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# A PROJECTION BASED REGULARIZED APPROXIMATION METHOD FOR ILL-POSED OPERATOR EQUATIONS

L. GRAMMONT , M.T. NAIR

**Abstract:** Problem of solving Fredholm integral equations of the first kind is a prototype of an ill-posed problem of the form  $T(x) = y$ , where  $T$  is a compact operator between Hilbert spaces. Regularizations and discretizations of such equations are necessary for obtaining stable approximate solutions for such problems. For ill-posed integral equations, a quadrature based collocation method has been considered by Nair (2012) for obtaining discrete regularized approximations. As a generalization of that, a projection collocation method has been studied in 2016. In both of the considered methods, the operator  $T$  is approximate by a sequence of finite rank operators. In the present paper, the authors choose to approximate  $TT^*$  by finite rank operators. It is found that in some cases, the derived estimates are improvements over the previous estimates.

**Keywords:** First kind Fredholm integral equations, Inverse problems, Ill-posed problems, Projection, Tikhonov regularization.

## 1. INTRODUCTION

Let  $X$  and  $Y$  be Hilbert spaces and  $T : X \rightarrow Y$  be a bounded linear operator with its range  $R(T)$  not closed in  $Y$ . We would like to obtain stable approximate solutions for the ill-posed operator equation

$$(1) \quad Tx = y,$$

where  $y \in Y$ . We know that (see [4]), for  $x \in X$ ,

$$\begin{aligned} T^*Tx = T^*y &\iff \|Tx - y\| = \inf_{u \in X} \|Tu - y\| \\ &\iff y \in D(T^\dagger) := R(T) + R(T)^\perp, \end{aligned}$$

where  $T^\dagger : D(T^\dagger) \rightarrow X$  denotes the Moore-Penrose inverse of  $T$ . Recall that  $T^\dagger$  is a closed operator, and it is continuous iff  $R(T)$  is closed. It is also known that (see [4]) for  $y \in D(T^\dagger)$ ,  $x^\dagger := T^\dagger y$  is the unique element in  $X$  such that

$$\|x^\dagger\| = \inf\{\|x\| : T^*Tx = T^*y\},$$

and it is known as the generalized solution of (1).

Since  $R(T)$  is not closed,  $T^\dagger$  is not continuous, and hence,  $\tilde{y}$  close to  $y \in D(T^\dagger)$  does not imply that  $\tilde{y} \in D(T^\dagger)$  and even if  $\tilde{y} \in D(T^\dagger)$ ,  $T^\dagger\tilde{y}$  need not be close to  $T^\dagger y$ . Therefore, it is necessary to use a regularized version of (1) for obtaining stable approximate solutions. In this note we shall use the Tikhonov regularization of (1) and then consider a projection method for obtaining approximations for the regularized solutions under the assumption that  $y \in D(T^\dagger)$ .

We may recall that the Tikhonov regularization of (1) is the equation

$$(2) \quad (T^*T + \alpha I)x_\alpha = T^*y$$

for  $\alpha > 0$  (cf. [1, 2, 4]). Clearly, since  $T^*T$  is a positive self-adjoint operator, (2) is uniquely solvable for every  $y \in Y$  and

$$y \mapsto x_\alpha := (T^*T + \alpha I)^{-1}T^*y$$

is a continuous linear operator. Further, using spectral theory, it can be shown that (see [4]) if  $y \in D(T^\dagger)$ , then

$$(3) \quad \|x^\dagger - x_\alpha\| = \alpha\|(T^*T + \alpha I)^{-1}x^\dagger\| \rightarrow 0 \quad \text{as } \alpha \rightarrow 0.$$

It is also known that (see [4])  $x_\alpha \in X$  is the unique element at which the function

$$x \mapsto \|Tx - y\|^2 + \alpha\|x\|^2$$

is minimized.

