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**Abstract**

In this paper we investigate the left invertibility problem for a class of nonlinear time-delay systems. In both cases of time delay systems with and without internal dynamics the invertibility conditions are given. A new approach based on the use of higher order sliding mode observer is developed for finite-time left invertibility and for asymptotic left inversion. Causal and non-causal estimation of the unknown inputs are respectively discussed. The results are illustrated by numerical examples in order to show the efficiency of the method and its limits.

*Key words:* Left inversion, nonlinear system, nonlinear time-delay system, asymptotic left inversion, internal dynamics.

## 1 Introduction

Time delay systems represent one of the most studied class of systems in control theory. Since the 60', many different problems are studied such as stability and stabilization of time delay systems, observation and observer design, parameter identification, etc. In the present work we are interested in the left invertibility problem of time delay systems with internal dynamics. The problem of unknown inputs recovering from the outputs is crucial. Such a problem has attracted the interest of the control community since it has direct applications in many domains, such as data secure transmission where the unknown input is the message, and fault detection and isolation where the fault is the unknown input. In fact, left invertibility problem has been studied since at least forty five years ago in linear control theory [33], [34] and thirty five year ago in nonlinear control theory [15], [35]. Most of those works are in the context of nonlinear systems without time delays, or linear systems with commensurate delays [2], [41].

In the literature, an important tool based on non-commutative ring, proposed in [36], is used to analyze nonlinear time delay systems in algebraic framework. Using this framework, many notions are extended to the case of nonlinear time delay systems and many results have already been obtained and published [40], [37]. In the context of constant time delay, the notions of Lie derivatives and relative degree are defined, and the differences between causal and non-causal invertibility are clarified in [38]. The canonical form of invertibility is also given in [39], and in [42] a method for estimating the unknown inputs is proposed. However, the algorithm for left invertibility proposed in [39] was only for system without internal dynamics (see also [4] and [13] and their references). For system with or without time-delay, the main difficulty when the internal dynamics appears is to estimate the state of such dynamics. One interesting solution in order to overcome such a difficulty is to allow the derivative of the unknown inputs [31] with a geometrical approach and with an algebraic one. If, however, the input derivatives are not possible, then it is necessary to compute and analyze the internal dynamics.

For nonlinear systems without delay, if the vector fields associated to the inputs verify involutivity property, then the internal dynamics does not depend on the unknown input. However, this rule is not valid for nonlinear systems with time delay. In order to analyze the internal dynamics and estimate its state, one solution is to rewrite this internal dynamics such that its dynamics become independent of the unknown input. Thus in the paper we will adapt a new way to determine the internal dynamics for the nonlinear systems with time-delay [1]. This method is based on the finite-time convergence by using the existing observer for time-delay

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system in the literature [3], [12], [14], [16], [17], [20] and the high order sliding mode proposed in [22], [23] (see also [10] for unknown input observer) is applied in this paper.

As a continuity of our preliminary work in [19], this paper provides a method for finite-time and asymptotic left invertibility for nonlinear time delay systems with and without internal dynamics. This method is based on a finite-time estimation of the state variables using a higher order sliding mode observer and on an asymptotic estimation of the internal dynamics. The causality of the unknown input estimation is given. Finally, in order to illustrate the result, numerical examples are provided.

## 2 Algebraic framework and notations

Consider the following class of multi-input multi-output nonlinear time-delay systems:

$$\begin{cases} \dot{x} = f(x(t-j\tau)) + \sum_{i=1}^{m} g_i(x(t-j\tau))u_i(t), \\ y = h(x(t-j\tau)), \\ x = \psi(t), u(t) = \varphi(t), \\ t \in \left[-s\tau, 0\right], \end{cases} \tag{1}$$

where $x \in W \subset \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ is the vector of the unknown inputs and $y \in \mathbb{R}^p$ is the output, with $p \geq m$. $\tau$ represents the basic commensurate time delay, and it is assumed in this paper that $f$, $g_i$ and $h$ are all meromorphic functions, where $f(x(t-j\tau))$, $g_i(x(t-j\tau))$, and $h(x(t-j\tau))$ denote functions depending on $x(t)$, $x(t-\tau)$, $x(t-2\tau)$ until $x(t-s\tau)$.

This class of nonlinear time-delay systems is widely studied in the literature, and a lot of tools are used to study their properties (see for instance [4], [13], [32] and references herein), among which we are interested in the use of algebraic tools to develop the invertibility conditions of system (1) when its internal dynamics are not vanished. Since the delay is assumed to be commensurate, this paper utilizes the algebraic framework, developed in [8], [25], [27], to study the structure of nonlinear time-delay systems. Under this framework, defining $\mathscr{K}$ as a field of meromorphic functions of finite number of variables of the form $\{x_i(t-j\tau), i \in [1,n], j \in [0,s]\}$ such that $\mathscr{E}$ represents the vector space over $\mathscr{K}: \mathscr{E} = span_{\mathscr{K}}\{d\varsigma : \varsigma \in \mathscr{K}\}$ and $\tau$ represents the basic commensurate time delay. After, in [36] this algebraic framework is generalized and the algebraic properties of the field $\mathscr{K}$ are studied.

Denote the operator $\delta$ as a backward shift operator, which means $\delta^j\varsigma(t) = \varsigma(t-j\tau)$ and $\delta^j(a(t)d\varsigma(t)) = a(t-j\tau)d\varsigma(t-j\tau)$. With this operator, we can then define the following set of polynomials $\mathscr{K}(\delta]$: $a(\delta] = \sum_{j=0}^{r_a} a_j\delta^j, a_j \in \mathscr{K}$. The addition for the entries in $\mathscr{K}(\delta]$ is defined as usual, but its multiplication is given by the following criteria:

$$a(\delta]b(\delta] = \sum_{k=0}^{r_a+r_b} \sum_{i+j=k}^{i\leq r_a, j\leq r_b} a_j(t)b_i(t-j\tau)\delta^k. \tag{2}$$

With the standard differential operator $d$, denote by $\mathscr{M}$ the left module over $\mathscr{K}(\delta]$:

$$\mathscr{M} = span_{\mathscr{K}(\delta]}\{d\varsigma, \varsigma \in \mathscr{K}\}. \tag{3}$$

Hence, $\mathscr{K}(\delta]$ is a non-commutative ring satisfying the associative law, and it has been proved in [18] and [36] that it is a left Ore ring, which enables us to define the rank conception.

In order to take into account the non-causal case, following the same idea, we can introduce as well the forward shift operator $\nabla$ such that $\nabla^j x(t) = x(t+j\tau)$ (see also [9]), which means as well $\nabla x(t) = \delta^{-1}x(t)$. In the following, for the sake of simplicity, a function $f(x,\delta,\nabla)$ simply means it is a function of $x$, and the backward and forward value of $x$.

Thanks to the above algebraic framework the nonlinear time-delay system (1) can be represented in a compact algebraic form as follows:

$$\begin{cases} \dot{x} = f(x,\delta) + \sum_{i=1}^{m} G_i(x,\delta)u_i(t), \\ y = h(x,\delta), \\ x = \psi(t), u(t) = \varphi(t), \\ t \in \left[-s\tau, 0\right], \end{cases} \tag{4}$$

where the notation $f(x,\delta)$ means $f(x,\delta) = f(x, x(t-\tau), \ldots, x(t-s\tau)))$ and the same is considered for $G(x,\delta) = (G_1(x,\delta), \ldots, G_m(x,\delta))$ and $h(x(t),\delta)$, $G_i = \sum_{j=0}^{s} g_i^j \delta^j$ with entries belonging to $\mathscr{K}(\delta]$.

Let us now give some definitions, which will be used in the sequel to develop our main results.

**Definition 1 (Left invertibility)** *System* (4) *is said to be asymptotically left invertible (or equivalently, u can be asymptotically estimated) if*

$$\|u(t) - \hat{u}(t)\| = 0 \quad when \quad t \to \infty \tag{5}$$

*with* $\hat{u}(t) = \varphi(\hat{\xi}, \delta, \nabla) \in \mathbb{R}^m$ *and* $\dot{\hat{\xi}} = \bar{f}(\hat{\xi}, y, \delta) \in \mathbb{R}^{\bar{n}}$ *with some user chosen functions* $\varphi$ *and* $\bar{f}$ *with proper dimension* $\bar{n}$. *It is finite-time left invertible (or equivalently, u can be finite-time estimated) if there exist* $T > 0$ *such that*

$$\|u(t) - \hat{u}(t)\| = 0, \quad \forall t > T. \tag{6}$$

**Definition 2 (Causality)** *System* (4) *is said to be causally asymptotically (or finite-time) left invertible if it is asymptotically (or finite-time) left invertible with $\hat{u}(t) = \varphi(\hat{\xi}, \delta)$. Otherwise, it is said to be non-causally asymptotically (or non-causally finite-time) left invertible .*

Let $f(x(t-j\tau))$ and $h(x(t-j\tau))$ for $0 \le j \le s$ be an $n$ and $p$ dimensional vector field respectively, with $f_r$ the entries of $f$ belonging to $\mathscr{K}$ for $1 \le r \le n$ and $h_i \in \mathscr{K}$ for $1 \le i \le p$, with $p \ge m$, and then $\frac{\partial h_i}{\partial x} := \left[\frac{\partial h_i}{\partial x_1}, \cdots, \frac{\partial h_i}{\partial x_n}\right] \in \mathscr{K}^{1\times n}(\delta]$ where for $1 \le r \le n$, $\frac{\partial h_i}{\partial x_r} := \sum_{j=0}^{s} \frac{\partial h_i}{\partial x_r(t-j\tau)}\delta^j \in \mathscr{K}(\delta]$, we can define the Lie derivative in the algebraic framework as follows: $L_f h_i := \sum_{r=1}^{n}\sum_{j=0}^{s} \frac{\partial h_i}{\partial x_r(t-j\tau)}\delta^j f_r$ and $L_G h_i := \sum_{r=1}^{m}\sum_{j=0}^{s} \frac{\partial h_i}{\partial x_r(t-j\tau)}\delta^j G_r \delta^j$.

