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BIJECTIONS OF GEODESIC LAMINATION SPACE PRESERVING LEFT HAUSDORFF CONVERGENCE

KEN'ICHI OHSHIKA AND ATHANASE PAPADOPOULOS

ABSTRACT. We prove a rigidity result for the action of the mapping class group on the space of geodesic laminations of a closed hyperbolic surface of genus $g \geq 2$ equipped with the left Hausdorff topology.

Classification AMS: 37E30, 57M99.

1. INTRODUCTION

Let S be a closed orientable surface with a hyperbolic metric m . We let $\text{Mod}(M)$ the mapping class group of S , that is, the group of homotopy classes of orientation-preserving homeomorphisms of M and $\text{Mod}^*(M)$ the extended mapping class group of S , that is, the group of all homotopy classes of homeomorphisms of M .

We denote by d_m the distance function on S induced from m .

Definition 1.1. For an ordered pair of compact sets K, L in S , the *left Hausdorff distance* $d_{\bar{H}}(K, L)$ between them is defined by

$$d_{\bar{H}}(K, L) = \inf\{\epsilon \mid K \subset N_\epsilon(L)\},$$

where N_ϵ denotes the ϵ -neighbourhood with respect to d_m .

We denote by $\mathcal{GL}(S)$ the space of geodesic laminations of S . This space was introduced by Thurston in his lecture notes [3] (see Chapter 8, and in particular §8.1) and it plays a major role in his theory of 3-manifolds and Kleinian groups. It is classically equipped with more than one topology, and we mention in particular the so-called Thurston topology and the Hausdorff topology. In this paper, we are interested on a new topological structure on this space, namely, being a set of compact subspaces of S , $\mathcal{GL}(S)$ is equipped with an induced left Hausdorff distance function which we also denote by $d_{\bar{H}}$.

Before stating the main theorem, we introduce a definition. We consider a bijection $f : \mathcal{GL}(S) \rightarrow \mathcal{GL}(S)$.

Definition 1.2. We say that f preserves left Hausdorff convergence when for any sequence $\{\lambda_i \in \mathcal{GL}(S)\}$ and $\mu \in \mathcal{GL}(S)$, we have

$$d_{\bar{H}}(\lambda_i, \mu) \rightarrow 0 \Leftrightarrow d_{\bar{H}}(f(\lambda_i), f(\mu)) \rightarrow 0.$$

It is easy to see the following equivalence :

$$d_{\bar{H}}(\lambda, \mu) = 0 \Leftrightarrow d_{\bar{H}}(f(\lambda), f(\mu)) = 0.$$

We let $\text{Aut}(\mathcal{GL}(S))$ be the group of bijections of $\mathcal{GL}(S)$ that preserve left Hausdorff convergence. We have a natural homomorphism

$$\text{Mod}^*(M) \rightarrow \text{Aut}(\mathcal{GL}(S)).$$

The aim of this paper is to prove the following.

Theorem 1.1. *The natural homomorphism*

$$\text{Mod}^*(M) \rightarrow \text{Aut}(\mathcal{GL}(S))$$

is an isomorphism.

Our results should be compared with analogous results by Charitos-Papadoperakis-Papadopoulos [1] on $\mathcal{GL}(S)$ equipped with the Thurston topology. The arguments there are different but one can find a similar analysis of the action of a homeomorphism of $\mathcal{GL}(S)$ on various points of this space depending on the dynamics of the leaves of the lamination representing these points.

We note that it is possible to define a topology associated to the left Hausdorff distance by using the sets of the form $U_\epsilon = \{\mu \mid d_{\bar{H}}(\mu, \lambda) < \epsilon\}$ as a basis for a fundamental system of neighbourhoods of a lamination λ . The result of this paper can then be formulated in terms of homeomorphisms of $\mathcal{GL}(S)$ with respect to this topology.

2. ACTIONS ON CURVES

In this section, and until Section 5 included, we assume that f is a bijection of $\mathcal{GL}(S)$ preserving left Hausdorff convergence.

