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***Abstract*— In this paper, we study the input-output decoupling and linearization of nonlinear two-input two-output time-varying delay systems. When working with delay systems, two problems may arise when constructing a feedback transformation for which the input-output map of the feedback modified systems is linear. The first issue is the boundedness of the control and the second one is its causality. We develop an algorithm allowing the construction of a causal and bounded feedback which permits to solve the input-output decoupling and linearization problem. The idea of our algorithm is to introduce, at each step, when the input-output decoupling is not possible, an artificial delay for the input that appears "too early" in the system. To that end, we propose, at each step, a precise procedure for defining a simple feedback transformation.**

Keywords: nonlinear time-varying delay systems, input-output decoupling, input-output linearization, Lie derivative.

I. Introduction

Input-output decoupling and linearization is an important tool in nonlinear control theory which aims to construct an invertible feedback transformation for which the input-output map of the feedback modified systems is linear. This problem is well know for nonlinear control systems without delays (see, e.g., [2], [6], [9]) and various aspects of it have been studied in the literature using different approaches such as the algebraic approach (see, e.g., [4], [14]) or the geometric one (see, e.g., [3], [13]). These methods have been extended to encompass nonlinear control systems with multiple (but constant) delays in the state variables as well as in the input and output of the system (see, e.g., [1], [3], [4], for the algebraic approach, and [5], [11], [12], for the geometric one).

However, in the particular case of nonlinear time-varying delay control systems (which is the subject of this paper), the problem is still largely open. Our goal is thus to describe input-output decoupling and linearization for nonlinear time-varying delay systems and to understand the problems that may arise when constructing a feedback from an recursive equation which is often the case when dealing with delay systems. In this paper, we treat the two-input two-output case (see [7], [8] for a related result for single-input single-output systems).

The originality of this paper resides in the development of sufficient conditions for the solvability of the input-output decoupling and linearization problem through a constructive algorithm. The solution of the problem is constructed by applying a feedback transformation at each step of the algorithm. The main idea is to introduce an artificial delay for the input that appears "too early" in the system (see also [2], [10], for a related method for nonlinear systems without delays where the inputs that appear "too early" are precompensated). The feedback transformation (defined in a simple way at each step) is not unique and thus our algorithm gives sufficient conditions only depending on the choices made at each iteration for that feedback transformation.

The paper is organized as follows. In Section II, we present some notations and recall the definition of the Lie derivative for time-varying delay systems (introduced by the authors in [8]). We explain the boundedness and causality problems that we may encounter when constructing a feedback from an recursive equation and state a lemma guaranteeing the boundedness of the control. In Section III, we give the main results of the paper: we present and discuss in details each step of the input-output decoupling algorithm. We illustrate our results by several examples in Section IV. Finally, technical proofs are given in Section V.

II. Notations, definitions and problem statement

Throughout, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with norm $\|\cdot\|$ and $\mathbb{R}_+$ the set of non-negative real numbers. For a real matrix $A = (a_{ij})$, $1 \leq i \leq n$, $1 \leq j \leq m$, we define $\|A\|_{\sup} = \sup_{ij}(|a_{ij}|)$.

Definition 1 (δ-operator): Let $\theta : \mathbb{R} \mapsto (0, \bar{\theta}]$ be a sufficiently smooth time-varying delay function which is supposed to be known and satisfying $\frac{d\theta}{dt} \leq 1$. Let denote by $\bar{\theta} > 0$ its supremum, and consider the recursive relation

$$\tau_{i+1} = \tau_i - \theta \circ \tau_i, \quad \text{for } i \geq 0,$$

where $\tau_0(t) = t$. We denote by δ^i the time delay operator that shifts the time from t to $\tau_i(t)$, that is

$$\delta^0\sigma(t) = \sigma(t) \quad \text{and} \quad \delta^i\sigma(t) = \sigma(\tau_i(t)), \quad \text{for } i \geq 0,$$

where σ is defined on an interval containing $[t - i\bar{\theta}, t]$. The application of δ^i on a composed function is given by

$$\delta^i\varphi(t, \sigma(t)) = \varphi(\tau_i, \delta^i\sigma(t)) = \varphi(\tau_i, \sigma(\tau_i)), \quad \text{for } i \geq 0. \tag{1}$$

Applied on the product of two functions, the delay operator acts as follows

$$\delta^i\varphi(t) \cdot \sigma(t) = (\delta^i\varphi(t)) \cdot (\delta^i\sigma(t)), \quad \text{for } i \geq 0, \tag{2}$$

i.e., the delay spreads to the right. If parentheses are present, i.e., we have $(\delta^i\varphi(t))\sigma(t)$, then the delay affects only the first function (here φ).

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Finally, we introduce the δ-operator which is defined by

$$\delta\sigma(t) = (\delta^0\sigma(t), \cdots, \delta^q\sigma(t)), \tag{3}$$

where q is the maximal order of the delay operator acting on σ.

Remark 1: The condition $\frac{d\theta}{dt} \leq 1$ on the derivative of the delay function is important for causality reasons [1]. In this paper, the case $\frac{d\theta}{dt} = 1$ is also excluded because the proposed input-output decoupling algorithm uses a delayed input extension and for $\frac{d\theta}{dt} = 1$ this extension is not relevant, see Example 1 for more details. Therefore, from now on, we suppose that $\frac{d\theta}{dt} < 1$.