A prototype of the ill-posed equation (1) is the Fredholm integral equation of the first kind,

$$(4) \quad \int_0^1 k(s, t)x(t)dt = y(s), \quad s \in [0, 1],$$

where kernel  $k(\cdot, \cdot)$  is such that the map  $T$  defined by

$$(5) \quad (Tx)(s) := \int_0^1 k(s, t)x(t)dt, \quad s \in [0, 1],$$

is a compact operator between appropriate Hilbert spaces of functions on  $[0, 1]$ .

Discrete regularization methods associated with Tikhonov regularization consist in approximating the operator  $T^*T$  by  $A_n$ , a finite rank operator. In [5], the operator  $T$  in (5) is from  $L^2[0, 1]$  into itself with a continuous kernel. In this case,  $T^*T$  is approximated by  $T_n^*T_n$ , where  $T_n : L^2[0, 1] \rightarrow \mathbb{K}_w^n$  is defined by

$$T_n x = (T(x)(t_1), T(x)(t_2), \dots, T(x)(t_n)), \quad x \in L^2[0, 1],$$

corresponding to a partition  $t_1, \dots, t_n$  of  $[a, b]$ , and  $\mathbb{K}_w^n = \mathbb{K}^n$  with a weighted inner product with weights  $w_i, i = 1, \dots, n$ . Here,  $\mathbb{K}$  is the field  $\mathbb{R}$  of real numbers or the field  $\mathbb{C}$  of complex numbers.

In [6], the author considers a more general discrete regularization method for the equation (1), approximating  $T^*$  by  $T_n^*T_n$ , and obtained the approximation for  $x_\alpha$  as  $x_{\alpha,n}$  and the corresponding estimate for the discretization error as

$$(6) \quad \|x_\alpha - u_{\alpha,n}\| \leq \frac{\|T^*T - T_n^*T_n\|}{\alpha} \|x^\dagger - x_\alpha\|.$$

The methods in [6] includes  $T_n = \pi_n T$ , where  $(\pi_n)$  is a sequence of finite rank operators which converges to the identity operator pointwise, so that

$$\|T^*T - T_n^*T_n\| \leq (\|T\| + \|T_n\|)\|(I - \pi_n)T\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

and also the method in [5], proving  $T_n^*T_n = F_n T$ , where  $F_n$  is the Nyström approximation of  $T^*$  associated to a convergent quadrature rule, namely,

$$F_n(x)(s) = \sum_{i=1}^n w_j k(t_j, s)x(t_j).$$

As  $\|x_\alpha - x^\dagger\|$  is the regularization error and can not be improved, our aim in this paper is to improve the discretization error  $\|x_\alpha - u_{\alpha,n}\|$ . For this purpose, we propose a projection based regularized approximation  $x_{\alpha,n}$  for which the an estimate for the error  $\|x_\alpha - x_{\alpha,n}\|$  can be better than the estimates in (6)(see Theorem 3).

It is also to be mentioned that the assumption  $y \in R(T)$  is used for the error analysis in [5] and [6]. In this paper, we derive the error estimates using the general case of  $y \in D(T^\dagger) = R(T) + R(T)^\perp$ .

## 2. THE PROJECTION METHOD

Let  $x_\alpha$  be as in equation (2). Since  $A := TT^* : Y \rightarrow Y$  is also positive self-adjoint operator, for every  $\alpha > 0$ , the operator  $A + \alpha I : Y \rightarrow Y$  is bijective and  $(A + \alpha I)^{-1}$  is a bounded operator. Let  $v_\alpha$  be the unique the element in  $Y$  satisfying the equation

$$(7) \quad (A + \alpha I)v_\alpha = y.$$

It can be seen that

$$(8) \quad x_\alpha := T^*v_\alpha.$$

Motivated by this observation, we consider the following method for obtaining approximations for  $x_\alpha$  for each  $\alpha > 0$ :

For each  $n \in \mathbb{N}$ , let  $\pi_n : Y \rightarrow Y$  be a continuous linear projection operator. In applications each  $\pi_n$  may be of finite rank. Let  $v_{\alpha,n}$  be the *Kantorovich approximation* (cf. [4])  $v_\alpha$ , that is,  $v_{\alpha,n}$  satisfies the equation

$$(\pi_n A + \alpha I)v_{\alpha,n} = y$$

uniquely. Of course, one has to impose some condition on  $(\pi_n)$  so that that the above equation is uniquely solvable. We assume throughout that

$$\|(I - \pi_n)A\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

which will ensure the unique solvability of (10). In this regard we state the following theorem which is helpful in deriving the error estimate as well. Its proof is also given for the sake of completion of exposition.

