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Generalized Pareto optimum and semi-classical spinors

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Abstract. In 1971, S.Smale presented a generalization of Pareto optimum he called the critical Pareto set. The underlying motivation was to extend Morse theory to several functions, i.e. to find a Morse theory for m differentiable functions defined on a manifold M of dimension ℓ. We use this framework to take a 2×2 Hamiltonian $\mathcal{H} = \mathcal{H}(p) \in C^\infty(T^*\mathbf{R}^2)$ to its normal form near a singular point of the Fresnel surface. Namely we say that $\mathcal{H}$ has the Pareto property if it decomposes, locally, up to a conjugation with regular matrices, as $\mathcal{H}(p) = u'(p)C(p)(u'(p))^*$, where $u : \mathbf{R}^2 \to \mathbf{R}^2$ has singularities of codimension 1 or 2, and $C(p)$ is a regular Hermitian matrix ("integrating factor"). In particular this applies in certain cases to the matrix Hamiltonian of Elasticity theory and its (relative) perturbations of order 3 in momentum at the origin.

1. Matrix valued Hamiltonian systems and crossing of modes

Here we recall some well known facts about generic normal forms for a matrix Hamiltonian near a crossing of modes. Consider a real valued symmetric Hamiltonian $\mathcal{H} \in C^\infty(T^*\mathbf{R}^d) \otimes \mathbf{R}^n$. We may also replace $\mathcal{H}(x,\xi)$ by a more general symbol (in a suitable class) with asymptotic sum $\mathcal{H}(x,\xi;h) = \mathcal{H}_0(x,\xi) + h\mathcal{H}_1(x,\xi) + \cdots$, and consider its semi-classical Weyl quantization

$$\mathcal{H}^w(x, hD_x; h)U(x;h) = (2\pi h)^{-d} \int\int e^{i(x-y)\eta/h}\mathcal{H}(\frac{x+y}{2}, \eta; h)U(y)\, dy\, d\eta$$

This defines an operator essentially self-adjoint on the space of square-integrable "spinors" $L^2(\mathbf{R}^d) \otimes \mathbf{C}^n$. For simplicity we keep denoting by $\mathcal{H}$ the principal symbol $\mathcal{H}_0$ which actually plays the main role in the present analysis.

Let $\Delta(x,\xi) = \det \mathcal{H}(x,\xi)$, and $N = \{(x,\xi) \in T^*\mathbf{R}^d, \Delta(x,\xi) = 0\}$ be the characteristic variety.

At points of N, the polarisation space $\ker H(x,\xi)$ has positive dimension $k \geq 1$.

For $k = 1$, $\{(x,\xi) \in N : d\Delta \neq 0\}$ is a smooth hypersurface, and modulo an elliptic factor, we can reduce $\mathcal{H}$ to a scalar symbol with simple characteristics; the corresponding operator, of real principal type, can again be locally conjugated to hD_{x_1} by an elliptic Fourier integral operator (FIO); this induces well-known polarization properties for solutions of the original system.

Effects which are truly specific for systems occur therefore when $k \geq 2$; let $\Sigma = \{(x,\xi) \in N : d\Delta = 0\}$ be the singular part of N.

For generic Hamiltonian $\mathcal{H}$, Σ is a stratified set. Splitting off elliptic summands, H can be (generically) reduced to a 2×2 system. For a 2×2 symmetric system $\begin{pmatrix} a+f & b \\ b & -a+f \end{pmatrix}$, one

should generically move 2 parameters to bring it to a matrix with an eigenvalue of multiplicity 2, and one more to bring this eigenvalue to 0; thus the top (larger) stratum of Σ is of codim 3. On directions transverse to Σ, N looks like a quadratic cone. In the next item we review some known results, making these observations more precise.

a) Classical and semi-classical normal forms for generic H

We first mention Arnold's [1] normal forms for (generic) $\Delta = \det \mathcal{H}$ near the stratum of Σ of codim 3: at a conical point (i.e. where $a = b = c = 0$), there are symplectic coordinates (p, q) such that modulo an elliptic factor, we can bring Δ to one of the following expressions :

$$\Delta(x,\xi) \equiv p_1 q_1 - p_2^2 \quad \text{hyperbolic conical point}$$
$$\Delta(x,\xi) \equiv p_1^2 + q_1^2 - p_2^2 \quad \text{elliptic conical point}$$