The conventional concepts like Lie derivative and relative degree have been generalized to nonlinear time-delay systems by [4], [11], [28], [29] and the following adapted definitions using the above algebraic framework are recalled (for more details see [25]).

**Definition 3 (Relative degree)** *System* (4) *has the relative degree $(\nu_1, \cdots, \nu_p)$ in an open set $W \subseteq \mathbb{R}^n$ if the following conditions are satisfied for $1 \le i \le p$:*

*(1) for all $x \in W$, $L_{G_j} L_f^r h_i = 0$ for all $1 \le j \le m$ and $0 \le r < \nu_i - 1$;*
*(2) there exists $x \in W$ such that $\exists j \in \{1, \cdots, m\}$, $L_{G_j} L_f^{\nu_i - 1} h_i \neq 0$.*

*If the first condition is satisfied for all $r \ge 0$ and some $i \in \{1, \cdots, p\}$, we set $\nu_i = \infty$.*

**Definition 4 (Observability indices)** *System* (4) *has the observability indices $(k_1, \cdots, k_p)$, where*

*$k_i = card\{d_k \ge i, 1 \le k \le n\}$ for $1 \le k \le p$ with $d_1 = rank_{\mathscr{K}(\delta]}\mathscr{F}_1$, and*

*$d_k = rank_{\mathscr{K}(\delta]}\mathscr{F}_k - rank_{\mathscr{K}(\delta]}\mathscr{F}_{k-1}$ for $2 \le k \le n$,*

*where $\mathscr{F}_k := span_{\mathscr{K}(\delta]}\left\{dh, dL_f h, \cdots, dL_f^{k-1} h\right\}$.*

Since we use the non-commutative algebra, for any matrix with entries belonging to $\mathscr{K}(\delta]$ the following definition that has been introduced in [25] will be used in the sequel.

**Definition 5 (Unimodular matrix)** *The matrix $A \in \mathscr{K}^{n\times p}(\delta]$ is said to be left (or right) unimodular over $\mathscr{K}(\delta]$ if there exists $A_L^{-1} \in \mathscr{K}^{p\times n}(\delta]$ with $p \le n$ (or $A_R^{-1} \in \mathscr{K}^{p\times n}(\delta]$ with $n \le p$), such that $A_L^{-1}A = \mathrm{I}_p$ ( or $AA_R^{-1} = \mathrm{I}_n$). It is said to be unimodular over $\mathscr{K}(\delta]$ if $A_L^{-1} = A_R^{-1}$.*

With the above definitions, in the next section, we start by providing left invertibility conditions for the simplest case of time-delay systems without internal dynamics.

## 3 Left inversion without internal dynamics

In this section, a canonical form is given for the nonlinear time-delay system. Sufficient conditions for left invertibility are deduced based on the obtained canonical form. Before this, let us recall the following result stated in [39].

**Theorem 1** *[39] For $1 \le i \le p$, denote by $k_i$ the observability indices and $\nu_i$ the relative degree index for $y_i$ of system* (4)*, and note $\rho_i = \min(k_i, \nu_i)$. Then there exists a mapping $\begin{bmatrix}\xi^T & \zeta^T\end{bmatrix}^T = \phi(x, \delta) \in \mathscr{K}^{n\times 1}$, such that system* (4) *can be transformed into the following form:*

$$\begin{cases} \dot{\xi}_i = A_i \xi_i + B_i \mu_i, \\ \dot{\zeta} = \bar{\alpha}(x,\delta) + \bar{\beta}(x,\delta) u, \\ y_i = C_i \xi_i, \end{cases} \tag{7}$$

*where $A_i$, $B_i$ and $C_i$ are of the Brunovsky form, $\xi_i = \begin{bmatrix}\xi_{i,1}, & \xi_{i,2}, & \cdots, & \xi_{i,\rho_i}\end{bmatrix}^T$ with $\xi_{i,j} = L_f^{j-1} h_i$, and $\mu_i = L_f^{\rho_i} h_i(x,\delta) + \sum_{j=1}^{m} L_{G_j} L_f^{\rho_i - 1} h_i(x,\delta) u_j \in \mathscr{K}$, $\bar{\alpha} \in \mathscr{K}^{l\times 1}$, $\bar{\beta} \in \mathscr{K}^{l\times 1}$ where $l = n - \sum_{i=1}^{p} \rho_i$. Moreover, if $k_i < \nu_i$, one has $\mu_i = L_f^{\rho_i} h_i = L_f^{k_i} h_i$.*

**Remark 1** *In Theorem 1, $\phi(x,\delta)$ is a mapping. In order to provide sufficient conditions for the invertibility of nonlinear time-delay systems in the rest of our paper we consider the following assumptions :*

*(1) $\phi(x,\delta)$ is a change of coordinates.*
*(2) $z = \phi(x,\delta)$ guarantees the equivalence between system* (4) *and the observer canonical form*

$$\begin{cases} \dot{\xi}_i = A_i \xi_i + B_i \mu_i, \\ \dot{\zeta} = \alpha(\xi,\zeta,\delta) + \beta(\xi,\zeta,\delta) u, \\ y_i = C_i \xi_i, \end{cases} \tag{8}$$

*where $\alpha \in \mathscr{K}^{l\times 1}$, and $\beta \in \mathscr{K}^{l\times 1}$.*

One may remark that Assumption 2 in Remark 1 can be justified by the recent result proposed in [5]. In that paper, it has been shown that even thought the change of coordinates is bicausal additional conditions are required in order to guarantee the equivalence between a nonlinear time-delay system and an observer canonical form. Necessary and sufficient conditions have been proposed in that paper for the case of single-input single-output nonlinear time-delay systems. Providing such conditions for the case of nonlinear multi-input multi-output time-delay systems may be considered in our future works.

Now, consider the dynamics $\dot{\zeta}$ and the last equation in $\dot{\xi}_i$ in (8) which can be rewritten as:

$$\begin{cases} \mathscr{H}(y_i^{(\rho_i)}) = \Psi(x,\delta) + \Gamma(x,\delta)u, \\ \dot{\zeta} = \alpha(\xi,\zeta,\delta) + \beta(\xi,\zeta,\delta)u, \end{cases} \tag{9}$$

where

$$\mathscr{H}(y_i^{(\rho_i)}) = \left[y_1^{(\rho_1)}, \cdots, y_p^{(\rho_p)}\right]^T$$
$$\Psi(x,\delta) = \left[L_f^{\rho_1} h_1, \cdots, L_f^{\rho_p} h_p\right]^T$$

and

$$\Gamma(x,\delta) = \begin{bmatrix} L_{G_1} L_f^{\rho_1-1} h_1 & \cdots & L_{G_m} L_f^{\rho_1-1} h_1 \\ \vdots & \ddots & \vdots \\ L_{G_1} L_f^{\rho_p-1} h_p & \cdots & L_{G_m} L_f^{\rho_p-1} h_p \end{bmatrix} \tag{10}$$

and let us define the set

$$\Phi := \{dh_1, \cdots, dL_f^{\rho_1-1} h_1, \cdots, dh_p, \cdots, dL_f^{\rho_p-1} h_p\}. \tag{11}$$

Noticing that system (8) contains an internal dynamics $\dot{\zeta}$, the following theorem deals with the case where the internal dynamics is vanished that can be considered as a trivial case.

**Theorem 2** *Suppose that the change of coordinates $\phi(x,\delta)$ in Theorem 1 is bicausal, then system* (4) *(or system* (8)*) is causally finite-time left invertible if $rank_{\mathscr{K}(\delta]}\Phi = n$ and $\Gamma(x,\delta)$ is left unimodular over $\mathscr{K}(\delta]$, where $\Gamma(x,\delta)$ and $\Phi$ are defined in* (10) *and* (11)*, respectively.*

**Proof.** Since the change of coordinates $\phi(x,\delta)$ is bicausal, it implies that there exists $\phi^{-1} \in \mathscr{K}^{n\times 1}$ such that $x = \phi^{-1}(\xi,\zeta,\delta)$. Then, if $rank_{\mathscr{K}(\delta]}\Phi = n$, we have $\dim\zeta = 0$, i.e. no internal dynamics appears in (8). It implies as well that $x = \phi^{-1}(\xi,\delta)$, depending only on $\xi$ and delays. Since $\xi_{i,j} = L_f^{j-1} h_i = y_i^{(j)}$ which is function of outputs and its derivatives, thus $x$ is causally observable. In this case, (9) can be rewritten as:

$$\Gamma(x,\delta)u = \mathscr{H}(y_i^{(\rho_i)}) - \Psi(x,\delta)$$

where both $\mathscr{H}$ and $\Psi$ are known since $x$ is causally observable. If $\Gamma(x,\delta)$ is left unimodular over $\mathscr{K}(\delta]$ which implies that $rank_{\mathscr{K}(\delta]}\Gamma(x,\delta) = m$, then according to Definition 5, there exists $\Gamma_L^{-1}(x,\delta)$ such that

$$u = \Gamma_L^{-1}(x,\delta)\left(\mathscr{H}(y_i^{(\rho_i)}) - \Psi(x,\delta)\right)$$

which implies that system (4) is causally left invertible.

In this case, we can obtain the estimation of $u$ as follows:

$$\hat{u} = \bar{\Gamma}_L^{-1}(\hat{x},\delta)\left(\mathscr{H}(\hat{y}_i^{(\rho_i)}) - \Psi(\hat{x},\delta)\right). \tag{12}$$

where $\hat{x}$ represents the estimation of $x$, which in fact can be estimated by using many existing methods, like sliding mode observer, algebraic observer, etc. $\hat{y}_i^{(\rho_i)}$ is the estimation of $y_i^{(\rho_i)}$ which however might be realized by the finite-time differentiators, such as algebraic one and High-Order Sliding Mode differentiator [1], [10], [22], [23], [24]. Since $\hat{x}$ and $\hat{y}_i^{(\rho_i)}$ can be estimated in finite time $T$ by the higher order sliding mode observer and following Definition 1, we can conclude that the unknown input $u$ is causally finite-time estimated with $\hat{u}$ is given from (12). ■

The following section will consider the non-trivial case, where $rank_{\mathscr{K}(\delta]}\Phi < n$ with $\Phi$ being defined in (11), i.e. the internal dynamics $\dot{\zeta}$ in (8) is not vanished. In this case, the invertibility of the unknown inputs depends on this internal dynamics.

