

Collective branch in the continuum of BCS superconductors and superfluid Fermi gases

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We demonstrate the existence of a collective excitation branch in the pair-breaking continuum of superfluid Fermi gases and BCS superconductors, as suggested by Littlewood and Varma in 1982. We analytically continue the RPA equation on the collective mode energy through its branch cut associated with the continuum, and obtain the full complex dispersion relation, including in the strong coupling regime. For $\Delta/\mu > 1.210$ (very close to unitarity in a superfluid Fermi gas), where Δ is the order parameter and μ the chemical potential, the real part of the branch is wholly within the band gap $[0, 2\Delta]$. In the long wavelength limit, the branch varies quadratically with the wave number, with a complex effective mass that we compute analytically. This contradicts the result of Littlewood and Varma that prevailed so far.

Introduction – Systems with a macroscopic coherence between pairs of fermions exhibit in their excitation spectrum a pair-breaking continuum, whose energy is greater than twice the order parameter Δ . This is particularly the case of superconductors and cold gases of spin-1/2 fermionic atoms. The collective behavior of these systems at energies lower than 2Δ is known: it is characterized by a bosonic excitation branch, of phononic start in neutral gases and plasmonic start in superconductors with Coulomb interactions [1]. The dispersion relation of this branch was calculated in the Random Phase Approximation (RPA) [2, 3] and its existence experimentally confirmed [4–6].

Conversely, the existence of a collective mode *inside* the pair-breaking continuum remains an open question that attracts a strong interest because of an analogy often suggested with the Higgs mode in field theory [7]. We identify two major shortcomings [8–10] in the existing theoretical treatment: (i) it is based on a particle-hole symmetry hypothesis and neglects the coupling between the amplitude and phase of the order parameter, which restricts it to the weak coupling regime, (ii) it is limited to long wavelengths. These shortcomings are prejudicial as they raise doubts about the very existence of this second collective mode [11], in particular at zero wave vector [12].

In this article, we clarify considerably the description of the collective modes of the continuum. By analytically continuing the order parameter Gaussian fluctuations matrix we reveal a pole in the propagator of Δ below the branch cut associated to the continuum, for positive chemical potential $\mu > 0$ and non-zero wave number only. We obtain the full dispersion relation of this mode completely accounting for amplitude-phase coupling. This allows us to deal with the strong coupling regime; in particular we show that the real part of the branch is fully in the band gap $[0, 2\Delta]$ when $\Delta > 1.210\mu$. In the weak

coupling and long wavelength limit, we disagree sharply with the prediction commonly accepted in the literature [10], in particular for the damping rate that, we find, has a quadratic start at low wave number, rather than a linear one. All our predictions are based on the RPA for contact interactions. This theory describes qualitatively well both cold Fermi gases in the BEC-BCS crossover and BCS superconductors (assuming that Coulomb interactions have no effect on the amplitude modes [10]), and is a prerequisite for any more realistic description of interactions.

The branch that we find describes the collective behavior of the pairs following an excitation of their internal degrees of freedom; its frequency is thus not trivially the edge of the continuum $2\Delta/\hbar$, as for the “Higgs oscillation” observed [12–18] at zero wave vector. It is observable in a superfluid Fermi gas using momentum-resolved Bragg spectroscopy, in the same way that Ref. [6] observed the Bogoliubov-Anderson mode. The pole in the analytic continuation results in a broadened peak at energy above 2Δ in the order parameter response function.

Fluctuations of the order parameter – We consider a homogeneous system of fermions of mass m , spin $\sigma = \uparrow / \downarrow$ and chemical potential μ , with attractive contact interactions. At zero temperature $T = 0$, the fluctuations of the order parameter Δ around its equilibrium value admit eigenmodes, which are the collective modes of the system. Expanding to second order in amplitude $\delta\lambda$ and phase $\delta\theta$ fluctuations yields the Gaussian action [19]

$$\mathcal{S} = \mathcal{S}_0 + \int d\omega \int d^3q (-i\Delta\delta\theta^* \quad \delta\lambda^*) M(\omega, \mathbf{q}) \begin{pmatrix} i\Delta\delta\theta \\ \delta\lambda \end{pmatrix} \quad (1)$$