We study input-output decoupling of two-input two-output nonlinear time-varying delay system of the form

$$\begin{aligned}
\dot{x}(t) &= f(\delta x(t), t) + g_1(\delta x(t), t)u_1(t) + g_2(\delta x(t), t)u_2(t),\\
y_i(t) &= h_i(\delta x(t), t), \quad i = 1, 2,\\
x(s) &= \zeta(s), \quad \forall s \in [-\mu, 0],\\
u_i(s) &= \psi_i(s), \quad \forall s \in [-\mu, 0], \quad i = 1, 2,
\end{aligned} \tag{4}$$

where $x(t) \in \mathbb{R}^n$ (or, more generally, $x(t) \in X$, where X is an open subset of $\mathbb{R}^n$) is the state of the system at time t, the vector fields $f, g_1, g_2 : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \to \mathbb{R}^n$ and the functions $h_i : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \to \mathbb{R}$, for $i = 1, 2$, are sufficiently smooth. The integer q corresponds to the maximal delay order explicitly involved in f, g_1, g_2 and h (but it does not mean that f, g_1, g_2 and h have necessarily the same delay orders). According to (3) and Remark 1, $\delta x(t) = (\delta^0 x(t), \cdots, \delta^q x(t))$ denotes the δ-operator associated to a sufficiently smooth time-varying delay function $\theta : \mathbb{R} \to (0, \bar{\theta}]$ satisfying $\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) < 1$, for all $t \in \mathbb{R}$, and where $\bar{\theta}$ is a positive real number.

The initial condition ζ belongs to $\mathcal{C}([-\mu, 0], \mathbb{R}^n)$, the Banach space of continuous functions from $[-\mu, 0]$ into $\mathbb{R}^n$, where μ is a sufficiently large integer, and the input $u : [-\mu, +\infty) \to \mathbb{R}^2$ is a Lebesgue measurable function. We thus suppose that ζ and ψ are known on a sufficiently large interval and, moreover, that they satisfy the differential equations of (4) on that interval. We also assume that system (4) is forward complete. These assumptions guarantee the existence of solutions on $[-\mu, +\infty)$ for each determined u.

In the case of the input-output decoupling problem, the output is connected to the control only indirectly through the state. To achieve input-output decoupling and linearization, we must find a direct relation between the inputs and the outputs of the system. This may be done by successive differentiation of the outputs h_i until the inputs appears in the resulting derivative equations. An important tool when differentiating the outputs is the Lie derivative. We will next recall the definition of the Lie derivative for time-varying delay systems (introduced by the authors in [8]).

[1]Without that assumption, we would obtain a contradiction with the fact that information available at t is not available at $t + \epsilon$.

Definition 2 (Lie derivative): Let $f : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \to \mathbb{R}^n$ be a smooth vector field whose components are functions of $(\delta x(t), t)$, and $h : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \to \mathbb{R}$ a real valued function of $(\delta x(t), t)$. The Lie derivative of h along f at $(\delta x(t), t)$ is defined as

$$L_f h(\delta x(t), t) = \sum_{i=0}^{q} \frac{\partial h}{\partial \delta^i x}\dot{\tau}_i \delta^i f(\delta x(t), t) + \frac{\partial h}{\partial t}(\delta x(t), t). \tag{5}$$

Observe that by taking the Lie derivative of h along f, we introduce new delays (via the term $\delta^i f$). As for system (4), in the above definition q denotes the maximal delay order explicitly involved in f or h, but the number of new delays introduced in $L_f h$ is related to the maximal delay order appearing in h only.

Since $L_f h$ is a real-valued function with delays, the above operation can be recursively repeated for higher order as

$$L_f^k h(\delta x(t), t) = \sum_{i=0}^{kq} \frac{\partial L_f^{k-1} h}{\partial \delta^i x}\dot{\tau}_i \delta^i f(\delta x(t), t) + \frac{\partial L_f^{k-1} h}{\partial t}(\delta x(t), t),$$

for $k \geq 2$. We also need to define a reduced Lie derivative.

Definition 3 (Reduced Lie derivative): Let $g : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \to \mathbb{R}^n$ be a smooth vector field and $h : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \to \mathbb{R}$ a real valued function. The reduced Lie derivative[2] of h along g is defined as

$$\overline{L}_g h(\delta x(t), t) = \sum_{i=0}^{q} \frac{\partial h}{\partial \delta^i x}\dot{\tau}_i \delta^i g(\delta x(t), t). \tag{6}$$

Contrary to (5), the partial derivative of h with respect to t is not present in (6). If h does not depend explicitly on t, then, obviously, $\overline{L}_g h$ coincide with $L_g h$. Notice also that, even if h does not depend explicitly on t, then $L_f h$ and $\overline{L}_g h$ will depend explicitly on t through the terms $\dot{\tau}_i$. The reduced Lie derivative will be always associated to the control vector fields g_1 and g_2.