## 4 Left invertibility with internal dynamics

This section will firstly discuss how to iteratively obtain a new canonical form for the non-trivial case and reduce the internal dynamics dimension. Then, both conditions of finite-time and asymptotic left inversion are provided. A method for asymptotic left invertibility is proposed. Using a high order sliding mode observer the state variables and their derivatives are estimated. An asymptotic estimator is proposed to estimate the internal dynamics. Since we deal with the invertibility problem of time-delay systems, causality conditions of the unknown input estimation are given as well.

### *4.1 Iterative canonical form*

In this part we consider system (4) (or equivalently system (8)) when $rank_{\mathscr{K}(\delta]}\Phi < n$ or $rank_{\mathscr{K}(\delta]}\Gamma < m$ where $\Gamma$ and $\Phi$ are defined in (10) and (11), respectively. In this case, the idea is to iteratively apply the algorithm developed in [38] to generate new virtual outputs for the systems, which are combinations of $y$, its derivatives and their backward shifts, with which the dimension of the internal dynamics can be reduced. This operation is iterated until $rank_{\mathscr{K}(\delta]}\Phi = n$ or $rank_{\mathscr{K}(\delta]}\Gamma = m$.

At the beginning of iteration, we initially note $k = 0$, $\mathscr{Y}_k = y$ with $y$ being defined in (4) and $\Phi_k = \Phi$ with $\Phi$ being defined in (11). Without loss of generality, suppose $rank_{\mathscr{K}(\delta]}\Phi_k = n_k < n$. Then, note

$$\pounds_k := span_{\mathbb{R}[\delta]}\left\{h_j, \cdots, L_f^{\rho_j-1} h_j\right\} \text{ for } 1 \le j \le \dim\mathscr{Y}_k \tag{13}$$

where $\mathbb{R}[\delta]$ is the commutative ring of polynomials in $\delta$ with coefficients belonging to the field $\mathbb{R}$, and let $\pounds_k(\delta]$ be the set of polynomials in $\delta$ with coefficients over $\pounds_k$. The module spanned by element of $\Phi_k$ over $\pounds_k(\delta]$ is defined as:

$$\Omega_k = span_{\pounds_k(\delta]}\{\varsigma, \varsigma \in \Phi_k\} \tag{14}$$

Denote

$$\mathscr{G} = span_{\mathbb{R}[\delta]}\{G_1, \dots, G_m\},$$

where $G_i$ is given in (4), and define the *left* annihilator:

$$\mathscr{G}_k^{\perp} := span_{\pounds_k(\delta]}\{\omega \in \mathscr{M} \mid \omega\beta = 0, \forall \beta \in \mathscr{G}\} \tag{15}$$

where $\mathscr{M}$ is defined in (3). Define

$$H_k = span_{\mathbb{R}[\delta]}\left\{\omega \in \mathscr{G}_k^{\perp} \cap \Omega_k | \omega f \notin \pounds_k\right\} \tag{16}$$

finally, according to the algorithm developed in [38], if $rank_{\mathscr{K}} H_k = l_k$, then there

$$\bar{y}_{k,i} = \bar{h}_{k,i} = \omega_i f \mod \pounds_k \text{ for } 1 \le i \le l_k$$

are independent of $\mathscr{Y}_k$.

Combing these new virtual outputs with the existed ones, we obtain

$$\mathscr{Y}_{k+1} = [\mathscr{Y}_k^T, \bar{y}_{k,1}, \cdots, \bar{y}_{k,l_k}]^T \in \mathbb{R}^{p+\sum_{i=0}^k l_i} \tag{17}$$

based on which we can compute $\rho_i$ for $1 \le i \le p + \sum_{i=0}^k l_i$ as the minimum value between the observability indices and the relative degree index for each output in $\mathscr{Y}_{k+1}$.

At the $(k+1)$th iteration, similar to (11), we can calculate $\Phi_{k+1}$. If $rank_{\mathscr{K}(\delta]}\Phi_{k+1} = n_{k+1} = n$, then Theorem 2 can be used to judge whether the unknown input is left invertible or not. If, however, $rank_{\mathscr{K}(\delta]}\Phi_{k+1} = n_{k+1} < n$, then we can follow the same procedure presented in this subsection to compute $\pounds_{k+1}$ as (13), $\Omega_{k+1}$ as (14), $\mathscr{G}_{k+1}^{\perp}$ as (15), $H_{k+1}$ as (16) and $\mathscr{Y}_{k+2}$ as (17).

Iteratively applying the above algorithm to system (4), we can continuously expand the observation space (or equivalently reduce the dimension of unobservable space which yields the internal dynamics), since $\mathscr{Y}_k \subseteq \mathscr{Y}_{k+1}$. The algorithm stops the iteration until $H_{k+1} = \emptyset$, or equivalently $\mathscr{Y}_k \equiv \mathscr{Y}_{k+1}$, i.e. it is not any more possible to reduce the unobservable space.

Suppose that after $\gamma$ iterations, we have $\mathscr{Y}_\gamma \equiv \mathscr{Y}_{\gamma+1}$. Note all outputs (including original and generated virtual ones) as $\bar{y}_i = \bar{h}_i$ for $1 \le i \le p + \sum_{k=0}^{\gamma} l_k$ where $l_k$ is the number of new virtual outputs generated during the $k$th iteration, and $\bar{h}_i$ represents the $i$th entry of $\mathscr{Y}_\gamma$. After $\gamma$ iterations, we obtain

$$\Phi_\gamma = \{d\bar{h}_i, \cdots, dL_f^{\rho_i - 1}\bar{h}_i\} \text{ for } 1 \le i \le p + \sum_{k=0}^{\gamma} l_k \tag{18}$$

Finally, with the deduced the change of coordinates as $\left[z^T,\ \eta^T\right]^T = \phi_\gamma(x,\delta)$, the canonical form can be written as follows:

$$\begin{cases} \dot{z}_{i,j} = z_{i,j+1}, \text{ for } 1 \le i \le p + \sum_{k=0}^{\gamma} l_k \text{ and } 1 \le j \le \rho_i - 1, \\ \dot{z}_{i,\rho_i} = b_i(x,\delta) + \sum_{j=1}^{m} a_{i,j}(x,\delta)u_j, \text{ for } 1 \le i \le p + \sum_{k=0}^{\gamma} l_k, \\ \dot{\eta} = f_\gamma(x,\delta,u), \\ \bar{y}_i = z_{i,1}, \end{cases} \tag{19}$$

with $z = [z_1^T, \cdots, z_{p+\sum_{k=0}^{\gamma} l_k}^T]^T$, $z_i = \left[z_{i,1},\ \cdots,\ z_{i,\rho_i}\right]^T$, $z_{i,j} = \bar{y}_i^{(j-1)} = L_f^{j-1}\bar{h}_i$, $a_{i,j}(x,\delta) = L_{g_j}L_f^{\rho_i-1}\bar{h}_i$ and $b_i(x,\delta) = L_f^{\rho_i}\bar{h}_i$ where $\rho_i$ is the minimum value of the observability index and relative degree for the $i$th entry of $\mathscr{Y}_\gamma$.

Using this reformulation, the problem of left invertibility of nonlinear time-delay systems with internal dynamics is studied in what follows.

### 4.2 Left inversion

Before developing our main results, the following assumption is imposed.

**Assumption 1** *Suppose that the change of coordinates $\left[z^T,\ \eta^T\right]^T = \phi_\gamma(x,\delta)$, which transforms system* (4) *into the canonical form* (19)*, is bicausal, i.e. there exists $x = \phi_\gamma^{-1}(z,\eta,\delta)$.*

**Remark 2** *Generally, the bicausality of change of coordinates is asked when study time-delay system in order to guarantee the equivalence between the original system and the transformed one. Note that $z$ in $\phi_\gamma(z,\eta,\delta)$ for* (19) *is uniquely determined, but $\eta$ can be freely chosen. Some results have been stated in the literature on how to choose $\eta$ such that the change of coordinates is bicausal [26, 6].*

The second and the third equation in (19) can be rewritten in the following compact way:

$$\begin{cases} \mathscr{H}_\gamma(\bar{y}_i^{(\rho_i)}) = \Psi_\gamma(x,\delta) + \Gamma_\gamma(x,\delta)u, \\ \dot{\eta} = f_\gamma(x,u,\delta), \end{cases} \tag{20}$$

The following will focus on the non trivial case where $rank_{\mathscr{K}(\delta]}\Phi_\gamma < n$ and $rank_{\mathscr{K}(\delta]}\Gamma_\gamma(x,\delta) = m$. It is evident that, if Assumption 1 is satisfied, then (20) can be written as

$$\begin{cases} \mathscr{H}_\gamma = \Psi_\gamma(z,\eta,\delta) + \Gamma_\gamma(z,\eta,\delta)u, \\ \dot{\eta} = f_\gamma(z,\eta,u,\delta), \end{cases} \tag{21}$$

Depending on how the internal dynamics involved in equation (21), two cases can be distinguished in the following subsections.

#### 4.2.1 First case: Causal finite-time left invertibility

**Theorem 3** *Suppose that Assumption 1 is satisfied, then system* (4) *(or equivalently system* (19)*) is causally finite-time left invertible if* $\frac{\partial \mathscr{H}_\gamma}{\partial \eta} = 0$ *and* $\Gamma_\gamma$ *is left unimodular over* $\mathscr{K}(\delta]$.