Lemma 2.1. *For any simple closed geodesic c , its image $f(c)$ is again a simple closed geodesic.*

This is derived from the following characterisation of simple closed geodesics in terms of $d_{\bar{H}}$.

Lemma 2.2. *Let c be a simple closed geodesic, and suppose that $d_{\bar{H}}(\lambda_i, c) \rightarrow 0$ for $\{\lambda_i\} \subset \mathcal{GL}(S)$. Then $\lambda_i = c$ for large i .*

Conversely if $d_{\bar{H}}(\lambda_i, \mu) \rightarrow 0$ implies that $\lambda_i = \mu$ for large i , then μ is a simple closed geodesic.

Proof. The former half is evident. For the latter half, let μ be a geodesic lamination satisfying the condition in the statement. Let μ_0 be a minimal component of μ . Then we have $d_{\bar{H}}(\mu_0, \mu) = 0$. Therefore by the condition, $\mu_0 = \mu$. If μ is not simple geodesic, the minimal component can be approximated by a sequence of simple closed geodesics c_i in the Hausdorff topology, and hence we have $d_{\bar{H}}(c_i, \mu) \rightarrow 0$, again contradicting the condition. Thus the only possibility is that μ is a simple closed geodesic. \square

We next show that the inclusion relation between geodesic laminations is preserved by f .

Lemma 2.3. *Suppose that $\lambda \subset \mu$ for two geodesic laminations. Then $f(\lambda) \subset f(\mu)$.*

Proof. This is because $\lambda \subset \mu \Leftrightarrow d_{\bar{H}}(\lambda, \mu) = 0$. □

A geodesic lamination consisting of a collection of disjoint simple closed geodesics on S whose number of components is ≥ 1 will be called a *multicurve*. (We regard a closed geodesic also as a multicurve.) We can characterise multicurves as follows.

Lemma 2.4. *A geodesic lamination μ is a multicurve, but not a simple closed geodesic if and only if the following holds.*

- (a) *If $d_{\bar{H}}(\lambda_i, \mu) \rightarrow 0$, then λ_i is contained in μ for large i .*
- (b) *μ coincides with the union of geodesic laminations properly contained in μ .*

Remark 2.1. The second condition is necessary. The first condition is also satisfied by a union of a simple closed geodesic and one single non-compact isolated leaf spiralling around it from one side.

Proof. Again, it is evident that multicurves which are not simple closed geodesics satisfy these two conditions. If μ satisfies these two conditions, then as was shown in the proof of Lemma 2.2, μ cannot have a minimal component which is not a simple closed geodesic. Condition (b) implies that μ is the union of its minimal components, and that there are more than one components. Therefore μ is a multicurve which is not a simple closed geodesic. □

By Lemmas 2.2 and 2.4, f takes any multicurve to a multicurve.

Now we show that f preserves the number of components for multicurves.

Lemma 2.5. *A multicurve μ with two components is characterised by the property that ‘if μ contains λ then either λ is a simple closed geodesic or $\mu = \lambda$ ’. Therefore f preserves this property.*

Inductively, a multicurve μ with n components is characterised by the property that ‘if μ contains λ , then λ is a multicurve with at most $n - 1$ components or $\lambda = \mu$.’ This is also preserved by f .

We note also that n simple closed geodesics are pairwise disjoint if and only if there exists a multicurve containing all of them. Therefore the disjointness is also preserved by f . Combining these, we see that f induces an automorphism on the curve complex $\mathcal{C}(S)$ of A . By Ivanov’s theorem [2], this implies that there is a homeomorphism of S inducing the same map as f on $\mathcal{C}(S)$. Thus we have the following.