**Proposition 1.** *For  $\alpha > 0$ , let  $n \in \mathbb{N}$  be such that  $\|(I - \pi_n)A\| \leq \frac{\alpha}{2}$ . Then we have the following.*

(i)  $I - (I - \pi_n)A(A + \alpha I)^{-1}$  is bijective and

$$\|(I - (I - \pi_n)A(A + \alpha I)^{-1})^{-1}\| \leq 2.$$

(ii)  $\pi_n A + \alpha I$  is bijective and

$$\|(\pi_n A + \alpha I)^{-1}\| \leq \frac{2}{\alpha}.$$

*Proof.* (i) Since  $A$  is self-adjoint,  $\|(A + \alpha I)^{-1}\| \leq \frac{1}{\alpha}$ . Now, note that

$$\begin{aligned} \pi_n A + \alpha I &= (A + \alpha I) - (I - \pi_n)A \\ &= [I - (I - \pi_n)A(A + \alpha I)^{-1}](A + \alpha I). \end{aligned}$$

Since

$$\|(I - \pi_n)A(A + \alpha I)^{-1}\| \leq \|(I - \pi_n)A\| \|(A + \alpha I)^{-1}\| \leq \frac{1}{2}.$$

Hence, by a well known result from functional analysis (see e.g. [3], pp 317), the operator  $I - (I - \pi_n)A(A + \alpha I)^{-1}$  is bijective and  $\|(I - (I - \pi_n)A(A + \alpha I)^{-1})^{-1}\| \leq 2$ .

(ii) By (i), the operator  $[I - (I - \pi_n)A(A + \alpha I)^{-1}](A + \alpha I)$ , that is, the operator  $\pi_n A + \alpha I$  is bijective and

$$(\pi_n A + \alpha I)^{-1} = (A + \alpha I)^{-1}[I - (I - \pi_n)A(A + \alpha I)^{-1}]^{-1}.$$

Thus,

$$\|(\pi_n A + \alpha I)^{-1}\| \leq \|(A + \alpha I)^{-1}\| \|[I - (I - \pi_n)A(A + \alpha I)^{-1}]^{-1}\| \leq \frac{2}{\alpha}.$$

This completes the proof. □

For  $\alpha > 0$ , let  $n_\alpha \in \mathbb{N}$  be such that

$$(9) \quad \|(I - \pi_n)A\| \leq \frac{\alpha}{2} \quad \forall n \geq n_\alpha.$$

Then, by Proposition 1, equation

$$(10) \quad (\pi_n A + \alpha I)v_{\alpha,n} = y$$

is uniquely solvable for all  $n \geq n_\alpha$ . Motivated by the relation (8), we define

$$x_{\alpha,n} := T^*v_{\alpha,n}, \quad n \geq n_\alpha.$$

As already mentioned, in practice, the projections  $\pi_n$  may be of finite rank. However, the solution  $v_{\alpha,n}$  of equation (10) is in an infinite dimensional space. So, in order to obtain  $v_{\alpha,n}$  using a system in a finite dimensional setting, we observe first that

$$v_{\alpha,n} = \frac{1}{\alpha}(y - \pi_n A v_{\alpha,n}).$$

Now, applying  $\pi_n A$  on both sides of (10), we obtain

$$(\pi_n A + \alpha I)\pi_n A v_{\alpha,n} = \pi_n A y$$

so that

$$w_{\alpha,n} := \pi_n A v_{\alpha,n}$$

belongs to  $R(\pi_n)$ , and

$$(11) \quad (\pi_n A + \alpha I)w_{\alpha,n} = \pi_n A y.$$

Thus we have the following algorithm.