**Proof.** If $\frac{\partial \mathscr{H}_\gamma}{\partial \eta} = 0$, then the first equation of (20) becomes

$$\mathscr{H}_\gamma(\bar{y}_i^{(\rho_i)}) = \Psi_\gamma(z,\delta) + \Gamma_\gamma(z,\delta)u \tag{22}$$

Therefore, both $\Psi_\gamma(z,\delta)$ and $\Gamma_\gamma(z,\delta)$ are completely known since they are independent of the internal dynamics. Since $z$ is causally observable (due to the fact that $\phi_\gamma(x,\delta)$ is bi-causal), so if $\Gamma_\gamma$ is left unimodular over $\mathscr{K}(\delta]$ which implies there exists $\Gamma_{\gamma,L}^{-1}(z,\delta)$ such that

$$u = \Gamma_{\gamma,L}^{-1}(z,\delta)[\mathscr{H}_\gamma(\bar{y}_i^{(\rho_i)}) - \Psi_\gamma(z,\delta)]$$

Since $z$ is function of outputs and its derivative, which can be obtained by using finite-time observer, therefore system (4) is causally finite-time left invertible. ■

#### 4.2.2 Second case: Causal asymptotic left invertibility

In this section, we consider the case where the input in (20) explicitly depends on the internal dynamics $\eta$. In order to estimate the unknown input, it is necessary to estimate the internal dynamics. Suppose that Assumption 1 is satisfied, i.e. $x = \phi_\gamma^{-1}(z,\eta,\delta)$. If $\Gamma_\gamma(x,\delta)$ defined in (20) is left unimodular over $\mathscr{K}(\delta]$, then the unknown input $u$ can be expressed as function of $x$, its derivatives and delays. Replacing $u$ and $x$ by the variables $z$ and $\eta$, finally the second equation of (21) can be written as

$$\dot{\eta} = f_\gamma(\eta,z,u,\delta) = \bar{f}_\gamma(\eta,z,\dot{z},\delta). \tag{23}$$

for which the following assumption is imposed.

**Assumption 2** *It is assumed that there exists a Lyapunov function* $V(e_\eta)$ *with* $e_\eta = \eta - \hat{\eta}$ *such that for all* $\delta$ *and* $z$ *the following inequality:*

$$\dot{V}(e_\eta) = \frac{\partial V}{\partial e_\eta}[\bar{f}_\gamma(\eta,z,\dot{z},\delta) - \bar{f}_\gamma(\hat{\eta},z,\dot{z},\delta)] < 0 \tag{24}$$

*is satisfied.*

Based on the above assumption, we are able to state the following result.

**Theorem 4** *System* (4) *is causally asymptotically left invertible if Assumptions 1 and 2 are satisfied, and* $\Gamma_\gamma(x,\delta)$ *defined in* (20) *is left unimodular over* $\mathscr{K}(\delta]$.

**Proof.** Since system (4) is equivalent to system (19), we need only to prove the above result for system (19). It is clear that $z$ in system (19) is observable, therefore we can use the existing method to estimate $z$ and its derivative in a finite time $T$, i.e. $\hat{z}(t) - z(t) = 0$ for all $t \geq T$.

Consider now the dynamics $\eta$ in system (20). If $\Gamma_\gamma(x,\delta)$ defined in (20) is left unimodular over $\mathscr{K}(\delta]$, then we have

$$u = \Gamma_{\gamma,L}^{-1}(x,\delta)[\mathscr{H}_\gamma(\bar{y}_i^{(\rho_i)}) - \Psi_\gamma(x,\delta)]$$

Since Assumption 1 is assumed to be satisfied, then there exists $x = \phi_\gamma^{-1}(z,\eta,\delta)$ such that the unknown input $u$ can be written as

$$u = \Gamma_{\gamma,L}^{-1}(z,\eta,\delta)[\mathscr{H}_\gamma(\bar{y}_i^{(\rho_i)}) - \Psi_\gamma(z,\eta,\delta)] \tag{25}$$

which however depends as well on the unobservable variable $\eta$. By inserting the above equation into (20), we get (23).

If Assumption 2 is satisfied, then the following estimator:

$$\dot{\hat{\eta}} = \bar{f}_\gamma(\hat{\eta},\hat{z},\dot{\hat{z}},\delta) \tag{26}$$

can guarantee that $\lim_{t\to\infty}(\hat{\eta}(t) - \eta(t)) = 0$. According to Definition 1, we prove that system (4) is causally asymptotically left invertible. ■

### 4.3 Observer design

As we have explained, the estimation of the unknown input is based on the finite-time estimation of the variables $z$ and their derivatives, which is also used to estimate the internal dynamics $\eta$. In what follows, we use the high order sliding mode observer [23] to estimate $z$ and its derivatives:

$$\begin{cases} \dot{\hat{z}}_{i,j} = \hat{z}_{i,j+1} - \lambda_{i,j}|\hat{z}_{i,j} - z_{i,j}|^{\frac{\rho_i}{\rho_i+1}} sign(\hat{z}_{i,j} - z_{i,j}), \\ \dot{\hat{z}}_{i,j+1} = \hat{z}_{i,j+2} - \lambda_{i,j+1}|\hat{z}_{i,j+1} - \dot{\hat{z}}_{i,j}|^{\frac{\rho_i-1}{\rho_i}} sign(\hat{z}_{i,j+1} - \dot{\hat{z}}_{i,j}), \\ \vdots = \vdots \\ \dot{\hat{z}}_{i,\rho_i-1} = \hat{z}_{i,\rho_i} - \lambda_{i,\rho_i-1}|\hat{z}_{i,\rho_i-1} - \dot{\hat{z}}_{i,\rho_i-2}|^{\frac{1}{2}} sign(\hat{z}_{i,\rho_i-1} - \dot{\hat{z}}_{i,\rho_i-2}), \\ \dot{\hat{z}}_{i,\rho_i} = -\lambda_{i,\rho_i} sign(\hat{z}_{i,\rho_i} - \dot{\hat{z}}_{i,\rho_i-1}). \end{cases} \tag{27}$$

The convergence of the above observer has already been demonstrated in [23] and [24]. It has been shown that, if the $r$th derivative of the output has a Lipschitz constant $L > 0$ and by choosing $\lambda > L$, then we have $\hat{z} = z$ after a finite time $T$. Thus, the internal dynamics $\eta$ can be asymptotically estimated by using the estimator (26).

If all conditions of Theorem 4 are satisfied, by replacing $z$, $\dot{z}$ and $\eta$ by their estimations $\hat{z}$, $\dot{\hat{z}}$ and $\hat{\eta}$, the unknown input can be asymptotically estimated by the following algebraic equation:

$$\hat{u} = \left[\Gamma_\gamma(\hat{z},\hat{\eta},\delta)\right]_L^{-1}\left(\mathscr{H}_\gamma(\hat{\bar{y}}_i^{(\rho_i)}) - \Psi_\gamma(\hat{z},\hat{\eta},\delta)\right). \tag{28}$$

where $\Gamma_\gamma$, $\mathscr{H}_\gamma$ and $\Psi_\gamma$ are given in (25).

*4.4 Non-causal left invertibility*

Theorem 4 indicates that the causality of the left invertibility depends on the matrix $\Gamma_\gamma$. If this matrix is not left unimodular over $\mathscr{K}(\delta]$, then the estimation of $u$ might be non causal. By introducing the forward shift operator $\nabla$, and denote now $\bar{\mathscr{K}}$ as the field of functions of finite number of variables from $\{x_i(t-j\tau)\}$ for $i\in[1,n]$, $j\in[-m,m]$, then we can define the field $\bar{\mathscr{K}}(\delta,\nabla]$ whose entry is of the following form:

$$a(\delta,\nabla] = \sum_{k=0}^{r_a} a_k\delta^k + \sum_{l=0}^{r_{\bar{a}}} \bar{a}_l\nabla^\ell \tag{29}$$

where all the coefficients belong to $\bar{\mathscr{K}}$. We can see that $\mathscr{K}\subseteq\bar{\mathscr{K}}$ and $\mathscr{K}(\delta]\subseteq\bar{\mathscr{K}}(\delta,\nabla]$. Then, for $\bar{\mathscr{K}}(\delta,\nabla]$, the addition is as usual, but the multiplication is given by the following relation:

$$\begin{aligned} a(\delta,\nabla]b(\delta,\nabla] = & \sum_{i=0}^{r_a}\sum_{j=0}^{r_b} a_i\delta^i b_j\delta^{i+j} + \sum_{i=0}^{r_a}\sum_{j=1}^{r_{\bar{b}}} a_i\delta^i\bar{b}_j\delta^i\nabla^j \\ & + \sum_{i=1}^{r_{\bar{a}}}\sum_{j=0}^{r_b} \bar{a}_i\nabla^i b_j\nabla^i\delta^j + \sum_{i=1}^{r_{\bar{a}}}\sum_{j=1}^{r_{\bar{b}}} \bar{a}_i\nabla^i\bar{b}_j\nabla^{i+j}. \end{aligned}$$

For any $a(\delta,\nabla]\in\bar{\mathscr{K}}(\delta,\nabla]$ defined in (29), note its maximum degree relative to $\nabla$ as follows:

$$\deg_\nabla^{\max} a(\delta,\nabla] = \max_{0\le l\le r_{\bar{a}}}\{\bar{a}_l\neq 0\}. \tag{30}$$

Finally, we have the following result.

**Corollary 1** *System* (4) *is causally asymptotically left invertible if the following two conditions:*

*(i)* $\Gamma_\gamma(x,\delta)\in\mathscr{K}^{(p+l_\gamma)\times m}(\delta]$ *in* (20) *is left unimodular over* $\bar{\mathscr{K}}(\delta,\nabla]$ *and Assumptions 1 and 2 are satisfied;*

*(ii) for any entry* $\beta_{i,j}(\delta,\nabla)$ *of* $\left[\Gamma_\gamma(x,\delta)\right]_L^{-1}$, $\deg_\nabla^{\max}\beta_{i,j}(\delta,\nabla)=0$ *for all* $1\le i\le p+\sum_{k=0}^{\gamma} l_k$ *and* $1\le j\le m$*;*

*are satisfied. If only the condition (i) is satisfied, then it is non-causally asymptotically left invertible.*

**Proof.** The proof of this Corollary is quite similar to the one of Theorem 4, thus the following will just explain the main difference.

