped the (Lebesgue) measure  $dx$  in the integrals.

We introduce the following assumption which will be needed later on.

**Assumption 2.5.** The eigenvalue  $\mu_2(\Omega_+, \mathbf{n}\mathfrak{n})$  is simple.

Note that Remark 2.3-(4) tells us that Assumption 2.5 is satisfied for domains  $\Omega$  with the  $\mathcal{G}_0$  symmetry.

**Remark 2.6.** Provided that Assumption 2.5 is satisfied, there is a  $D$ -symmetric eigenfunction of  $-\Delta$  in  $\Omega$ , associated with  $\nu^+(\Omega)$ . Such an eigenfunction is uniquely determined, up-to-sign, by the condition  $\int_{\Omega} \phi_\Omega^2 = 1$ . As we will see in Lemma 2.15, one can actually make a unique choice of  $\phi_{\Omega_i}$  along a path of domains.

<sup>2</sup>By this, we mean ‘‘up to multiplication by a nonzero scalar’’.



**2.2. Preliminary estimates.** We shall now examine how the eigenvalues  $\nu^\pm(\Omega)$ , and the corresponding eigenfunctions, vary with the domain  $\Omega \in \mathcal{L}_M$ . For this purpose, and following [28], we introduce the following distance in the class  $\mathcal{L}_M$ ,

$$(2.12) \quad d_r(\Omega_1, \Omega_2) = \|\rho_1 - \rho_2\|_\infty,$$

if the domains are defined by the functions  $\rho_1$  and  $\rho_2$  respectively, as in (2.5).

Note that this distance is bigger than the Hausdorff distance between open sets contained in a given compact ball  $D$ ,

$$(2.13) \quad d_H(\Omega_1, \Omega_2) := d^H(D \setminus \Omega_1, D \setminus \Omega_2).$$

Here,

$$(2.14) \quad d^H(K_1, K_2) := \max \left\{ \sup_{x \in K_1} \inf_{y \in K_2} d(x, y), \sup_{x \in K_2} \inf_{y \in K_1} d(x, y) \right\},$$

is the Hausdorff distance between the compact sets  $K_1$  and  $K_2$ , and  $d(x, y)$  is the Euclidean distance between the points  $x, y \in \mathbb{R}^2$ .

Note that the distance defined in (2.13) does not depend on the choice of the compact  $D$ , once it contains both  $\Omega_1$  and  $\Omega_2$ .

**Notation.** In the sequel,  $|\Omega|$  denotes the area of a domain  $\Omega$ . We will also use the following convention. We use constants  $C_i, i \in \mathbb{N}$  in the statements, and local constants  $C_{i,j}, i, j \in \mathbb{N}$  inside the proofs. Note that the constants are not numbered linearly. When a constant appears, we mention which parameters it depends upon.

**Lemma 2.7.** *There exists a constant  $C_1(M)$  such that, for any domains  $\Omega_1, \Omega_2 \in \mathcal{L}_M$ ,*

$$(2.15) \quad |\Omega_1 \setminus \Omega_2| \leq C_1(M) d_r(\Omega_1, \Omega_2).$$

*Proof.* It suffices to notice that

$$\Omega_1 \setminus \Omega_2 = \{(r, \theta) \mid \rho_2(\theta) \leq r < \rho_1(\theta)\},$$

and to compute the area in polar coordinates.  $\square$

**Lemma 2.8.** *There exists a constant  $C_2(M)$  such that, for any  $\Omega \in \mathcal{L}_M$ ,*

$$(2.16) \quad \max\{\nu_2(\Omega), \nu^+(\Omega), \nu^-(\Omega)\} \leq C_2(M).$$

*Proof.* Since  $\Omega \in \mathcal{L}_M$ , condition (2.2) is satisfied. We then have,

$$\begin{cases} \nu_2(\Omega) \leq \delta_2(\Omega) < \delta_2(B(M^{-1})), \\ \nu^+(\Omega) = \mu_2(\Omega_+, \mathbf{nn}) \leq \delta_2(\Omega_+) \leq \delta_2(B(M^{-1}) \cap \{u > 0\}), \\ \nu^-(\Omega) = \mu_1(\Omega_+, \mathbf{nd}) \leq \delta_1(\Omega_+) \leq \delta_1(B(M^{-1}) \cap \{u > 0\}), \end{cases}$$

where we have used  $\delta$ 's to denote Dirichlet eigenvalues.  $\square$

**Proposition 2.9.** *There exists a constant  $C_3(M)$  such that, for any  $\Omega \in \mathcal{L}_M$ , the normalized eigenfunction  $\psi_\Omega$  (defined in Remark 2.4), and the normalized eigenfunction  $\phi_\Omega$  (defined in Remark 2.6), belong to the Sobolev space  $H^2(\Omega)$ , with corresponding Sobolev norm less than or equal to  $C_3(M)$ ,*

$$(2.17) \quad \|\psi_\Omega\|_{H^2(\Omega)} + \|\phi_\Omega\|_{H^2(\Omega)} \leq C_3(M).$$

*Proof.* We sketch the proof in Appendix A. The point we want to stress here, is that the bound is uniform with respect to the domains in  $\mathcal{L}_M$ .  $\square$

**Remark 2.10.** *The  $H^2$  estimates in this proposition hold for convex domains. For more general Lipschitz domains, there are only  $H^s$  estimates, with  $s = \frac{3}{2}$  in [28], or  $s < \frac{3}{2}$  in [42]. A counter-example is given in [21].*

**Proposition 2.11** (Extension theorem). *For any domain  $\Omega \in \mathcal{L}_M$ , there exists a linear extension operator  $\mathbb{E}_\Omega$ , such that for any  $s > 0$ ,*

$$\mathbb{E}_\Omega : H^s(\Omega) \rightarrow H^s(\mathbb{R}^n),$$

*and there exists a positive constant  $C_4(M, s)$ , such that, for all  $u \in H^s(\Omega)$ ,*

$$(2.18) \quad \begin{cases} \|\mathbb{E}_\Omega(u)\|_{H^s(\mathbb{R}^n)} \leq C_4(M, s)\|u\|_{H^s(\Omega)}, \\ \mathbb{E}_\Omega(u)|_\Omega = u \text{ almost everywhere,} \\ \mathbb{E}_\Omega(u) \text{ is } D\text{-}(anti)\text{symmetric, if } u \text{ is.} \end{cases}$$

*Furthermore, one can choose  $\mathbb{E}_\Omega(u)$  with compact support in  $B(2M)$ .*

*Proof.* This proposition follows from Theorem 5 in [43, Chap. VI.3] and interpolation. We again point out that the constant  $C_4(M, s)$  is uniform in  $\mathcal{L}_M$ .  $\square$

Finally, we mention the classical Sobolev embedding theorem, in the form that we will use. Recall that  $B(R)$  is the open ball with center the origin, and radius  $R$  in  $\mathbb{R}^2$ .

**Proposition 2.12.** *For all  $\alpha \in [0, 1[$ , the space  $H^2(B(R))$  embeds continuously in  $C^{0,\alpha}(\overline{B}(R))$ . The space  $H^1(B(R))$  embeds continuously in  $L_p(B(R))$  for all  $p \geq 2$ . In particular, for any  $s$ ,  $1 \leq s < 2$ , and for any  $v \in H^2(B(R))$ , we have  $v \in C^{0,s-1}(\overline{B}(R))$ ,  $dv \in L_{\frac{2}{2-s}}(B(R), \mathbb{R}^2)$ , and there exists a constant  $C_5(R, s)$ , such that*

$$(2.19) \quad \|v\|_{L_\infty(B(R))} + \|dv\|_{L_{\frac{2}{2-s}}(B(R))} \leq C_5(R, s)\|v\|_{H^2(B(R))}.$$

*Proof.* See [21], Theorem 1.4.4.1, and equations (1,4,4,3)–(1,4,4,6), for the statements, and Adams [2], Chap. IV and V, for the proofs.  $\square$

**Assumption 2.13.** *From now on, we choose some  $s_0 \in ]1, 2[$ , and use the notation,*

$$p_0 := p(s_0) = \frac{2}{2 - s_0}, \text{ and } q_0 := q(s_0) = s_0 - 1 > 0.$$

**2.3. Properties of  $\nu^+(\Omega)$  and  $\phi_\Omega$ .** In this section, we are interested in how the  $D$ -symmetric eigenfunction  $\phi_\Omega$  changes along a deformation  $\Omega_t$  of the domain. Note that in [28], Jerison and Nadirashvili consider the  $D$ -anti-invariant eigenfunctions, in the context of the ‘‘hot spots’’ conjecture. Although we do not need it, we provide a proof of the anti-invariant case in Appendix C.

**Lemma 2.14.** *There exists a constant  $C_{20}(M, s_0)$  such that, for any domains  $\Omega_1, \Omega_2 \in \mathcal{L}_M$ ,*

$$(2.20) \quad \left| \nu^+(\Omega_1) - \nu^+(\Omega_2) \right| \leq C_{20} d_r(\Omega_1, \Omega_2)^{q_0}.$$

*Proof.* For the proof, we use the following notation:  $\lambda_i = \nu^+(\Omega_i)$ ;  $\phi_i = \phi_{\Omega_i}$  is a normalized  $D$ -invariant eigenfunction of  $-\Delta$  in  $\Omega_i$ , belonging to  $\nu^+(\Omega_i)$ , in particular we have  $\int_{\Omega_i} \phi_i = 0$ ;  $\Phi_i = \mathbb{E}_{\Omega_i}(\phi_{\Omega_i})$  is a  $D$ -invariant extension of  $\phi_{\Omega_i}$ , given by Proposition 2.11. We also introduce the function  $\Theta_2$  such that

$$(2.21) \quad \Theta_2 = \Phi_2 - |\Omega_1|^{-1} \int_{\Omega_1} \Phi_2,$$

so that  $\int_{\Omega_1} \Theta_2 = 0$ , and  $d\Theta_2 = d\Phi_2$ .

Then,

$$(2.22) \quad \int_{\Omega_1} \Theta_2^2 = \int_{\Omega_1} \Phi_2^2 - |\Omega_1|^{-1} \left( \int_{\Omega_1} \Phi_2 \right)^2.$$

Writing

$$\int_{\Omega_1} \Phi_2 = \int_{\Omega_2} \Phi_2 + \int_{\Omega_1 \setminus \Omega_2} \Phi_2 - \int_{\Omega_2 \setminus \Omega_1} \Phi_2,$$

using the fact that  $\int_{\Omega_2} \Phi_2 = \int_{\Omega_2} \phi_2 = 0$ , and Lemma 2.7, we obtain,

$$\left| \int_{\Omega_1} \Phi_2 \right| \leq \|\Phi_2\|_\infty (|\Omega_1 \setminus \Omega_2| + |\Omega_2 \setminus \Omega_1|)$$

so that there exists a constant  $C_{20,1}(M, s_0)$  such that

$$(2.23) \quad \left| \int_{\Omega_1} \Phi_2 \right| \leq C_{20,1} d_r(\Omega_1, \Omega_2).$$

We also have

$$\int_{\Omega_1} \Theta_2^2 = \int_{\Omega_2} \Phi_2^2 + \int_{\Omega_1 \setminus \Omega_2} \Phi_2^2 - \int_{\Omega_2 \setminus \Omega_1} \Phi_2^2 - |\Omega_1|^{-1} \left( \int_{\Omega_1} \Phi_2 \right)^2.$$

Using the same arguments as above, as well as (2.2), we obtain that there exists a constant  $C_{20,2}(M, s_0)$  such that

$$(2.24) \quad 1 - C_{20,2} d_r(\Omega_1, \Omega_2) \leq \int_{\Omega_1} \Theta_2^2 \leq 1 + C_{20,2} d_r(\Omega_1, \Omega_2).$$

Similarly, we write

$$(2.25) \quad \int_{\Omega_1} |d\Phi_2|^2 = \int_{\Omega_2} |d\Phi_2|^2 + \int_{\Omega_1 \setminus \Omega_2} |d\Phi_2|^2 - \int_{\Omega_2 \setminus \Omega_1} |d\Phi_2|^2.$$

Because  $(d\Phi_2)|_{\Omega_2} = d\phi_2$ , the first integral in the right-hand side is equal to  $\lambda_2$ . Letting  $\Omega$  be either  $\Omega_1 \setminus \Omega_2$ , or  $\Omega_2 \setminus \Omega_1$ , we can write

$$(2.26) \quad \int_{\Omega} |d\Phi_2|^2 \leq \left( \int_{\Omega} |d\Phi_2|^{2/(2-s_0)} \right)^{2-s_0} |\Omega|^{q_0},$$

with the notation of Assumption 2.13.

As above, recalling that  $d\Theta_2 = d\Phi_2$ , we conclude that there exists a constant  $C_{20,3}(M, s_0)$  such that

$$(2.27) \quad \int_{\Omega_1} |d\Theta_2|^2 \leq \lambda_2 + C_{20,3} d_r(\Omega_1, \Omega_2)^{q_0}.$$

By symmetry between  $\lambda_1$  and  $\lambda_2$ , this completes the proof of the lemma.  $\square$

We now consider a family  $\{\Omega_t\}_{0 \leq t \leq a}$  of domains in the class  $\mathcal{L}_M$ . We use the notation,

$$(2.28) \quad \Omega_{t,+} := \Omega_t \cap \{u > 0\},$$

and we decompose the boundary  $\partial\Omega_{t,+}$  into two parts,  $\partial\Omega_t \cap \{u > 0\}$  and  $D \cap \Omega_{t,+}$ . We assume furthermore that the domains  $\Omega_t$  satisfy the Assumption 2.5, i.e., that the eigenvalues  $\nu^+(\Omega_t)$ , or equivalently the eigenvalues  $\mu_2(\Omega_{t,+}, \mathbf{nn})$ , are simple.

Call  $\phi_t$  an eigenfunction associated with  $\nu^+(\Omega_t)$ , with  $L_2$ -norm 1. It is uniquely defined up to sign. Denote its extension  $\mathbb{E}_{\Omega_t}(\phi_t)$  by  $\Phi_t$  (see, Proposition 2.11). Recall that  $\phi_t$  and  $\Phi_t$  are both symmetric with respect to  $D$ .

We also use the notation,

$$(2.29) \quad \begin{cases} \lambda_t := \nu^+(\Omega_t) = \mu_2(\Omega_{t,+}, \mathbf{nn}), \\ \mu_0 := \mu_3(\Omega_{0,+}, \mathbf{nn}), \end{cases}$$

Observe that Assumption 2.5 on  $\Omega_0$  implies that

$$(2.30) \quad \lambda_0 < \mu_0.$$

**Lemma 2.15.** *Let  $\{\Omega_t\}_{0 \leq t \leq a}$  be a family of domains in the class  $\mathcal{L}_M$ , satisfying Assumption 2.5. Assume that  $d_r(\Omega_t, \Omega_0)$  tends to zero when  $t$  tends to zero.*

- (1) *For  $d_r(\Omega_t, \Omega_0)$  small enough, the function  $\phi_t$  can be uniquely defined by the normalization*

$$\int_{\Omega_t} \phi_t^2 = 1 \text{ and } \int_{\Omega_t \cap \Omega_0} \phi_t \phi_0 > 0.$$

- (2) When  $t$  tends to zero,  $\Phi_t|_{\Omega_0}$  tends to  $\phi_0$  in  $L_2(\Omega_0)$ . Furthermore, the family  $\Phi_t$  is relatively compact in  $C^{0,s_0-1}(\mathbb{R}^2)$ , and weakly compact in  $H^2(\mathbb{R}^2)$ .
- (3) For any  $k \in \mathbb{N}$ , and for any compact  $K \subset \Omega_0$ , the functions  $\Phi_t$  tend to  $\phi_0$  in  $C^k(K)$ .