Corollary 2.6. *Let f be a bijection on $\mathcal{GL}(S)$ preserving left Hausdorff convergence. Then there is a homeomorphism $g : S \rightarrow S$ such that f and g induce the same simplicial automorphism on $\mathcal{C}(S)$.*

Ivanov's theorem also shows that this homeomorphism g is unique up to isotopy provided that $\text{genus}(S) \geq 3$. When $\text{genus}(S) = 2$, there are two choices of isotopy classes whose difference is represented by a hyperelliptic involution. Indeed, the hyperelliptic involution ι acts on $\mathcal{C}(S)$ trivially, and hence g and $\iota \circ g$ induce the same action on $\mathcal{C}(S)$.

3. APPROXIMABLE LAMINATIONS

From now on, f is as before a bijection of $\mathcal{GL}(S)$ preserving left Hausdorff convergence, and g denotes an automorphism of S inducing the same map as f on the curve complex $\mathcal{C}(S)$. For any geodesic ℓ on S , we abuse the symbol $g(\ell)$ to denote the geodesic homotopic to $g(\ell)$. In this way, we regard g as acting on $\mathcal{GL}(S)$.

Definition 3.1. We say that a geodesic lamination μ is approximable when there is a sequence of multicurves c_i which converges to μ in the (ordinary) Hausdorff topology. We denote by $\mathcal{AL}(S)$ the subset of approximable laminations.

Lemma 3.1. *If λ is a union of its minimal components, then it is approximable.*

Proof. First suppose that λ is minimal. Take a leaf l of λ . For each positive integer n , we choose an arc a_n on l with length greater than n whose endpoints can be joined by a geodesic arc b_n transverse to l of length less than $1/n$ and such that the endpoints of the arc a_n arrive on different sides of b_n . Since l is dense in λ , the closed geodesic c_n homotopic to $a_n \cup b_n$ converges to λ in the Hausdorff topology.

In the general case, we can take a sequence of closed geodesics $\{c_i^j\}$ as above for each minimal component λ_j so that the $c_i^j \cap c_i^{j'}$ if $j \neq j'$. Then the union $\cup_j c_i^j$ converges to λ as $i \rightarrow \infty$ in the Hausdorff topology. \square

Since inclusion is preserved by f by Lemma 2.3, any minimal lamination is mapped to a minimal lamination by f .

Lemma 3.2. *If μ is an approximable lamination, then there is a sequence of multicurves $\{c_i\}$ with the following properties.*

- (i) $d_{\bar{H}}(c_i, \mu) \rightarrow 0$.
- (ii) Any λ such that $d_{\bar{H}}(c_i, \lambda) \rightarrow 0$ contains μ .

Proof. Let $\{c_i\}$ be a sequence of multicurves converging to μ in the Hausdorff topology. Then by the definition of the Hausdorff topology and $d_{\bar{H}}$, we have (i) and (ii). \square

Corollary 3.3. *If μ is an approximable lamination, then $f(\mu) = g(\mu)$.*

Proof. Take c_i as in Lemma 3.2. Then $d_{\bar{H}}(f(c_i), f(\mu)) \rightarrow 0$ since f preserves left Hausdorff convergence. By Lemma 2.3, if $d_{\bar{H}}(f(c_i), \lambda) \rightarrow 0$, then λ contains $f(\mu)$. Since $g(c_i) = f(c_i)$, we have $d_{\bar{H}}(f(c_i), g(\mu)) \rightarrow 0$, and hence

$g(\mu)$ contains $f(\mu)$. Since g is a homeomorphism of S , it also preserves left Hausdorff convergence and inclusion. Thus, by exchanging the roles of f and g , $f(\mu)$ contains $g(\mu)$. \square

4. NON-COMPACT ISOLATED LEAVES

Definition 4.1. Let l be a non-compact isolated leaf of a geodesic lamination λ . There are one or two minimal components onto which the two ends of l spiral. We call these components the *limit components* of l and denote them by $L^+(l), L^-(l)$. These two limit components may coincide. We note that limit components are minimal components, and hence are contained in $\mathcal{AL}(S)$.