When $d \geq 3$ Braam and Duistermaat [6] [7] have further brought $\mathcal{H}$ to its normal form near the stratum of Σ of codim 3 at a conical point : Modulo elliptic summands, we can reduce to the case where $\mathcal{H}$ is a 2×2 real symmetric matrix, and in the sense of quadratic forms, i.e. by replacing $\mathcal{H}$ by $\mathcal{A}^*\mathcal{H}\mathcal{A}$, where $\mathcal{A}$ is elliptic, there are symplectic coordinates (p, q) such that

$$\mathcal{A}^*\mathcal{H}\mathcal{A} = \begin{pmatrix} -p_1 + p_2 & q_1 p_3 \\ q_1 p_3 & p_1 + p_2 \end{pmatrix} \text{ hyperbolic conical point}$$
$$\mathcal{A}^*\mathcal{H}\mathcal{A} = \begin{pmatrix} p_1 - p_2 & q_2 p_3 \\ q_2 p_3 & p_1 + p_2 \end{pmatrix} \text{ elliptic conical point}$$

Moreover these normal forms extend naturally to quantization of Weyl symbols, composing $\mathcal{A}$ by a suitable FIO $\mathcal{U}$ microlocally defined near the conical point, and associated with the canonical change of coordinates $(x, \xi) \mapsto (q, p)$. Namely, in the hyperbolic case

$$\mathcal{U}^*(\mathcal{A}^*\mathcal{H}\mathcal{A})^w\mathcal{U} = \begin{pmatrix} -hD_{q_1} + hD_{q_2} & q_1 hD_{q_3} \\ q_1 hD_{q_3} & hD_{q_1} + hD_{q_2} \end{pmatrix}$$

and similarly in the elliptic case.

Normal forms are useful for studying the Hamiltonian flow of $\mathcal{H}(x, \xi)$ or finding quasi-modes for $\mathcal{H}(x, hD_x)$ in terms of classical functions. These normal forms are structurally stable, i.e. they are not affected by small perturbations of a generic $\mathcal{H}$ in the $C^\infty(T^*\mathbf{R}^d)\otimes\mathbf{R}^n$ topology. Even more precise results are available in 1-D, see e.g. [15] in the framework of Born-Oppenheimer approximation, or [8], [3] for Bogoliubov-de Gennes Hamiltonian.

b) Particular cases: $k = 2$, codim 3 singularities

In many physical situations however, the genericity assumption is not fulfilled, and special reductions should be carried out :

• *Conical refraction in 3-D* Recall from [11] in dimension $d = 3$ that the normal form is given by the symmetric matrix

$$\mathcal{H} = \begin{pmatrix} p_1 + p_2 & p_3 \\ p_3 & p_1 - p_2 \end{pmatrix} \tag{1}$$

which is independent of space variables. This is not structurally stable, because the singular part Σ of N is involutive.

• *Graphene Hamiltonian in 2-D.* The Hamiltonian is a complex Hermitian matrix [9]:

$$\mathcal{H}(p) = \begin{pmatrix} 0 & f(p_1,p_2) \\ \overline{f}(p_1,p_2) & 0 \end{pmatrix} = |\lambda(p)| \begin{pmatrix} 0 & e^{-i\theta(p)} \\ e^{-i\theta(p)} & \end{pmatrix} \tag{2}$$

Here $p = (p_1, p_2) \in \mathbf{R}^2$ are quasi-momenta, and the eigenvalues are explicitely given by

$$\lambda(p) = \pm t\sqrt{3 + 2\cos(\sqrt{3}p_1 a) + 4\cos(\sqrt{3}p_2 a/2)\cos(3p_2 a/2)}$$

Energy vanishes at Dirac points (2 Dirac points per hexagonal cell). The linearization at the Dirac point is of the form

$$\mathcal{H}(p) = v \begin{pmatrix} 0 & k_1 - ik_2 \\ k_1 + ik_2 & 0 \end{pmatrix}, \quad v = 3t/(2a) \tag{3}$$

c) Particular case: higher order singularities

Here we are concerned in the case where Δ vanishes of order 4 at the conical point (leaving open the case $k = 3$). The physical example is the Hamiltonian quadratic in the momenta from Elasticity Theory [5] in (2+1)-D, that is obtained in the following way.