For system (19), if $\Gamma_\gamma(x,\delta)$ is unimodular over $\bar{\mathscr{K}}(\delta,\nabla]$, then from (20), we have

$$u = \left[\Gamma_\gamma(x,\delta)\right]_L^{-1}\left(\mathscr{H}_\gamma(\bar{y}_i^{(\rho_i)}) - \Psi_\gamma(x,\delta)\right)$$

where $\left[\Gamma_\gamma(x,\delta)\right]_L^{-1}$ now might contain $\nabla$. Therefore, system (4) is non-causally asymptotically left invertible if the condition (i) in Corollary 1 is satisfied.

However, for any entry $\beta_{i,j}(\delta,\nabla)$ of $\left[\Gamma_\gamma(x,\delta)\right]_L^{-1}$, if $\deg_\nabla^{\max}\beta_{i,j}(\delta,\nabla)=0$ for all $1\le i\le p+\sum_{k=0}^{\gamma}l_k$ and $1\le j\le m$, which implies that $\left[\Gamma_\gamma(x,\delta)\right]_L^{-1}\in\mathscr{K}^{(p+\sum_{k=0}^{\gamma}l_k)\times m}(\delta]$, then $\Gamma_\gamma(x,\delta)$ is left unimodular over $\mathscr{K}(\delta]$. Therefore, according to Theorem 4, we can conclude that system (4) is causally asymptotically left invertible. ■

In the following section, we provide numerical examples to illustrate the cases of causal asymptotic left inversion of nonlinear time-delay systems with internal dynamics.

## 5 Illustrative examples

*5.1 Academic example*

Consider the following nonlinear time-delay system:

$$\begin{cases} \dot{x}_1 = x_1 - x_3, \\ \dot{x}_2 = -\delta x_2 + x_4 + \delta u_1, \\ \dot{x}_3 = 2x_1 - 2x_3 + \delta x_1 - u_1, \\ \dot{x}_4 = -x_4 - x_1^2 + x_1x_3 + x_5 + u_2, \\ \dot{x}_5 = -x_5 + (\delta x_1)^2 u_2, \\ y_1 = x_1,\ y_2 = x_2, \\ \delta^j x_i = x_i(t-j\tau) \quad \text{where} \quad \tau = 0.5s, \end{cases}$$

for which one can check that $\nu_1=k_1=\rho_1=2$ and $\nu_2=k_2=\rho_2=1$ and $\Phi=\{dx_1,\ d(x_1-x_3),\ dx_2\}$ thus we have $rank_{\mathscr{K}}\Phi = 3 < n = 5$.

**Step 1: Iterative canonical form**

Following the first step of the algorithm proposed in [38], we obtain $\Gamma = \begin{pmatrix} 1 & 0 \\ \delta & 0 \end{pmatrix}$ which is not of full rank over $\mathscr{K}(\delta]$ since $rank_{\mathscr{K}(\delta]}\Gamma = 1 < m = 2$, then the unknown input $u_2$ can not be estimated. To solve this problem, we have to iterate one more step the algorithm given in [38] to generate the virtual output.

Following the procedure described in section 4.1, initially set $k=0$, $\mathscr{Y}_0 = y$, $\Phi_0=\Phi$, then we calculate

$\pounds_0 = span_{\mathbb{R}[\delta]}\{x_1,\, x_1 - x_3,\, x_2\}$, and define the module spanned by element of $\Phi$ over $\pounds(\delta]$ as follows:

$$\Omega_0 = span_{\pounds_0(\delta]}\{dx_1,\, dx_2,\, dx_3\}.$$

Since

$$\mathscr{G} = span_{\mathbb{R}[\delta]}\{(0,\delta,-1,0,0)^T,(0,0,0,1,(\delta x_1)^2)^T\}$$

then, its *left* annihilator:

$$\mathscr{G}_0^\perp = span_{\pounds_0(\delta]}\{dx_1,\, dx_2 + \delta dx_3,\, (\delta x_1)^2 dx_4 - dx_5\}$$

thus, we have

$$\Omega_0 \cap \mathscr{G}_0^\perp = span_{\pounds_0(\delta]}\{dx_1,\, dx_2 + \delta dx_3\}.$$

which enables us to get

$$H_0 = span_{\mathbb{R}[\delta]}\{dx_2 + \delta dx_3\}$$

since $\omega = dx_2 + \delta dx_3 \in \Omega_0 \cap \mathscr{G}_0^\perp$, $\omega f \notin \pounds_0$ and $\omega f = -\delta x_2 + x_4 + \delta[2x_1 - 2x_3 + \delta x_1]$. Therefore, we can obtain 1 the new virtual output

$$\bar{y}_{1,1} = \omega f \mod \pounds_0 = x_4$$

which can be determined as follows:

$$\bar{y}_{1,1} = x_4 = \dot{y}_2 + \delta y_2 - \delta^2 y_1 - \delta \dot{y}_1 - \delta \ddot{y}_1.$$

Combine the original and virtual outputs together as

$$\mathscr{Y}_1 = [y_1, y_2, \bar{y}_{1,1}]^T$$

then we can calculate again the observability indices and relative degrees for each output: $\nu_1 = k_1 = \rho_1 = 2$ and $\nu_2 = k_2 = \rho_2 = 1$, $\nu_3 = k_3 = \rho_3 = 1$ and $\Phi_1 = \{dx_1,\, d(x_1 - x_3),\, dx_2,\, dx_4\} \Rightarrow rank_{\mathscr{K}(\delta]}\Phi_1 = 4 < n = 5$. Following the same procedure, we have

$$\begin{aligned}
&\pounds_1 = span_{\mathbb{R}[\delta]}\{x_1,\, x_1 - x_3,\, x_2\, x_4\}\\
&\Omega_1 = span_{\pounds_0(\delta]}\{dx_1,\, dx_2,\, dx_3,\, dx_4\}\\
&\mathscr{G}_1^\perp = span_{\pounds_0(\delta]}\{dx_1,\, dx_2 + \delta dx_3,\, (\delta x_1)^2 dx_4 - dx_5,\, dx_4\}
\end{aligned}$$

which finally yields $H_1 = \emptyset$, therefore we have $\mathscr{Y}_2 = \mathscr{Y}_1$, and the algorithm stops to be iterated. Consequently, the internal dynamic $\eta = x_5$ has the dimension $\dim \eta = 1$. Define the following change of coordinates:

$$z = \phi_1(x,\delta) = (z_{1,1},\, z_{1,2},\, z_{2,1},\, z_{3,1}, \eta)^T = (x_1,\, x_1 - x_3,\, x_2,\, x_4, \eta)^T$$

which is bicausal over $\mathscr{K}(\delta]$, since:

$$x = \phi_1^{-1}(z,\delta) = (z_{1,1},\, z_{1,2},\, z_{2,1},\, z_{3,1}, \eta)^T.$$

From Section 3, the system is rewritten as follows:

$$\begin{cases}
\dot{z}_{1,1} = z_{1,2},\\
\dot{z}_{1,2} = -z_{1,2} - \delta z_{1,1} + u_1,\\
\dot{z}_{2,1} = -\delta z_{2,1} + z_{3,1} + \delta u_1,\\
\dot{z}_{3,1} = -z_{3,1} - z_{1,1}z_{1,2} + \eta + u_2,\\
\dot{\eta} = -\eta + (\delta z_{1,1})^2 u_2,\\
\bar{y}_1 = z_{1,1},\ \bar{y}_2 = z_{2,1},\ \bar{y}_3 = z_{3,1}.
\end{cases}$$

**Step 2: Left inversion**

From the above canonical form, i.e. after $\gamma = 1$ iteration of the algorithm proposed in [38], the matrix $\Gamma_\gamma$ is directly obtained as:

$$\Gamma_\gamma = \begin{bmatrix} 1 & 0\\ \delta & 0\\ 0 & 1 \end{bmatrix}$$

which becomes full column rank over $\mathscr{K}(\delta]$. It can be clearly seen that it is left unimodular over $\mathscr{K}(\delta]$ with the left inverse

$$[\Gamma_\gamma]_L^{-1} = \begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 1 \end{bmatrix}.$$

We can then obtain the following internal dynamics:

$$\dot{\eta} = -\eta + (\delta z_{1,1})^2(\dot{z}_{3,1} + z_{3,1} + z_{1,1}z_{1,2} - \eta). \tag{31}$$

Consider the following estimator for the internal dynamics:

$$\dot{\hat{\eta}} = -\hat{\eta} + (\delta \hat{z}_{1,1})^2(\dot{\hat{z}}_{3,1} + \hat{z}_{3,1} + \hat{z}_{1,1}\hat{z}_{1,2} - \hat{\eta}). \tag{32}$$

The estimation error $e_\eta = \eta - \hat{\eta}$ is then

$$\dot{e}_\eta = -e_\eta - (\delta z_{1,1})^2\eta + (\delta \hat{z}_{1,1})^2\hat{\eta} + F(e), \tag{33}$$

with $F(e) = (\delta z_{1,1})^2(\dot{z}_{3,1} + z_{3,1} + z_{1,1}z_{1,2}) - (\delta \hat{z}_{1,1})^2(\dot{\hat{z}}_{3,1} + \hat{z}_{3,1} + \hat{z}_{1,1}\hat{z}_{1,2})$. By choosing the following Lyapunov function:

$$V(e_\eta) = \frac{1}{2}e_\eta^T e_\eta,$$

we obtain:

$$\dot{V} = -e_\eta^2 - e_\eta((\delta z_{1,1})^2\eta - (\delta \hat{z}_{1,1})^2\hat{\eta}) + e_\eta F(e).$$

From the finite-time observer designed hereafter, there exists $T > 0$ such that for all $t > T$, $e = 0$, then $F(e) = 0$ and $(\delta \hat{z}_{1,1})^2 = (\delta z_{1,1})^2$ thus $\dot{V} = -e_\eta^2 - (\delta z_{1,1})^2 e_\eta^2 < 0$, therefore Assumption 2 is verified. Then, all conditions in Theorem 4 are satisfied and the studied system is causally asymptotically left invertible, and the unknown inputs are given by:

$$\begin{cases}
u_1 = \dot{z}_{1,2} + z_{1,2} + \delta z_{1,1},\\
u_2 = \dot{z}_{3,1} + z_{3,1} + z_{1,1}z_{1,2} - \eta,
\end{cases}$$

where $u_2$ can be asymptotically estimated since $\hat{\eta}$ converges asymptotically to $\eta$ and $u_1$ can be estimated in finite-time.