Lemma 4.1. *Let f be a bijection of $\mathcal{GL}(S)$ preserving left Hausdorff convergence. Let λ be a geodesic lamination and suppose that λ has a non-compact isolated leaf ℓ . Let $L^+(\ell), L^-(\ell)$ (possibly the same) be the limit components of ℓ . Then $f(\lambda)$ contains $f(L^+(\ell)), f(L^-(\ell))$ as minimal components and an isolated leaf having $f(L^+(\ell)), f(L^-(\ell))$ as its limit components.*

Proof. Suppose that ℓ has two limit components $L^+(\ell), L^-(\ell)$, and consider the geodesic lamination $L^+(\ell) \cup L^-(\ell) \cup \ell$, which we denote by L . Then L is a sublamination of λ . By Lemma 2.3, $f(\lambda)$ contains $f(L)$. On the other hand, since f is induced by a homeomorphism g of S on $\mathcal{AL}(S)$, $f(L^+(\ell) \cup L^-(\ell)) = g(L^+(\ell)) \cup g(L^-(\ell))$ is the union of two minimal components, which we denote by L_f^+, L_f^- . Since $L^+(\ell) \cup L^-(\ell)$ is the union of all minimal components of L and since this property is preserved by f , L_f^+, L_f^- are the minimal components of $f(L)$.

By our definition, L contains $L^+(\ell) \cup L^-(\ell)$ properly and it is minimal among all laminations containing $L^+(\ell) \cup L^-(\ell)$. This property must be preserved by f . Therefore $f(L) \setminus (L_f^+ \cup L_f^-)$ contains only one leaf, and it is a non-compact isolated leaf, which we denote by ℓ_f . If ℓ_f has only one of L_f^+, L_f^- , say L_f^+ , as its limit component, then we have proper inclusions $f(L_f^+) \subsetneq f(L_f^+) \cup \ell_f \subsetneq f(L)$. By applying f^{-1} to these inclusions, we have a geodesic lamination L' such that $L_f^+ \subsetneq L' \subsetneq L$. This implies that $L' = L^+(\ell) \cup L^-(\ell)$ by our definition of ℓ . This is a contradiction since we would have then $L_f^+ \cup L_f^- = f(L') = f(L_f^+) \cup \ell_f$.

Thus, we have shown that ℓ_f has both L_f^+ and L_f^- as limit components. Since ℓ_f is contained in $f(L) \subset f(\lambda)$, we are done in this case. The same kind of argument works also in the case when ℓ has only one limit component. \square

5. FINITE LAMINATIONS

Definition 5.1. A geodesic lamination is called *finite* when all its minimal components are simple closed geodesics.

We next refine Lemma 4.1 in the case when λ is a finite lamination to show that $f(\lambda)$ contains a leaf 'homotopic' to $g(\ell)$ (or $\iota \circ g(\ell)$ when $\text{genus}(S) = 2$) as unique non-compact isolated leaf.

Definition 5.2. Let ℓ be a non-compact isolated leaf of a finite lamination $\lambda \in \mathcal{GL}(S)$ with limit components L^+, L^- , which may coincide. Then the homotopy class of ℓ is defined to be the homotopy class of $\ell \setminus (A(L^+) \cup A(L^-))$ on $S \setminus (A(L^+) \cup A(L^-))$ where A denotes a thin annular neighbourhood which is taken to be pairwise disjoint for the minimal components of λ .

Lemma 5.1. *Let ℓ be a non-compact isolated leaf of a finite lamination $\lambda \in \mathcal{GL}(S)$. Then there is a leaf of $f(\lambda)$ which has the same limit components and the same homotopy class as $g(\ell)$ when $\text{genus}(S) \geq 3$. When $\text{genus}(S) = 2$ the leaf is homotopic to either $g(\ell)$ or $\iota \circ g(\ell)$, where ι is a hyperelliptic involution.*