On the set of maps $\Gamma : \mathbf{R}_t \times \mathbf{R}^2_x \to \mathbf{R}^2, (t, x, y) \mapsto (\phi(x, y, t), \psi(x, y, t))$ with Sobolev regularity $H^1(\mathbf{R}^3)$, we consider the Lagrangian density:

$$\mathcal{L}(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} \phi_t \\ \psi_t \end{pmatrix}, \begin{pmatrix} \phi_x \\ \psi_x \end{pmatrix}, \begin{pmatrix} \phi_y \\ \psi_y \end{pmatrix}) = T - V = \frac{1}{2}(\phi_t^2 + \psi_t^2) - \frac{1}{2}(\phi_x^2 + \psi_y^2) - \frac{1}{2}A(x,y)(\phi_y + \psi_x)^2$$

Euler-Lagrange equations from the variational principle $\delta \int \mathcal{L}\, dt\, dx\, dy = 0$ lead to extremal curves (rays) in the (t, x, y)-space, and the set of points (t, x, y) connected to the origin in $\mathbf{R}^3$ by such a ray is called the "world front". A matter of interest are the singularities of the world front.

Applying Fourier transformation with respect to (x, y, t)

$$\phi \mapsto \widehat{\phi},\ \phi_x \mapsto \xi\widehat{\phi},\ \phi_y \mapsto \eta\widehat{\phi},\ \phi_t \mapsto \tau\widehat{\phi},\ \text{etc.}\ldots$$

switches from Lagrangian formulation to Hamiltonian formulation and leads to a Pseudo-differential Hamiltonian system (as a quadratic form in $(\widehat{\phi}, \widehat{\psi})$) with principal symbol

$$\mathcal{H} = \begin{pmatrix} \tau^2 & 0 \\ 0 & \tau^2 \end{pmatrix} - \begin{pmatrix} \xi^2 + A(x,y)\eta^2 & A(x,y)\xi\eta \\ A(x,y)\xi\eta & A(x,y)\xi^2 + \eta^2 \end{pmatrix} \tag{4}$$

Genericity properties for this Hamiltonian imply as above that the singular set Σ is of codimension 3, so that it can be brought to one of Arnold-Braam-Duistermaat normal forms (in 2-D) near Σ. However, genericity breaks down in the case of constant coefficient $A(x, y) =$ Const., which justify a direct approach.

In particular, for $A = \frac{1}{2}$, the spatial part has determinant

$$\Delta(\xi, \eta) = \frac{1}{2}(\xi^2 + \eta^2)^2$$

which vanishes of order 4 at $(\xi, \eta) = 0$.

Our purpose, precisely in this case, is to provide a normal form for $\mathcal{H}$ near $\tau = \xi = \eta = 0$. This could be achieved by a straightforward diagonalisation of $\mathcal{H}$, but our method carries naturally to (relative) perturbations of this Hamiltonian, depending on (ξ, η) alone; more naively this example is intended as an illustration of the role played by the generalized Pareto set.

2. Generalized Pareto optimum

A central problem in Differential Calculus consists in maximizing a function: Morse theory on a smooth manifold provides a globalization of this problem.

Economists are rather concerned in maximizing simultenaously several "utility functions", obtaining this way the notion of Pareto optimum in a free exchange economy.

In 1971, S.Smale presented a generalization of Pareto optimum he called the *critical Pareto set* [16]. The main mathematical motivation is to find a Morse theory for m differentiable functions defined on a manifold M of dimension ℓ. Note that this problem is distinct from this of relative extrema of a single function, which is relevant to Lagrange multipliers.

a) "Classical" Pareto optimum in a free exchange economy

Consider a free exchange economy consisting in m consumers $i = 1, \cdots, m$, and for each i, let $x_i = (x_i^{(1)}, \cdots, x_i^{(J)})$ represent the (positive) amount of goods $j = 1, \cdots, J$, with $x_i^{(j)} \geq 0$. We define the "commodity space" as $P = \{x = (x^{(1)}, \cdots, x^{(J)}) \in \mathbf{R}^J; x^{(j)} \geq 0\}$. An unrestricted state of the economy is a point $\vec{x} = (x_1, \cdots, x_m) \in P^m$, but we may restrict to the affine space $M = \{\vec{x} \in P^m : \sum_{i=1}^m x_i^{(j)} = C^{(j)}, j = 1, \cdots, J\}$ with total ressource $C^{(j)}$ of good j.