Similarly to control systems without delays, the time-derivative of a function $z = \varphi(\delta x(t), t)$, where $\varphi : \mathbb{R}^{n(q+1)} \times \mathbb{R}_+ \to \mathbb{R}$, can be written using the Lie derivatives as follows:

$$\dot{z} = L_f \varphi + \bar{L}_{g_1}\varphi u_1 + \bar{L}_{g_2}\varphi u_2.$$

Recall that the operator δ spreads to the right, see equation (2), thus $\bar{L}_{g_1}\varphi u_1$ involves as many delayed controls as the delays present in the function φ. Indeed, denote by $m \geq 0$ the highest delay order of φ, then, according to the definition of the reduced Lie derivative, we have:

$$\bar{L}_{g_1}\varphi u_1 = \sum_{i=0}^{m} \frac{\partial \varphi}{\partial \delta^i x}\dot{\tau}_i(\delta^i g_1(x, \delta)) \cdot (\delta^i u_1).$$

Therefore, the reduced Lie derivatives $\bar{L}_{g_1}\varphi$ associated to the control vector field g_1 can be seen as a δ-polynomial and can be developed with respect to the δ^i-operator as follows:

$$\bar{L}_{g_1}\varphi = \sum_{i=0}^{m} a_1^i(x, \delta, t)\delta^i,$$

[2] As indicated by the notations, in our study, the reduced Lie derivative (6) will always be computed for the control vector fields g_1 and g_2, while the Lie derivative (5) will always be associated to the drift f.

where

$$a_1^i(x,\delta,t) = \frac{\partial \varphi}{\partial \delta^i x}\dot{\tau}_i(\delta^i g_1(x,\delta)),\ 0 \le i \le m.$$

The same remark holds for $\bar{L}_{g_2}\varphi$.

For each output h_i, $1 \le i \le 2$, we define its relative degree ρ_i as follows

$$\begin{cases} \bar{L}_{g_1}L_f^\ell h_i(\delta x(t),t) = \bar{L}_{g_2}L_f^\ell h_i(\delta x(t),t) = 0, \\ \text{for } 1 \le \ell \le \rho_i - 2, \forall (\delta x(t),t) \in \mathbb{R}^{n(q+1)} \times \mathbb{R}_+, \\ (\overline{L}_{g_1}L_f^{\rho_i-1}h_i, \overline{L}_{g_2}L_f^{\rho_i-1}h_i)(\delta x(t),t) \neq (0,0). \end{cases}$$

Assumption 1: In this paper, we do not deal with singularities, i.e., when we say that a function does not vanish, we mean that it is nonzero for any $t \ge 0$.

Since our system is not necessarily autonomous, the relative degree may depend on time, but here, due to Assumption 1, this case cannot occur.

By $L_f^\rho h$ we will mean the vector of two smooth functions whose i-entry is $L_f^{\rho_i}h_i$. We introduce the (2×2)-decoupling matrix

$$\mathcal{A}(\delta] = \begin{pmatrix} \bar{L}_{g_1}L_f^{\rho_1-1}h_1 & \bar{L}_{g_2}L_f^{\rho_1-1}h_1 \\ \bar{L}_{g_1}L_f^{\rho_2-1}h_2 & \bar{L}_{g_2}L_f^{\rho_2-1}h_2 \end{pmatrix}.$$

Following the above observations, the decoupling matrix can be seen as a δ-polynomial and can be developed with respect to the δ^i-operator as follows:

$$\mathcal{A}(\delta] = \mathcal{A}^0(\delta x(t),t)\delta^0 + \cdots + \mathcal{A}^m(\delta x(t),t)\delta^m, \quad (7)$$

where the coefficients $\mathcal{A}^\ell$, $0 \le \ell \le m$, are (2×2)-real matrices

$$\mathcal{A}^\ell(\delta x(t),t) = \begin{pmatrix} a_{11}^\ell(\delta x(t),t) & a_{12}^\ell(\delta x(t),t) \\ a_{21}^\ell(\delta x(t),t) & a_{22}^\ell(\delta x(t),t) \end{pmatrix}, \quad (8)$$

with a_{ij}^ℓ, $1 \le i,j \le 2$, being the coefficient of the term δ^ℓ in the δ-polynomial $L_{g_j}L_f^{\rho_i-1}h_i$, and m being the maximal delay order involved in the δ-polynomials $L_{g_j}L_f^{\rho_i-1}h_i$, for $1 \le i,j \le 2$.

Definition 4: Consider a δ-polynomial of the form $\mathcal{B}(\delta] = \sum_{i=0}^m \mathcal{B}^i(\delta x(t),t)\delta^i$, whose coefficients are (2×2)-real matrices. We call minimal degree of $\mathcal{B}$ the order of its first coefficient non identically zero, i.e., the integer $0 \le k \le m$ such that

$$\mathcal{B}^k(\delta x(t),t) \neq 0, \quad (9)$$

$$\mathcal{B}^i(\delta x(t),t) \equiv 0, \qquad \forall\ 0 \le i < k, \quad (10)$$

where 0 denotes the (2×2) zero-matrix. The integer m is called the maximal degree of $\mathcal{B}$ (we clearly suppose $\mathcal{B}^m(\delta x(t),t) \neq 0$).

Remark 2: We denote by j the minimal degree of the δ-polynomial $\mathcal{A}$, given by (7)-(8) and associated to the decoupling matrix of system (4).

Definition 5: We say that the problem of input-output decoupling is solvable for system (4) if each output h_i admits a finite relative degree ρ_i and if there exist a bounded and causal feedback $u(t)$ and an integer k verifying

$$\mathcal{A}(\delta]u(t) = -L_f^\rho h + \delta^k v,\ t \ge \tau_j^{-1}(0),\ \text{and } k \ge j, \quad (11)$$

where j is the minimal degree of $\mathcal{A}$. If such $u(t)$ exists then the feedback modified system satisfies

$$y_i^{(\rho_i)} = \delta^k v_i,\ \text{for } 1 \le i \le 2,$$

where v is the new control (assigned with respect to the properties that we want to achieve). Moreover, the system is said input-output decoupled and linearizable with delay if $k > 0$ (resp., without delay if $k = 0$).