**Step 3: Observer design and inputs estimation**

As it can be seen the internal dynamics (31) is function of the states and their derivatives, which are estimated using the following High-Order Sliding Mode observer:

$$\begin{cases} \dot{\hat{z}}_{1,1} = \hat{z}_{1,2} - \lambda_{1,1}|\hat{z}_{1,1} - y_1|^{\frac{2}{3}} sign(\hat{z}_{1,1} - y_1), \\ \dot{\hat{z}}_{1,2} = \hat{z}_{1,3} - \lambda_{1,2}|\hat{z}_{1,2} - \dot{\hat{z}}_{1,1}|^{\frac{1}{2}} sign(\hat{z}_{1,2} - \dot{\hat{z}}_{1,1}), \\ \dot{\hat{z}}_{1,3} = -\lambda_{1,3} sign(\hat{z}_{1,3} - \dot{\hat{z}}_{1,2}), \\ \dot{\hat{z}}_{2,1} = \hat{z}_{2,2} - \lambda_{2,1}|\hat{z}_{2,1} - y_2|^{\frac{1}{2}} sign(\hat{z}_{2,1} - y_2), \\ \dot{\hat{z}}_{2,2} = -\lambda_{2,2} sign(\hat{z}_{2,2} - \dot{\hat{z}}_{2,1}). \end{cases} \tag{34}$$

As (34) is a finite-time observer, thus there exists $T$ such that for all $t > T$ we have $\hat{z}_{1,1} = y_1$, $\hat{z}_{1,2} = \dot{y}_1$, $\hat{z}_{1,3} = \ddot{y}_1$, $\hat{z}_{2,1} = y_2$, and $\hat{z}_{2,2} = \dot{y}_2$. Then, they are replaced in the expression of $\bar{y}_1$, and its derivative is generated using the following differentiator:

$$\begin{cases} \dot{\hat{z}}_{3,1} = \hat{z}_{3,2} - \lambda_{3,1}|\hat{z}_{3,1} - \bar{y}_1|^{\frac{1}{2}} sign(\hat{z}_{3,1} - \bar{y}_1), \\ \dot{\hat{z}}_{3,2} = -\lambda_{3,2} sign(\hat{z}_{3,2} - \dot{\hat{z}}_{3,1}). \end{cases}$$

Finally, the unknown inputs can be causally estimated from the following algebraic relation:

$$\begin{cases} \hat{u}_1 = \hat{z}_{1,3} + \hat{z}_{1,2} - \delta \hat{z}_{1,1}, \\ \hat{u}_2 = \hat{z}_{3,2} + \hat{z}_{3,1} + \hat{z}_{1,1}\hat{z}_{1,2} - \hat{\eta}, \end{cases}$$

where $\hat{\eta}$ is given in (32).

The simulation has been realized for $\tau = 0.5s$, the initial conditions $z_0 = (1,1,3,2,1)^T$ for $t \in [-0.5, 0]$, the initial conditions of the observer are fixed to zero for $t \in [-0.5, 0]$ and its gains are: $\lambda_{1,1} = 5$, $\lambda_{1,2} = 3$,$\lambda_{1,3} = 1$, $\lambda_{2,1} = 5$, $\lambda_{2,2} = 3$, $\lambda_{3,1} = 12$, and $\lambda_{3,2} = 10$. The unknown inputs to estimate are $u_1 = 0.45 sin(t)$ and $u_2 = 0.45 cos(t)$. The results are represented in Fig. 1-7, where it appears clearly that $T = 4s$, and after that the unknown inputs Fig. 6-7 are recovered.

From Figures 1-4 we remark that the states variables are estimated efficiently in finite-time. The estimated states are used to estimate the internal dynamics asymptotically (see Figure5) which is used to estimate the unknown inputs. From Figures 6 and 7 we can observe that the unknown input $u_1$ is recovered in finite-time and $u_2$ is just asymptotically estimated thing which explained by the fact that $u_2$ depends on the internal dynamic which is asymptotically estimated.

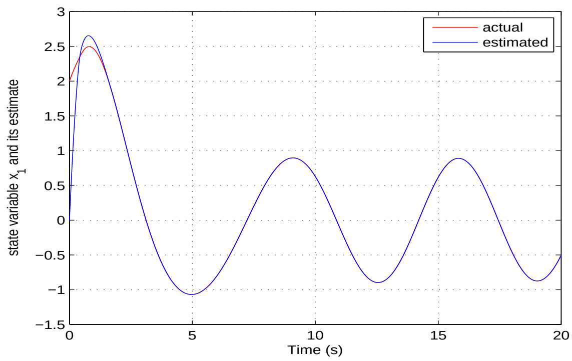


Fig. 1. $x_1$ and its estimate

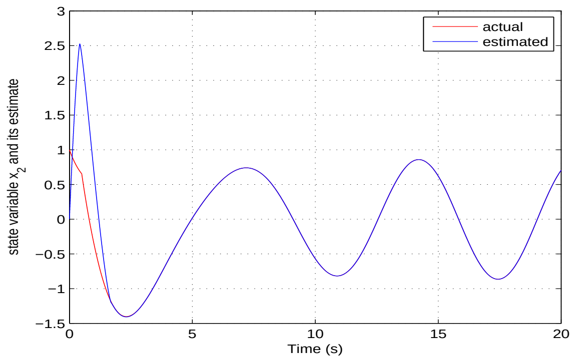


Fig. 2. $x_2$ and its estimate

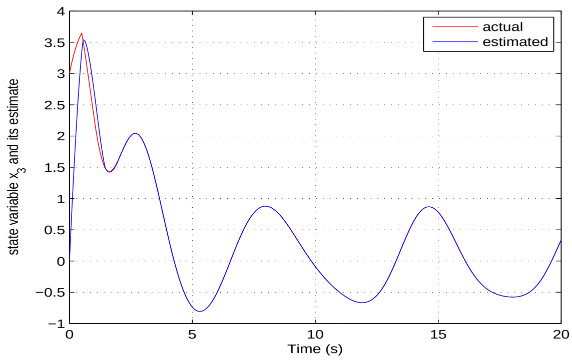


Fig. 3. $x_3$ and its estimate

### 5.2 *Academic example: Left invertibility when the change of coordinates is not bicausal*

Let us consider the following nonlinear time-delay system

$$\begin{cases} \dot{x}_1 = -\delta x_1 + x_2, \\ \dot{x}_2 = -\delta x_3 + \delta u_1, \\ \dot{x}_3 = \delta x_1 - \delta x_3 - 3x_3 + u_2, \\ \dot{x}_4 = -x_4 - 2\delta x_4 + x_5 + u_1, \\ \dot{x}_5 = -25x_5 + 3\delta x_1 + u_1, \\ y_1 = \delta x_1, \ y_2 = x_3, \ y_3 = x_4, \\ \delta^j x_i = x_i(t - j\tau), \text{ where } \tau = 0.5s. \end{cases} \tag{35}$$

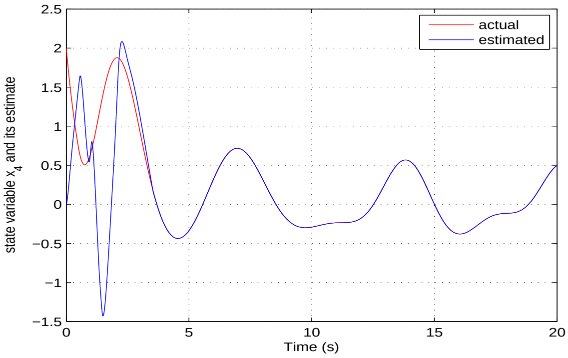


Fig. 4. $\bar{y}_1$ and its estimate

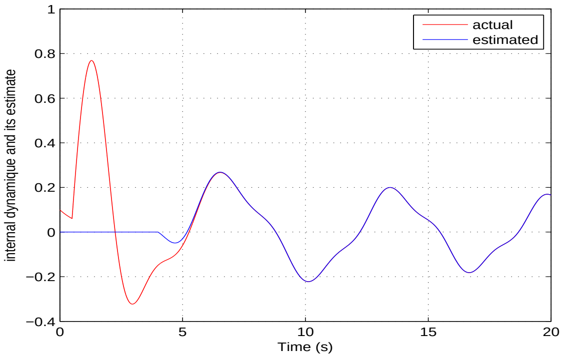


Fig. 5. $\eta$ and its estimate

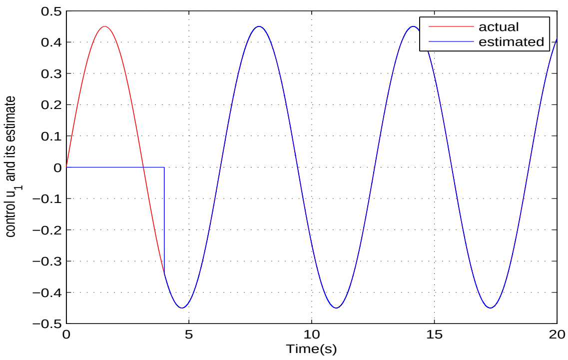


Fig. 6. The unknown input $u_1$ and its estimate

**Step 1: canonical form**

One can check that $\nu_1 = k_1 = \rho_1 = 2$, $\nu_2 = k_2 = \rho_2 = 1$, $\nu_3 = k_3 = \rho_3 = 1$, and $\Phi = \{\delta dx_1, -\delta^2 dx_1 + \delta dx_2, dx_3, dx_4\}$. Thus, we have $rank_{\mathscr{K}} \Phi = 4 < n = 5$. By checking the iteration of the algorithm (the concrete calculation is omitted since it follows the same procedure presented before), it can be shown that $H_0 = \emptyset$, thus it is not possible to generate new virtual outputs. Therefore, we chose the following change of coordinates:

$$\phi(x,\delta) = [\xi^T, \eta^T]^T = [\delta x_1, -\delta^2 x_1 + \delta x_2, x_3, x_4, x_5]^T. \quad (36)$$

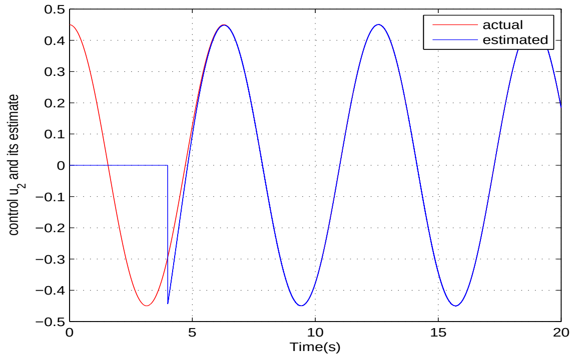


Fig. 7. The unknown input $u_2$ and its estimate

One can remark that the inverse $\phi^{-1}(\xi,\eta,\delta)$ is given by

$$x = \phi^{-1}(\xi,\eta,\delta,\nabla) = [\nabla\xi_1, \nabla\xi_2 + \nabla\xi_1, \xi_3, \xi_4, \eta]^T. \quad (37)$$

Thus, the change of coordinates $\phi(x,\delta)$ is not bicausal. Assumption 1 is not satisfied.