The fluctuation matrix M is symmetric and gives access to the propagator of the Δ field through a mere inversion. The equation giving the energy $z_{\mathbf{q}}$ of the collective modes

as a function of their wave vector \mathbf{q} is then

$$\det M(z, \mathbf{q}) = 0 \quad (2)$$

Since the order parameter Δ describes pair condensation, the coefficients of its fluctuation matrix contain an integral over the internal wave vector \mathbf{k} of the pairs, involving $\xi_k = \hbar^2 k^2 / 2m - \mu$ and $E_k = \sqrt{\xi_k^2 + \Delta^2}$, the dispersion relations of free fermions and BCS quasiparticles respectively, as well as the energy $E_{\mathbf{k}\mathbf{q}} = E_{\mathbf{k}+\mathbf{q}/2} + E_{\mathbf{k}-\mathbf{q}/2}$ of a pair of quasiparticles of total wave vector \mathbf{q} :

$$M_{\pm\pm}(z, \mathbf{q}) = \int \frac{d^3k}{2} \left[\frac{(W_{\mathbf{k}\mathbf{q}}^\pm)^2}{z - E_{\mathbf{k}\mathbf{q}}} - \frac{(W_{\mathbf{k}\mathbf{q}}^\pm)^2}{z + E_{\mathbf{k}\mathbf{q}}} + \frac{1}{E_{\mathbf{k}}} \right] \quad (3)$$

$$M_{+-}(z, \mathbf{q}) = \int \frac{d^3k}{2} W_{\mathbf{k}\mathbf{q}}^+ W_{\mathbf{k}\mathbf{q}}^- \left[\frac{1}{z - E_{\mathbf{k}\mathbf{q}}} + \frac{1}{z + E_{\mathbf{k}\mathbf{q}}} \right] \quad (4)$$

where the indices $+$ and $-$ refer to phase and amplitude fluctuations respectively and we introduce the notation $(W_{\mathbf{k}\mathbf{q}}^\pm)^2 = (E_{\mathbf{k}+\mathbf{q}/2} E_{\mathbf{k}-\mathbf{q}/2} + \xi_{\mathbf{k}+\mathbf{q}/2} \xi_{\mathbf{k}-\mathbf{q}/2} \pm \Delta^2) / (2E_{\mathbf{k}+\mathbf{q}/2} E_{\mathbf{k}-\mathbf{q}/2})$.¹ Note that Eq. (2) is found also in the RPA [1, 20, 21], by a diagrammatic resummation [3] or by linearization of the time-dependent BCS equations [22].

Since Eq. (2) is invariant under the change of z to $-z$, we can restrict ourselves to $\text{Re } z \geq 0$. The matrix M then has a branch cut for $z \in \mathcal{C}_{\mathbf{q}} = \{E_{\mathbf{k}\mathbf{q}}, \mathbf{k} \in \mathbb{R}^3\}$, originating in the terms having $z - E_{\mathbf{k}\mathbf{q}}$ in the denominator in (3 - 4). As such, Eq (2) has at most one solution for fixed \mathbf{q} : it is real, below the continuum $\hbar\omega_{\mathbf{B},\mathbf{q}} < \min \mathcal{C}_{\mathbf{q}}$, and corresponds to the bosonic Anderson-Bogoliubov branch [3]. This is not what we are looking for here. The collective modes we want to characterize are *inside* the continuum, that is, *a priori* for $\text{Re } z_{\mathbf{q}} > \min \mathcal{C}_{\mathbf{q}}$. As in the textbook problem of a single atom coupled to the electromagnetic field [23], the correct way to obtain a solution to (2) in the presence of the continuum is to analytically continue the matrix M through its branch cut [8]. This is an opportunity for us to recall the beautiful prescription of Nozières [24, 25] on how to analytically continue a function of the form

$$f(z) = \int_{-\infty}^{+\infty} d\omega \frac{\rho(\omega)}{z - \omega}, \quad (5)$$

analytic for $\text{Im } z \neq 0$ but exhibiting a branch cut on the real axis, wherever the spectral density $\rho(\omega)$ is non-zero. The non-analytic contribution to $M_{\sigma\sigma'}$, with $\sigma, \sigma' = \pm$, is naturally cast into this form with the spectral densities

$$\rho_{\sigma\sigma'}(\omega, \mathbf{q}) = \int \frac{d^3k}{2} W_{\mathbf{k}\mathbf{q}}^\sigma W_{\mathbf{k}\mathbf{q}}^{\sigma'} \delta(\hbar\omega - E_{\mathbf{k}\mathbf{q}}) \quad (6)$$

¹ with $W_{\mathbf{k}\mathbf{q}}^+ > 0$ for all \mathbf{k} and $W_{\mathbf{k}\mathbf{q}}^- > 0$ if and only if $k^2 > 2m\mu/\hbar^2 - q^2/4$.