Proof. Construct a pants decomposition by disjoint simple closed geodesics in $S \setminus (L^+ \cup L^- \cup \ell)$, so that $L^+ \cup L^-$ is contained in only one pair of pants if $\text{genus}(S) \geq 3$, and denote it by C . We have $f(C \cup L^+ \cup L^-) = g(C \cup L^+ \cup L^-)$. Since $C \cup L^+ \cup L^- \cup \ell$ is a geodesic lamination, $f(C \cup L^+ \cup L^- \cup \ell)$ is also a geodesic lamination, which implies that $f(\lambda)$ contains a non-compact isolated leaf l' disjoint from $g(C \cup L^+ \cup L^-)$ with limit components $g(L^+), g(L^-)$. In the case when $\text{genus}(S) \geq 3$, since there is only one pair of pants in $S \setminus g(C \cup L^+ \cup L^-)$ whose frontier contain $g(L^+) \cup g(L^-)$, this pair of pants must contain l' , and hence is homotopic to $g(\ell)$.

In the case when $\text{genus}(S) = 2$, it is possible that l' is contained in the pair of pants lying on the opposite side of $g(L^+ \cup L^- \cup c)$ from the one containing $g(\ell)$. In this case $\iota(l')$ is homotopic to $g(\ell)$. \square

Next, we shall take into account the direction to which a non-compact isolated leaf spirals around simple closed geodesics.

Definition 5.3. We call a non-compact isolated leaf ℓ of a finite lamination *unapproachable* when it has only one limit component and if it spirals around this component from its two sides in the same direction. Otherwise, ℓ is called *approachable*.

Lemma 5.2. *Let ℓ be an approachable non-compact isolated leaf of a finite geodesic lamination λ . Then there is an approximable finite lamination λ' containing ℓ but no other non-compact isolated leaf homotopic to ℓ .*

Proof. Let L^+, L^- be the limit components of ℓ , which may coincide. We can regard ℓ as obtained from an arc a with endpoints lying on $L^+ \cup L^-$ by spiralling it around L^+ and L^- . We extend a to a simple closed curve c so that the endpoints of a constitute essential intersection of c with $L^+ \cup L^-$, without adding an arc parallel to a . By performing Dehn twists around L^+ and L^- on c infinitely many times and taking the Hausdorff limit to the same direction as the spiralling of ℓ , we get an approximable lamination as

we wanted. (Since ℓ is approachable, we can realise ℓ by an infinite iteration of Dehn twists.) \square

In the case when $\text{genus}(S) = 2$ we shall need another lemma.

Lemma 5.3. *Suppose that $\text{genus}(S) = 2$, and let ℓ and ℓ' be two approachable non-compact isolated leaves of a finite lamination which has the following properties.*

- (a) ℓ has two distinct limit components L^+ and L^- .
- (b) ℓ' has either one or two limit components. If ℓ' has only one limit component, then its ends spiral around the limit component from the same side.
- (c) One of the limit components L^+ of ℓ is also a limit component of ℓ' whereas L^- is not.
- (d) The leaves ℓ and ℓ' spiral around L^+ from the same side.

Then, there is a approximable geodesic lamination λ' containing $\mu_\ell \cup \mu_{\ell'}$ and having a leaf which intersects both $\iota(\ell)$ and $\iota(\ell')$ transversely, where ι denotes a hyperelliptic involution.

Proof. If ℓ' has two limit components, let L' be the limit component other than L^+ . If ℓ' has only one limit component, choose a closed geodesic disjoint from $L^+ \cup \ell \cup L^- \cup \ell'$, and let it be L' . (By the property (b), such a closed geodesic exists.) Then $L^+ \cup L^- \cup L'$ decompose S into two pairs of pants, P and P' . By the properties (b) and (d), ℓ and ℓ' are contained in the same pair of pants, say P . Now we can extend $\mu_\ell \cup \mu_{\ell'}$ to a geodesic lamination as we wanted by adding a leaf in P' which intersects $\iota(\ell), \iota(\ell')$ transversely choosing the spiralling directions appropriately. \square