*Proof of Assertion (1).* We begin as in the proof of Lemma 2.14. For the time being,  $\phi_t$  is well-defined up to sign. Let

$$(2.31) \quad \Theta_t = \Phi_t - |\Omega_0|^{-1} \int_{\Omega_0} \Phi_t,$$

so that  $\int_{\Omega_0} \Theta_t = 0$ , and  $d\Theta_t = d\Phi_t$ . Furthermore, the function  $\Theta_t$  is  $D$ -symmetric.

Then,

$$(2.32) \quad \int_{\Omega_0} \Theta_t^2 = \int_{\Omega_0} \Phi_t^2 - |\Omega_0|^{-1} \left( \int_{\Omega_0} \Phi_t \right)^2.$$

We introduce the notation,

$$\delta(t) = d_r(\Omega_t, \Omega_0).$$

The constants  $C_{25,i}$  which appear below only depend on  $M$  and  $s_0$ .

Since  $\int_{\Omega_t} \phi_t = 0$ , we conclude as in the proof of Lemma 2.14 that there exist constants  $C_{25,1}$  and  $C_{25,2}$  such that,

$$(2.33) \quad \begin{cases} \left| \int_{\Omega_0} \Phi_t \right| \leq C_{25,1} \delta(t), \\ \left| \int_{\Omega_0} \Phi_t^2 - 1 \right| \leq C_{25,2} \delta(t). \end{cases}$$

Using the condition (2.2), it follows that there exist constants  $C_{25,3}, \dots, C_{25,5}$ , such that

$$(2.34) \quad 1 - C_{25,3} \delta(t) \leq \int_{\Omega_0} \Theta_t^2 \leq 1 + C_{25,3} \delta(t),$$

and, using Lemma 2.14,

$$(2.35) \quad \int_{\Omega_0} |d\Theta_t|^2 \leq \lambda_t + C_{25,4} \delta(t)^{q_0} \leq \lambda_0 + C_{25,5} \delta(t)^{q_0}.$$

Define the function

$$(2.36) \quad \Sigma_t := \Theta_t - \left( \int_{\Omega_0} \Theta_t \phi_0 \right) \Phi_0.$$

Then  $\Sigma_t$  is  $D$ -symmetric and satisfies

$$(2.37) \quad \int_{\Omega_0} \Sigma_t = 0 \text{ and } \int_{\Omega_0} \Sigma_t \phi_0 = 0.$$

It follows from our assumptions and notation that,

$$(2.38) \quad \int_{\Omega_0} |d\Sigma_t|^2 \geq \mu_0 \int_{\Omega_0} \Sigma_t^2,$$

$$(2.39) \quad \int_{\Omega_0} \Sigma_t^2 = \int_{\Omega_0} \Theta_t^2 - \left( \int_{\Omega_0} \Theta_t \phi_0 \right)^2.$$

Using the fact that  $(d\Phi_0)|_{\Omega_0} = d\phi_0$ , and the variational definition of  $(\lambda_0, \phi_0)$ , we also have

$$(2.40) \quad \int_{\Omega_0} |d\Sigma_t|^2 = \int_{\Omega_0} |d\Theta_t|^2 - \lambda_0 \left( \int_{\Omega_0} \Theta_t \phi_0 \right)^2.$$

From (2.40) and the estimates on  $\Theta_t$ , there exists a constant  $C_{25,6}$  such that

$$(2.41) \quad \int_{\Omega_0} |d\Sigma_t|^2 \leq \lambda_0 - \lambda_0 \left( \int_{\Omega_0} \Theta_t \phi_0 \right)^2 + C_{25,6} \delta(t).$$

From (2.37), (2.39) and (2.41), it follows that there exist constants such that

$$(2.42) \quad \int_{\Omega_0} |d\Sigma_t|^2 \geq \mu_0 \left\{ 1 - C_{25,7} \delta(t) - \left( \int_{\Omega_0} \Theta_t \phi_0 \right)^2 \right\}.$$

and

$$(2.43) \quad \left| \int_{\Omega_0} \Theta_t \phi_0 \right| \geq 1 - \frac{(1 + \mu_0) C_{25,8}}{\mu_0 - \lambda_0} \delta(t)^{q_0}.$$

From (2.43), we deduce that for  $\delta(t)$  small enough, the integral  $\int_{\Omega_0} \Theta_t \phi_0$  is not zero. Note that  $\int_{\Omega_0} \Theta_t \phi_0 = \int_{\Omega_0} \Phi_t \phi_0$ . Write

$$\int_{\Omega_0} \Phi_t \phi_0 = \int_{\Omega_0 \cap \Omega_t} \phi_t \phi_0 + \int_{\Omega_0 \setminus \Omega_t} \Phi_t \phi_0,$$

and note that the second term tends to zero with  $\delta(t)$ . It follows that  $\int_{\Omega_0 \cap \Omega_t} \phi_t \phi_0 \neq 0$ , provided that  $\delta(t)$  is small enough. This means that we can choose the sign of  $\phi_t$  such that  $\int_{\Omega_0 \cap \Omega_t} \phi_t \phi_0 > 0$ , provided that  $\delta(t)$  is small enough. This proves the first assertion.

*Proof of Assertion (2).* We now assume  $\delta(t)$  to be small enough, so that we can uniquely determine the eigenfunction  $\phi_t$  by  $\|\phi_t\|_{L_2(\Omega_t)} = 1$ , with  $\int_{\Omega_0 \cap \Omega_t} \phi_t \phi_0 > 0$ . More precisely, by (2.43), there exists a constant  $C_{25,9}(M, s_0, \lambda_0, \mu_0)$  such that

$$(2.44) \quad \int_{\Omega_0} \Phi_t \phi_0 \geq 1 - C_{25,9} \delta(t)^{q_0}.$$

Using (2.33), (2.44), and the fact that  $\phi_0$  is normalized, there exists a constant  $C_{25,10}(M, s_0, \lambda_0, \mu_0)$  such that

$$(2.45) \quad \int_{\Omega_0} (\Phi_t - \phi_0)^2 \leq C_{25,10} \delta(t)^{q_0}.$$

It follows that the functions  $\Phi_t$  tend to  $\phi_0$  in  $L_2(\Omega_0)$ .

The family  $\{\phi_t, t \geq 0\}$  is uniformly bounded in the  $H^2(\Omega_t)$  (Proposition 2.9), and hence the family  $\{\Phi_t, t \geq 0\}$  is uniformly bounded in  $H^2(\mathbb{R}^2)$ , with compact support in  $B(2M)$  (Proposition 2.11). It follows that it is relatively compact in  $C^{0, s_0-1}(\mathbb{R}^2)$ , and weakly compact in  $H^2(\mathbb{R}^2)$ . The second assertion follows.

Let  $k$  be an integer, and let  $K \subset \Omega_0$  be any compact subset. For  $t$  small enough, we have  $K \subset \Omega_t$ . By interior regularity,  $\Phi_t|_K = \phi_t|_K$  is

uniformly bounded in  $C^{k+1}(K)$  norm, and hence admits a convergent subsequence  $\Phi_{t_j}$  in  $C^k(K)$ . Inequality (2.45) shows that the limit of this subsequence must be  $\phi_0$ . It follows that  $\Phi_{t_j}$  converges to  $\phi_0$  in  $C^k(K)$ . Because the limit is independent of the subsequence, it follows that  $\phi_t$  tends to  $\phi_0$  in  $C^k(K)$ .  $\square$

**Remark.** Here is an alternative argument for the last assertion, which gives a stronger control of the convergence.

Let  $\chi_1, \chi_2 \in C_0^\infty(\Omega_0)$  such that  $\chi_2 = 1$  on  $\text{supp}(\chi_1)$ . We have

$$\begin{aligned} \Delta\chi_1(\phi_t - \phi_0) &= [\Delta, \chi_1](\chi_2(\phi_t - \phi_0)) - \chi_1(\lambda_t\phi_t - \lambda_0\phi_0) \\ (2.46) \quad &= [\Delta, \chi_1](\chi_2(\phi_t - \phi_0)) - \chi_1\lambda_t(\phi_t - \phi_0) \\ &\quad - \chi_1(\lambda_t - \lambda_0)\phi_0. \end{aligned}$$

Applying  $(I - \Delta)^{-\frac{1}{2}}$  to this equality, and using Lemma 2.14 and (2.45), we get

$$\|\chi_1(\phi_t - \phi_0)\|_{H^1} \leq C\delta(t)^{\frac{q_0}{2}}.$$

Hence, for any compact  $K \subset \Omega_0$ , we have

$$\|\phi_t - \phi_0\|_{H^1(K)} \leq C\delta(t)^{\frac{q_0}{2}}.$$

Similarly, starting from (2.46), given any  $k \in \mathbb{N}$ , and any compact  $K$ , we obtain,

$$\|\phi_t - \phi_0\|_{H^k(K)} \leq C(k, K)\delta(t)^{\frac{q_0}{2}}.$$

### 3. DOMAINS WITH THE SYMMETRY OF AN EQUILATERAL TRIANGLE

**3.1. Preparation.** Let  $\mathcal{T}_e$  be the equilateral triangle, with vertices at  $(0, 0)$ ,  $(1, 0)$  and  $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ . The symmetry group of  $\mathcal{T}_e$  is generated by the mirror symmetries with respect to the side bisectors, see Table 3.1.

Up to scaling, the positive first Dirichlet eigenfunction of  $\mathcal{T}_e$  is given by the formula (see [11]),

$$(3.1) \quad \xi_1^\circ(x, y) := \sin\left(\frac{4\pi y}{\sqrt{3}}\right) + \sin\left(2\pi\left(x - \frac{y}{\sqrt{3}}\right)\right) - \sin\left(2\pi\left(x + \frac{y}{\sqrt{3}}\right)\right),$$

which can also be written

$$(3.2) \quad \xi_1^\circ(x, y) = 4 \sin\left(\frac{2\pi y}{\sqrt{3}}\right) \sin\left(\pi\left(x - \frac{y}{\sqrt{3}}\right)\right) \sin\left(\pi\left(x + \frac{y}{\sqrt{3}}\right)\right).$$

**Proposition 3.1.** *The function  $\xi_1^\circ$  is positive in the interior of  $\mathcal{T}_e$ . It has a unique critical point at  $(\frac{1}{2}, \frac{\sqrt{3}}{6})$ , the centroid of the triangle. For  $0 < c < \max_{\mathcal{T}_e} \xi_1^\circ$ , the level curves  $\{\xi_1^\circ = c\}$  are smooth convex curves which have the same symmetries as  $\mathcal{T}_e$ .*

To prove this proposition, we use the following lemma [29].

**Lemma 3.2.** *Let  $\Omega$  be a convex bounded open set in  $\mathbb{R}^2$ . Let  $u$  be a positive, superharmonic function ( $\Delta u < 0$ ) in  $\Omega$ . If  $\det \text{Hess}(\log(u))$ , the determinant of the Hessian of the function  $\log(u)$ , is positive, then the super-level sets  $\{u > c\}$  are (strictly) convex.*

*Proof of the lemma.* Let  $v := \log(u)$ . Then,

$$u^2 \Delta v = u \Delta u - |du|^2.$$

Since  $u$  is positive and superharmonic, it follows that  $\Delta v < 0$ , so that  $\text{Hess}(v)$  has at least one negative eigenvalue. On the other hand, since we work in dimension 2, the positivity of  $\det \text{Hess}(v)$  implies that both eigenvalues of  $\text{Hess}(v)$  have the same sign. It follows that both eigenvalues are negative, and hence that  $\text{Hess}(v)$  is negative definite. The function  $u$  is (strictly) log-concave, and the lemma follows.  $\square$

*Proof of the proposition.* It is easy to see that the only critical points of the function  $\xi_1^\circ$  in the closed triangle are the vertices and the centroid. This function is invariant under the mirror symmetries with respect to the side bisectors of the triangle, and under the rotations with center the centroid, and angles  $\pm \frac{2\pi}{3}$ . It follows that its level sets have the same symmetries. Clearly,  $\xi_1^\circ$  is positive and superharmonic. It remains to show that  $\det \text{Hess}(\log \xi_1^\circ)$  is positive. This can be done by brute force. Let  $\xi := \log(\xi_1^\circ)$ . A Maple-aided computation gives,

$$(3.3) \quad \begin{cases} \det \text{Hess}(\xi) = \frac{4\pi^4}{3} \frac{N(\xi)}{D(\xi)}, \text{ with} \\ N(\xi) = 2 - 2 \cos\left(\frac{2\pi y}{\sqrt{3}}\right) \cos\left(\pi\left(x - \frac{y}{\sqrt{3}}\right)\right) \cos\left(\pi\left(x + \frac{y}{\sqrt{3}}\right)\right), \\ D(\xi) = (\xi_1^\circ)^2. \end{cases}$$

The proof of Proposition 3.1 is complete.  $\square$

**Notation.** We shall now work with the equilateral triangle  $\mathcal{T}_0$ , with vertices  $A = (-\frac{1}{2}, -\frac{\sqrt{3}}{6})$ ,  $B = (\frac{1}{2}, -\frac{\sqrt{3}}{6})$  and  $C = (0, \frac{\sqrt{3}}{3})$ , and centroid  $O = (0, 0)$ . Making the change of coordinates  $x = \frac{1}{2} + u$  and  $y = \frac{\sqrt{3}}{6} + v$ , in  $\xi_1^\circ$ , we obtain a first Dirichlet eigenfunction for  $\mathcal{T}_0$ ,

$$(3.4) \quad \varphi_1^\circ(u, v) = 4 \sin \frac{\pi}{3} (1 + 2\sqrt{3}v) \sin \frac{\pi}{3} (1 - 3u + \sqrt{3}v) \sin \frac{\pi}{3} (1 - 3u - \sqrt{3}v).$$

Define the function,

$$(3.5) \quad f_0(u, v) := (1 + 2\sqrt{3}v) (1 + 3u - \sqrt{3}v) (1 - 3u - \sqrt{3}v).$$

**Proposition 3.3.** *The function  $f_0$  is positive in the interior of  $\mathcal{T}_0$ . It has a unique critical point at  $O$ , the centroid of the triangle. For  $0 < c < 1$ , the level curves  $\{f_0 = c\}$  are smooth convex curves which have the same symmetries as  $\mathcal{T}_e$ .*



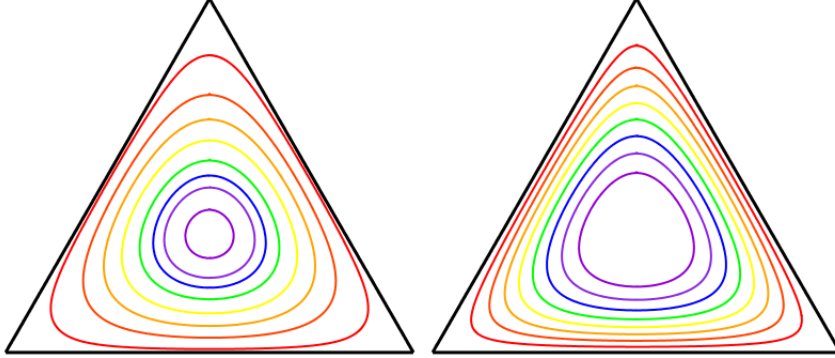


FIGURE 3.1. Level sets of  $\varphi_1^0$  (left) and  $f_0$  (right)

*Proof.* The first two assertions are clear. The function  $f_0$  is clearly invariant under the symmetries of  $\mathcal{T}_0$ , so are its level sets. The computation gives  $\Delta f_0 = -36$ , so that  $f_0$  is superharmonic. Let  $g := \log(f_0)$ . Define the functions  $A_{uu}$ ,  $A_{uv}$  and  $A_{vv}$  by the formulas

$$A_{uu} = f_0^2 \frac{\partial^2 g}{\partial u^2}, \text{ etc.}$$

Then,

$$f_0^4 \det \text{Hess}(g) = A_{uu} A_{vv} - (A_{uv})^2.$$

A Maple-aided computation gives,

$$A_{uu} A_{vv} - (A_{uv})^2 = 324 f_0^2 (1 + 6u^2 + 6v^2),$$

so that

$$(3.6) \quad \det \text{Hess}(g)(u, v) = 324 \frac{1 + 6u^2 + 6v^2}{f_0^2(u, v)}.$$

This completes the proof of Proposition 3.3.  $\square$

**Remark 3.4.** *The proof of the convexity of the level sets of a first Dirichlet eigenfunction of a convex domain, in any dimension, seems to go back to [14, Theorem 6.1] (without clearly stated assumptions on the domain). For a general 2-dimensional convex domain, see [15, Corollary 4.6] where a stronger result is proved, namely, the level sets are strictly convex. The proof in [1] seems to only apply to 2-dimensional domains whose boundary has a strictly positive curvature.*

**Remark 3.5.** *Recall that the torsion or warping function  $f_\Omega$  is the solution of  $\Delta u = -1$  in  $\Omega$ , and  $u|_\Omega = 0$ . The function  $f_0$  is (up to scaling) the warping function of the equilateral triangle, see [25, Section 7]. The square root of the warping function  $f_\Omega$  is known to be strictly concave: see [32, Theorem 4.1] for a general convex domain; see [29] and [30], Example 3.4 page 120, for domains  $\Omega$  which are sufficiently smooth, and strictly convex.*

3.2. **Domains with  $\mathcal{G}_0$ -symmetry.** Recall that  $\mathcal{T}_0$  is the equilateral triangle with vertices  $A$ ,  $B$ , and  $C$ , and centroid  $O$ . Call  $D_A$ ,  $D_B$  and  $D_C$  the bisectors of its sides. The coordinates are chosen so that  $D_C = \{u = 0\}$ , see Figure 3.2.