**Algorithm:**

- (1) Solve equation (11):  $(\pi_n A + \alpha I)w_{\alpha,n} = \pi_n A y$ .
- (2) Compute:  $v_{\alpha,n} = \frac{1}{\alpha}(y - w_{\alpha,n})$ .
- (3) Compute:  $x_{\alpha,n} := T^*v_{\alpha,n}$ .

We observe that for defining  $x_{\alpha,n}$ , it is required to solve the equation (11), that is,

$$(\pi_n A + \alpha I)w_{\alpha,n} = \pi_n A y,$$

and its unique solvability is assured under condition (9). However, if  $\pi_n$  are orthogonal projections, then the solvability of (11) is ensured without using condition (9), and thus  $x_{\alpha,n} := T^*v_{\alpha,n}$  are well defined for all  $n \in \mathbb{N}$ . This is a simple consequence of the following general result.

**Theorem 2.** *Let  $\alpha > 0$  and let  $(\pi_n)$  is a sequence of orthogonal projections on  $Y$ . Then for every  $(n, \psi) \in \mathbb{N} \times Y$ , there exists a unique  $\varphi_{\alpha,n} \in Y$  such that*

$$(\pi_n A + \alpha I)\varphi_{\alpha,n} = \pi_n \psi.$$

*Proof.* Note that for  $(n, \psi) \in \mathbb{N} \times Y$  and  $\varphi \in Y$ ,

$$(\pi_n A + \alpha I)\varphi = \pi_n \psi \iff (\pi_n A \pi_n + \alpha I)\varphi = \pi_n \psi.$$

Note that, since  $\pi_n$  is an orthogonal projection and  $A$  is a positive self adjoint operator, the operator  $\pi_n A \pi_n$  is also a positive self adjoint operator. Hence, for each  $\alpha > 0$ ,  $\pi_n A \pi_n + \alpha I$  bijective and has continuous inverse. This completes the proof.  $\square$

Next theroem gives an error estimate which is the main result of this paper. We shall be using the following easily verifiable reations:

$$(12) \quad (TT^* + \alpha I)^{-1}T = T(T^*T + \alpha I)^{-1},$$

$$(13) \quad (T^*T + \alpha I)^{-1}T^* = T^*(TT^* + \alpha I)^{-1}.$$

Also, we shall make use of the estimates (see [4], pp 156)

$$(14) \quad \|(T^*T + \alpha I)^{-1}T^*T\| \leq 1, \quad \|(TT^* + \alpha I)^{-1}T\| \leq \frac{1}{2\sqrt{\alpha}}.$$

**Theorem 3.** For  $\alpha > 0$ , let  $n \in \mathbb{N}$  be such that  $\|(I - \pi_n)A\| \leq \frac{\alpha}{2}$  and let  $y \in Y$ . Let  $v_{\alpha,n} \in Y$  be as in (10) and let  $x_{\alpha,n} := T^*v_{\alpha,n}$ . Suppose  $y \in D(T^\dagger)$ . Then

$$\|x_\alpha - x_{\alpha,n}\| \leq \frac{1}{\sqrt{\alpha}} \min\{\varepsilon_{\alpha,n}^{(1)}, \varepsilon_{\alpha,n}^{(2)}, \varepsilon_{\alpha,n}^{(3)}\},$$

where

$$\begin{aligned} \varepsilon_{\alpha,n}^{(1)} &:= \|(I - \pi_n)T\| \|x^\dagger\|, \\ \varepsilon_{\alpha,n}^{(2)} &:= \frac{1}{2\sqrt{\alpha}} \|(I - \pi_n)A\| \|x^\dagger\| \\ \varepsilon_{\alpha,n}^{(3)} &:= \frac{1}{\alpha} \|(A - \pi_n A)T\| \|x^\dagger - x_\alpha\|. \end{aligned}$$

Further, we have the following.