Nevertheless, using the change of coordinates (36), system (35) can be rewritten as

$$\begin{cases} \dot{\xi}_1 = \xi_2, \\ \dot{\xi}_2 = -\delta\xi_2 - \delta^2\xi_3 + \delta^2 u_1, \\ \dot{\xi}_3 = -3\xi_3 - \delta\xi_3 + \xi_1 + u_2, \\ \dot{\xi}_4 = -\xi_4 - 2\delta\xi_4 + \eta + u_1, \\ \dot{\eta} = -25\eta + 3\xi_1 + u_1, \end{cases} \quad (38)$$

where $\eta = x_5$ represents the internal dynamic.

**Step 2: Left inversion**

From the above canonical form, the matrix $\Gamma$ is directly obtained as:

$$\Gamma = \begin{bmatrix} \delta^2 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which is full column rank over $\mathscr{K}(\delta]$. It can be clearly seen that it is left unimodular over $\mathscr{K}(\delta]$ with the left inverse

$$[\Gamma]_L^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

which yields the following internal dynamic:

$$\dot{\eta} = -26\eta + \bar{f}(\xi,\dot{\xi},\delta) \quad (39)$$

where $\bar{f}(\xi,\dot{\xi},\delta) = 3\xi_1 + \dot{\xi}_4 + \xi_4 + 2\delta\xi_4$.

In order to estimate $\eta$ we consider the following estimator

$$\dot{\hat{\eta}} = -26\hat{\eta} + \bar{f}(\hat{\xi}, \dot{\hat{\xi}}, \delta), \tag{40}$$

with $\bar{f}(\hat{\xi}, \dot{\hat{\xi}}, \delta) = 3\hat{\xi}_1 + \dot{\hat{\xi}}_4 + \hat{\xi}_4 + 2\delta\hat{\xi}_4$.

The estimation error $e_\eta = \eta - \hat{\eta}$ is then

$$\dot{e}_\eta = -26e_\eta + F(e), \tag{41}$$

with $F(e) = \bar{f}(\xi, \dot{\xi}, \delta) - \bar{f}(\hat{\xi}, \dot{\hat{\xi}}, \delta)$ and $e = \xi - \hat{\xi}$. By choosing the following Lyapunov function: $V(e_\eta) = \frac{1}{2}e_\eta^2$, we obtain: $\dot{V} = -26e_\eta^2 + e_\eta F(e)$. From the finite-time observer designed as in (27), there exists $T > 0$ such that for all $t > T$, $e = 0$, then $F(e) = 0$. Then, $\dot{V} = -26e_\eta^2 < 0$, which implies that Assumption 2 is verified.

One can remark that even though the change of coordinates $\phi(x, \delta)$ is not bicausal (Assumption 1 is not verified), system (35) is causally asymptotically left invertible. Indeed, the unknown inputs are given by:

$$\begin{cases} \hat{u}_1 = \dot{\hat{\xi}}_4 + \hat{\xi}_4 + 2\delta\hat{\xi}_4 - \hat{\eta}, \\ \hat{u}_2 = \dot{\hat{\xi}}_3 - \hat{\xi}_1 + \delta\hat{\xi}_3 + 3\hat{\xi}_3, \end{cases}$$

where $\hat{u}_1$ and $\hat{u}_2$ are the estimates of $u_1$ and $u_2$ respectively. $\hat{\xi}$ is the estimation of $\xi$ computed in a finite time using the high order sliding mode observer and $\hat{\eta}$ is the estimation of $\eta$ obtained using the asymptotic estimator given in (40). One can remark that $u_1$ is causally asymptotically estimated and $u_2$ is estimated causally in a finite time even though the change of coordinates $\phi(x, \delta)$ defined in (36) is not bicausal.

Simulations are performed for $\tau = 0.5s$, where the initial conditions are $x_0 = \begin{bmatrix} 2, & 3, & 1, & 2, 0.1 \end{bmatrix}^T$ for $t \in [-0.5, 0]$, the initial conditions of the observer are fixed to zero for $t \in [-0.5, 0]$ and its gains are: $\lambda_{1,1} = 12$, $\lambda_{1,2} = 7$, $\lambda_{1,3} = 3$, $\lambda_{2,1} = 12$, $\lambda_{2,2} = 2$, $\lambda_{3,1} = 15$, and $\lambda_{3,2} = 7$. The unknown inputs to estimate are $u_1 = -0.2\sin(t)$ and $u_2 = -0.5cos(0.5t) + sin(0.5t)$. The results are represented in Figures 8-10, where it can be seen clearly that the internal dynamics and the unknown inputs can be asymptotically estimated.

### 5.3 *Biological example: Application to Crosstalk between ERK and STAT5a interaction*

Let us consider the case where the interaction between ERK and STAT5a happens in a non-homogeneous medium which can be the case in the presence of cancer [7], [30], whose model can be written as follows [21]:

$$\begin{cases} \dot{x}_1 = c_0 x_1 x_3 + c_2' x_2 - u_1, \\ \dot{x}_2 = c_0 \delta x_1 \delta x_3 - c_2 x_2 + u_1, \\ \dot{x}_3 = -c_1 x_1 x_3 + c_3' x_4 + u_2, \\ \dot{x}_4 = c_1 \delta x_1 \delta x_3 - c_3 x_4 - u_2, \end{cases} \tag{42}$$

where $x_1$, $x_2$, $x_3$, and $x_4$ are denoting concentrations of ERK-inactive, ERK-active, STAT5a-unphosphorylated and STAT5a-phosphorylated, respectively. $c_0$ and $c_1$ are propor-

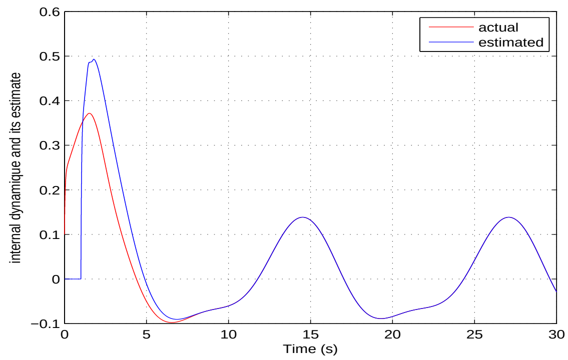


Fig. 8. $\eta_2$ and its estimate

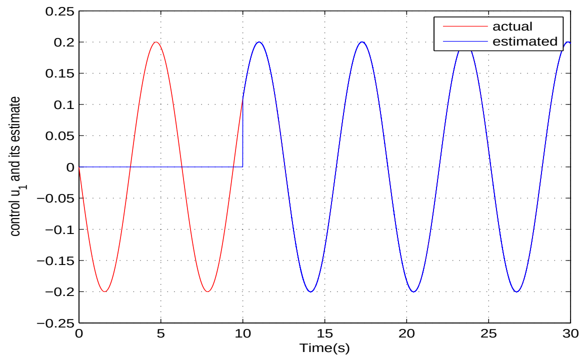


Fig. 9. The unknown input $u_1$ and its estimate

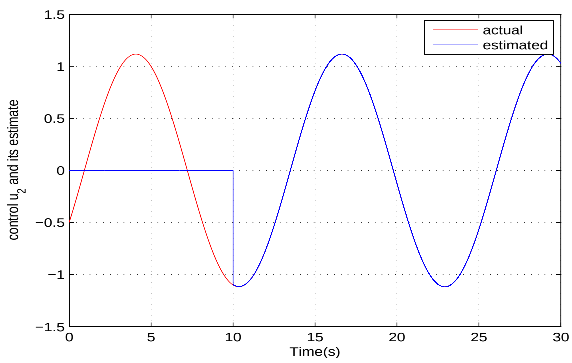


Fig. 10. The unknown input $u_2$ and its estimate

tional to the frequency of collisions of ERK and STAT5a protein molecules and present rate constant of reactions of associations and $c_2$, $c_2^{'}$, $c_3^{'}$ and $c_3$ are constants of exponential growths and disintegrations. In this example, we consider the case $c_3^{'} < c_3$ and $c_2^{'} < c_2$ which in fact is related to the presence of some special cancers [30]. The inputs $u_1$ and $u_2$ are inhibitor and activator sources respectively, and they are the unknown inputs to be estimated in this example.

**Step 1: Canonical form**

We consider that the concentrations of ERK-inactive and STAT5a-unphosphorylated are available to measurements, therefore we have

$$\begin{cases} y_1 = x_1, \\ y_2 = x_3, \end{cases} \tag{43}$$

one can check that $\nu_1 = k_1 = \rho_1 = 1$ and $\nu_2 = k_2 = \rho_2 = 1$ and $\Phi = \{dx_1,\ dx_3\}$ thus we have $rank_{\mathscr{K}}\Phi = 2 < n = 4$. By checking the iteration of the algorithm (the concrete calculation is omitted since it follows the same procedure presented before), it can be shown that $H_0 = \emptyset$, thus it is not possible to generate new virtual outputs.