Remark 3: In equation (11), we consider $\delta^k v$ instead of v. This is simply because we do not request to know v in the future. Indeed, if we replace $\delta^k v$ by v in (11), in the particular case where v is of the form $v = \psi(\delta x(t),t)$, advances appear in (11) if the lowest delay order in v is smaller than j. Another question is why we do not simply take $\delta^j v$ instead of $\delta^k v$, where $k \ge j$. This point will be clarified later in the algorithm (where transformations involving some artificial delays will be applied). Nevertheless, even if we take $\delta^k v$ instead of v, advances may derive from the $L_f^\rho h$ term. That case is excluded by the causality conditions of our algorithm, see relations (14) and (15).

III. Main result: input-output decoupling algorithm

Two problems arise when constructing a feedback u from an equation of the form (11). Since u is described by a recursive equation[3], the first problem is its boundedness and the second one is its causality.

The following lemma gives conditions that guarantees that the feedback u solution of an equation of the form

$$\mathcal{B}(\delta]u(t) = \alpha(\delta x(t),t) + \delta^k v, \quad (12)$$

for a given δ-polynomial $\mathcal{B}(\delta]$, with minimal degree k, and a given vector α, stays bounded when $\alpha(\delta x(t),t) + \delta^k v(t)$ is bounded, for $t \ge \tau_k^{-1}(0)$.

Lemma 1: Let $\mathcal{B}(\delta] = \sum_{i=k}^m \mathcal{B}^i(\delta x(t),t)\delta^i$ be a δ-polynomial, with $k < m$, where k (resp., m) is its minimal (resp., maximal) degree, and suppose that the matrix $\mathcal{B}^k$ is invertible for all $t \ge \tau_k^{-1}(0)$. Assume that v is such that $\alpha(\delta x(t),t)+\delta^k v(t)$ is bounded over $[\tau_k^{-1}(0),+\infty)$ and $u(t)$ is bounded over the interval $[-m\bar{\theta},0]$. If there exists a constant $\gamma > 1$ such that

$$\sup_{t \ge \tau_k^{-1}(0)} \left\| (\mathcal{B}^k)^{-1}\mathcal{B}^i \right\|_{\sup} \le \frac{1}{2\gamma(m-k)}, \quad \forall i > k, \quad (13)$$

then, for every $t \ge 0$, we have

$$\|u(t)\| \le \frac{\gamma}{\gamma-1} \sup_{s \ge \tau_k^{-1}(0)} \left\| (\mathcal{B}^k)^{-1}\Big(\alpha(\delta x(s),s) + \delta^k v(s)\Big) \right\| + \varepsilon(t),$$

[3]By "recursive" we point out that the construction of u requires a recursive prediction of the values of v (which is chosen as a suitable function of the state and the desired trajectory in order to achieve the desired behavior) over the intervals $[0,\tau_j^{-1}(0)]$ and $[\tau_i^{-1}(0),\tau_{i+1}^{-1}(0)]$, for $i \ge j$, the integer j being the minimal degree of $\mathcal{A}(\delta]$, see Definition 4 and Remark 2.

where $\varepsilon(t)$ tends to 0 when t tends to $+\infty$.

Next, we present and discuss the main result of this paper which is a constructive input-output decoupling algorithm. Recall that $\mathcal{A}(\delta]$ is the δ-polynomial relative to (7)-(8) and that we denote by j (resp., by m) its minimal (resp., maximal) degree. The principal observation on which the algorithm is based is the fact that if the matrix $\mathcal{A}^j$ is invertible and satisfies Lemma 1, then a bounded feedback can be constructed from equation (11), with $k = j$, in order to decouple and linearize the system. But, in general, $\mathcal{A}^j$ does not need to be invertible. In that case, the idea is to transform the δ-polynomial $\mathcal{A}(\delta]$ via some invertible feedback transformations in such a way that its first nonzero matrix is invertible and satisfies Lemma 1. To that end, an invertible feedback transformation will be involved at each step of the algorithm. The algorithm stops if we manage to construct a new δ-polynomial with the desired properties or after $m - j$ iterations (and in that case the system cannot be decoupled and linearized with the proposed choices of transformations).

We would like to stress the fact that our algorithm proposes sufficient conditions only, depending on certain choices made at each iteration, and if we are not able to decouple and linearize the system (via a bounded and causal feedback) with those choices, this does not necessarily mean that the system cannot be decoupled and linearized.

Algorithm and comments

Figure 1 summarizes the input-output decoupling algorithm. We comment below in details each of its steps (in particular, the construction of the transformations T and R, see Figure 1).