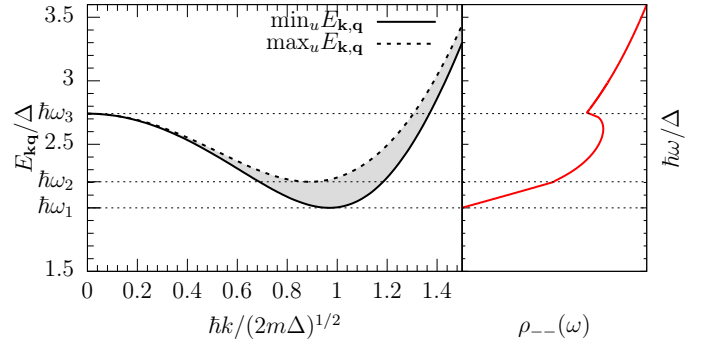


FIG. 1: As a function of k , the interval between $\min_u E_{\mathbf{k}\mathbf{q}}$ (reached for $u = 0$, solid line on the left) and $\max_u E_{\mathbf{k}\mathbf{q}}$ (reached for $u = \pm 1$, dashed line on the left) determines an energy band (gray area) in which the resonance $\hbar\omega = E_{\mathbf{k}\mathbf{q}}$ occurs for at least one value of $u = \cos(\mathbf{k}, \mathbf{q})$ in $[-1, 1]$. For fixed ω , the integration interval over k in (6) is read horizontally; as a function of ω , its structure undergoes 3 transitions in ω_1 , ω_2 and ω_3 (dotted lines), which result in angular points in the spectral density, here the example of ρ_{--} (red solid line on the right). In this figure, $\mu/\Delta = 1$ and $\hbar q/\sqrt{2m\Delta} = 0.5$.

The analytic continuation of f from the upper half-complex plane to the lower half-plane, through an interval $[\omega_1, \omega_2]$ of the branch cut where ρ is analytic, is simply

$$f_{\downarrow}(z) = \begin{cases} f(z) & \text{if } \text{Im } z > 0 \\ f(z) - 2i\pi\rho(z) & \text{if } \text{Im } z \leq 0 \end{cases} \quad (7)$$

where $z \mapsto \rho(z)$ is the analytic continuation of ρ for $\text{Im } z \neq 0$. This is readily demonstrated by writing $\rho(\omega) = (\rho(\omega) - \rho(z)) + \rho(z)$ in (5).

To carry out the analytic continuation of M , we then study the function $\omega \mapsto \rho_{\sigma\sigma'}$ first on the real axis, and search for singularities. For that, we calculate the integral over \mathbf{k} in (6) in a spherical frame of axis \mathbf{q} . The integral over the azimuthal angle disappears by rotational invariance and we use the Dirac δ to perform the integral over the polar angle parametrized by $u = \mathbf{k} \cdot \mathbf{q}/kq$. The remaining integral over k is then restricted to a domain represented on Fig. 1, whose form depends on ω . When $\mu > 0$ the BCS excitation branch has its minimum in $k_0 = \sqrt{2m\mu/\hbar^2}$; then, for $0 < q < 2k_0$ the function $\omega \mapsto \rho_{\sigma\sigma'}$ has three angular points related to a configuration change of the integration domain, which divides the real axis in four distinct sectors (see Fig. 1). (i) For $\omega < \omega_1 = 2\Delta/\hbar$, the resonance condition $\hbar\omega = E_{\mathbf{k}\mathbf{q}}$ is never satisfied, so that $\rho_{\sigma\sigma'}(\omega < \omega_1) = 0$. This sector is outside the branch cut $\mathcal{C}_{\mathbf{q}}$. (ii) For $\omega_1 < \omega < \omega_2$ the resonance is reached for $k \in [k_1, k_2]$, with

$$\frac{\hbar k_{1,2}}{(2m)^{1/2}} = \sqrt{\mu - \frac{\hbar^2 q^2}{8m} \pm \frac{\sqrt{\hbar^2 \omega^2 - 4\Delta^2}}{2}} \quad (8)$$