By choosing the change of coordinates: $[\xi^T, \eta^T]^T = [x_1, x_3, x_2, x_4]^T$, system (42) can be rewritten as

$$\begin{cases} \dot{\xi}_1 = c_0\xi_1\xi_2 + c_2^{'}\eta_1 - u_1, \\ \dot{\xi}_2 = -c_1\xi_1\xi_2 + c_3^{'}\eta_2 + u_2, \\ \dot{\eta}_1 = c_0\delta\xi_1\delta\xi_2 - c_2\eta_1 + u_1, \\ \dot{\eta}_2 = c_1\delta\xi_1\delta\xi_2 - c_3\eta_2 - u_2, \end{cases} \tag{44}$$

where $\eta = [\eta_1, \eta_2]^T$ constitutes the internal dynamic.

**Step 2: Left inversion**

From the above canonical form, the matrix $\Gamma$ is directly obtained as:

$$\Gamma = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is full column rank over $\mathscr{K}(\delta]$. It can be clearly seen that it is left unimodular over $\mathscr{K}(\delta]$ with the left inverse

$$[\Gamma]_L^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

which yields the following internal dynamic:

$$\dot{\eta} = \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} = A\eta + \bar{f}(\xi, \dot{\xi}, \delta) \tag{45}$$

where $A = \begin{bmatrix} c_2' - c_2 & 0 \\ 0 & c_3^{'} - c_3 \end{bmatrix}$

and $\bar{f}(\xi, \delta, \dot{\xi}) = \begin{bmatrix} c_0\delta\xi_1\delta\xi_2 - \dot{\xi}_1 - c_0\xi_1\xi_2 \\ c_1\delta\xi_1\delta\xi_2 - \dot{\xi}_2 - c_1\xi_1\xi_2 \end{bmatrix}$.

In order to estimate $\eta$ we consider the following estimator

$$\dot{\hat{\eta}} = A\hat{\eta} + \bar{f}(\hat{\xi}, \dot{\hat{\xi}}, \delta), \tag{46}$$

with $\bar{f}(\hat{\xi}, \delta, \dot{\hat{\xi}}) = \begin{bmatrix} c_0\delta\hat{\xi}_1\delta\hat{\xi}_2 - \dot{\hat{\xi}}_1 - c_0\hat{\xi}_1\hat{\xi}_2 \\ c_1\delta\hat{\xi}_1\delta\hat{\xi}_2 - \dot{\hat{\xi}}_2 - c_1\hat{\xi}_1\hat{\xi}_2 \end{bmatrix}$.

The estimation error $e_\eta = \eta - \hat{\eta}$ is then

$$\dot{e}_\eta = Ae_\eta + F(e), \tag{47}$$

with $F(e) = \bar{f}(\xi, \dot{\xi}, \delta) - \bar{f}(\hat{\xi}, \dot{\hat{\xi}}, \delta)$ and $e = \xi - \hat{\xi}$. By choosing the following Lyapunov function: $V(e_\eta) = \frac{1}{2}e_\eta^T e_\eta$, we obtain: $\dot{V} = e_\eta^T Ae_\eta + e_\eta^T F(e)$. From the finite-time observer designed as in (27), there exists $T > 0$ such that for all $t > T$, $e = 0$, then $F(e) = 0$. Since $c_3^{'} < c_3$ and $c_2^{'} < c_2$ in matrix $A$, we obtain $\dot{V} = e_\eta^T Ae_\eta < 0$, which implies that Assumption 2 is verified. Then, all conditions in Theorem 4 are satisfied and the studied system is causally asymptotically left invertible. Indeed, the unknown inputs are given by:

$$\begin{cases} u_1 = -\dot{\xi}_1 - c_0\xi_1\xi_2 + c_2^{'}\eta_1, \\ u_2 = \dot{\xi}_2 + c_1\xi_1\xi_2 - c_3^{'}\eta_2, \end{cases}$$

where $u_1$ and $u_2$ can be asymptotically estimated since $\xi$ can be estimated in a finite time via the high order sliding observer and $\eta$ can be asymptotically estimated by (46).

The simulation has been realized for $\tau = 0.5s$, where the initial conditions $x_0 = \begin{bmatrix} 0.01,\ 0.02,\ 0.01,\ 0.03 \end{bmatrix}^T$ for $t \in [-0.5, 0]$, the initial conditions of the observer are fixed to zero for $t \in [-0.5, 0]$ and its gains are: $\lambda_{1,1} = 7$, $\lambda_{1,2} = 3,5, \lambda_{1,3} = 1$, $\lambda_{2,1} = 8$, $\lambda_{2,2} = 4$. The unknown inputs to estimate are $u_1 = 0,1 - 0,1\sin(3t)$ and $u_2 = 0,1\cos(3t)$. We considered the case where $c_0 = 1$, $c_1 = 0,5$, $c_2 = 10$, $c_3 = 30$, $c_3^{'} = c_2$ and $c_2^{'} = 0,5c_2$. The results are represented in Fig. 11-14, where it can be seen clearly that the internal dynamics and the unknown inputs can be asymptotically estimated.

## 6 Conclusion

This paper has provided a solution based on high-order sliding mode observer for the left invertibility problem of nonlinear time-delay system with unknown inputs. Both cases

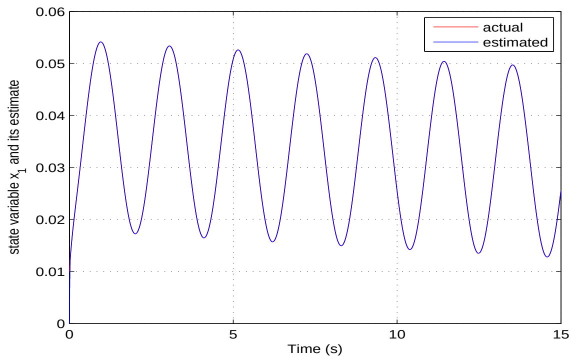


Fig. 11. $x_1$ and its estimate

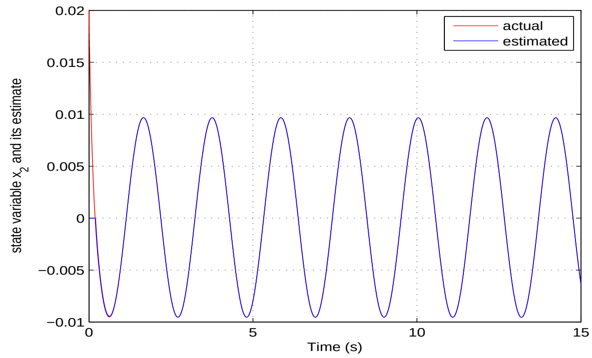


Fig. 12. $\eta_1$ and its estimate

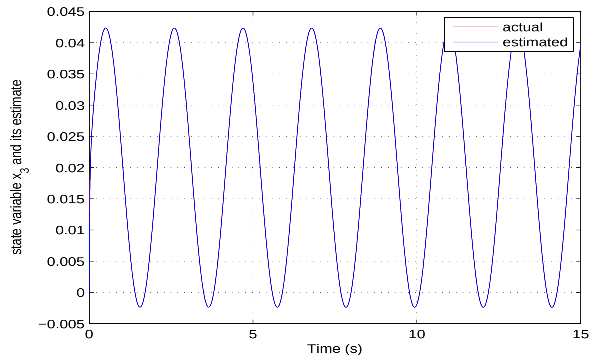


Fig. 13. $x_3$ and its estimate

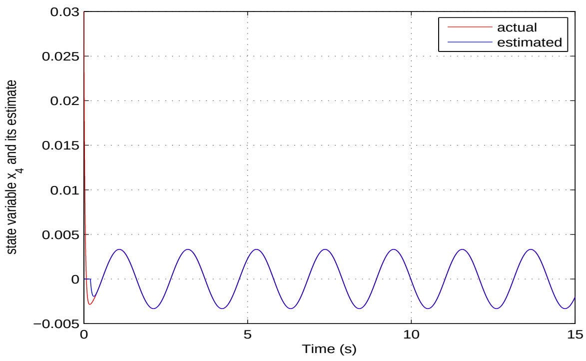


Fig. 14. $\eta_2$ and its estimate

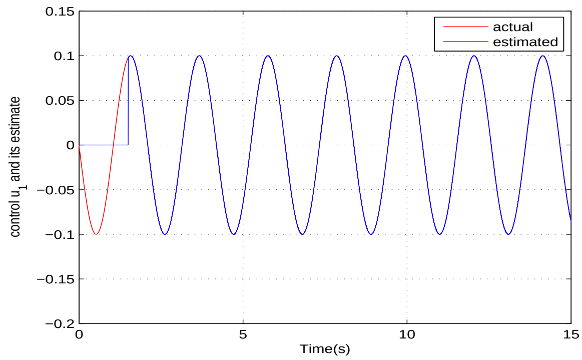


Fig. 15. The unknown input $u_1$ and its estimate

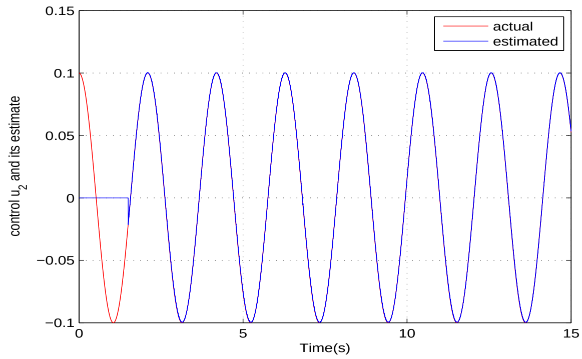


Fig. 16. The unknown input $u_2$ and its estimate

of time delay systems with and without internal dynamics are considered. Finite-time and asymptotic left invertibility are studied and conditions for the causality of the unknown input estimation are provided. In addition, academic and biological examples are given in order to show the feasibility of the proposed method.

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