The isometry group of  $\mathcal{T}_0$  is the group

$$(3.7) \quad \mathcal{G}_0 = \{I, D_A, D_B, D_C, R, R^2\}.$$

where  $D_A$ , is the mirror symmetry with respect to the bisector  $D_A$ ,  $R$  the rotation with center  $O$  and angle  $\frac{2\pi}{3}$ ,  $\dots$ , see Table 3.1.

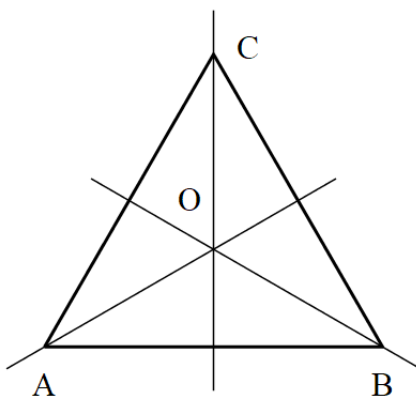
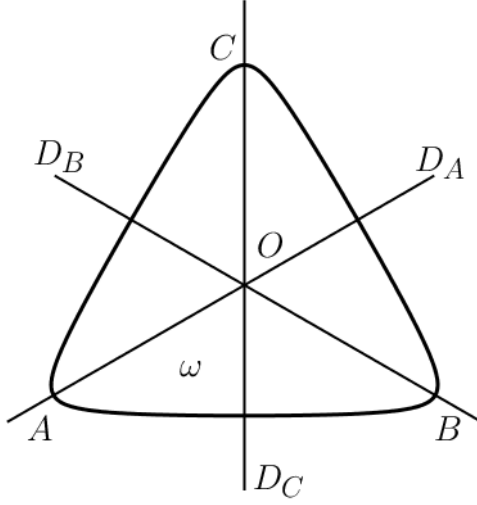


FIGURE 3.2. The equilateral triangle  $\mathcal{T}_0$

$f \backslash g$	$I$	$D_A$	$D_B$	$D_C$	$R$	$R^2$
$f \circ g$						
$I$	$I$	$D_A$	$D_B$	$D_C$	$R$	$R^2$
$D_A$	$D_A$	$I$	$R$	$R^2$	$D_B$	$D_C$
$D_B$	$D_B$	$R^2$	$I$	$R$	$D_C$	$D_A$
$D_C$	$D_C$	$R$	$R^2$	$I$	$D_A$	$D_B$
$R$	$R$	$D_C$	$D_A$	$D_B$	$R^2$	$I$
$R^2$	$R^2$	$D_B$	$D_C$	$D_A$	$I$	$R$

TABLE 3.1. The group  $\mathcal{G}_0$

To construct smooth counterexamples to ECP, the idea is to start from the equilateral triangle, and to consider the class  $\mathcal{L}_{M,0}$  of domains  $\Omega$

FIGURE 3.3. A domain  $\Omega$  in the class (3.8)

with the following properties,

$$(3.8) \quad \begin{cases} \Omega \in \mathcal{L}_M \text{ for some } M, \\ \Omega \text{ admits } \mathcal{G}_0 \text{ as symmetry group,} \end{cases}$$

see Figure 3.3.

The super-level sets  $\{x \in \mathcal{T}_0 \mid \varphi_1^0 > c\}$  of the first Dirichlet eigenfunction, and the super-level sets  $\{x \in \mathcal{T}_0 \mid f_0 > c\}$  of the function  $f_0$  provide examples of  $C^\infty$  convex domains  $\Omega$  with the symmetry group  $\mathcal{G}_0$ , see Figure 3.1. Another example is the equilateral triangle with rounded corners,  $\mathcal{T}_{0,a}$ : replace each corner by an arc of circle centered on the corresponding bisector, and tangent to the sides, with radius  $a$ . This yields a convex domain, with  $C^1$ , piecewise  $C^2$ , boundary, with symmetry group  $\mathcal{G}_0$ .

One can actually show that these families of domains belong to the class  $\mathcal{L}_{M,0}$  for some  $M > 0$ , see (3.8).

In Section 4, we shall consider yet another family. We will prove that it is indeed in the class  $\mathcal{L}_M$  for some  $M$ .

We conclude this section with a spectral property of the domains in the class  $\mathcal{L}_{M,0}$ .

**Proposition 3.6.** *Let  $\Omega$  be a domain in the class (3.8). Then, the first Neumann eigenvalues of  $\Omega$  satisfy*

$$(3.9) \quad 0 = \nu_1 < \nu_2 = \nu_3 < \nu_4 \leq \dots$$

*More precisely, the eigenspace  $\mathcal{E}(\nu_2)$  admits a basis  $\{\phi, \psi\}$  such that  $D_C^* \phi = \phi$ , and  $D_C^* \psi = -\psi$ . Furthermore,  $\mathcal{Z}(\phi) \cap D_C = \{O\}$ , and  $\mathcal{Z}(\psi) = D_C \cap \mathcal{T}_0$ .*

*Proof.* (i) Let  $D := D_C$ , and define the set of functions

$$(3.10) \quad \mathcal{S}_\sigma := \{\varphi \mid D^* \varphi = \sigma \varphi\}, \quad \sigma \in \{+, -\}.$$

Because  $D$  is an isometry,  $D^*$  leaves  $\mathcal{E}(\nu_2)$  globally invariant, and the eigenspace decomposes as

$$(3.11) \quad \mathcal{E}(\nu_2) = (\mathcal{E}(\nu_2) \cap \mathcal{S}_+) \oplus (\mathcal{E}(\nu_2) \cap \mathcal{S}_-).$$

Because the rotation  $R$  is an isometry,  $R^*$  leaves  $\mathcal{E}(\nu_2)$  globally invariant, and so does the map

$$(3.12) \quad T := R^* - R^{*2}$$

which commutes with  $\Delta$ .

(ii) It is easy to see that  $D^* \circ T = -T \circ D^*$ , so that

$$(3.13) \quad T(\mathcal{E}(\nu_2) \cap \mathcal{S}_\pm) \subset \mathcal{E}(\nu_2) \cap \mathcal{S}_\mp,$$

$$(3.14) \quad \ker(T) = \ker(R^* - I),$$

and that

$$(3.15) \quad \mathcal{S}_\sigma \cap \ker(T) = \{\varphi \mid D_M^* \varphi = \sigma \varphi, \forall M \in \{A, B, C\}\}.$$

(iii) We claim that  $\dim \mathcal{E}(\nu_2) > 1$ .

Assume on the contrary that  $\dim \mathcal{E}(\nu_2) = 1$ , and let  $0 \neq \varphi \in \mathcal{E}(\nu_2)$ .

Because  $R$  is an isometry and  $R^3 = I$ , we first observe that

$$R^* \varphi = \varphi.$$

Secondly, the nodal set  $\mathcal{Z}(\varphi)$  cannot contain a closed curve. Indeed,  $\varphi$  would otherwise have a nodal domain  $\omega_1$  strictly contained in  $\Omega$ . This would imply that  $\nu_2(\Omega) = \nu_2 = \delta_1(\omega_1) > \delta_1(\Omega)$ , where  $\delta_1$  denote the first Dirichlet eigenvalue, and where we have used the monotonicity of Dirichlet eigenvalues with respect to domain inclusion. On the other hand, according to Pólya [40] and Szegő [46],  $\nu_2(\Omega) < \delta_1(\Omega)$ , a contradiction.

Finally, because  $\Omega$  is simply-connected,  $\mathcal{Z}(\varphi)$  would be a simple curve meeting  $\partial\Omega$  at exactly two points. This now contradicts the fact that  $\mathcal{Z}(\varphi)$  is invariant under the rotation  $R$ .

(iv) We now analyze  $\mathcal{E}(\nu_2)$ . There are two cases.

*Case 1.* There exists some  $0 \neq \psi \in \mathcal{E}(\nu_2) \cap \mathcal{S}_-$ . Then  $\psi$  vanishes on  $D_C$ , and cannot vanish elsewhere by Courant's nodal domain theorem. In this case,  $\psi \notin \ker(T)$ , otherwise, as explained in (iii),  $\psi$  would also be anti-invariant with respect to the bisectors  $D_A, D_B$ , and  $\psi$  would have more nodal domains than permitted by Courant's theorem. It follows that  $\phi := T(\psi)$  is not zero, belongs to  $\mathcal{E}(\nu_2) \cap \mathcal{S}_+$ , and is linearly independent from  $\psi$ .

*Case 2.* There exists some  $\xi \notin \mathcal{E}(\nu_2) \cap \mathcal{S}_-$ . In this case,  $\phi := \frac{1}{2}(\xi + D^* \xi)$  is not zero, and belongs to  $\mathcal{E}(\nu_2) \cap \mathcal{S}_+$ . The arguments in (iii) prove

that  $\phi \notin \ker(T)$ . Then  $\psi = T(\phi)$  is not zero, belongs to  $\mathcal{E}(\nu_2) \cap \mathcal{S}_-$ , and is linearly independent from  $\phi$ .

In both cases, we conclude that there exist linearly independent functions  $\phi \in \mathcal{E}(\nu_2) \cap \mathcal{S}_+$  and  $\psi \in \mathcal{E}(\nu_2) \cap \mathcal{S}_-$ . It follows that both spaces have dimension at least 1, and we recover the fact that  $\mathcal{E}(\nu_2)$  has dimension at least 2.

(v) Any function  $0 \neq \psi \in \mathcal{E}(\nu_2) \cap \mathcal{S}_-$  vanishes on  $D_C$  and has precisely two nodal domains  $\Omega_{\pm}$  which are the connected components of  $\Omega \setminus D_C$ . Let  $\Omega_+ := \Omega \cap \{u > 0\}$ . The function  $\psi|_{\Omega_+}$  is the first eigenfunction of  $\Omega_+$  with Neumann boundary condition on  $\Omega_+ \cap \partial\Omega$ , and Dirichlet boundary condition on  $\Omega_+ \cap D_C$ . Such a function is unique up to scaling. This implies that  $\dim \mathcal{E}(\nu_2) \cap \mathcal{S}_- = 1$ . As we have seen, in step (iv), the map  $T$  is injective from  $\mathcal{E}(\nu_2) \cap \mathcal{S}_+$  to  $\mathcal{E}(\nu_2) \cap \mathcal{S}_-$ . It follows that  $\dim \mathcal{E}(\nu_2) \cap \mathcal{S}_+ = 1$  as well, and hence that  $\dim \mathcal{E}(\nu_2) = 2$ .

(v-bis) Here is an alternative argument to conclude that  $\dim \mathcal{E}(\nu_2) = 2$ , see also Remark 3.8.

*On the one hand*, we claim that  $\dim \mathcal{E}(\nu_2) \leq 3$ . Indeed, if the dimension were at least 4, we could find an eigenfunction  $\varphi \in \mathcal{E}(\nu_2)$  admitting at least three distinct zeroes on  $\partial\Omega$  and, using the fact that  $\Omega$  is simply-connected,  $\varphi$  would have at least 3 nodal domains, contradicting Courant's theorem.

Alternatively, we could find an eigenfunction  $\psi$  with a critical zero  $x_0$  of order at least two. By the local structure theorem, the nodal set  $\mathcal{Z}(\psi)$  would contain at least four semi-arcs issuing from  $x_0$ , and we would again arrive at a contradiction with Courant's theorem.

*On the other hand*, it follows from (i) that  $\mathcal{E}(\nu_2)$  does not contain any  $R$ -invariant eigenfunction. As a consequence, if  $u \in \mathcal{E}(\nu_2)$ , then  $u$  and  $R^*u$  span a two dimensional space  $\mathcal{E}^1(\nu_2)$  which is  $R^*$ -invariant because  $u + R^*u + R^{*2}u = 0$  (otherwise this function would be  $R^*$  invariant). If the dimension of  $\mathcal{E}(\nu_2)$  were bigger than 2, we could find another function  $v \in \mathcal{E}(\nu_2)$ , orthogonal to  $\mathcal{E}^1(\nu_2)$ . Repeating the previous argument with  $v$  instead of  $u$ , we would conclude that the dimension of  $\mathcal{E}(\nu_2)$  is at least 4, a contradiction.

(vi) Let  $0 \neq \phi \in \mathcal{E}(\nu_2) \cap \mathcal{S}_+$ . According the (iv) and (v), there exists  $\psi \in \mathcal{E}(\nu_2) \cap \mathcal{S}_-$  such that  $T(\psi) = \phi$ . The function  $\psi$  vanishes at  $O$ , and hence, so does the function  $\phi$  since  $O$  is the center of the rotation  $R$ .

We claim that  $\mathcal{Z}(\phi) \cap D_C = \{O\}$ , and that  $O$  is a regular point of  $\mathcal{Z}(\phi)$ . Indeed, we would otherwise have at least two arcs emanating from  $D_C$  into  $\Omega_+$ . Such arcs could be followed until they intersect, or until they reach the boundary of  $\Omega_+$ . Reasoning as we did in (v-bis), this would contradict either Courant's nodal domain theorem, or the closed nodal line theorem for  $\phi \in \mathcal{E}(\nu_2)$  explained in (i).