(iii) For $\omega_2 < \omega < \omega_3$, the resonance occurs for $k \in [k_1, k'_1] \cup [k'_2, k_2]$ where k_1 and k_2 are still given by (8) and

k'_1 and k'_2 are the real positive solutions of the polynomial equations

$$\tilde{\omega}^4 - 4\tilde{\omega}^2 P(\tilde{k}, \tilde{q}) + 4R^2(\tilde{k}, \tilde{q}) = 0 \quad \text{and} \quad \tilde{\omega}^2 - 2P(\tilde{k}, \tilde{q}) > 0 \quad (9)$$

with the polynomials $P(x, y) = (x^2 + y^2/4 - \mu/\Delta)^2 + x^2 y^2 + 1$ and $R(x, y) = 2xy(x^2 + y^2/4 - \mu/\Delta)$ and the notations $\tilde{k}^2 = \hbar^2 k^2/2m\Delta$, $\tilde{q}^2 = \hbar^2 q^2/2m\Delta$ and $\tilde{\omega} = \hbar\omega/\Delta$. (iv) For $\omega > \omega_3$, solutions k_1 and k'_1 are no longer real, and the integration interval is reduced to $[k'_2, k_2]$.

After this study the extension of $\rho_{\sigma\sigma'}$ to the complex plane is straightforward: we simply replace ω by z in Eqs. (8–9) giving the integration boundaries. $\rho_{\sigma\sigma'}(z)$ is then given by a line integral between complex boundaries. Now, using the Nozières prescription (7), we obtain an analytical continuation of M , where the angular points ω_1 , ω_2 and ω_3 become the branching points of three branch cuts.

Numeric study at arbitrary q – We find a solution $z_{\mathbf{q}} = \hbar\omega_{\mathbf{q}} - i\hbar\Gamma_{\mathbf{q}}/2$ to Eq. (2) in the analytic continuation through the interval $[\omega_1, \omega_2]$, which we identify as the energy of the “Littlewood-Varma” collective mode we seek. The dispersion relation $q \mapsto \omega_{\mathbf{q}}$ is represented on Fig. 2 for different pairing strength $\mu/\Delta = 1/10, 10$ and 100 ($1/k_F a \simeq 0.5, -1.5$ and -3.0 in a cold spin-1/2 Fermi gas with k_F the Fermi wave number and a the s -wave scattering length). Departing quadratically from its limit 2Δ in $q = 0$, the branch goes through a maximum of height proportional to Δ in the weak coupling regime $\Delta \ll \mu$, then plunges into the band gap $[0, 2\Delta]$ at $q = q_{\text{sup}}$. In the strong coupling regime $\Delta > \mu$, the domain $[0, q_{\text{sup}}]$ where the energy of the branch is greater than 2Δ shrinks, until its disappearance for $\mu/\Delta \simeq 0.8267$. Conversely, the damping rate $\Gamma_{\mathbf{q}}$ is a strictly increasing function of q , also starting quadratically from its zero limit in $q = 0$. This is in direct contrast with the commonly accepted prediction in the literature of a damping rate linear in q [10].

The branch disappears in $q = 2k_0$ (hence before the Bogoliubov-Anderson branch hits the continuum [3]) when the interval $[\omega_1, \omega_2]$ through which our analytic continuation passes is reduced to one point. Last, we exclude the existence of a branch of energy greater than $2\sqrt{\Delta^2 + \mu^2}$ in the BEC regime where $\mu < 0$ and where the three singularities of $\rho_{\sigma\sigma'}$ gather, $\omega_1 = \omega_2 = \omega_3 = 2\sqrt{(|\mu| + \hbar^2 q^2/2m)^2 + \Delta^2}/\hbar$.