1) Calculate the relative degree $\rho = (\rho_1, \rho_2)$ of system (4).
   a) If $\rho_1 + \rho_2 > n$, then the system cannot be decoupled and linearized (a part of the system is not controllable) and the algorithm stops.
   b) If $\rho_1 + \rho_2 \leq n$, then calculate the δ-polynomial $\mathcal{A}(\delta]$, its minimal (resp., maximal) degree j (resp., m).
2) If $j = m$, then compute $\det(\mathcal{A}^m)$.
   a) If $\det(\mathcal{A}^m) \neq 0$ then we can always construct from (11) a bounded feedback u. If in addition
   $$\frac{\partial L_f^\rho h}{\partial \delta^i x} \equiv 0 \text{ and } \frac{\partial \mathcal{A}^m}{\partial \delta^i x} \equiv 0, \qquad (14)$$
   for $0 \leq i \leq m-1$, then the feedback is also causal and the system is input-output decoupled and linearized and the algorithm stops.
   b) If $\det(\mathcal{A}^m) = 0$, then the system can never be input-output decoupled and linearized and the algorithm stops.
3) If $j < m$, then set $iter = 0$, $iter_{max} = m - j$ and go to the next step.
4) Compute the matrix $\mathcal{A}^j$ and its determinant.

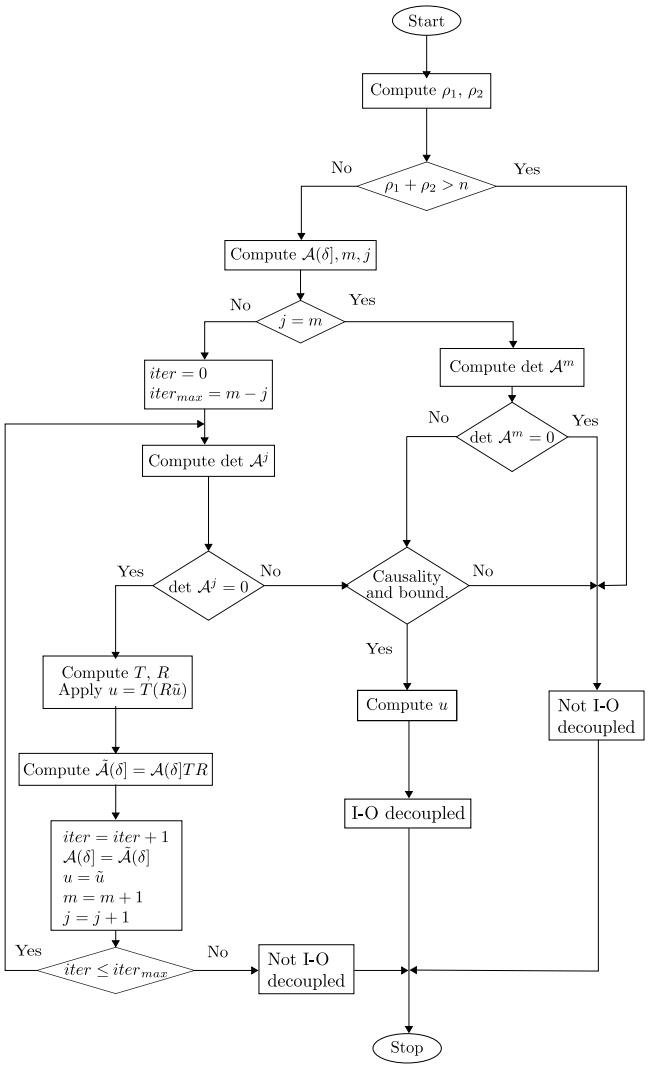

Fig. 1. Input-output decoupling algorithm

   a) If $\det(\mathcal{A}^j) \neq 0$, then check if Lemma 1 for $\mathcal{B}(\delta] = \mathcal{A}(\delta]$ is verified:
      – If (13) is satisfied, then we can always construct from (12), with $\mathcal{B}(\delta] = \mathcal{A}(\delta]$ and $\alpha = -L_f^\rho h$, a bounded feedback u. If in addition,
      $$\frac{\partial L_f^\rho h}{\partial \delta^i x} \equiv 0 \text{ and } \frac{\partial \mathcal{A}^\ell}{\partial \delta^i x} \equiv 0, \qquad (15)$$
      for $0 \leq i \leq j-1$, $j \leq \ell \leq m$, the feedback is also causal and the system is input-output decoupled and linearized and the algorithm stops.
      – If (13) is not satisfied then the system cannot be input-output decoupled and linearized and the algorithm stops.
   b) If $\det(\mathcal{A}^j) = 0$, then our goal is to transform the δ-polynomial $\mathcal{A}(\delta]$ via some invertible feedback transformations in such a way that its first nonzero matrix is invertible and satisfies Lemma 1. Define

the following invertible transformation

$$T = \begin{pmatrix} \alpha & \nu_1(\delta x(t), t) \\ \beta & \nu_2(\delta x(t), t) \end{pmatrix}, \qquad (16)$$

where the vector $\nu = (\nu_1, \nu_2)^T$ is such that

$$\delta^j \nu \in \text{Ker}(\mathcal{A}^j)$$

and α and β are constant parameters for which we take the simplest possible choice such that T is invertible, that is $(\alpha, \beta) = (1, 0)$ or $(\alpha, \beta) = (0, 1)$.