We have completed the proof of Proposition 3.6.  $\square$

**Remark 3.7.** *Note that the inequality  $\nu_2(\omega) < \delta_1(\omega)$  is valid for any sufficiently regular, bounded domain, without any convexity assumption. The fact that a second Neumann eigenfunction cannot have a closed nodal line motivated the “closed nodal line conjecture for a second Dirichlet eigenfunction”, see [38], last paragraph on page 466, and Conjecture 5, and [31].*

**Remarks 3.8.** *Concerning the multiplicity of  $\nu_2$ , we can mention the following.*

- (1) *According to [35, Remarks (2), p. 206], if  $\Omega$  is close enough to  $\mathcal{T}_0$  in the sense of the Hausdorff distance, then*

$$\dim \mathcal{E}(\nu_2(\Omega)) = \dim \mathcal{E}(\nu_2(\mathcal{T}_0)) = 2.$$

- (2) *For any smooth simply-connected domain  $\Omega$ ,  $\dim \mathcal{E}(\nu_2(\Omega)) \leq 3$ . This bound was first given by Cheng [16] for smooth simply-connected compact surfaces without boundary, see also the assertion in [27, line (-8), p. 1170]. In this latter paper, the authors indicate that the assumption that  $\Omega$  is smooth is probably too strong. The smoothness assumption is used to describe the local behaviour of the nodal set at a boundary point. In the non-smooth case, it might be possible to obtain a result on the local structure of the nodal set similar to the one described by Alessandrini [3] for the Dirichlet boundary condition. See also [23]*
- (3) *In [37, Theorem 2.3], Lin proved that the second Dirichlet eigen-space of a smooth convex domain has dimension at most 2.*

#### 4. THEOREM 1.2: APPLYING THE DEFORMATION ARGUMENT

**4.1. Main goals.** In this section, we apply the deformation technique of Section 2 to the restricted class of domains

$$(4.1) \quad \mathcal{L}_{M,0} := \{\Omega \in \mathcal{L}_M \mid \Omega \text{ admits } \mathcal{G}_0 \text{ as symmetry group}\},$$

and we give the proof of Theorem 1.2.

Let  $t$  be a nonnegative parameter. Introduce the function

$$(4.2) \quad f_{0,t}(u, v) = (1 + t + 2\sqrt{3}v)(1 + t + 3u - \sqrt{3}v)(1 + t - 3u - \sqrt{3}v).$$

When  $t = 0$ , we recover the function  $f_0$  defined by (3.5). When  $t > 0$ , the function  $f_{0,t}$  is a torsion function for the triangle  $\mathcal{T}_{0,t}$  obtained from  $\mathcal{T}_0$  by dilation of ratio  $(1 + t)$ . This equilateral triangle has vertices  $A_t = (-\frac{1+t}{2}, -\frac{\sqrt{3}(1+t)}{6})$ ,  $B = (\frac{1+t}{2}, -\frac{\sqrt{3}(1+t)}{6})$  and  $C = (0, \frac{\sqrt{3}(1+t)}{3})$ .

An immediate computation gives that

$$(4.3) \quad f_{0,t}(A) = f_{0,t}(B) = f_{0,t}(C) = t^2(3 + t).$$

**Definition 4.1.** Define the domain  $\Omega_t$  to be the super-level set

$$(4.4) \quad \Omega_t := \{f_{0,t} > t^2(3+t)\}.$$

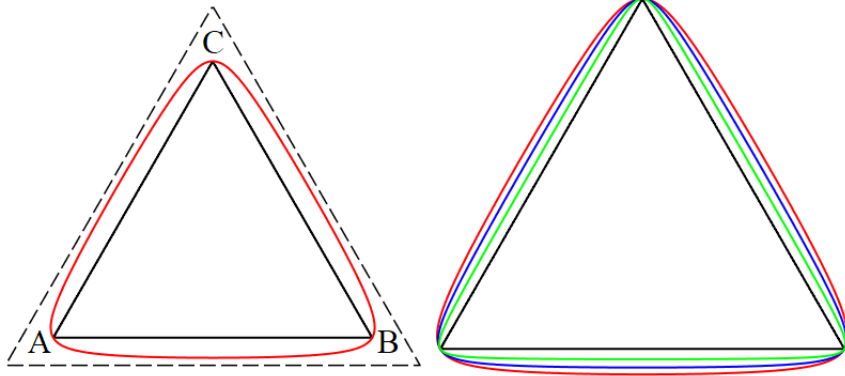


FIGURE 4.1. Domains  $\Omega_t$

The triangle  $\mathcal{T}_0$ , the triangle  $\mathcal{T}_{0,t}$  (dashed line), and a domain  $\Omega_t$  (red line) are displayed in Figure 4.1, left. The triangle  $\mathcal{T}_0$ , and domains  $\Omega_t$ , with  $t = 0.3$  (red),  $t = 0.2$  (blue), and  $t = 0.1$  (green), are displayed in Figure 4.1, right.

Let us summarize the properties of the domains  $\Omega_t$ .

**Proposition 4.2.** *The family of domains  $\{\Omega_t\}_{0 \leq t \leq \frac{1}{2}}$  has the following properties.*

- (1)  $\Omega_0 = \mathcal{T}_0$ .
- (2) For  $t > 0$ , the domain  $\Omega_t$  is convex, bounded, and open, with  $C^\infty$  boundary. Furthermore,  $\mathcal{T}_0 \subset \Omega_t$ , and  $A, B, C \in \partial\Omega_t$ .
- (3) The domain  $\Omega_t$  has the symmetry group  $\mathcal{G}_0$ .
- (4) The family  $\Omega_t$  is increasing, for  $0 < t_1 < t_2$ ,

$$\Omega_{t_1} \subset \Omega_{t_2}.$$

- (5) For  $0 \leq t \leq \frac{1}{2}$ , the domains  $\Omega_t$  belong to the class  $\mathcal{L}_M$  for some positive constant  $M$ .

*Proof.* Assertion (1) is obvious.

Assertion (2) follows from Proposition 3.3 by dilation of ratio  $(1+t)$ .

Assertion (3). By definition of  $\Omega_t$ , the vertices  $A, B$  and  $C$  belong to  $\partial\Omega_t$ . The inclusion (of open sets)  $\mathcal{T}_0 \subset \Omega_t$  follows from the convexity of  $\Omega_t$ .

Assertion (4). The domain  $\Omega_t$  can also be defined by  $\{g_t > 0\}$ , where

$$g_t(u, v) = f_{0,t}(u, v) - t^2(3+t) = f_0(u, v) + 3t - 9t(u^2 + v^2).$$

Let  $t_1 < t_2$ . To prove that  $\Omega_{t_1} \subset \Omega_{t_2}$ , it suffices to consider the points  $(u, v) \in \Omega_{t_1} \setminus \mathcal{T}_0$ . For such  $(u, v)$ , we have  $g_{t_1}(u, v) > 0$  and  $f_0(u, v) \leq 0$ .

This implies that

$$3t_1(1 - 3u^2 - 3v^2) > -f_0(u, v) \geq 0,$$

and hence that  $1 - 3u^2 - 3v^2 > 0$ . On the other hand, we have

$$g_{t_1}(u, v) - g_{t_2}(u, v) = 3(t_1 - t_2)(1 - 3u^2 - 3v^2) < 0,$$

i.e.,  $g_{t_2}(u, v) > 0$ , or  $(u, v) \in \Omega_{t_2}$ .

Assertion (5). Since  $\mathcal{T}_0 \subset \Omega_t \subset \mathcal{T}_{0,t}$ , the domains satisfy condition (2.2). It remains to show that they satisfy condition (2.5), i.e., that they can be defined in polar coordinates, as

$$\Omega_t = \{(r, \theta) \mid 0 \leq r < \rho(t, \theta)\}$$

where the functions  $\rho(t, \cdot)$  are uniformly Lipschitz. Due to rotational invariance, it suffices to look at the part of  $\partial\Omega_t$  contained in the sector  $BOC$ , see Figure 4.2. This part of the boundary is symmetric with respect to the bisector  $DA$ , so that it suffices to look at the sector  $BOa$ . With respect to the  $u$ -axis  $Ou$ , the angle  $\theta$  then varies from  $-\frac{\pi}{6}$  ( $OB$ ) to  $\frac{\pi}{6}$  ( $Oa$ ).

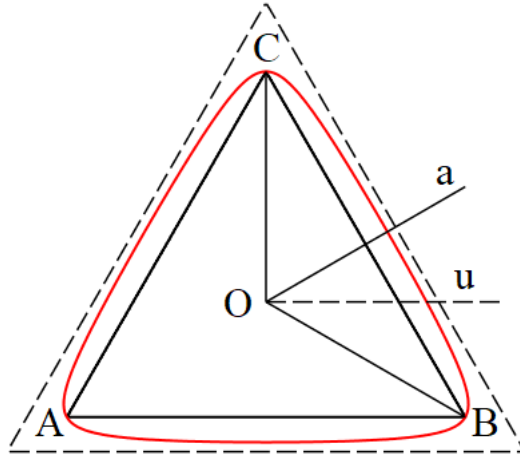


FIGURE 4.2.

Instead of polar coordinates  $(\rho, \theta)$ , we use “inverse” polar coordinates  $(r, \theta)$ , where  $r\rho \equiv 1$ . The inverse polar equation of the side  $BC$  of  $\mathcal{T}_0$ , is

$$(4.5) \quad r_A(\theta) = 2\sqrt{3} \cos\left(\theta - \frac{\pi}{6}\right), \quad \text{for } \theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right].$$

Let  $r = r(t, \theta)$  be the inverse polar equation of the arc  $BC \subset \partial\Omega_t$ . Because  $\mathcal{T}_0 \subset \mathcal{T}_{0,t}$ , we have

$$(4.6) \quad \frac{1}{1+t} r_A(\theta) \leq r(t, \theta) \leq r_A(\theta) \quad \text{for } \theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right].$$



Using the definition of  $\Omega_t$ , we also have that  $r(t, \theta)$  is a root of the equation

$$(4.7) \quad (1 + 3t)r^3 - 9(1 + t)r + 6\sqrt{3}\sin^3(\theta) - 18\sqrt{3}\sin(\theta)\cos(\theta) = 0.$$

or, equivalently,

$$(4.8) \quad (1 + 3t)r^3 - 9(1 + t)r - 6\sqrt{3}\sin(3\theta) = 0.$$

Looking at the global picture of  $f_{0,t}^{-1}(0)$ , it is easy to see that this equation has one simple root satisfying (4.6). Taking the derivative  $r_\theta$  with respect to  $\theta$ , we obtain,

$$(4.9) \quad \left((1 + 3t)r^2 - 3(1 + t)\right)r_\theta - 6\sqrt{3}\cos(3\theta) = 0.$$

Note that

$$(4.10) \quad (1 + 3t)r^3 - 3(1 + t)r = \left((1 + 3t)r^3 - 9(1 + t)r\right) + 6(1 + t)r,$$

so that

$$(4.11) \quad (1 + 3t)r^3 - 3(1 + t)r = 6\left((1 + t)r + \sqrt{3}\sin(3\theta)\right).$$

Using (4.6), we have

$$(4.12) \quad (1 + t)r + \sqrt{3}\sin(3\theta) \geq 2\sqrt{3}\cos\left(\theta - \frac{\pi}{6}\right) + \sqrt{3}\cos\left(3\left(\theta - \frac{\pi}{6}\right)\right),$$

and hence

$$(4.13) \quad (1 + 3t)r^3 - 3(1 + t)r \geq \sqrt{3}\cos\left(\theta - \frac{\pi}{6}\right)\left(4\cos^2\left(\theta - \frac{\pi}{6}\right) - 1\right).$$

It follows that  $r_\theta$  is positive in the interval  $]\frac{\pi}{6}, \frac{\pi}{6}[$ , and that

$$(4.14) \quad 0 \leq \frac{r_\theta(t, \theta)}{r(t, \theta)} \leq \tan\left(\frac{\pi}{6} - \theta\right) \leq \sqrt{3}.$$

Note that  $r(t, \theta) \geq 2\sqrt{3}$ . This proves that condition (2.5) is satisfied with  $M = \sqrt{3}$ .  $\square$

**4.2. Proof of Theorem 1.2.** The fact that the equilateral triangle  $\mathcal{T}_0$  provides a counterexample to  $\text{ECP}(\mathcal{T}_0, \mathbf{n})$  follows from the analysis of the level lines of the  $D$ -symmetric second Neumann eigenfunction  $\phi_{\mathcal{T}_0}$ , see [11, Section 6]. Some of the levels lines of  $\phi_{\mathcal{T}_0}$  are displayed in Figure 4.3.

Deform the domain  $\mathcal{T}_0 =: \Omega_0$  using Proposition 4.2. Denote the normalized  $D$ -symmetric eigenfunctions by  $\phi_t$ , and their extensions by  $\Phi_t$ .

The function  $\phi_0$  is such that  $\phi_0(C) > 0$ , and  $\phi_0(A) = \phi_0(B) < 0$ , see Figure 4.4. According to [11, Section 6], we now choose (and fix) some  $a > 0$ , such that  $\{\phi_0 + a = 0\}$  consists of two disjoint arcs, symmetric with respect to the side bisector  $D_C$  (blue arcs in the figure). We have  $\phi_0 + a > 0$  in the connected component of  $\mathcal{T}_0 \setminus \{\phi_0 + a = 0\}$  which contains  $O$ , and  $\phi_0 + a < 0$  in the two connected components

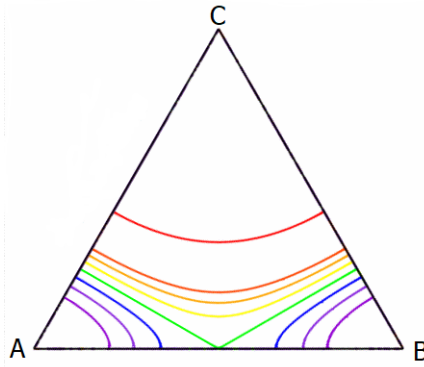


FIGURE 4.3. Level lines the second symmetric Neumann eigenfunction of the equilateral triangle

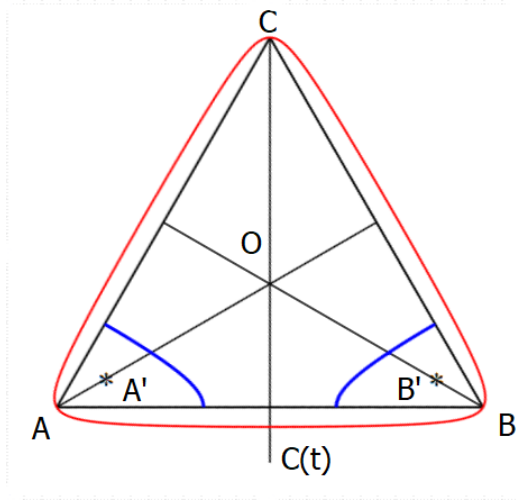


FIGURE 4.4. Proof of Theorem 1.2.

close to the vertices  $A$  and  $B$ . Choose  $A'$  and  $B'$  in these connected components. Note that  $\phi_0|_{D_C} + a > 0$ , and  $\phi_0(A') + a = \phi_0(B') + a < 0$ .

We now consider the family  $\Omega_t$ . Apply Lemma 2.15 to the family  $\phi_t$ , and get that for  $t$  sufficiently small

$$\phi_t(A') + a = \phi_t(B') + a < 0.$$

Call  $C(t)$  the intersection point of the bisector  $D_C$  with  $\partial\Omega_t$ , opposite to the vertex  $C$ .