Long wavelength limit – In this limit, we obtain several analytical results that corroborate our numerical study. We deal separately with the singular case $q = 0$, where the matrix $M(z, \mathbf{q} = 0)$ is expressible in terms of the complete elliptic functions of the first and third kind, $K(k)$ and $\Pi(n, k)$ [26]:

$$\begin{aligned} \text{th}\zeta \tilde{M}_{++}(z, 0) &= \frac{\tilde{M}_{--}(z, 0)}{\text{th}\zeta} = -\pi(2e^t)^{1/2}[F(\zeta) - F(-\zeta)] \\ \tilde{M}_{+-}(z, 0) &= -\pi(2e^t)^{1/2}[F(\zeta) + F(-\zeta)] \end{aligned} \quad (10)$$

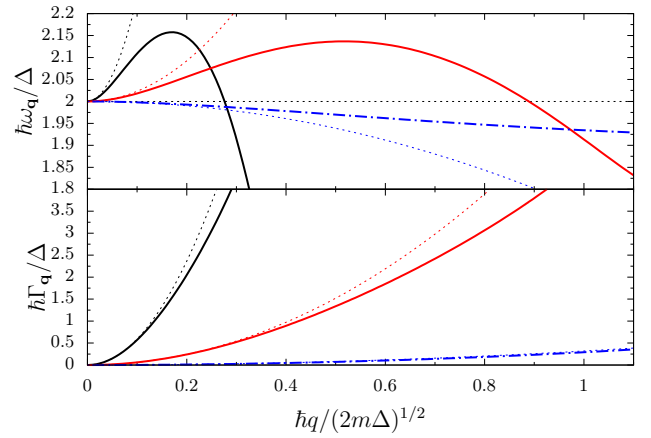


FIG. 2: Frequency (top) and damping rate (bottom) of the gapped continuum collective modes as functions of their wave number q for $\mu/\Delta = 100$ ($1/k_F a \simeq -3.0$, black solid curve), $\mu/\Delta = 10$ ($1/k_F a \simeq -1.5$, red solid curve) and $\mu/\Delta = 0.1$ ($1/k_F a \simeq 0.5$, blue dashed-dotted curve). The dotted curves show the behavior quadratic in q at low q obtained analytically from Eqs. (12)-(16).

with variables defined by $\mu/\Delta = \text{sh } t$ and $z = 2\Delta \text{ch } \zeta$, ζ constrained by $\text{Re } \zeta \geq 0$ and $-\pi/2 \leq \text{Im } \zeta \leq \pi/2$, and ²

$$F(\zeta) = (\text{sh } t + \text{sh } \zeta)[\Pi(e^{t+\zeta}, ie^t) - \Pi(-e^{t-\zeta}, ie^t)] + K(ie^t) \text{ch } \zeta \quad (11)$$

Eq. (2) then reads simply $F(\zeta)F(-\zeta) = 0$. The only solution to this equation, even after analytic continuation³ of $F(\zeta)$, is $\zeta = \pm i\pi/2$ (that is $z = 0$): this is the starting point of the phononic Anderson-Bogoliubov branch. Thus, the limit $\omega = 2\Delta/\hbar$ in $q = 0$ of the continuum collective branch is not a solution of (2): $F(\zeta)$ has a finite non-zero limit $F(0)$ when $\zeta \rightarrow 0$ with $\text{Im } z > 0$. For $q = 0$, there is therefore no collective mode of angular frequency $2\Delta/\hbar$, and the oscillations observed [15–18] at this frequency correspond simply to the edge of the pair-breaking continuum, as understood by Refs. [9, 12, 13]. Here the collective behavior manifests itself rather in the power-law damping of the oscillations [13, 14].

For small but not zero q , and $\mu > 0$, the resonance sector between $\hbar\omega_1 = 2\Delta$ and $\hbar\omega_2 = 2\Delta + \mu\hbar^2 q^2/2m\Delta + O(q^4)$ in Fig. 1 has a width $O(q^2)$ in energy, and $O(q)$ in the wave number k around the minimum $k_0 = \sqrt{2m\mu}/\hbar^2$ of the BCS branch. We then set

$$z_{\mathbf{q}} = 2\Delta + Z \frac{\hbar^2 q^2}{4m^*} + O(q^3) \quad \text{and} \quad k = k_0 + Kq \quad (12)$$

² Using the spectral densities $\rho_{\sigma\sigma'}(\omega)$, we express $M_{\sigma\sigma'}(z, 0)$ as an integral over ω . The change of variable $\hbar\omega = \Delta(x^2 + 1/x^2)$, $0 < x < 1$ leads to elliptic integrals (see §8.110 of [26]).