(i) If  $x^\dagger = T^*u$  for some  $u \in Y$ , then

$$\|x_\alpha - x_{\alpha,n}\| \leq \frac{\|A - \pi_n A\| \|u\|}{\sqrt{\alpha}}.$$

(ii) If  $x^\dagger = T^*Tv$  for some  $v \in X$ , then

$$\|x_\alpha - x_{\alpha,n}\| \leq \frac{\|(A - \pi_n A)T\| \|v\|}{\sqrt{\alpha}}.$$

*Proof.* Let  $v_\alpha$  be as in (8). Then we have

$$v_\alpha - v_{\alpha,n} = (A + \alpha I)^{-1}[I - (I - \pi_n)A(A + \alpha I)^{-1}]^{-1}(\pi_n A - A)(A + \alpha I)^{-1}y$$

so that

$$x_\alpha - x_{\alpha,n} = T^*(A + \alpha I)^{-1}[I - (I - \pi_n)A(A + \alpha I)^{-1}]^{-1}(\pi_n A - A)(A + \alpha I)^{-1}y.$$

By Proposition 1, we have  $\|[I - (I - \pi_n)A(A + \alpha I)^{-1}]^{-1}\| \leq 2$ . Hence, using the relation  $T^*(A + \alpha I)^{-1} = (T^*T + \alpha I)^{-1}T^*$  and the estimate (14), we have

$$\|x_\alpha - x_{\alpha,n}\| \leq \frac{\varepsilon_n(y)}{\sqrt{\alpha}},$$

where

$$\varepsilon_n(y) := \|(I - \pi_n)A(A + \alpha I)^{-1}y\|.$$

Since  $y \in D(T^\dagger)$ , we have  $T^*y = T^*Tx^\dagger$ . Hence,

$$A(A + \alpha I)^{-1}y = (A + \alpha I)^{-1}Ay = (TT^* + \alpha I)^{-1}TT^*Tx^\dagger.$$

Now, using the relations (12) and (13), we obtain

$$\begin{aligned} \varepsilon_n(y) &= \|(I - \pi_n)(TT^* + \alpha I)^{-1}TT^*Tx^\dagger\| \\ &= \|(I - \pi_n)T(T^*T + \alpha I)^{-1}T^*Tx^\dagger\| \\ &= \|(I - \pi_n)A(A + \alpha I)^{-1}Tx^\dagger\| \\ &= \|(A - \pi_n A)T(T^*T + \alpha I)^{-1}x^\dagger\| \end{aligned}$$

Hence, from (14) and (3), we obtain

$$\begin{aligned} \|(I - \pi_n)T(T^*T + \alpha I)^{-1}T^*Tx^\dagger\| &\leq \|(I - \pi_n)T\| \|x^\dagger\| \\ \|(I - \pi_n)A(A + \alpha I)^{-1}Tx^\dagger\| &\leq \frac{\|(I - \pi_n)A\| \|x^\dagger\|}{2\sqrt{\alpha}} \\ \|(A - \pi_n A)T(T^*T + \alpha I)^{-1}x^\dagger\| &\leq \frac{\|(A - \pi_n A)T\| \|x^\dagger - x_\alpha\|}{\alpha}. \end{aligned}$$

Therefore,

$$\varepsilon_n(y) \leq \min\{\varepsilon_{\alpha,n}^{(1)}, \varepsilon_{\alpha,n}^{(2)}, \varepsilon_{\alpha,n}^{(3)}\},$$

where

$$\begin{aligned} \varepsilon_{\alpha,n}^{(1)} &:= \|(I - \pi_n)T\| \|x^\dagger\|, \\ \varepsilon_{\alpha,n}^{(2)} &:= \frac{1}{2\sqrt{\alpha}} \|(I - \pi_n)A\| \|x^\dagger\| \\ \varepsilon_{\alpha,n}^{(3)} &:= \frac{1}{\alpha} \|(A - \pi_n A)T\| \|x^\dagger - x_\alpha\|. \end{aligned}$$