**Claim 1.** For  $t$  sufficiently small,  $\phi_t|_{[CC(t)]} + a > 0$ .

Indeed, we could otherwise find a sequence  $t_k$ , tending to zero, and a point  $m_k \in [CC(t_k)]$ , such that  $\phi_{t_k}(m_k) + a \leq 0$ . The family  $\Phi_{t_k}$  is bounded in  $H^2$  with compact support in  $B(0, 2M)$ . Hence, there exists a subsequence  $t'_j$  which tends to 0, and a function  $\Phi \in C^0(\mathbb{R}^2) \cap H^2(\mathbb{R}^2)$  such that  $m_k$  converges to some  $m \in [CC(0)]$  and  $\Phi_{t'_j}$  converges to  $\Phi$

uniformly in  $B(2M)$ , and in particular in  $\overline{\mathcal{T}_0}$ . Since, by Lemma 2.15,  $\Phi_{t_j}|_{\mathcal{T}_0}$  converges to  $\phi_0$  in  $\mathcal{D}'(\mathcal{T}_0)$ , it follows that  $\phi_0 = \Phi|_{\mathcal{T}_0}$  and this extends by continuity to  $\overline{\mathcal{T}_0}$ . In particular, we would get  $\Phi(m) + a = \phi_0(m) + a \leq 0$ . A contradiction.

The claim proves that for  $t$  small enough, the points  $A'$  and  $B'$  belong to distinct connected components of  $\Omega_t \setminus \{\phi_t + a = 0\}$ , so that  $\phi_t + a$  has at least three connected component (a “positive” one, and two “negative ones”).

We shall now prove that, for  $t$  small enough,  $\phi_t + a$  has exactly three nodal domains.

**Lemma 4.3.** *Let  $\{\varphi_n, n \geq 1\}$  be an orthonormal basis of eigenfunctions of the Neumann problem in a bounded domain  $\Omega$ , associated with the eigenvalues  $0 = \nu_1(\Omega) < \nu_2(\Omega) \leq \dots$ . Choose  $\varphi_1$  (a constant function) to be positive. Then, for any  $a > 0$ , the set  $\Omega \setminus \{\varphi_n + a\varphi_1 = 0\}$  has at most  $(n - 1)$  connected components in which  $\varphi_n + a$  is positive.*

**Remark 4.4.** *A statement analogous to Lemma 4.3, for the Dirichlet problem in  $\Omega$ , appears as Theorem 1 in [20]. The proof given by Gladwell-Zhu is similar to the proof of Courant’s nodal domain theorem, and turns out to apply to both the Dirichlet and the Neumann boundary conditions, hence to Lemma 4.3. The examples of rectangles with cracks in [11, Section 3] show that one can a priori not control the number of connected components of  $\Omega \setminus \{\varphi_n + a\varphi_1 = 0\}$  in which  $\varphi_n + a$  is negative.*

We proceed with the proof that, for  $t$  small enough,  $\phi_t + a$  has exactly three nodal domains. According Lemma 4.3, we have to prove that  $\{\phi_t + a < 0\}$  has at most two connected components. The proof goes as follows.

First, we observe that  $\phi_0$  is naturally defined as a trigonometric polynomial on all  $\mathbb{R}^2$ . Observe that for  $t$  small enough,  $\{\phi_0 + a = 0\} \cap \Omega_t$  consists of two symmetric curves crossing  $\partial\Omega_t$  transversally at the points  $ac(t), ab(t), ba(t), bc(t)$ . As  $t$  tends to 0, these points tend to the intersection points of  $\{\phi_0 + a = 0\}$  with  $\partial\mathcal{T}_0$ , see Figure 4.5.

For  $\epsilon > 0$  small enough, we introduce,

$$(4.15) \quad \Omega_-(a + \epsilon, \phi_0, t) := \{\phi_0 + a + \epsilon \leq 0\} \cap \Omega_t,$$

$$(4.16) \quad \Omega_+(a - \epsilon, \phi_0, t) := \{\phi_0 + a - \epsilon \geq 0\} \cap \Omega_t,$$

and

$$(4.17) \quad \Omega(a, \epsilon, \phi_0) := \{-\epsilon \leq \phi_0 + a \leq \epsilon\} \cap \Omega_t.$$

These domains are displayed respectively in green, blue, and white in Figure 4.6.

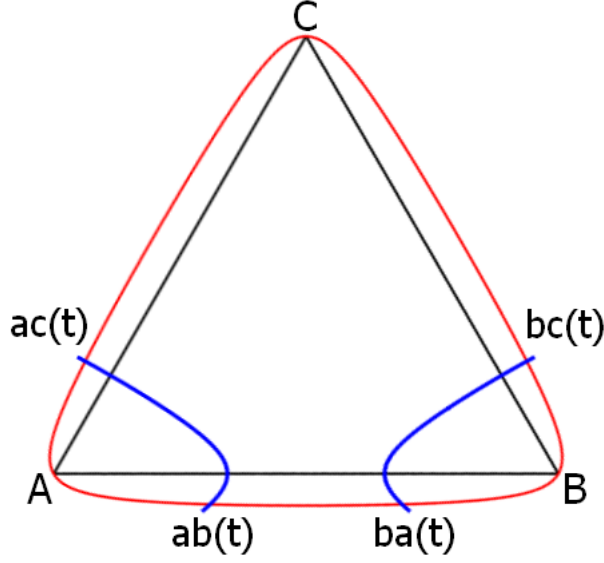


FIGURE 4.5.

**Claim 2.** For  $t$  small enough,

$$(4.18) \quad \begin{cases} \Omega_-(a + \epsilon, \phi_0, t) \subset \{\phi_t + a < 0\}, \\ \Omega_+(a - \epsilon, \phi_0, t) \subset \{\phi_t + a > 0\}. \end{cases}$$

Indeed, if the first inclusion were not true, there would exist a sequence  $t_n > 0$ , tending to 0, and  $x_n \in \Omega_{t_n}$ , such that  $\phi_{t_n}(x_n) + a \geq 0$  and  $\Phi_{t_n}$  bounded in  $H^2$ . As above, after extraction of a subsequence we can assume that  $x_n \rightarrow x_\infty$ , and that  $\Phi_{t_n}$  tends to  $\Phi$  in  $C^0$ . This implies the existence of  $x_\infty$  such that  $\Phi(x_\infty) = \phi_0(x_\infty) \geq -a$ . But  $x_\infty \in \Omega_-(a + \epsilon, \phi_0, 0)$  leading to a contradiction. The second inclusion can be proved in a similar way.

As a consequence, for  $t$  small enough, there are two symmetric components of  $\{\phi_t + a < 0\}$ , each one containing a component of  $\{\phi_0 + a + \epsilon \leq 0\} \cap \Omega_t$ . Furthermore, the “positive” component of  $\phi_t + a$  contains  $\Omega_+(a - \epsilon, \phi_0, t)$ .

We deduce from this localization, that a third “negative” connected component of  $\phi_t + a$ , if any, is necessarily contained in  $\Omega(a, \epsilon, \phi_0)$ , hence stays away from the vertices  $A$ ,  $B$  and  $C$ .

**Claim 3.** The only critical points of the function  $\phi_0$  in the square  $[-\frac{5}{8}, \frac{5}{8}] \times [-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{2}]$  are the vertices  $A, B, C$ , and the mid-point  $M_C$  of the side  $AB$ .

We refer to [9] for the explicit expression of the Neumann eigenvalues and eigenfunctions of the equilateral triangle  $\mathcal{T}_e$ . After translation and rotation, we find that the second Neumann eigenfunction of  $\mathcal{T}_0$ , which

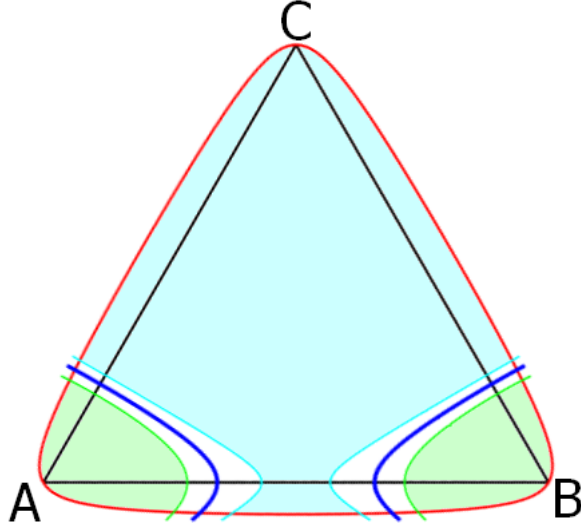


FIGURE 4.6. Localization

is symmetric with respect to  $D_C$  is given by the formula,

$$(4.19) \quad \phi_0(u, v) = a_0 \left( \cos \frac{4\pi u}{3} + \cos \frac{2\pi(1-u-\sqrt{3}v)}{3} + \cos \frac{2\pi(1+u-\sqrt{3}v)}{3} \right),$$

where  $a_0 \neq 0$  is a normalizing constant.

It follows that the critical points of  $\phi_0$  satisfy the equations,

$$(4.20) \quad \begin{cases} \sin \frac{2\pi u}{3} \left\{ \cos \frac{2\pi u(1-\sqrt{3}v)}{3} + 2 \cos \frac{2\pi u}{3} \right\} = 0, \\ \sin \frac{2\pi u(1-\sqrt{3}v)}{3} \cos \frac{2\pi u}{3} = 0. \end{cases}$$

The claim follows easily. It is also illustrated by Figure 4.7 which displays the triangle  $\mathcal{T}_0$ , the square  $[-\frac{5}{8}, \frac{5}{8}] \times [-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{2}]$ , the zero set of  $\partial_u \phi_0$  (green) and the zero set of  $\partial_v \phi_0$  (magenta).

**Claim 4.** For  $t$  small enough,  $\phi_t + a < 0$  has exactly two connected components.

For the proof, we proceed by contradiction. If not, there exists a sequence  $t_n \rightarrow 0$ , and a connected component  $\omega(t_n)$  of  $\phi_t + a < 0$ , which according to Claim 2 must be contained in  $\Omega(a, \varepsilon, \phi_0)$ .

Let  $x_n \in \omega(t_n)$  be the point at which  $\phi_{t_n}$  achieves its minimum in  $\omega(t_n)$ . We have necessarily  $\nabla \phi_{t_n}(x_n) = 0$ . After extraction of a subsequence if necessary, we can assume that  $x_n$  converged to some  $x_\infty$  which belongs to  $\overline{\mathcal{T}_0}$ , and satisfies  $-\varepsilon \leq \phi_0(x_\infty) + a \leq \varepsilon$ . There are two possibilities. If  $x_\infty \in \mathcal{T}_0$ , using Lemma 2.15, we get that  $\phi_{t_n}$  converges to  $\phi_0$  in a small ball around  $x_\infty$  in  $C^1$  sense, and this implies that  $\nabla \phi_0(x_\infty) = 0$ . A contradiction with Claim 3.

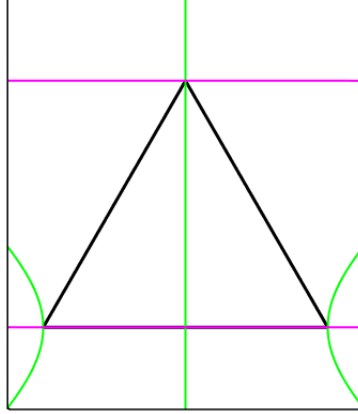


FIGURE 4.7. Localization of the critical points

The second possibility is that  $x_\infty \in \partial\mathcal{T}_0$ . Here, we have to use a uniform boundary regularity for the Neumann Laplacian in  $\Omega_t$  when we are far from  $A, B, C$ . We consider a small ball centered at  $\partial\mathcal{T}_0 \cap \{\phi_0 + a = 0\}$  of radius  $r(\epsilon)$  and containing  $\partial\mathcal{T}_0 \cap \{-2\epsilon \leq \phi_0 + a \leq 2\epsilon\}$  (hence  $x_\infty$ ). For each  $t > 0$ , we consider a function  $\chi(t, x)$  with support in the ball, equal to 1 in a fixed neighborhood of  $x_\infty$  and such that  $\partial_\nu \chi(t, x) = 0$  on  $\partial\Omega_t$ . It is easy to get such a function  $C^\infty$  in both variables  $t$  and  $x$  due to the uniform regularity of  $\partial\Omega(t)$  there (for  $t \in [0, t_0]$  with  $t_0 > 0$  small enough). We now consider  $\hat{\phi}_t := \chi(t, x)\phi_t$  in  $\Omega_t$ . This is a bounded family in  $H^2$ , and  $\hat{\phi}_t$  satisfies the Neumann condition.

We have

$$-\Delta \hat{\phi}_t = [-\Delta, \chi(t, \cdot)]\phi_t + \lambda_t \hat{\phi}_t.$$

The left hand side is uniformly bounded in  $H^1$ , and supported in the ball  $B(x_\infty, r(\epsilon))$ . We have a uniform (with respect to  $t$ ) regularity of this Neumann problem (with locally  $C^\infty$  boundary), and we get that the family  $\hat{\phi}_t$  is bounded in  $H^3(\Omega(t))$ .

We now extend it in a bounded family  $\hat{\Phi}_t \in H_0^3(B(0, 2M))$ . Coming back to our sequence  $\phi_{t_n}$ , we observe that in particular  $\hat{\Phi}_{t_n}$  is a bounded family in  $H_0^3(B(0, 2M))$ . Extracting a subsequence if necessary, we can assume that  $\hat{\Phi}_{t_n}$  converges in  $C^1(B(0, 2M))$  to  $\hat{\Phi}_\infty$ . Now we have  $\nabla \hat{\phi}_{t_n}(x_n)$  tends to  $\nabla \hat{\Phi}_\infty(x_\infty)$ . For  $n$  large enough  $\nabla \hat{\phi}_{t_n}(x_n) = 0$  which implies  $\nabla \hat{\Phi}_\infty(x_\infty) = 0$ . Looking at the restriction to  $\mathcal{T}_0$ , we also have  $\hat{\Phi}_\infty = \chi(0, \cdot)\phi_0$  in  $\mathcal{T}_0$  in  $\mathcal{D}'(\mathcal{T}_0)$ , which extends to  $\bar{\mathcal{T}}_0$  by continuity.

This implies  $0 = \nabla \hat{\Phi}_\infty(x_\infty) = \nabla \phi_0(x_\infty)$ , in contradiction with Claim 3.

*Note.* The preceding argument also shows that there cannot exist a second positive connected component for  $t > 0$  small enough (without making use of the theorem of Gladwell and Zhu).

## APPENDIX A. BOUNDS ON EIGENFUNCTIONS OF CONVEX DOMAINS

The purpose of this appendix is to prove the universal bounds on eigenfunctions of the Laplacian in a convex domain, which we need in Section 2. As a matter of fact, we consider both the Dirichlet, and the Neumann boundary conditions.

Such bounds are well-known when the domain is regular (say  $C^2$ ). A general convex domain has a Lipschitz boundary only. One difficulty comes from the fact that the various definitions of Sobolev spaces do not coincide in this case.

There is a huge literature on the subject, with many cross references. It is rather cumbersome to keep track of the dependence of the constants upon the domains, here in the class  $\mathcal{L}_M$ . In this appendix, we follow P. Grisvard's book [21]. More precisely, we adapt the proofs of Theorem 3.2.1.2 and 3.2.1.3 in [21] to our purpose.

We begin by recalling the main definitions and notation from [21] that we will use.

**A.1. Notation and definitions from [21].** We work in  $\mathbb{R}^n$ , with coordinates  $x = (x_1, \dots, x_n)$ , Euclidean norm  $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$ , and Lebesgue measure  $dx = dx_1 \cdots dx_n$ . We use the notation  $\mathbb{N}$  for the set of nonnegative integers, and  $\mathbb{N}^\bullet$  for the set of positive integers.