Each consumer is supposed to have an utility function $u_i : P \to \mathbf{R}$ (generally an homogenous function of $x^{(1)}, \cdots, x^{(J)}$). Thus consumer i prefers x_i to x_i' iff $u_i(x_i) > u_i(x_i')$. The level sets $u_i^{-1}(c)$, $c > 0$ are called "indifference surfaces".

One considers exchanges in M which will increase each u_i on M. A state $\vec{x} \in M$ is called "Pareto optimal" iff there is no $\vec{x}' \in M$ such that $u_i(x_i) \geq u_i(x_i')$ for all i, and $u_j(x_j) > u_j(x_j')$ for some j. If $\vec{x} \in M$ is not Pareto, $\vec{x}$ is not economically stable. For $m = 1$ Pareto optimum equals the usual notion of a (constrained) maximum of $u : M \to \mathbf{R}, x \mapsto u(x)$.

Physicists would replace everywhere the words "maximizing" by "minimizing", and "total resource" by "total energy".

b) Generalized Pareto optimum in the sense of Smale.

Here we do not only consider (joined) maxima, but also (joined) critical points.

Let M be a smooth manifold, dim $M = \ell$, and $u : M \to \mathbf{R}^m$ be m smooth functions defined by $u = (u_1, \cdots, u_m)$ (vector of "utilities").

Let $H(x) = (u'(x))^{-1}(\mathbf{R}_+^m)$, where $u'(x) : T_xM \to \mathbf{R}_+^m$ denotes the derivative (Jacobian) of u at $x \in M$, and $\mathbf{R}_+^m$ the set of $v = (v_1, \cdots, v_m) \in \mathbf{R}^m$ with $v_j > 0$ all j.

Definition 1 [16]: We call $\Theta = \{x \in M : H(x) = \emptyset\}$ the Pareto critical set.

Thus the relation $x \in \Theta$ means that there is no smooth curve $\gamma :]-\epsilon, \epsilon[\to M$ starting at x, and such that $t \mapsto u_i \circ \gamma(t)$ increases for all i's (gradient flow dynamics). For $m = 1$, Pareto critical set is just the set of critical points of u.

If u is a (single) Morse function on M, Θ is a discrete set, and the Hessian $u''(x)$ is defined intrinsically on Θ. For $m > 1$, Θ need not be discrete; but when it consists, as is usual, of a submanifold of dim $m - 1$, then $u''(x)$ is still intrinsically defined on Θ and valued in the 1-D space $\mathbf{R}^m/\mathrm{Ran}u'(x)$.

The open subset $\Theta_S \subset \Theta$ of stable points (classical Pareto set), which reduces for $m = 1$ to the set of local maxima of u, plays a special rôle.

• *Paradigm of Pareto critical in 2-D*: the immersed Klein bottle in $\mathbf{R}^3$.

The paradigm of a Morse function on a compact 2-manifold is the "height function" on the embedded torus, and its (Pareto) critical points are the minimum, 2 saddles, and maximum. Similarly Pareto critical set for 2 functions of 2 variables will be obtained from Klein bottle [Wan], by projecting the immersed bottle in $\mathbf{R}^3$ onto a suitable plane, and looking at the "joined extrema" of the coordinates functions on the projection plane. Another, more convenient immersion of Klein bottle in $\mathbf{R}^3$ (though not so easy to visualize) is the so called "figure eight" or "bagel" immersion, given by

$$\begin{aligned} x_1 &= (r + \cos\tfrac{\theta}{2}\sin v - \sin\tfrac{\theta}{2}\sin 2v)\cos\theta \\ x_2 &= (r + \cos\tfrac{\theta}{2}\sin v - \sin\tfrac{\theta}{2}\sin 2v)\sin\theta \\ x_3 &= \sin\tfrac{\theta}{2}\sin v + \cos\tfrac{\theta}{2}\sin 2v \end{aligned}$$

with $r > 2$ a parameter, $-\pi \le \theta < \pi, 0 \le v < 2\pi$ are the variables. It is obtained by gluing two Möbius bands along their edges. Then the map with typical critical Pareto set is given by $u = (\sqrt{x_1^2 + x_2^2}, x_3)$.