Proposition 5.4. *Let ℓ be an approachable non-compact isolated leaf whose limit components L^+, L^- are simple closed geodesics, which may coincide. Let μ_ℓ be the geodesic lamination $L^+ \cup \ell \cup L^-$. Then $f(\mu_\ell) = g(\mu_\ell)$ when $\text{genus}(S) \geq 3$. In the case when $\text{genus}(S) = 2$, we have either $f(\mu_\ell) = g(\mu_\ell)$ or $f(\mu_\ell) = \iota \circ g(\mu_\ell)$, and the alternative does not depend on ℓ .*

Proof. By Lemma 5.2, there is an approximable finite lamination λ containing μ_ℓ which does not have a leaf other than ℓ homotopic to ℓ . By Corollary 3.3, we have $f(\lambda) = g(\lambda)$. On the other hand, if $\text{genus}(S) \geq 3$, by Lemma 5.1 shows that $f(\mu_\ell)$ consists of $g(L^+) \cup g(L^-)$ and a non-compact isolated leaf homotopic to $g(\ell)$. Since $f(\mu_\ell)$ is contained in $f(\lambda) = g(\lambda)$, the only isolated leaf of $f(\mu_\ell)$ must coincide with $g(\ell)$. Thus we have completed the proof in the case when $\text{genus}(S) \geq 3$.

Suppose that $\text{genus}(S) = 2$. Then the same argument as in the case of $\text{genus}(S) \geq 3$ implies that $f(\mu_\ell)$ is either $g(\mu_\ell)$ or $\iota \circ g(\mu_\ell)$. We need to show that one of the alternatives holds for all μ_ℓ . First consider two isolated leaves ℓ and ℓ' as in the statement of Lemma 5.3, and consider λ' there. Since λ' has a leaf ℓ'' intersecting $\iota(\ell), \iota(\ell')$ transversely, if $f(\mu_\ell) = g(\mu_\ell)$, we cannot have $f(\mu_{\ell''}) = \iota \circ g(\mu_{\ell''})$, for both $f(\mu_\ell)$ and $f(\mu_{\ell'})$ are contained

in $f(\lambda')$, and hence we have $f(\mu_{\ell''}) = g(\mu_{\ell''})$, and by the same argument $f(\mu_{\ell'}) = g(\mu_{\ell'})$ holds. In the same way, we see that if $f(\mu_{\ell}) = \iota \circ g(\mu_{\ell})$, then we have $f(\mu_{\ell'}) = \iota \circ g(\mu_{\ell'})$. Thus one of the alternatives holds for both μ_{ℓ} and $\mu_{\ell'}$.

If ℓ has only one limit component and spirals around it from its both sides, we have $\iota(\mu_{\ell}) = \mu_{\ell}$. Therefore both alternatives hold for such a case, and this can be excluded from the argument. Since two pants decompositions of S can be joined by iterating elementary moves, and two ideal triangulations of a pair of pants has at least one common edge, for any given two non-compact isolated leaves ℓ and $\bar{\ell}$ of different laminations having the property (b) of Lemma 5.3, there is a sequence of non-compact isolated leaves $\ell = \ell_1, \ell_2, \dots, \ell_k = \bar{\ell}$ such that ℓ_j and ℓ_{j+1} satisfy the hypotheses of Lemma 5.3. Therefore, by repeating the argument in the previous paragraph, we see that if $f(\mu_{\ell}) = g(\mu_{\ell})$ then $f(\mu_{\bar{\ell}}) = g(\mu_{\bar{\ell}})$. Thus we have completed the proof. \square

Corollary 5.5. *For any finite geodesic lamination λ that does not contain any unapproachable non-compact isolated leaf, we have $f(\lambda) = g(\lambda)$ when $\text{genus}(S) \geq 3$. If $\text{genus}(S) = 2$, $f(\lambda) = g(\lambda)$ for any such λ or $f(\lambda) = \iota \circ g(\lambda)$ for any such λ .*