A.1.1. For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ , we denote by  $D^\alpha$  the derivative of order  $|\alpha| = \alpha_1 + \dots + \alpha_n$ ,

$$(A.1) \quad D^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n} .$$

A.1.2. Given  $\Omega \subset \mathbb{R}^n$  an open set,  $\mathcal{D}(\Omega)$  denotes the set of  $C^\infty$  functions with compact support contained in  $\Omega$ , and  $\mathcal{D}'(\Omega)$  denotes the set of distributions in  $\Omega$ .

A.1.3. Let  $s = m + \sigma$ , with  $m \in \mathbb{N}$ , and  $0 \leq \sigma < 1$ . Given an open set  $\Omega \subset \mathbb{R}^n$ , possibly equal to  $\mathbb{R}^n$ , the Sobolev space  $H^s(\Omega)$  is defined as the set of distributions  $u$  in  $\mathcal{D}'(\Omega)$ , whose derivatives  $D^\alpha u$  are in  $L_2(\Omega)$  (understood with respect to the Lebesgue measure  $dx$ ), for all  $\alpha$  such that  $|\alpha| \leq m$ , with the additional conditions, , when  $0 < \sigma < 1$ ,

$$(A.2) \quad \int_{\Omega \times \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{n+2\sigma}} < \infty ,$$

for all  $\alpha$  such that  $|\alpha| = m$ . The associated norms are given respectively by,

$$(A.3) \quad \begin{cases} \|u\|_{m,\Omega}^2 = \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha u|^2 , \\ \|u\|_{s,\Omega}^2 = \|u\|_{m,\Omega}^2 + \sum_{|\alpha|=m} \int_{\Omega \times \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{n+2\sigma}} , \end{cases}$$

when  $s = m, \sigma = 0$ , and  $s = m + \sigma, 0 < \sigma < 1$  respectively. See [21], Definition 1.3.1.1 ( $\Omega = \mathbb{R}^n$ ), and Definition 1.3.2.1 ( $\Omega \subset \mathbb{R}^n$  open subset), and the notation  $H^s(\Omega)$  for the spaces  $W_2^s(\Omega)$ .

A.1.4. Denote by  $\mathring{H}^s(\Omega)$  the closure of  $\mathcal{D}(\Omega)$ , with respect to the norm  $\|\cdot\|_{s,\Omega}$ , see [21], Definition 1.3.2.2.

A.1.5. Given a function  $f$  defined in the open set  $\Omega$ ,  $\tilde{f}$  denotes the extension of  $f$  to  $\mathbb{R}^n$ , by zero outside  $\Omega$ ,

$$(A.4) \quad \tilde{f} = \begin{cases} f, & \text{in } \Omega, \\ 0, & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

A.1.6. Let  $U$  be a distribution in  $\mathbb{R}^n$ , and let  $\Omega \subset \mathbb{R}^n$  be an open set. Define  $U|_\Omega$ , the restriction of  $U$  to  $\Omega$ , by

$$(A.5) \quad \langle U|_\Omega, \varphi \rangle = \langle U, \tilde{\varphi} \rangle,$$

for any  $\varphi \in \mathcal{D}(\Omega)$ .

A.1.7. For  $s > 0$ , and  $\Omega$  an open subset of  $\mathbb{R}^n$ , define the set of distributions,

$$(A.6) \quad H^s(\overline{\Omega}) = \{u \in \mathcal{D}'(\Omega) \mid \exists U \in H^s(\mathbb{R}^n), \text{ such that } u = U|_\Omega\},$$

with the norm

$$(A.7) \quad \|u\|_{s,\overline{\Omega}} = \inf \{\|U\|_{s,\mathbb{R}^n} \mid u = U|_\Omega\}.$$

The following inclusions hold,

$$(A.8) \quad \begin{cases} H^s(\overline{\Omega}) \subseteq H^s(\Omega), \text{ for any } s > 0, \\ \mathring{H}^m(\Omega) \subseteq H^m(\overline{\Omega}) \subseteq H^m(\Omega), \text{ for any } m \in \mathbb{N}^\bullet. \end{cases}$$

See [21], Definition 1.3.2.4 and Equations (1, 3, 2, 5) and (1, 3, 2, 6).

A.1.8. For any  $s > 0$ , and  $\Omega$  an open set, define the set of distributions

$$(A.9) \quad \widetilde{H}^s(\Omega) = \{u \in H^s(\Omega) \mid \tilde{u} \in H^s(\mathbb{R}^n)\},$$

with the norm

$$(A.10) \quad \|u\|_{s,\Omega}^{\sim} = \|\tilde{u}\|_{s,\mathbb{R}^n}.$$

The following inclusions hold,

$$(A.11) \quad \begin{cases} \widetilde{H}^s(\Omega) \subseteq H^s(\overline{\Omega}), \text{ for any } s > 0, \\ \mathring{H}^m(\Omega) \subseteq \widetilde{H}^m(\Omega), \text{ for any } m \in \mathbb{N}^\bullet. \end{cases}$$

See [21], Definition 1.3.2.5 and Equations (1, 3, 2, 8) and (1, 3, 2, 9). Note however, that it is not easy to control the norm of the inclusion  $\mathring{H}^s(\Omega) \subseteq \widetilde{H}^s(\Omega)$  when  $s$  is not an integer, see [21, p. 19].



### A.2. Key results from [21].

**Lemma A.1** ([21], Corollary 1.2.2.3). *A convex, bounded, open subset  $\Omega$  of  $\mathbb{R}^n$  has Lipschitz boundary  $\Gamma := \partial\Omega$  (in the sense that it is locally the graph of a Lipschitz function).*

As a matter of fact, one can say more: the Lipschitz constant of the domain can easily be bounded from above in terms of the inner radius and the diameter of  $\Omega$ , see [26, Theorem 2.4.7].

**Lemma A.2.** *Let  $\Omega$  be a convex, bounded, open subset in  $\mathbb{R}^n$ . Then, for any  $\varepsilon > 0$ , there exist two convex, bounded, open subsets  $\Omega_1, \Omega_2$ , in  $\mathbb{R}^n$ , with Lipschitz constant bounded from above by some constant  $B_L(\Omega, \varepsilon)$ , such that*

- (1)  $\Omega_1 \subset \Omega \subset \Omega_2$ ,
- (2)  $\Omega_j$  has  $C^2$  boundary  $\Gamma_j$ ,  $j = 1, 2$ ,
- (3)  $d_H(\Omega_1, \Omega_2) < \varepsilon$ ,

where  $d_H(\Omega_1, \Omega_2)$  denotes the Hausdorff distance between  $\Omega_1$  and  $\Omega_2$ .

In Appendix B, we give a proof of this Lemma in the special case of a domain in the class  $\mathcal{L}_M$ .

### A.3. Bounds on eigenfunctions, Neumann boundary condition.

**Theorem A.3.** *Let  $\Omega \subset \mathbb{R}^n$  be a convex, bounded, open subset, with boundary  $\Gamma := \partial\Omega$ , diameter bounded from above by some constant  $B_D$ , and Lipschitz constant bounded from above by some constant  $B_L$ .*

*Let  $\phi \in H^1(\Omega)$  be a variational eigenfunction of the Laplacian in  $\Omega$ , with the Neumann boundary condition,*

$$(A.12) \quad \int_{\Omega} \langle d\phi, d\varphi \rangle = \lambda \int_{\Omega} \phi \varphi$$

for all  $\varphi \in H^1(\Omega)$ . Then, there exists a constant  $C_n$ , such that,

$$(A.13) \quad \|\phi\|_{2,\Omega} \leq \lambda C_n \|\phi\|_{0,\Omega}.$$

**Remark.** The main point of the preceding proposition is that the constants are uniform in the class of domains  $\mathcal{L}_M$  we are interested in in this paper.

We adapt the proof of Theorem 3.2.1.3 in [21] (existence and uniqueness for the Neumann problem in  $\Omega$ ).

#### Grisvard's proof.

Let  $\phi \in H^1(\Omega)$  be a Neumann eigenfunction for  $-\Delta$  in  $\Omega$  associated with the eigenvalue  $\lambda$ . Without loss of generality, we may assume that  $\|\phi\|_{0,\Omega} = 1$ .

Fix some  $\mu > 0$  (for this whole subsection). Let  $f = (\mu + \lambda)\phi$ .

A.3.1. Using Lemma A.2, choose a sequence  $\{\Omega_m\}_{m \geq 1}$  of open convex subsets with  $C^2$  boundaries, and Lipschitz constants bounded from above by  $2B_D$ , such that  $\Omega \subset \Omega_m$ , and  $d_H(\Omega, \Omega_m)$  tends to zero when  $m$  tends to infinity.

A.3.2. According to Corollary 2.2.2.6 in [21], to each  $C^2$  convex set  $\Omega_m$ , there exists a (unique) function  $u_m \in H^2(\Omega_m)$  such that,

$$(A.14) \quad \begin{cases} (-\Delta + \mu)u_m &= \tilde{f} \text{ in } \Omega_m, \\ \gamma_m \frac{\partial u_m}{\partial n_e} &= 0 \text{ on } \Gamma_m, \end{cases}$$

where  $n_e$  denotes the outward unit normal to  $\Gamma_m$ , and where  $\tilde{f}$  denotes the extension of the function  $f = (\mu + \lambda)\phi$  by zero outside  $\Omega$ .

A.3.3. The a priori estimate, Theorem 3.1.3.3 in [21], applied to the  $C^2$  domain  $\Omega_m$  and to the function  $u_m \in H^2(\Omega_m)$  gives the existence of a constant  $C_5(\mu)$  such that

$$(A.15) \quad \|u_m\|_{2, \Omega_m} \leq C_5(\mu) \|(-\Delta + \mu)u_m\|_{0, \Omega_m} = \lambda C_5(\mu).$$

A.3.4. It follows that the sequence  $\{u_m|_\Omega\}$  is uniformly bounded by  $\lambda C_5(\mu)$  in  $H^2(\Omega)$ . Hence, there exist some  $u \in H^2(\Omega)$ , and a subsequence  $\{u_{(m,1)}\} \subset \{u_m\}$  such that  $u_{(m,1)}|_\Omega$  converges weakly to  $u$  in  $H^2(\Omega)$ , and strongly in  $H^1(\Omega)$  and in  $L_2(\Omega)$ . Furthermore,

$$(A.16) \quad \|u\|_{2, \Omega} \leq \lambda C_5(\mu).$$

A.3.5. **Claim.** The function  $u$  is a variational solution of the Neumann problem

$$(A.17) \quad \int_\Omega \langle du, dv \rangle + \mu \int_\Omega u v = \int_\Omega f v,$$

for all  $v \in H^1(\Omega)$ .

A.3.6. Assume the Claim is true. Since  $\mu > 0$ , the equality

$$(-\Delta + \mu)u = (-\Delta + \mu)\phi$$

implies that  $u = \phi$  and we conclude that

$$(A.18) \quad \|\phi\|_{2, \Omega} \leq \lambda C_5(\mu) \|\phi\|_{0, \Omega}.$$

This is the uniform bound we were looking for.

*Proof of the Claim.* Choose some  $v \in H^1(\Omega)$ . Since  $\Gamma$  is Lipschitz, we have the continuous linear extension operator  $\mathbb{E}_\Omega : H^1(\Omega) \rightarrow H^1(\mathbb{R}^n)$ , with norm controlled in terms of the Lipschitz constant of  $\Omega$ , such that  $V := \mathbb{E}_\Omega(v)$  satisfies  $V|_\Omega = v$ . Clearly  $V_{\Omega_m} \in H^1(\Omega_m)$ , and

$$(A.19) \quad \begin{aligned} \int_{\Omega_{(m,1)}} (\langle du_{(m,1)}, dV \rangle + \mu u_{(m,1)} V) &= \int_{\Omega_{(m,1)}} f V \\ &= \int_\Omega f v. \end{aligned}$$

Write

$$(A.20) \quad \int_{\Omega_{(m,1)}} u_{(m,1)} V - \int_{\Omega} u v = \int_{\Omega_{(m,1)} \setminus \Omega} u_{(m,1)} V + \int_{\Omega} (u_{(m,1)} - u) v$$

It follows that

$$(A.21) \quad \left| \int_{\Omega_{(m,1)}} u_{(m,1)} V - \int_{\Omega} u v \right| \leq \|u_{(m,1)}\|_{0,\Omega_{(m,1)}} \int_{\Omega_{(m,1)} \setminus \Omega} V^2 + \|u_{(m,1)} - u\|_{0,\Omega} \|v\|_{0,\Omega}.$$

It follows easily that the left-hand side tends to zero when  $m$  tends to infinity. A similar argument shows the

$$(A.22) \quad \int_{\Omega_{(m,1)}} \langle du_{(m,1)}, dV \rangle - \int_{\Omega} \langle du, dv \rangle$$

tends to zero. The claim is proved.

The proof of Theorem A.3, Neumann boundary condition, is complete.  $\square$

#### A.4. Bounds on eigenfunctions, Dirichlet boundary condition.

**Theorem A.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a convex, bounded, open subset, with boundary  $\Gamma := \partial\Omega$ , diameter bounded from above by some constant  $B_D$ , and Lipschitz constant bounded from above by some constant  $B_L$ .*

*Let  $\phi \in \mathring{H}^1(\Omega)$  be a variational eigenfunction of the Laplacian in  $\Omega$ , with the Dirichlet boundary condition,*

$$(A.23) \quad \int_{\Omega} \langle d\phi, d\phi \rangle = \lambda \int_{\Omega} \phi \varphi$$

*for all  $\varphi \in \mathring{H}^1(\Omega)$ . Then, there exists a constant,  $C_{\mathfrak{d}}(B_D, B_L)$ , which depends only on the indicated arguments, such that,*

$$(A.24) \quad \|\phi\|_{2,\Omega} \leq \lambda C_{\mathfrak{d}}(B_D, B_L) \|\phi\|_{0,\Omega}.$$

**Remark.** The main point of the preceding proposition is that the constants are uniform in the class of domains  $\mathcal{L}_M$  we are interested in in this paper.

We adapt the proof of Theorem 3.2.1.2 in [21] (existence and uniqueness for the Dirichlet problem in  $\Omega$ ).

#### Grisvard's proof.

Let  $\phi \in \mathring{H}^1(\Omega)$  be a Dirichlet eigenfunction of  $-\Delta$  in  $\Omega$ , associated with the eigenvalue  $\lambda$ . Without loss of generality, we may assume that  $\|\phi\|_{2,\Omega} = 1$ . Let  $f = \lambda \phi$ .

A.4.1. Using Lemma A.2, choose a sequence  $\{\Omega_m\}_{m \geq 1}$  of open convex subsets with  $C^2$  boundaries, and Lipschitz constants bounded from above independently of  $m$ , such that  $\Omega_m \subset \Omega$ , and  $d_H(\Omega_m, \Omega)$  tends to zero when  $m$  tends to infinity.

Apply Theorem 2.2.2.3 of [21] to the  $C^2$  convex domain  $\Omega_m$ . For each  $m \geq 1$ , there exists a (unique) function  $u_m \in H^2(\Omega_m) \cap \mathring{H}^1(\Omega_m)$  solving the Dirichlet problem

$$(A.25) \quad \begin{cases} -\Delta u_m &= f \text{ in } \Omega_m, \\ \gamma_m u_m &= 0 \text{ on } \Gamma_m, \end{cases}$$

where  $\gamma_m$  is the trace operator on  $\Gamma_m$ .