Remark 4: The transformation T changes the polynomial $\mathcal{A}(\delta]$ such that, after its application, the first non zero coefficient $\mathcal{A}^j$ has its second column identically zero. Notice also that it is not T that acts on $\mathcal{A}^j$, but $\delta^j T$ (hence the construction of T with the second column being, after the application of a delay of order j, a vector belonging to the kernel of $\mathcal{A}^j$). Notice that since $\mathcal{A}^j$ is not identically zero and $\det(\mathcal{A}^j) = 0$, the dimension of the kernel is one and the choice of the vector ν is unique up to a multiplicative function of $\delta(x)$ and t. So, defined as above, T is uniquely given up to a multiplicative function. In fact, any transformation of the form (16) with α and β being any functions of $\delta(x)$ and t such that T is invertible would give an identically zero second column for the new $\mathcal{A}^j$. Obviously the choice of such a transformation is not unique and an important question is how to chose it. Here, we propose to take for T the simplest possible choice: the parameters (α, β) are chosen constant and equal to (0, 1), if the original matrix $\mathcal{A}^j$ has its first column identically zero (and, in that case, T simply permutes the columns of $\mathcal{A}^j$) or equal to (1, 0), otherwise.

Remark 5: The parameters involved in T play an important role for the boundedness and the causality of the control. They have to be chosen such that condition (13) is satisfied for the first (new) δ-polynomial whose first nonzero matrix is invertible (obtained, if such δ-polynomial exists, at step 5) of the algorithm after a certain number of iterations).

5) Introduce the following feedback transformation $u = T(R\tilde{u})$, where

$$R = \begin{pmatrix} \delta^1 & 0 \\ 0 & 1 \end{pmatrix},$$

that transforms the polynomial $\mathcal{A}(\delta]$ into:

$$\tilde{\mathcal{A}}(\delta] = \mathcal{A}(\delta]TR = \mathcal{A}^j(\delta^j T)R\delta^j + \cdots + \mathcal{A}^m(\delta^m T)R\delta^m,$$

which can also be written as

$$\tilde{\mathcal{A}}(\delta] = \tilde{\mathcal{A}}^{j+1}\delta^{j+1} + \cdots + \tilde{\mathcal{A}}^{m+1}\delta^{m+1},$$

with

$$\begin{aligned} \tilde{\mathcal{A}}^{i+1} &= \left(\mathcal{A}^i(\delta^i T)\right)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad + \left(\mathcal{A}^{i+1}(\delta^{i+1}T)\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

for $j \leq i \leq m$ and where $\mathcal{A}^{m+1}$ is the zero matrix. Once the feedback transformation is applied and $\tilde{\mathcal{A}}(\delta]$ computed, go to the next step.

Remark 6: The transformation R simply introduces an artificial delay in the first control (after the application of T, that control corresponds to the input that appears "to early" into the system), the second one remaining unchanged. The first nonzero coefficient of the new δ-polynomial $\tilde{\mathcal{A}}(\delta]$ is no longer the coefficient of order j, but the one of order $(j + 1)$ (so the minimal degree of the new polynomial increases from j to $j + 1$) and the new corresponding matrix $\tilde{\mathcal{A}}^{j+1}$, inherits the first column from $\mathcal{A}^j$ and its second column from $\mathcal{A}^{j+1}(\delta^{j+1}T)$. By introducing an artificial delay in the system, also the maximal degree of the δ-polynomial increases.

6) Increase the number of iterations: $iter = iter + 1$. Set $\mathcal{A}(\delta] = \tilde{\mathcal{A}}(\delta]$, $j = j + 1$, $m = m + 1$, and $u = \tilde{u}$.
   a) If $iter > iter_{max}$, then the system cannot be decoupled and linearized with those choices of transformations T.
   b) If $iter \leq iter_{max}$, then return to step 4).

Remark 7: Notice that, now, step 4) (in particular checking Lemma 1) is applied on the new matrix $\tilde{\mathcal{A}}^{j+1}$ that depend on the original one $\mathcal{A}^j$, but also on the transformation T applied previously. It is thus clear that the boundedness of the control depends on the choice of T (or, equivalently, on the choice of the parameters α, β and of the vector ν).

IV. Examples

Example 1: Consider the following two-input two-output time-varying delay system

$$\begin{aligned} \dot{x}_1 &= au_1 + bu_2 \\ \dot{x}_2 &= cu_1 + du_2 \\ \dot{x}_3 &= x_1 - \delta^1 x_2 \\ h_1 &= x_1, \\ h_2 &= x_3, \end{aligned} \qquad (17)$$

where the constant parameters $a, b, c, d \in \mathbb{R}$, are such that $a \neq 0$ and $ad - bc \neq 0$ (otherwise the vector fields g_1 and g_2 would be colinear and the system would be, in fact, a one-input system), with initial conditions $x(t) = \zeta(t)$ and $u(t) = \psi(t), \forall t \in [-\bar{\theta}, 0]$. The relative degree is $(\rho_1, \rho_2) = (1, 2)$. Knowing that $\rho_1 + \rho_2 = 3$, then step 1)-b) of the algorithm is applied and the δ-polynomial $\mathcal{A}(\delta]$ of system (17) should be computed. A straightforward calculation leads to

$$\mathcal{A}(\delta] = \mathcal{A}^0\delta^0 + \mathcal{A}^1\delta^1,$$

where

$$\mathcal{A}^0 = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \quad \text{and} \quad \mathcal{A}^1 = \dot{\tau}_1 \begin{pmatrix} 0 & 0 \\ -c & -d \end{pmatrix}.$$