Θ is a stratified set, consisting in a finite number of segments (codimension 1 strata) where $\mathrm{rank}(u') = 1$, terminating at codimension 2 strata (isolated points) where $\mathrm{rank}(u') = 0$.

We will only consider $\ell = m$ which leads to the simplest topology.

• *Elementary Pareto sets*: the case of quadratic polynomials in 2-D.

As in the one-dimensional case, quadratic polynomials in $\mathbf{R}^\ell$ provide useful examples of maps with a typical critical Pareto set. For $m = \ell = 2$ we may take

$$\begin{aligned} u_1(x) = \tfrac{1}{2}(x_1^2 + x_2^2),\ u_2(x) = \tfrac{1}{2}(x_1^2 + x_1x_2 + x_2^2), \quad \Theta = \{0\} \\ u_1(x) = x_1x_2,\ u_2(x) = \tfrac{1}{2}(x_1^2 - x_2^2), \quad \Theta = \{0\} \\ u_1(x) = \tfrac{1}{2}(x_1^2 + x_2^2),\ u_2(x) = \tfrac{1}{2}(x_1^2 - x_2^2), \quad \Theta = \{x_1 = 0\} \end{aligned}$$

More generally, the critical Pareto set of two elliptic or two hyperbolic quadratic linearly independent polynomials reduces to the origin, and to a line for a "mixed" pair.

3. Matrix valued Hamiltonian systems with the Pareto property

Since we are interested in 2-spinors, we work with $m = \ell = 2$ and $\mathcal{H}(x,p) = \mathcal{H}(p)$.

Definition 2: Let $\mathcal{H} \in C^\infty(\mathbf{R}^2) \otimes \mathbf{R}^2 \otimes \mathbf{R}^2$ be a (real) Hermitian matrix. We say that $\mathcal{H}$ has the *Pareto property* iff there exists a smooth map (but with possibly degenerate derivative) $u : \mathbf{R}^2 \to \mathbf{R}^2$, and a (regular) Hermitian matrix $C(p)$ such that, locally, and up to conjugation with an elliptic factor of the form $\mathcal{H}(p) \mapsto \mathcal{A}(p)^*\mathcal{H}(p)\mathcal{A}(p)$, we have:

$$\mathcal{H}(p) = u'(p)C(p)(u'(p))^*$$

Remark: A hint on this definition is the following: let $p \in \mathbf{R}^2$ such that $u_1'(p) = 0$, then the "pure classical state" $\binom{1}{0} \in N(p)$, and similarly for $\binom{0}{1}$ when $u_2'(p) = 0$. So any "classical state" $\binom{x}{y}$ is a superposition of "classical pure states".

These decompositions are local and sometimes can be checked only in the sense of germs. If exact, we say that $\mathcal{H}$ is "integrable in Pareto sense". Only in 1-D problems, one can consider general Hamiltonians of the form $H(x,p)$.

4. Pareto property and the quadratic Hamiltonian of Elasticity Theory

Because Hamiltonians depending on p alone are not structurally stable, few Hamiltonian systems verify Pareto property. We have :

Theorem 1: For $A = 1/2$, the quadratic Hamiltonian of Elasticity Theory (4) is integrable in Pareto sense: with $u_1(\xi,\eta) = \frac{1}{2\sqrt{2}}(\xi^2 - \eta^2)$, $u_2(\xi,\eta) = \frac{1}{\sqrt{2}}\xi\eta$, we have $\mathcal{H}(\xi,\eta) = u'(\xi,\eta)\mathrm{diag}(2,1)(u'(\xi,\eta))^*$ with $u'(u')^* = \frac{1}{2}(\xi^2+\eta^2)\mathrm{Id}$. Let $\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, we have the "skew-diagonalization"

$$\frac{1}{2}\sigma_2(\mathcal{H}(\xi,\eta) - \tau^2)\sigma_2^* = u'(\xi,\eta)\big(\mathrm{diag}(\frac{1}{2},1) - (\xi^2+\eta^2)^{-1}\tau^2\mathrm{Id}\big)(u'(\xi,\eta))^* \tag{5}$$