Apply the a priori inequality, Theorem 3.1.3.1 of [21], to each  $u_m$ ,  $m \geq 1$ . There exists a constant  $C_1(B_D)$  such that, for all  $m \geq 1$ ,

$$(A.26) \quad \|u_m\|_{2, \Omega_m} \leq C_1(B_D) \|\Delta u_m\|_{0, \Omega_m} \leq \lambda C_1(B_D).$$

A.4.2. For each  $m \geq 1$ , consider the function  $\widetilde{u}_m$  (extension of  $u_m$  by zero outside  $\Omega_m$ ). Since  $u_m \in H^1(\Omega_m)$  and  $\gamma_m u_m = 0$ , Theorem 1.5.1.5 of [21] (with  $k = 1, \ell = 0$ ), implies that  $\widetilde{u}_m \in H^1(\mathbb{R}^n)$ , and

$$(A.27) \quad \|\widetilde{u}_m\|_{1, \mathbb{R}^n} = \|u_m\|_{1, \Omega_m} \leq \lambda C_1(B_D).$$

For any  $1 \leq i, j \leq n$ , we also have, trivially,

$$(A.28) \quad \|D_i D_j u_m\|_{0, \Omega_m} \leq \lambda C_1(B_D),$$

and hence

$$(A.29) \quad \|\widetilde{D_i D_j u_m}\|_{0, \mathbb{R}^n} = \|D_i D_j u_m\|_{0, \Omega_m} \leq \lambda C_1(B_D).$$

It follows that there exist  $U \in H^1(\mathbb{R}^n)$ , and  $V_{i,j} \in L_2(\mathbb{R}^n)$ ,  $1 \leq i, j \leq n$ , and a subsequence  $\{u_{(m,1)}\} \subset \{u_m\}$  such that

$$(A.30) \quad \begin{cases} \widetilde{u_{(m,1)}} &\rightharpoonup U \text{ weakly in } H^1(\mathbb{R}^n), \\ \widetilde{D_i D_j u_{(m,1)}} &\rightharpoonup V_{i,j} \text{ weakly in } L_2(\mathbb{R}^n). \end{cases}$$

A.4.3. **Claim.** The restriction  $u := U|_{\Omega}$  satisfies

$$(A.31) \quad \begin{cases} -\Delta u &= \lambda \phi \text{ in } \Omega, \\ \gamma u &= 0 \text{ on } \Gamma, \end{cases}$$

where  $\gamma u$  is the trace of  $u$  on  $\Gamma$ .

*Proof of the claim.* Since  $U \in H^1(\mathbb{R}^n)$ , we have  $u \in H^1(\Omega)$ . Since  $\text{supp}(\widetilde{u}_m) \subset \overline{\Omega}$ , we also have  $\text{supp}(U) \subset \overline{\Omega}$ , and hence  $\tilde{u} = U$ , so that  $\tilde{u} \in \widetilde{H}^1(\mathbb{R}^n)$ . Since  $u \in H^1(\Omega)$  and  $\tilde{u} \in H^1(\mathbb{R}^n)$ , using Corollary 1.5.1.6 in [21], and the fact that  $\Gamma$  is Lipschitz (Lemma A.1), we have  $\gamma u = 0$  on  $\Gamma$ , i.e.,  $u \in \mathring{H}^1(\Omega)$ .

To show that  $-\Delta u = f$  in  $\mathcal{D}'(\Omega)$ , we choose some  $\varphi \in \mathcal{D}(\Omega)$ . Then, for  $m \geq m_0$  for some  $m_0 = m_0(\varphi)$ , we have  $\text{supp}(\varphi) \subset \Omega_{(m,1)}$ . One can then write

$$(A.32) \quad \begin{aligned} \int_{\Omega} f \varphi &= \int_{\Omega_{(m,1)}} f \varphi = - \int_{\Omega_{(m,1)}} \varphi \Delta u_{(m,1)} \\ &= \int_{\Omega_{(m,1)}} \langle du_{(m,1)}, d\varphi \rangle = \int_{\Omega} \langle \widetilde{du_{(m,1)}}, d\varphi \rangle \\ &\rightarrow \int_{\Omega} \langle du, d\varphi \rangle. \end{aligned}$$

A.4.4. Since  $u \in \mathring{H}^1(\Omega)$  and  $-\Delta u = f = \lambda\phi$  in  $\mathcal{D}'(\Omega)$ , it follows that  $u = \phi$ .

A.4.5. An argument similar to the preceding one, with  $D_i \widetilde{D_j u_{(m,1)}}$  shows that  $D_i D_j u = V_{i,j}$  in  $\mathcal{D}'(\Omega)$ , so that  $u \in H^2(\Omega)$ .

### An extra argument to prove the uniform bound.

A.4.6. **Claim.** There exists a constant  $C_{\mathfrak{d}}(B_D, B_D)$  such that

$$(A.33) \quad \|u\|_{2,\Omega} \leq \lambda C_{\mathfrak{d}}(B_D, B_D) \|u\|_{0,\Omega}.$$

Take the above sequence  $\{u_m\}$ , and define  $w_m := \mathbb{E}_{\Omega_m}(u_m)$ , where  $\mathbb{E}_{\Omega_m}$  is the extension operator given by Proposition 2.9: this is a continuous linear operator  $\mathbb{E}_{\Omega_m} : H^2(\Omega_m) \rightarrow H^2(\mathbb{R}^n)$ , whose norm is controlled in terms of the Lipschitz constant  $B_L$ : there exists a constant  $C_2(B_L)$  such that, uniformly in  $m$ ,

$$(A.34) \quad \|w_m\|_{2,\mathbb{R}^n} \leq C_2(B_L) \|u_m\|_{2,\Omega_m} \leq \lambda C_2(B_L) C_1(B_D),$$

so that  $\{w_m\}$  is uniformly bounded in  $H^2(\mathbb{R}^n)$ . It follows that there exists  $W \in H^2(\mathbb{R}^n)$ , and a subsequence  $\{u_{(m,2)}\} \subset \{u_{(m,1)}\}$  such that  $u_{(m,2)}$  converges weakly to  $W$  in  $H^2(\mathbb{R}^n)$ . This implies that

$$(A.35) \quad \|W\|_{2,\mathbb{R}^n} \leq \lambda C_2(B_L) C_1(B_D).$$

Using the same limiting trick as above, we can show that  $W|_{\Omega} = u$  in  $\mathcal{D}'(\Omega)$ .

Finally, we can conclude that

$$(A.36) \quad \|\phi\|_{2,\Omega} \leq \lambda C(B_L) C_1(B_D) \|\phi\|_{0,\Omega}.$$

This finishes the proof of Theorem A.4. □

## APPENDIX B. PROOF OF LEMMA A.2

In this appendix, we sketch of proof of Lemma A.2, in the particular case of a domain in  $\mathcal{L}_M$ .

Assume that the domain  $\Omega$  contains 0, and is defined in polar coordinates,

$$\Omega = \{(r, \theta) \mid 0 \leq r \leq \rho(\theta)\}.$$

When  $\rho$  defines a simple closed  $C^2$  curve, a necessary and sufficient condition for  $\Omega$  to be convex is that the signed curvature of the curve does not change sign, or equivalently that  $\rho^2 + 2(\rho')^2 - \rho\rho'' \geq 0$ , see [13],

Theorem 9.6.2 and § 8.4.14.2. Defining  $R := \frac{1}{\rho}$ , the condition becomes  $R'' + R \geq 0$ .

When  $\rho$  is Lipschitz, the corresponding condition is that  $R'' + R \geq 0$  is the sense of distributions. Indeed, according to [18, Chap. 3.4] (this is a result due to H. Rademacher [41]), if  $\Omega$  is a convex domain defined by the equation  $\{(r, \theta) \mid 0 \leq r < \frac{1}{R(\theta)}\}$ , then

$$(B.1) \quad \begin{vmatrix} R(\theta) & \cos \theta & \sin \theta \\ R(\phi) & \cos \phi & \sin \phi \\ R(\psi) & \cos \psi & \sin \psi \end{vmatrix} \begin{vmatrix} 1 & \cos \theta & \sin \theta \\ 1 & \cos \phi & \sin \phi \\ 1 & \cos \psi & \sin \psi \end{vmatrix} \geq 0,$$

for all  $\theta, \phi$  and  $\psi$ . For any  $h$  and  $\phi$ , choose  $\theta = \phi - h$ , and  $\psi = \phi + h$ . Applying Condition (B.1) to these values, one finds that

$$(B.2) \quad 2 \sin^2(h) (1 - \cos h) [R(\phi - h) + R(\phi + h) - 2 \cos(h) R(\phi)] \geq 0,$$

for any  $h$  and  $\phi$ . Letting  $h$  tend to zero, this condition implies that  $R'' + R \geq 0$  in the sense of distributions.

We now consider domains defined by an equation

$$(B.3) \quad \Omega = \{(r, \theta) \mid 0 \leq r < \frac{1}{R(\theta)}\}.$$

Fix some  $\varepsilon > 0$ .

Let  $\Omega_{a,\varepsilon}$  denote the domain defined by the equation  $\{0 \leq r < \frac{1+\varepsilon}{R}\}$ , with  $\varepsilon$  small enough. Then,  $\Omega \subset \Omega_{a,\varepsilon}$ .

By convolution, we can find a smooth function  $R_\varepsilon$  such that

$$\frac{R}{1 + 3\varepsilon/2} < R_\varepsilon := R \star \varphi_\varepsilon < \frac{R}{1 + \varepsilon/2},$$

for some mollifier  $\varphi_\varepsilon$ .

Define the domain  $\Omega_{2,\varepsilon}$  by

$$\Omega_{2,\varepsilon} := \{(r, \theta) \mid 0 \leq r < \frac{1}{R_\varepsilon(\theta)}\}.$$

Then, clearly  $\Omega \subset \Omega_{2,\varepsilon}$ .

There exists some  $M > 0$ , such that

$$B(2M^{-1}) \subset \Omega \subset B(M).$$

and, taking  $\varepsilon$  small enough, we have

$$B(M^{-1}) \subset \Omega_{2,\varepsilon} \subset B(2M).$$

The fact that  $\Omega$  is convex implies that  $R''(\theta) + R(\theta) \geq 0$  in the sense of distributions. The same relation holds, in the classical sense, for the function  $R_\varepsilon$ ,

$$R_\varepsilon''(\theta) + R_\varepsilon(\theta) \geq 0,$$

so that the domain  $\Omega_\varepsilon$  is convex too.

We can now apply Theorem 2.4.7 in [26] to conclude that the domains  $\Omega_{2,\varepsilon}$  have a uniform Lipschitz bound.

APPENDIX C. DEFORMATION ARGUMENT: PROPERTIES OF  $\nu^-(\Omega)$   
AND  $\psi_\Omega$

In this appendix, in analogy with Section 2, we look at the deformation of the  $D$  anti-symmetric eigenfunction  $\psi_\Omega$  along a path  $\{\Omega_t\}$  of domains in the class  $\mathcal{L}_M$ . This case was studied by Jerison and Nadirashvili, [28, Lemma 2.5], in the context of the “hot spots” conjecture.

**Lemma C.1.** *There exists a constant  $C_{10}(M, s_0)$  such that, for any domains  $\Omega_1, \Omega_2 \in \mathcal{L}_M$ ,*

$$(C.1) \quad \left| \nu^-(\Omega_1) - \nu^-(\Omega_2) \right| \leq C_{10}(M, s_0) d_r(\Omega_1, \Omega_2)^{q_0}.$$

*Proof.* For the proof, we use the following notation:  $\lambda_i = \nu^-(\Omega_i)$ ,  $\psi_i = \psi_{\Omega_i}$  and  $\Psi_i = \mathbb{E}_{\Omega_i}(\psi_{\Omega_i})$ . We use the estimate provided by Proposition 2.11, and the fact that  $\Psi_2$  is  $D$ -antisymmetric. We can write,

$$\begin{aligned} \int_{\Omega_1} \Psi_2^2 &= \int_{\Omega_1 \cap \Omega_2} \Psi_2^2 + \int_{\Omega_1 \setminus \Omega_2} \Psi_2^2, \text{ and} \\ \int_{\Omega_2} \Psi_2^2 &= \int_{\Omega_1 \cap \Omega_2} \Psi_2^2 + \int_{\Omega_2 \setminus \Omega_1} \Psi_2^2, \end{aligned}$$

so that

$$- \int_{\Omega_2 \setminus \Omega_1} \Psi_2^2 \leq \int_{\Omega_1} \Psi_2^2 - 1 \leq \int_{\Omega_1 \setminus \Omega_2} \Psi_2^2,$$

where we have used the normalization  $\int_{\Omega_2} \psi_2^2 = 1$ , and the fact that  $\Psi_2|_{\Omega_2} = \psi_2$ . Using (2.15), and (2.17)–(2.19), we conclude that there exists a constant  $C_{10,1}(M, s_0)$  such that

$$(C.2) \quad 1 - C_{10,1} d_r(\Omega_1, \Omega_2) \leq \int_{\Omega_1} \Psi_2^2 \leq 1 + C_{10,1} d_r(\Omega_1, \Omega_2).$$

Using a similar argument, we write

$$\int_{\Omega_1} |d\Psi_2|^2 = \int_{\Omega_2} |d\Psi_2|^2 - \int_{\Omega_2 \setminus \Omega_1} |d\Psi_2|^2 + \int_{\Omega_1 \setminus \Omega_2} |d\Psi_2|^2,$$

so that there exists a constant  $C_{10,2}(M, s_0)$  such that

$$(C.3) \quad \int_{\Omega_1} |d\Psi_2|^2 \leq \lambda_2 + C_{10,2} d_r(\Omega_1, \Omega_2)^{q_0},$$

where we have used the fact that  $d\Psi_2|_{\Omega_2} = d\psi_2$ , and applied Hölder’s inequality,

$$\int_{\Omega_1 \setminus \Omega_2} |d\Psi_2|^2 \leq \left( \int_{\Omega_1 \setminus \Omega_2} |d\Psi_2|^{2/(2-s_0)} \right)^{2-s_0} \left( \int_{\Omega_1 \setminus \Omega_2} 1 \right)^{q_0},$$

together with the normalization  $\int_{\Omega_2} \Psi_2^2 = 1$  and the inequalities (2.15), (2.17)–(2.19).

Recall that  $\Psi_2$  is antisymmetric. Using the min-max,

$$\lambda_1 \leq \frac{\int_{\Omega_1} |d\Psi_2|^2}{\int_{\Omega_1} \Psi_2^2},$$

inequalities (C.2) and (C.3), and Lemma 2.8, we conclude that there exists  $C_{10,3}(M, s_0)$  such that

$$\lambda_1 \leq \lambda_2 + C_{10,3} d_r(\Omega_1, \Omega_2)^{q_0}.$$

The lemma follows by symmetry between  $\lambda_1$  and  $\lambda_2$ .  $\square$

We now consider a family  $\{\Omega_t\}_{0 \leq t \leq a}$  of domains in the class  $\mathcal{L}_M$ . We denote the uniquely defined normalized eigenfunction  $\psi_{\Omega_t}$ , see Remark 2.4, by  $\psi_t$ , and its extension  $\mathbb{E}_{\Omega_t}(\psi_t)$  by  $\Psi_t$ . The functions  $\Psi_t$  are anti-symmetric with respect to  $D$ . We use the notation  $\lambda_t = \nu^-(\Omega_t)$ , and we observe that these eigenvalues are uniformly bounded from above in the class  $\mathcal{L}_M$ , independently of  $t$ , see Lemma 2.8.