Thus, we obtain

$$\|x_\alpha - x_{\alpha,n}\| \leq \frac{\varepsilon_n(y)}{\sqrt{\alpha}} \leq \frac{1}{\sqrt{\alpha}} \min\{\varepsilon_{\alpha,n}^{(1)}, \varepsilon_{\alpha,n}^{(2)}, \varepsilon_{\alpha,n}^{(3)}\}.$$



If, in addition,  $x^\dagger = T^*u$  for some  $u \in Y$ , then we have

$$\begin{aligned}\varepsilon_n(y) &= \|(\pi_n T - T)(T^*T + \alpha I)^{-1}T^*Tx^\dagger\| \\ &= \|(\pi_n A - A)(A + \alpha I)^{-1}Au\| \\ &= \|\pi_n A - A\| \|u\|.\end{aligned}$$

Hence, we obtain (i). In case  $x^\dagger = T^*Tv$  for some  $v \in X$ , then

$$\begin{aligned}\varepsilon_n(y) &= \|(\pi_n T - T)(T^*T + \alpha I)^{-1}T^*Tx^\dagger\| \\ &= \|(\pi_n T - T)(T^*T + \alpha I)^{-1}T^*TT^*Tv\| \\ &= \|(\pi_n TT^* - TT^*)(TT^* + \alpha I)^{-1}TT^*Tv\| \\ &= \|(\pi_n A - A)T\| \|v\|\end{aligned}$$

Thus, the proof is complete.  $\square$

**Corollary 4.** *Let  $y \in D(T^\dagger)$ . For each  $n \in \mathbb{N}$ , let  $\alpha_n > 0$  be such that*

$$\max\{2\|A - \pi_n A\|, \|(A - \pi_n A)T\|^{2/3}\} \leq \alpha_n.$$

*Then the following are true.*

(i) *For each  $n \in \mathbb{N}$ , there exists a unique  $v_n \in Y$  such that*

$$(\pi_n A + \alpha_n I)v_n = y.$$

(ii) *For each  $n \in \mathbb{N}$ , if  $x_n \in X$  is such that  $(T^*T + \alpha_n I)x_n = T^*y$  and if  $\tilde{x}_n := T^*v_n$ , then*

$$\|x^\dagger - \tilde{x}_n\| \leq 2\|x^\dagger - x_n\|.$$

*Proof.* (i) Since  $\|A - \pi_n A\| \leq \alpha_n/2$ , by Proposition 1, there exists a unique  $v_n \in Y$  such that  $(\pi_n A + \alpha I)v_n = y$ .

(ii) By Theorem 3 and by the definition of  $\alpha_n$ , we obtain

$$\|x_n - \tilde{x}_n\| \leq \frac{\|(\pi_n A - A)T\|}{\alpha_n^{3/2}} \|x^\dagger - x_n\| \leq \|x^\dagger - x_n\|.$$

Hence,

$$\|x^\dagger - \tilde{x}_n\| \leq \|x^\dagger - x_n\| + \|x_n - \tilde{x}_n\| \leq 2\|x^\dagger - x_n\|.$$

Thus, the proof is complete.  $\square$

**Corollary 5.** *For each  $n \in \mathbb{N}$ , let  $x_n$  and  $\tilde{x}_n$  be as in Corollary 4.*

- (1) *If  $y \in D(T^\dagger)$  and  $x^\dagger \in R(T^*)$ , then  $\|x^\dagger - \tilde{x}_n\| = O(\sqrt{\alpha_n})$ .*
- (2) *If  $y \in D(T^\dagger)$  and  $x^\dagger \in R(T^*T)$ , then  $\|x^\dagger - \tilde{x}_n\| = O(\alpha_n)$ .*

## 3. COMPARISON WITH PREVIOUS METHODS

Let us recall the approximation  $u_{\alpha,n}$  of (1) proposed in [6], namely,

$$u_{\alpha,n} = T_n^* w_{\alpha,n},$$

where  $T_n = \pi_n T$  and  $w_{\alpha,n}$  is the solution of

$$(T_n T_n^* + \alpha I) w_{\alpha,n} = \pi_n y,$$

This approximation includes some of the existing projection based methods but also a quadrature based collocation method considered by Nair in [5]. In this connection, we make a few observations.