From this we can construct by inverse Fourier transformation semi-classical spinors ${}^t(\phi,\psi)(x,y;\tau)$ that verify near "Helmholtz equation" corresponding to (4), τ^2 standing for the energy parameter. We write $\widetilde{\phi}$ for the Fourier transform w.r.t the space variables. There are two linearly independent solutions ${}^t(\phi_1,\psi_1)(x,y;\tau)$ of the equation

$$\big(\mathrm{diag}(\frac{1}{2},1) - (\xi^2+\eta^2)^{-1}\tau^2\big)^t(\widetilde{\phi}_1,\widetilde{\psi}_1) = 0$$

given by

$$\phi_1(x,y,\tau) = J_0(\sqrt{x^2+y^2}\frac{\sqrt{2}\tau}{h}), \quad \psi_1 = J_0(\sqrt{x^2+y^2}\frac{\tau}{h}) \tag{6}$$

and ϕ,ψ are derived from (5) by convolution integrals. From this it is standard to deduce the spectral properties of $\mathcal{H}$.

• *Relative stability of the Pareto property*

Theorem 2: Let

$$\mathcal{H}(\xi,\eta) = \begin{pmatrix} \xi^2 + \frac{1}{2}\eta^2 & \frac{1}{2}\xi\eta \\ \frac{1}{2}\xi\eta & \frac{1}{2}\xi^2 + \eta^2 \end{pmatrix} + \mathcal{O}((\xi,\eta)^3)$$

Then $\mathcal{H}(\xi,\eta)$ has the Pareto property (at least in the sense of germs at 0), with $C(\xi,\eta) = C_0 + \mathcal{O}(\xi,\eta)$, $u(\xi,\eta) = u_0(\xi,\eta) + \mathcal{O}((\xi,\eta)^2)$. Moreover there is a skew-diagonalization of type (5), and we can find a set of solutions to the "Helmholtz equation" as in (6).

The proof goes as in Poincaré-Dulac theorem [2]: Namely we seek for a "new" u of the form $v = u + f$, with $f = \mathcal{O}(|\xi,\eta|^3)$, and a "new" C of the form $\mathrm{diag}(2,1) + \begin{pmatrix} a & b \\ b & d \end{pmatrix}$. The upper-left matrix element of $v'\mathcal{H}(v')^*$ is given by $\sqrt{2}L_1 f_1 + (\xi^2 + \frac{\eta^2}{2}) + \frac{1}{2}(a\xi^2 - 2b\xi\eta + d\eta^2)$, with $L_1 = 2x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$, which has a resonance 2:1. Its lower-right matrix element is given by $\sqrt{2}L_2 f_2 + (\eta^2 + \frac{\xi^2}{2}) + \frac{1}{2}(a\eta^2 + 2b\xi\eta + d\xi^2)$ with $L_2 = 2y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. The off-diagonal terms involve the term $L_1 f_2 + L_2 f_1$. It turns out that we can solve (at least perturbatively) this system, the coefficients $a = a_1\xi + a_2\eta + \cdots$, $b = b_1\xi + b_2\eta + \cdots$, $d = d_1\xi + d_2\eta + \cdots$ being determined to fulfill the compatibility conditions. The skew-diagonalization of type (5) follows from the fact that $v'(\xi,\eta)(v'(\xi,\eta))^* = \frac{1}{2}(\xi^2+\eta^2)^2(\mathrm{Id} + \mathcal{O}(|\xi,\eta|)$ is close to a multiple of Id. We can still construct semi-classical solutions, and their asymptotics in h for small E, obtained by varying the argument of the Bessel functions.

5. Conclusion

We have made an attempt to extend the notion of "non-degenerate critical point" for a scalar Hamiltonian to the notion of "Pareto critical set" Θ for a 2×2 Hamiltonian system. We focussed to the case where Θ reduces to a point. Our analysis applies to the Hamiltonian $\mathcal{H}(p)$ quadratic in momenta arising in Elasticity theory, for a particular value of the coefficients. It allows to account for the spectral properties of $\mathcal{H}$ together with the semi-classical spectral asymptotics of (relative, i.e. depending again only on p) perturbations of $\mathcal{H}$ near $E = 0$. These Hamiltonians present a codimension 3 singularity of order 4 at $\Theta = \{0\}$. It seems difficult to extend this purely algebraic approach for general Hamiltonians of the form $\mathcal{H}(x, p)$: namely, when trying a map $u : \mathbf{R}^4 \to \mathbf{R}^2$, eigenvectors of $\mathcal{H}$ are given from those of $C(x, p)$ by applying a pseudo-inverse of $(u')^*$, which is not uniquely defined.

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