Proof. By Lemma 3.1 and Corollary 3.3, f and g coincide on the minimal components of λ , and by Lemma 2.3, the number of the non-compact isolated leaves of $f(\lambda)$ is the same as that of λ . Let ℓ be a non-compact isolated leaf of λ , which is approachable by assumption. By Proposition 5.4, we have $f(\mu_{\ell}) = g(\mu_{\ell})$ (or $f(\mu_{\ell}) = \iota \circ g(\mu_{\ell})$ when $\text{genus}(S) = 2$), and since $f(\lambda)$ contains $f(\mu_{\ell})$, it must have $g(\ell)$ (or $\iota \circ g(\ell)$ when $\text{genus}(S) = 2$) as a non-compact isolated leaf. Since this holds for every non-compact isolated leaf, $f(\lambda)$ contains all non-compact isolated leaves of $g(\lambda)$ (or $\iota \circ g(\lambda)$ when $\text{genus}(S) = 2$). Since $f(\lambda)$ and $g(\lambda)$ have the same number of such leaves, which is equal to the number of non-compact isolated leaves of λ , we have $f(\lambda) = g(\lambda)$ (or $f(\lambda) = \iota \circ g(\lambda)$ when $\text{genus}(S) = 2$).

In the case when $\text{genus}(S) = 2$, by Proposition 5.4 either $f(\mu_{\ell}) = g(\mu_{\ell})$ for all λ and ℓ or $f(\mu_{\ell}) = \iota \circ g(\mu_{\ell})$ for all λ and ℓ . This shows the second sentence of our corollary. \square

Now we turn to unapproachable non-compact isolated leaves.

Lemma 5.6. *Let ℓ be an unapproachable non-compact isolated leaf of a finite geodesic lamination, and L its (unique) limit component. Then for $\mu_{\ell} = L \cup \ell$, we have $f(\mu_{\ell}) = g(\mu_{\ell})$ if $\text{genus}(S) \geq 3$. In the case when $\text{genus}(S) = 2$, we have either $f(\mu_{\ell}) = g(\mu_{\ell})$ or $f(\mu_{\ell}) = \iota \circ g(\mu_{\ell})$, and the alternative does not depend on ℓ , nor whether ℓ is unapproachable or approachable.*

Proof. Take a simple closed geodesic d in $S \setminus \mu_{\ell}$, and two approachable non-compact isolated leaves ℓ_1 and ℓ_2 as follows.

- 1 ℓ_1 and ℓ_2 are disjoint, and are contained in $S \setminus (\mu_{\ell} \cup d)$.
- 2 For $j = 1, 2$, the ends of ℓ_j spiral around d and L .

3 ℓ_1 and ℓ_2 spiral around L from opposite sides of L .

Set ν_ℓ to be $\mu_\ell \cup d \cup \ell_1 \cup \ell_2$, and ν'_ℓ to be $d \cup L \cup \ell_1 \cup \ell_2$.

Suppose that $\text{genus}(S) \geq 3$ for the moment. By Corollary 5.5, we have $f(\nu'_\ell) = g(\nu'_\ell)$. By Lemma 5.1, $f(\nu_\ell)$ has a non-compact isolated leaf ℓ' homotopic to $g(\ell)$. Since ℓ' is disjoint from $f(\nu'_\ell)$, which must be contained in $f(\nu_\ell)$, the direction of spiralling is the same as $g(\ell)$ at both ends, and hence $\ell' = g(\ell)$. Thus we have $f(\nu_\ell) = g(\nu_\ell)$.

Next suppose that $\text{genus}(S) = 2$. By the same argument as in the case of $\text{genus}(S) \geq 3$, if $f(\nu'_\ell) = g(\nu'_\ell)$, we have $f(\nu_\ell) = g(\nu_\ell)$. Otherwise, we have $f(\nu_\ell) = \iota \circ g(\nu_\ell)$. Since one of the alternatives holds for all ν'_ℓ by Corollary 5.5, we see that the alternative does not depend on ℓ . \square

Now we can prove the following.