³ The branch cut $[2\Delta, +\infty[$ (respectively $[2\sqrt{\Delta^2 + \mu^2}, +\infty[$) in z translates into a branch $[0, +\infty[$ in ζ for $\mu > 0$ (respectively $[-t, +\infty[$ for $\mu < 0$).

with $m^* = m\Delta/2\mu$ the effective mass of the BCS branch near its minimum. We thus focus our attention on the domain of wave vectors where the denominator in (3,4) is of order q^2 :

$$z - E_{\mathbf{k}\mathbf{q}} = z - 2\Delta - \frac{\hbar^2 q^2}{m_*} (K^2 + u^2/4) + O(q^3) \quad (13)$$

Now, using the expansions of the numerator amplitudes $W_{\mathbf{k}\mathbf{q}}^+ \sim 1$ and $W_{\mathbf{k}\mathbf{q}}^- \sim \hbar^2 k_0 q K / m\Delta$, and performing the integral over the angular variable u before that over K we obtain without difficulty the analytic expressions for $\text{Im } z > 0$:

$$\tilde{M}_{++}(z, \mathbf{q}) = -\frac{i\pi^2 k_\Delta}{q} \text{asin} \frac{1}{\sqrt{Z}} + O(1) \quad (14)$$

$$\tilde{M}_{--}(z, \mathbf{q}) = -\frac{i\pi^2 \mu q}{2\Delta k_\Delta} \left[\sqrt{Z-1} + Z \text{asin} \frac{1}{\sqrt{Z}} \right] + O(q^2) \quad (15)$$

where we introduce $k_\Delta = \sqrt{2m\Delta/\hbar^2}$ and the dimensionless integrals $\tilde{M}_{\sigma\sigma'} = M_{\sigma\sigma'}\Delta/k_\Delta^3$. Since the divergence of M_{++} of order $1/q$ is compensated by the suppression of M_{--} linear in q , the finite non-zero limit (10) of M_{+-} in $q = 0$, $\hbar\omega = 2\Delta$ suffices. All that remains is to insert expressions (10,14,15) in the RPA equation (2) and analytically continue the product $M_{++}M_{--}$ through its branch cut $[0, 1]$ in Z (corresponding to the segment $[\hbar\omega_1, \hbar\omega_2]$ in z) with the substitutions $\text{asin} 1/\sqrt{Z} \rightarrow \pi - \text{asin} 1/\sqrt{Z}$ and $\sqrt{Z-1} \rightarrow -\sqrt{Z-1}$ to obtain an explicit yet transcendental equation on Z :

$$\left[\pi - \text{asin} \frac{1}{\sqrt{Z}} \right] \left[Z \left(\pi - \text{asin} \frac{1}{\sqrt{Z}} \right) - \sqrt{Z-1} \right] + \frac{2}{\pi^4 \mu} \left(\frac{\hbar^2}{2m} \right)^3 M_{+-}^2(2\Delta, 0) = 0 \quad (16)$$

The continuation is for the entire lower half-plane, including $\text{Re } z < 2\Delta$ ($\text{Re } Z < 0$). The unique solution of Eq. (16) faithfully reproduces the coefficient of q^2 in Fig. 2; it is represented in Fig. 3 as a function of μ/Δ . The real part changes sign for $\mu/\Delta \simeq 0.8267$, which confirms the existence of a branch of energy below 2Δ in the strong coupling regime. One should not overinterpret the existence of solutions below 2Δ : they remain separated from the segment $[0, 2\Delta]$ of the real axis by a branch cut, so that their effect is felt near the real axis only at energies greater than 2Δ .

In the weak coupling limit $\mu/\Delta \rightarrow +\infty$, M_{+-} tends to zero because of particle-hole symmetry $k \leftrightarrow k_0 - k$, so the RPA equation reduces to $M_{++}M_{--} = 0$ for $q \neq 0$, and Eq. (16) to its Z -dependent first line. As suggested in [10], the Littlewood-Varma collective mode is then a pure amplitude mode (a root of M_{--}), while the phononic phase mode is given by $M_{++} = 0$. However, we obtain a very different dispersion relation

$$z_{\mathbf{q}} \underset{\mu/\Delta \rightarrow +\infty}{\underset{q \rightarrow 0}{\simeq}} 2\Delta + (0.2369 - 0.2956i) \frac{\hbar^2 q^2}{4m^* \Delta} \quad (17)$$