Recall that, according to our assumptions $\dot{\theta}(t) < 1$, then $\dot{\tau}_1 = 1 - \dot{\theta}(t) \neq 0$ and, thus, the maximal degree of $\mathcal{A}(\delta]$ is $m = 1$. The minimal degree j, defined by Definition 4 (and Remark 2), is equal to 0. Remark that if the case $\dot{\theta}(t) = 1$ were considered, then we would have $j = m = 0$ and, by step 2), the system could not be input-output decoupled and linearized. Knowing that $j < m$, we skip step 2) and set $iter = 0$. It is obvious that $\det \mathcal{A}^0 = 0$ and then step 4)-b) is used. Let

$$T = \begin{pmatrix} 1 & -b \\ 0 & a \end{pmatrix}$$

which is invertible because a is supposed to be non zero. We can thus apply $u = T(R\tilde{u})$ which transforms the δ-polynomial $\mathcal{A}(\delta]$ into:

$$\tilde{\mathcal{A}}(\delta] = \tilde{\mathcal{A}}^1\delta^1 + \tilde{\mathcal{A}}^2\delta^2,$$

where

$$\tilde{\mathcal{A}}^1 = \begin{pmatrix} a & 0 \\ a & \dot{\tau}_1(bc - ad) \end{pmatrix} \quad \text{and} \quad \tilde{\mathcal{A}}^2 = \dot{\tau}_1 \begin{pmatrix} 0 & 0 \\ -c & 0 \end{pmatrix}.$$

We set $iter = 1$ and return to step 4)-a). Since $a(bc-ad) \neq 0$ (by hypothesis), the new matrix $\mathcal{A}^j$ (which is the old $\tilde{\mathcal{A}}^1$) is of full rank and it is thus possible to compute $\tilde{u}$ from

$$\tilde{\mathcal{A}}^1\delta^1\tilde{u} = \delta^1 v - \tilde{\mathcal{A}}^2\delta^2\tilde{u}, \quad t \geq \tau_1^{-1}(0).$$

It only remains to verify if the boundedness condition (13) holds (notice that here we do not have any causality problem since all the matrices are constant and $L_f^\rho h = 0$). To that end, we have to check if the (constant) parameters a, b, c and d satisfy the following inequality

$$\left| \frac{c}{bc - ad} \right| \leq \frac{1}{2}. \tag{18}$$

In that case, the boundedness condition (13) is verified and the system is input-output decoupled and linearized. Observe that, if inequality (18) does not hold, then one can never find $\gamma > 1$ satisfying (13) of Lemma 1.

Example 2: Consider the following example

$$\begin{array}{rcl} \dot{x}_1 & = & \cos(\delta^1 x_1)u_1 - \sin(\delta^1 x_1)u_2 \\ \dot{x}_2 & = & \sin(\delta^1 x_1)u_1 + \cos(\delta^1 x_1)u_2 \\ \dot{x}_3 & = & \delta^1 x_2 \\ h_1 & = & x_1, \\ h_2 & = & x_3, \end{array} \tag{19}$$

defined on $X =]-\frac{\pi}{2}, \frac{\pi}{2}[\times\mathbb{R}\times\mathbb{R}$, with initial conditions $x(t) = \zeta(t)$ and $u(t) = \psi(t), \forall t \in [-\bar{\theta}, 0]$. Here the relative degree is $(\rho_1, \rho_2) = (1, 2)$. Since $\rho_1 + \rho_2 = 3$, we apply step 1)-b) which leads to the δ-polynomial $\mathcal{A}(\delta]$:

$$\mathcal{A}(\delta] = \mathcal{A}^0\delta^0 + \mathcal{A}^1\delta^1,$$

with

$$\mathcal{A}^0 = \begin{pmatrix} \cos(\delta^1 x_1) & -\sin(\delta^1 x_1) \\ 0 & 0 \end{pmatrix}$$

and

$$\mathcal{A}^1 = \dot{\tau}_1 \begin{pmatrix} 0 & 0 \\ \sin(\delta^2 x_1) & \cos(\delta^2 x_1) \end{pmatrix}.$$

From step 4)-b), see equation (16), the matrix T can be chosen of the form:

$$T = \begin{pmatrix} 1 & \sin(\delta^1 x_1) \\ 0 & \cos(\delta^1 x_1) \end{pmatrix} \tag{20}$$

and is invertible on X. By applying the transformation $u = T(R\tilde{u})$, the δ-polynomial $\mathcal{A}(\delta]$ becomes:

$$\tilde{\mathcal{A}}(\delta] = \tilde{\mathcal{A}}^1\delta^1 + \tilde{\mathcal{A}}^2\delta^2,$$

with

$$\tilde{\mathcal{A}}^1 = \begin{pmatrix} \cos(\delta^1 x_1) & 0 \\ 0 & \dot{\tau}_1 \end{pmatrix}$$

and

$$\tilde{\mathcal{A}}^2 = \dot{\tau}_1 \begin{pmatrix} 0 & 0 \\ \sin(\delta^2 x_1) & 0 \end{pmatrix}.$$

The matrix $\tilde{\mathcal{A}}^1$ is everywhere invertible on X, and it is thus possible to find, on X, a feedback $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)^T$, which satisfies the following equation:

$$\tilde{\mathcal{A}}^1\delta^1\tilde{u} = \delta^1 v - \tilde{\mathcal{A}}^2\delta^2\tilde{u}, \quad t \geq \tau_1^{-1}(0). \tag{21}$$

In order to conclude that the system is, indeed, input-output decoupled (via a bounded and causal transformation), Lemma 1 should also be verified.