(a) Recall the estimate (6) proved in [6], namely,

$$\|x_\alpha - u_{\alpha,n}\| \leq \frac{\|T^* T - T_n^* T_n\|}{\alpha} \|x^\dagger - x_\alpha\|.$$

If  $\pi_n$  is an orthogonal projection, then  $T_n^* T_n = T^* \pi_n T$  so that

$$\|T^* T - T_n^* T_n\| = \|T^*(I - \pi_n)T\| \leq \|(I - \pi_n)T\|^2$$

. Thus, we have

$$\|x_\alpha - u_{\alpha,n}\| \leq \left( \frac{\|(I - \pi_n)T\|}{\sqrt{\alpha}} \right)^2 \|x^\dagger - x_\alpha\|.$$

Thus, if  $\pi_n$  are orthogonal projections, then the estimate

$$(15) \quad \|x_\alpha - x_{\alpha,n}\| \leq \frac{1}{\sqrt{\alpha}} \|(I - \pi_n)T\| \|x^\dagger\|$$

results from Theorem 3 is not as good as that in (6) whenever

$$\frac{\|(I - \pi_n)T\|}{\sqrt{\alpha}} < 1.$$

If  $\pi_n$  are not orthogonal, then with  $T_n = \pi_n T$ , the (6) is

$$\|x_\alpha - u_{\alpha,n}\| \leq \frac{\|T^* T - T^* \pi_n^* \pi_n T\|}{\alpha} \|x^\dagger - x_\alpha\|.$$

Thus, in this case, the estimate in (15) is better than (6) if and only if

$$\frac{\|(I - \pi_n)T\|}{\sqrt{\alpha}} \|x^\dagger\| \leq \frac{\|T^* T - T^* \pi_n^* \pi_n T\|}{\alpha} \|x^\dagger - x_\alpha\|$$

if and only if

$$\|(I - \pi_n)T\| \|x^\dagger\| \leq \frac{\|T^* T - T^* \pi_n^* \pi_n T\|}{\sqrt{\alpha}} \|x^\dagger - x_\alpha\|.$$

Therefore, the it is possible that the the estimate in (15) is of better order than (6).

This is the case if

$$\alpha_n := \|T^* T - T^* \pi_n^* \pi_n T\|^2 \quad \text{and} \quad \|(I - \pi_n)T\| = o(\|x^\dagger - x_{\alpha_n}\|).$$

The rate  $\|(I - \pi_n)T\| = o(\|x^\dagger - x_{\alpha_n}\|)$  is possible if  $x^\dagger$  is less smooth.

(b) Another estimates from Theorem 3 is

$$(16) \quad \|x_\alpha - x_{\alpha,n}\| \leq \frac{1}{\alpha^{3/2}} \|(A - \pi_n A)T\| \|x^\dagger - x_\alpha\|.$$

This case estimate is better than (6) if and only if

$$\frac{1}{\alpha^{3/2}} \|(A - \pi_n A)T\| \|x^\dagger - x_\alpha\| \leq \frac{\|T^*T - T^*\pi_n^*\pi_n T\|}{\alpha} \|x^\dagger - x_\alpha\|$$

if and only if

$$\frac{1}{\sqrt{\alpha}} \|(A - \pi_n A)T\| \leq \|T^*T - T^*\pi_n^*\pi_n T\|$$

if and only if

$$\frac{1}{\sqrt{\alpha}} \|(I - \pi_n)T(T^*T)\| \leq \|T^*(I - \pi_n^*\pi_n)T\|.$$

This can happen if for example  $\|(I - \pi_n)T(T^*T)\| = o(\|T^*(I - \pi_n^*\pi_n)T\|)$  and if we take

$$\sqrt{\alpha_n} \geq \frac{\|(I - \pi_n)T(T^*T)\|}{\|T^*(I - \pi_n^*\pi_n)T\|}.$$

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