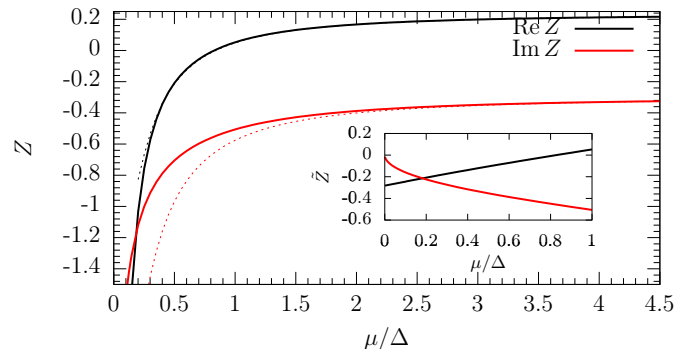


FIG. 3: The real (black solid curve) and imaginary (red solid curve) parts of the dimensionless coefficient Z of q^2 in the energy $z_{\mathbf{q}}$ of the “Littlewood-Varma” mode are represented as functions of μ/Δ . The dashed curves give the asymptotic expansion in the weak coupling limit $\mu/\Delta \rightarrow +\infty$: $Z = Z_0 - \frac{2Z_0^2}{Z_0-1} \left(\frac{\Delta}{\pi\mu} \right)^2 \ln^2 \frac{\Delta}{8\mu e} + \dots$ with $Z_0 = 0.2369 - 0.2956i$. The inset shows the rescaled coefficient $\tilde{Z} = Z\mu/\Delta = Zm/2m^*$ which admits the finite real limit $\tilde{Z}_\infty = -16K^2(i)/\pi^4 \simeq -0.2823$ in the strong coupling limit $\mu/\Delta \rightarrow 0^+$, its imaginary part tending to zero like $-12K(i)(\mu/\Delta)^{1/2}/\pi^3$.

where even the real part of Z differs from the $2/3$ of [10].

Our calculation shows the limits of the usual analogy with Higgs modes in field theory: although it is also a gapped amplitude mode in the weak coupling limit, the collective mode, here immersed in a continuum, is obtained only after a non-perturbative treatment of the coupling to fermionic degrees of freedom; impossible therefore to obtain it from a low-energy effective action as suggested sometimes [7, 27].

Observability in the response functions – The collective branch that we predict appears in the response functions of the system. We show in Fig. 4 the example of the order parameter response function $|N/\det M|(\omega + i0^+, \mathbf{q})$ with $N = (M_{++} + M_{--})/2 + M_{+-}$, obtained from the amplitude and phase response functions by the change of variable $\delta\Delta = \delta\lambda + \Delta i\delta\theta$. At low q , when the mode is weakly damped, the branch obtained in the analytic continuation remarkably predicts the position of a broadened peak in the response function. We note with great interest that Ref. [6] observes such a peak in its measurements of the density-density response function. This may well be our collective branch, particularly if the q^2 dependence of the peak location at low q was confirmed.

Conclusion – We have established on solid theoretical foundations the existence of a collective branch inside the pair-breaking continuum of a BCS superconductor and superfluid Fermi gas, and we have fully characterized its dispersion relation and damping rate in the RPA. We thus give a clear and rigorous answer to an old condensed matter problem. Our branch appears clearly in the response functions of the system and can therefore

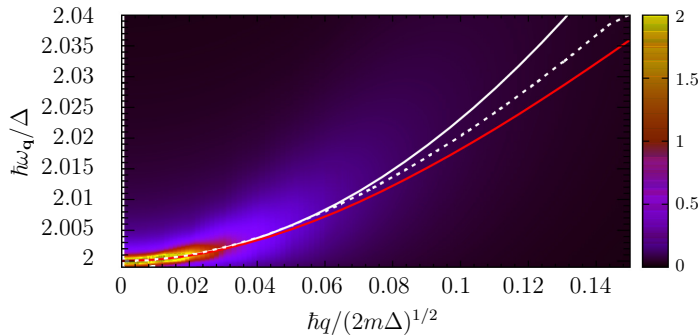


FIG. 4: The intensity of the pairing field response function is represented in colors and arbitrary units as a function of q and ω in the BCS regime ($\mu/\Delta = 10$). The top of a peak in the function (white dashed curve) is well reproduced at low q by the collective branch of the analytic continuation (red solid curve and white solid curve for its quadratic approximation (12)).

be measured in a cold atom gas by Bragg spectroscopy.

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