**Lemma C.2.** *Let  $\{\Omega_t\}_{0 \leq t \leq a}$  be a family of domains in the class  $\mathcal{L}_M$ , and use the preceding notation. If  $d_r(\Omega_t, \Omega_0)$  tends to zero when  $t$  tends to zero then, for any  $k \in \mathbb{N}$ , the function  $\Psi_t$  tends to  $\psi_0$  in  $C^k$  on compact subsets of  $\Omega_0$ .*

*Proof.* From the proof of Lemma C.1, there exist constants  $C_{15,i}$ ,  $1 \leq i \leq 4$ , depending only on  $M$  and  $s_0$ , such that

$$(C.4) \quad \begin{aligned} 1 - C_{15,1} d_r(\Omega_t, \Omega_0) &\leq \int_{\Omega_0} \Psi_t^2 \leq 1 + C_{15,2} d_r(\Omega_t, \Omega_0), \text{ and} \\ \int_{\Omega_0} |d\Psi_t|^2 &\leq \lambda_t + C_{15,3} d_r(\Omega_t, \Omega_0)^{q_0} \leq \lambda_0 + C_{15,4} d_r(\Omega_t, \Omega_0)^{q_0}. \end{aligned}$$

We need the following claim.

**Claim.** There exist constants  $C_{15,5}(M, s_0, \Omega_0)$  and  $C_{15,6}(M, s_0, \Omega_0)$  such that

$$(C.5) \quad \begin{cases} \left| \int_{\Omega_0} \Psi_t \psi_0 \right| \geq 1 - C_{15,5} d_r(\Omega_t, \Omega_0)^{q_0}, \\ \left| \int_{\Omega_0 \cap \Omega_t} \Psi_t \psi_0 \right| \geq 1 - C_{15,5} d_r(\Omega_t, \Omega_0)^{q_0}, \text{ and} \\ \left| \int_{\Omega_0} \Psi_t \psi_0 - \int_{\Omega_0 \cap \Omega_t} \Psi_t \psi_0 \right| \leq C_{15,6} d_r(\Omega_t, \Omega_0)^{q_0}. \end{cases}$$

Assume that the claim is true. From the normalization of  $\psi_0$  and  $\psi_t$ , and the fact that they are anti-symmetric, we have

$$\int_{\Omega_0 \cap \Omega_t} \Psi_t \psi_0 = \int_{\Omega_0 \cap \Omega_t} \psi_t \psi_0 = 2 \int_{\Omega_{0,+} \cap \Omega_{t,+}} \psi_t \psi_0 > 0.$$

By (C.5), we can take  $d_r(\Omega_t, \Omega_0)$  small enough, so that  $\int_{\Omega_0} \Psi_t \psi_0 > 0$ . This also implies that there exists a constant  $C_{15,7}(M, s_0, \Omega_0)$  such that

$$(C.6) \quad \int_{\Omega_0} \Psi_t \psi_0 \geq 1 - C_{15,7} d_r(\Omega_t, \Omega_0)^{q_0}.$$



It then follows that there exists a constant  $C_{15,8}(M, s_0, \Omega_0)$  such that,

$$(C.7) \quad \int_{\Omega_0} (\Psi_t - \psi_0)^2 \leq C_{15,8} d_r(\Omega_t, \Omega_0)^{q_0},$$

provided that  $d_r(\Omega_t, \Omega_0)$  is small enough. In particular,  $\Psi_t$  tends to  $\psi_0$  in  $L_2(\Omega_0)$ .

Let  $k$  be an integer, and let  $K \subset \Omega_0$  be any compact subset. For  $t$  small enough, we have  $K \subset \Omega_t$ . By interior regularity,  $\Psi_t$  is uniformly bounded in  $C^{k+1}(K)$  norm, and hence admits a convergent subsequence  $\Psi_{t_j}$  in  $C^k(K)$ . Inequality (C.7) shows that the limit of this subsequence must be  $\psi_0$ . It follows that  $\Psi_t$  converges to  $\psi_0$  in  $C^k(K)$ .

*Remark.* One can actually give a more precise result, as in Lemma 2.15.

*Proof of the claim.* Define

$$\Theta_t := \Psi_t - \left( \int_{\Omega_0} \Psi_t \psi_0 \right) \Psi_0.$$

Because  $\Psi_t$  and  $\Psi_0$  are  $D$ -antisymmetric, so is  $\Theta_t$ . Clearly, we have  $\int_{\Omega_0} \Theta_t \psi_0 = 0$ , because  $\psi_0$  is normalized.

Define  $\mu_0 := \mu_2(\Omega_{0,+}, \mathbf{n}\mathfrak{d})$ . Then,

$$\lambda_0 = \nu^-(\Omega_0) = \mu_1(\Omega_{0,+}, \mathbf{n}\mathfrak{d}) < \mu_2(\Omega_{0,+}, \mathbf{n}\mathfrak{d}) = \mu_0,$$

where we have decomposed  $\partial\Omega_{0,+}$  into two parts,  $\partial\Omega_0 \cap \{u > 0\}$ , with the Neumann boundary condition, and  $D \cap \Omega_{0,+}$ , with the Dirichlet boundary condition, as in (2.7).

It follows that

$$(C.8) \quad \int_{\Omega_0} |d\Theta_t|^2 \geq \mu_0 \int_{\Omega_0} \Theta_t^2.$$

We have,

$$(C.9) \quad \begin{cases} \int_{\Omega_0} \Theta_t^2 = \int_{\Omega_0} \Psi_t^2 - \left( \int_{\Omega_0} \Psi_t \psi_0 \right)^2, \\ \int_{\Omega_0} |d\Theta_t|^2 = \int_{\Omega_0} |d\Psi_t|^2 - \lambda_0 \left( \int_{\Omega_0} \Psi_t \psi_0 \right)^2, \end{cases}$$

where we have used the definition of  $\psi_0$  as a variational eigenfunction.

Using (C.4), we deduce from (C.8) and (C.9)

$$(C.10) \quad \begin{cases} \int_{\Omega_0} |d\Theta_t|^2 \leq \lambda_0 \left( 1 - \left( \int_{\Omega_0} \Psi_t \psi_0 \right)^2 \right) + C_{15,4} d_r(\Omega_t, \Omega_0)^{q_0}, \\ \int_{\Omega_0} |d\Theta_t|^2 \geq \mu_0 \left( \int_{\Omega_0} \Psi_t^2 - \left( \int_{\Omega_0} \Psi_t \psi_0 \right)^2 \right), \\ \int_{\Omega_0} |d\Theta_t|^2 \geq \mu_0 \left( 1 - \left( \int_{\Omega_0} \Psi_t \psi_0 \right)^2 \right) - \mu_0 C_{15,1} d_r(\Omega_t, \Omega_0). \end{cases}$$

It follows that there exists a constant  $C_{15,9}(M, s_0, \Omega_0)$  such that

$$(C.11) \quad \begin{cases} (\mu_0 - \lambda_0) \left( \int_{\Omega_0} \Psi_t \psi_0 \right)^2 \geq (\mu_0 - \lambda_0) - C_{15,9} d_r(\Omega_t, \Omega_0)^{q_0}, \\ \left| \int_{\Omega_0} \Psi_t \psi_0 \right| \geq 1 - \frac{C_{15,9}}{\mu_0 - \lambda_0} d_r(\Omega_t, \Omega_0)^{q_0}. \end{cases}$$

Furthermore, we can write

$$(C.12) \quad \int_{\Omega_0} \Psi_t \psi_0 = \int_{\Omega_0 \cap \Omega_t} \Psi_t \psi_0 + \int_{\Omega_0 \setminus \Omega_t} \Psi_t \psi_0,$$

where the second term can be bounded from above in absolute value by some  $C_{15,12}(M) d_r(\Omega_t, \Omega_0)$ . The other estimates in the claim follow easily. The proof of Lemma C.2 is complete.  $\square$

## REFERENCES

- [1] A. Acker, L.E. Payne, and G. Philippin. On the convexity of the fundamental mode in the clamped membrane problem, and the existence of convex solutions in a related free boundary problem. *Journal of Applied Mathematics and Physics (ZAMP)* 32 (1981) 683-694 (1981). [16](#)
- [2] R. A. Adams. *Sobolev spaces*. Academic Press, 1975. [9](#)
- [3] G. Alessandrini. Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains. *Comment. Math. Helv.* 69 (1994) 142-154. [21](#)
- [4] V. Arnold. The topology of real algebraic curves (the works of Petrovskii and their development). *Uspekhi Math. Nauk.* 28:5 (1973) 260-262. Russian, translated in [6]. [41](#)
- [5] V. Arnold. Topological properties of eigenoscillations in mathematical physics. *Proc. Steklov Inst. Math.* 273 (2011) 25-34. [3](#)
- [6] V. Arnold. Topology of real algebraic curves (works of I.G. Petrovskii and their development). Translated from [4] by Oleg Viro. In *Collected works, Volume II. Hydrodynamics, Bifurcation*. [3](#), [41](#)
- [7] P. Bérard and B. Helffer. Nodal sets of eigenfunctions, Antonie Stern's results revisited. *Séminaire de Théorie spectrale et géométrie*. 32 (2014-2015) 1-37. [2](#)
- [8] P. Bérard and B. Helffer. Edited extracts from Antonie Stern's thesis. *Séminaire de Théorie spectrale et géométrie*. 32 (2014-2015) 39-72. [2](#)
- [9] P. Bérard and B. Helffer. Courant-sharp eigenvalues for the equilateral torus, and for the equilateral triangle. *Letters in Math. Physics* 106 (2016). [27](#)
- [10] P. Bérard and B. Helffer. Sturm's theorem on zeros of linear combinations of eigenfunctions. [arxiv:1706.08247](#). [2](#)
- [11] P. Bérard and B. Helffer. On Courant's nodal domain property for linear combinations of eigenfunctions. [arXiv:17050.3731](#). [3](#), [4](#), [14](#), [24](#), [26](#)
- [12] P. Bérard and B. Helffer. More counterexamples to the Extended Courant Property (ECP). [arXiv:1803.00449](#). [3](#)
- [13] M. Berger and B. Gostiaux. *Differential geometry: manifolds, curves, and surfaces*. Graduate Texts in Math. 115. Springer, 1988. [36](#)
- [14] H.J. Brascamp and E.H. Lieb. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for Log concave functions, and with an application to the diffusion equation. *Journal of Functional Analysis* 22 (1976) 366-389. [16](#)
- [15] L. A. Caffarelli and A. Friedman. Convexity of solutions of semilinear elliptic equations. *Duke Math. J.* 52:2 (1985) 431-456. [16](#)

- [16] S-Y. Cheng. Eigenfunctions and nodal sets. *Comment. Math. Helv.* 51 (1976) 43-55. [21](#)
- [17] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. 1.* First English edition. Interscience, New York 1953. [2](#), [3](#)
- [18] H. Eggleston. *Convexity.* Cambridge University Press, 1958. [37](#)
- [19] F. Gantmacher and M. Krein. *Oscillation matrices and kernels and small vibrations of mechanical systems.* Revised edition. AMS Chelsea Publishing. [3](#)
- [20] G. Gladwell and H. Zhu. The Courant-Herrmann conjecture. *Z. Angew. Math. Mech. (ZAMM)* 83:4 (2003) 275-281. [3](#), [26](#)
- [21] P. Grisvard. *Elliptic problems in nonsmooth domains.* Pitman, 1985. [9](#), [30](#), [31](#), [32](#), [33](#), [34](#), [35](#)
- [22] D.A. Gudkov. The topology of real projective algebraic varieties. *Russian Math. Surveys* 29:4 (1974) 1-79. [3](#)
- [23] B. Helffer, T. Hoffmann-Ostenhof, and S. Terracini. Nodal domains and spectral minimal partitions. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 26 (2009), 101–138. [21](#)
- [24] B. Helffer and M. Persson-Sundqvist. Nodal domains in the square – The Neumann case. *arXiv:1410.6702v3. Mosc. Math. J.* 15 (2015) 455-495. [2](#)
- [25] B. Helffer and M. Persson-Sundqvist. On the semi-classical analysis of the groundstate energy of the Dirichlet Pauli operator. *arXiv:1605.04193.* [16](#)
- [26] A. Henrot and M. Pierre. *Variation et optimisation des formes—une analyse géométrique—* *Mathématiques & Applications* 48. Springer-SMAI (2005). [6](#), [32](#), [38](#)
- [27] T. Hoffmann-Ostenhof, P. Michor, and N. Nadirashvili. Bounds on the multiplicity of eigenvalues for fixed membranes. *Geom. Funct. Anal. (GAFA)* 9:6 (1999) 1169-1188. [7](#), [21](#)
- [28] D. Jerison and N. Nadirashvili. The “hot spots” conjecture for domains with two axes of symmetry. *J. Amer. Math. Soc.* 13:4 (2000) 741-772. [4](#), [5](#), [8](#), [9](#), [10](#), [38](#)
- [29] B. Kawohl. When are superharmonic functions concave? Applications to the St. Venant torsion problem and to the fundamental mode of the clamped membrane. *J. Appl. Math. Mech. (ZAMM)* 64 (1984) T364-T366. [14](#), [16](#)
- [30] B. Kawohl. *Rearrangements and convexity of level sets in PDE.* *Lecture Notes in Math.* 1150. Springer, 1985. [16](#)
- [31] J. B. Kennedy. A toy Neumann analogue of the nodal line conjecture. *arXiv:1706.10037.* [21](#)
- [32] A. Kennington. Power concavity and boundary value problems. *Indiana Univ. Math. J.* 34:3 (1985) 687-704. [16](#)
- [33] N. Kuznetsov. On delusive nodal sets of free oscillations. *Newsletter of the European Mathematical Society* 96 (2015) 34-40. [3](#)
- [34] R. S. Laugesen and B. A. Siudeja. Triangles and other special domains. In “*Shape optimization and spectral theory*”, A. Henrot Ed., Chap. 6, 149–200. De Gruyter, 2017. [7](#)
- [35] H. A. Levine and H. F. Weinberger. Inequalities between Dirichlet and Neumann eigenvalues. *Arch. Rat. Mech. Anal.* 94 (1986) 193-208. [21](#)
- [36] J. Leydold. On the number of nodal domains of spherical harmonics. *Topology* 35 (1996) 301-321. [3](#)
- [37] C-S. Lin. On the second eigenfunction of the Laplacian in  $\mathbb{R}^2$ . *Commun. Math. Phys.* 111 (1987) 161-166. [21](#)
- [38] L.E. Payne. Isoperimetric inequalities and their applications. *SIAM Review* 9:3 (1967) 453–488. [21](#)

- [39] Å. Pleijel. Remarks on Courant's nodal theorem. *Comm. Pure. Appl. Math.* 9 (1956) 543-550. [2](#)
- [40] G. Pólya. Remarks on the foregoing paper. *J. Math. Phys. Mass. Inst. Techn.* 31 (1952) 55-57. [19](#)
- [41] H. Rademacher. Über eine funktionale Ungleichung in der Theorie der konvexen Körper. *Math. Z.* 13 (1922) 18-27. [37](#)
- [42] G. Savaré. Regularity results for elliptic equations in Lipschitz domains. *J. Funct. anal.* 152 (1998) 176-201. [9](#)
- [43] E.M. Stein. *Singular integrals and differentiability properties of functions.* Princeton University Press, 1970. [9](#)
- [44] C. Sturm. Mémoire sur les équations différentielles linéaires du second ordre. *Journal de mathématiques pures et appliquées (Journal de Liouville)* 1 (1836) 106-186. [2](#)
- [45] C. Sturm. Mémoire sur une classe d'équations à différences partielles. *Journal de mathématiques pures et appliquées (Journal de Liouville)* 1 (1836) 373-444. [2](#)
- [46] G. Szegő. Inequalities for certain eigenvalues of a membrane of given area. *J. Rat. Mech. Anal.* 3 (1954) 343-356. [19](#)
- [47] O. Viro. Construction of multi-component real algebraic surfaces. *Soviet Math. dokl.* 20:5 (1979) 991-995. [3](#)
- [48] M. Wuertz. *The implicit function theorem for Lipschitz functions and applications.* Master Thesis, University of Missouri, 2008. [6](#)

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