Proposition 5.7. *We have $f(\lambda) = g(\lambda)$ for all finite geodesic laminations if $\text{genus}(S) \geq 3$. We have $f(\lambda) = g(\lambda)$ for all finite geodesic laminations or $f(\lambda) = \iota \circ g(\lambda)$ for all geodesic laminations if $\text{genus}(S) = 2$.*

Proof. We first assume that $\text{genus}(S) \geq 3$. Let λ' be the union of the minimal components and the approachable non-compact isolated leaves of λ . By Corollary 5.5, $f(\lambda') = g(\lambda')$, and hence $f(\lambda)$ contains $g(\lambda')$. Now, let ℓ be an unapproximable non-compact isolated leaf of λ . By Lemma 5.6, $f(\lambda)$, which contains $f(\mu_\ell) = g(\mu_\ell)$, must contain $g(\ell)$. Since this holds for every unapproximable non-compact isolated leaf of λ , $f(\lambda)$ contains $g(\lambda)$. Since f preserves the inclusions, the number of the leaves of $f(\lambda)$ is the same as that of λ , hence as that of $g(\lambda)$. Therefore, the only possibility is $f(\lambda) = g(\lambda)$.

Now we turn to the case when $\text{genus}(S) = 2$. In this case, we have $f(\lambda') = g(\lambda')$ or $f(\lambda') = \iota \circ g(\lambda')$. If the first possibility holds, this must hold for all λ' , and also we have $f(\mu_\ell) = g(\mu_\ell)$. Therefore $f(\lambda) = g(\lambda)$ for every finite geodesic lamination λ . Similarly, if $f(\lambda') = \iota \circ g(\lambda')$, then this holds for all λ' , and hence $f(\lambda) = \iota \circ g(\lambda)$ for every finite geodesic lamination λ . \square

6. PROOF OF THE MAIN THEOREM

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We first show that if $f : \mathcal{GL}(S) \rightarrow \mathcal{GL}(S)$ is a bijection preserving left Hausdorff convergence, then there is an extended mapping class h inducing the same bijection on $\mathcal{GL}(S)$.

Let λ be a geodesic lamination. Since finite laminations are dense in $\mathcal{GL}(S)$ with the Hausdorff topology, there is a sequence of finite laminations $\{\mu_i\}$ converging to λ . By Proposition 5.7, we have $f(\mu_i) = h(\mu_i)$ for a homeomorphism $h : S \rightarrow S$. (This is either g or $\iota \circ g$ in Proposition 5.7.) Since h is a homeomorphism, $h(\lambda)$ coincides with the Hausdorff limit μ_∞ of $h(\mu_i) = f(\mu_i)$. Since f preserves left Hausdorff convergence, $f(\lambda)$ contains

the Hausdorff limit μ_∞ . As was seen before, f preserves the number of minimal components and the number of non-compact isolated leaves. Thus, the only possibility is $f(\lambda) = \mu_\infty$, which is equal to $h(\lambda)$.

Thus, the natural homomorphism $\text{Mod}^*(S) \rightarrow \text{Aut}(\mathcal{GL}(S))$ is surjective.

For $\text{genus}(S) \geq 3$, this homomorphism is injective since if two extended mapping classes induce the same bijection on $\mathcal{GL}(S)$, they induce the same action on the curve complex $\mathcal{C}(S)$, and we know by Ivanov's result [2] that the natural homomorphism $\text{Mod}^*(S) \rightarrow \mathcal{C}(S)$ is injective.

It remains to consider the case $\text{genus}(S) = 2$. We know that in this case, if a homeomorphism h of S induces the identity map on the curve complex $\mathcal{C}(S)$, then h is either homotopic to the identity or to the hyperelliptic involution ι of S . But the hyperelliptic involution does not induce the identity map on $\mathcal{GL}(S)$. To see this, take a geodesic pair of pants decomposition of S which is invariant by ι up to homotopy, and complete it to a geodesic lamination by adding leaves which spiral along the three pants curves in a way that is not invariant by the hyperelliptic involution ι . Thus, ι does not induce the identity map on $\mathcal{GL}(S)$. This completes the proof. \square

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