As already explained, the transformation T is not unique and some choices (maybe more complicated than that proposed by our algorithm) may lead to a simpler equation for $\tilde{u}$. For instance, consider the following transformation

$$T = \begin{pmatrix} \cos(\delta^1 x_1) & \sin(\delta^1 x_1) \\ -\sin(\delta^1 x_1) & \cos(\delta^1 x_1) \end{pmatrix}$$

which is invertible on X (in fact, the above T is invertible for all x_1 and, contrary to choice (20) given by the algorithm, this T would work even if the original system evolves on $\mathbb{R}^3$ instead of X). By applying $u = T(R\tilde{u})$, we obtain a simpler δ-polynomial (than for the previous T):

$$\tilde{\mathcal{A}}(\delta] = \tilde{\mathcal{A}}^1\delta^1,$$

with

$$\tilde{\mathcal{A}}^1 = \begin{pmatrix} 1 & 0 \\ 0 & \dot{\tau}_1 \end{pmatrix}.$$

Now, the control $\tilde{u}$ has to be constructed from:

$$\tilde{\mathcal{A}}^1\delta^1\tilde{u} = \delta^1 v, \quad t \geq \tau_1^{-1}(0),$$

and, in this case, $\tilde{u}$ is always bounded and causal (and, contrary to choice (20) of T, proposed by the algorithm, we do not need to verify Lemma 1).

V. Proof of Lemma 1

The proof of Lemma 1 follows the same lines as those of [8, Lemma 1], the only difference being that we work with matrix equations instead of scalar ones. For sake of completeness, we present the principal points of the proof.

Under the assumption that $\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) < 1$ and the definition of $\tau_k(\cdot)$, one can always define a new time-scale allowing to shift the time $t \mapsto \tilde{t}$ with $\tilde{t} = \tau_k(t)$ (see the proof of [8, Lemma 1], for more details). Thus, equation (12) can be written as

$$\mathcal{B}^k u(\tilde{t}) + \cdots + \mathcal{B}^m \delta^{m-k} u(\tilde{t}) = \alpha(\tilde{t}) + v(\tilde{t}), \qquad (22)$$

where the argument $(\delta x(\tilde{t}), \tilde{t})$ have been omitted for the $\mathcal{B}^i$'s. In order to simplify the notations, we will write t instead of $\tilde{t}$.

One can prove (see [8, Lemma 1]) the existence of a strictly increasing sequence $(t_l)_{l\geq 0}$ such that

$$t_0 = 0, \qquad t_l = t_{l+1} - \theta(t_{l+1}), \qquad \forall l \geq 0,$$

and $t_l \to +\infty$ with l.

Denote

$$M = \sup_{t\geq 0} \left\| \left(\mathcal{B}^k\right)^{-1} \left(\alpha(t) + v(t)\right) \right\|.$$

Since τ_i, $1 \leq i \leq m$, is an increasing function, over $[t_0, t_1]$, we have

$$\tau_1(t) \in [\tau_1(t_0), \tau_1(t_1)] = [-\theta(0), 0] \subset [-m\bar{\theta}, 0],$$

and we can easily verify that

$$\tau_i(t) \in [\tau_i(t_0), \tau_i(t_1)] \subset [-m\bar{\theta}, 0], \quad \forall 1 < i \leq m-k.$$

Then, from relation (22) and from the fact that there exists (by hypothesis of Lemma 1) a constant $\gamma > 1$ such that relation (13) holds for all $t \geq 0$ (thus for $t \in [t_0, t_1]$ also), it follows that, for every $t \in [t_0, t_1]$, we have

$$\begin{aligned} \|u(t)\| &\leq \sum_{l=k+1}^{m} \|\left(\mathcal{B}^k\right)^{-1} \mathcal{B}^l u(\tau_{l-k}(t))\| + M \\ &\leq 2 \sum_{l=k+1}^{m} \|\left(\mathcal{B}^k\right)^{-1} \mathcal{B}^l\|_{\sup} \sup_{s\leq t_0} \|u(s)\| + M \\ &\leq \frac{1}{\gamma} \sup_{s\leq t_0} \|u(s)\| + M. \end{aligned}$$

Using the same arguments as above, for every $t \in [t_1, t_2]$, we obtain

$$\begin{aligned} \|u(t)\| &\leq 2 \sum_{l=k+1}^{m} \|\left(\mathcal{B}^k\right)^{-1} \mathcal{B}^l\|_{\sup} \sup_{s\leq t_1} \|u(s)\| + M \\ &\leq \left(\frac{1}{\gamma}\right)^2 \sup_{s\leq t_0} \|u(s)\| + \left(1 + \frac{1}{\gamma}\right) M. \end{aligned}$$

By an induction argument, for $l \geq 2$, we deduce that over $[t_l, t_{l+1}]$, we have

$$\|u(t)\| \leq \left(\frac{1}{\gamma}\right)^l \sup_{s\leq t_0} \|u(s)\| + M \cdot \sum_{i=0}^{l} \left(\frac{1}{\gamma}\right)^i.$$

Recall that the constant γ is such that $\gamma > 1$, thus, we deduce that for any $\varepsilon > 0$, there exists an integer l_ϵ such that, for $t \geq t_{l_\epsilon}$, we have

$$\|u(t)\| \leq \varepsilon + \frac{\gamma}{\gamma - 1} M.$$

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