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# ON POINCARÉ AND LOGARITHMIC SOBOLEV INEQUALITIES FOR A CLASS OF SINGULAR GIBBS MEASURES

DJALIL CHAFAÏ AND JOSEPH LEHEC

ABSTRACT. This note, mostly expository, is devoted to Poincaré and logarithmic Sobolev inequalities for a class of singular Boltzmann–Gibbs measures. Such measures allow to model one-dimensional particles with confinement and singular pair interaction. The functional inequalities come from the convexity of confinement and interaction. We prove optimality in the case of quadratic confinement by using a factorization of the measure. This optimality phenomenon holds for all beta Hermite ensembles including the Gaussian unitary ensemble, a famous exactly solvable model of random matrix theory. We further explore exact solvability by reviewing the relation to Dyson–Ornstein–Uhlenbeck diffusion dynamics which admits the Hermite–Lassalle orthogonal polynomials as a complete set of eigenfunctions. We also discuss the McKean–Vlasov mean-field limit of the dynamics, as well as the consequence of the logarithmic Sobolev inequality in terms of concentration of measure for Lipschitz functions such as maxima and linear statistics.

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## 1. INTRODUCTION

The aim of this note is first to provide synthetic exposition gathering material from several distant sources, and second to provide extensions and also novelty about optimality.

Let  $n \geq 2$  be an integer that plays the role of a dimension. Let  $V : \mathbb{R} \rightarrow \mathbb{R}$  be a function called the “localization potential” and let  $W : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  be a function called the “interaction potential”. The “energy” of a “configuration”  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  is

$$U(x) = \sum_{i=1}^n V(x_i) + \sum_{i < j} W(x_i - x_j) = U_V(x) + U_W(x) \in \mathbb{R} \cup \{+\infty\}.$$

Throughout the article we assume that  $W$  is a convex function whose domain equals  $(0; +\infty)$ . In particular we have  $W \equiv +\infty$  on  $(-\infty; 0]$ . As a result the energy  $U(x)$  of a

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configuration  $x \in \mathbb{R}^n$  is finite if and only if it belongs to the ‘‘Weyl chamber’’

$$D = \{x \in \mathbb{R}^n : x_1 > \cdots > x_n\}.$$

Assuming that

$$Z_\mu = \int_{\mathbb{R}^n} e^{-U(x_1, \dots, x_n)} dx_1 \cdots dx_n < \infty$$

we define a probability measure  $\mu$  on  $\mathbb{R}^n$  by

$$\mu(dx) = \frac{e^{-U(x_1, \dots, x_n)}}{Z_\mu} dx. \quad (1.1)$$

The support of  $\mu$  is  $\overline{D} = \{x \in \mathbb{R}^n : x_1 \geq \cdots \geq x_n\}$ . Let us denote by  $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$  the Euclidean norm of  $x \in \mathbb{R}^n$ . Note that if

$$W(u) = \begin{cases} -\beta \log u, & \text{if } u > 0 \\ +\infty & \text{otherwise} \end{cases} \quad (1.2)$$

where  $\beta$  is a positive parameter, and if  $X$  is a random vector of  $\mathbb{R}^n$  distributed according to  $\mu$ , then for every  $\sigma > 0$ , the scaled random vector  $\sigma X$  follows the law  $\mu$  with same  $W$  but with  $V$  replaced by  $V(\cdot/\sigma)$ .

Following Edelman [25], the beta Hermite ensemble  $\mu$  corresponds to the case

$$V(u) = n \frac{u^2}{2},$$

and  $W$  given by (1.2). In this case  $\mu$  rewrites using a Vandermonde determinant as

$$d\mu(x) = \frac{e^{-\frac{\beta}{2}|x|^2}}{Z_\mu} \prod_{i < j} (x_i - x_j)^\beta \mathbb{1}_{\{x_1 \geq \cdots \geq x_n\}} dx. \quad (1.3)$$

The normalizing constant  $Z_\mu$  can be explicitly computed in terms of Gamma functions by reduction to a classical Selberg integral, but this is useless for our purposes in this work. The Gaussian unitary ensemble (GUE) of Dyson [28] corresponds to  $\beta = 2$ , namely

$$d\mu(x) = \frac{e^{-\frac{\beta}{2}|x|^2}}{Z_\mu} \prod_{i < j} (x_i - x_j)^2 \mathbb{1}_{\{x_1 \geq \cdots \geq x_n\}} dx. \quad (1.4)$$

Note that on  $\mathbb{R}^n$  the density of the beta Hermite ensemble (1.3) with respect to the Gaussian law  $\mathcal{N}(0, \frac{1}{n}I_n)$  is equal up to a multiplicative constant to  $\prod_{i < j} (x_i - x_j)^\beta$  times the indicator function of the Weyl chamber. The cases  $\beta = 1$  and  $\beta = 4$  are known as the Gaussian orthogonal ensemble (GOE) and the Gaussian symplectic ensemble (GSE).

Let  $L^2(\mu)$  be the Lebesgue space of measurable functions from  $\mathbb{R}^n$  to  $\mathbb{R}$  which are square integrable with respect to  $\mu$ . Let  $H^1(\mu)$  be the Sobolev space of functions in  $L^2(\mu)$  with weak derivative in  $L^2(\mu)$  in the sense of Schwartz distributions.

We provide in Section 2 some useful or beautiful facts about (1.1), (1.3), and (1.4).

**1.1. Functional inequalities and concentration of measure.** Throughout the article, given  $\rho \in \mathbb{R}$ , we say that a function  $\phi: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  is  $\rho$ -convex if  $x \mapsto \phi(x) - \rho|x|^2/2$  is convex. In particular a 0-convex function is just a convex function. An equivalent condition is

$$\phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\phi(x) + \lambda\phi(y) - \frac{\rho\lambda(1-\lambda)}{2}|y-x|^2$$

for all  $x, y \in \mathbb{R}^n$  and  $\lambda \in [0, 1]$ . If  $\phi$  is  $\mathcal{C}^2$ -smooth on its domain then this is yet equivalent to  $\text{Hess}(\phi) \geq \rho I_n$  as quadratic forms, pointwise. Given  $f \in L^2(\mu)$  we denote

$$\text{var}_\mu(f) = \int_{\mathbb{R}^n} f^2 d\mu - \left( \int_{\mathbb{R}^n} f d\mu \right)^2$$

the variance of  $f$ . And if  $f \geq 0$

$$\text{ent}_\mu(f) = \int_{\mathbb{R}^n} f \log f \, d\mu - \left( \int_{\mathbb{R}^n} f \, d\mu \right) \log \left( \int_{\mathbb{R}^n} f \, d\mu \right)$$

denotes its entropy.

**Theorem 1.1** (Poincaré inequality). *Let  $\mu$  be as in (1.1). Assume that  $W$  is convex and that  $V$  is  $\rho$ -convex for some positive  $\rho$ . Then for all  $f \in H^1(\mu)$ ,*

$$\text{var}_\mu(f) \leq \frac{1}{\rho} \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu.$$

*This holds in particular with  $\rho = n$  for the beta Hermite ensemble (1.3) for all  $\beta > 0$ .*

**Theorem 1.2** (Log-Sobolev inequality). *Let  $\mu$  be as in (1.1). Assume that  $W$  is convex and that  $V$  is  $\rho$ -convex for some positive  $\rho$ . Then for all  $f \in H^1(\mu)$ ,*

$$\text{ent}_\mu(f^2) \leq \frac{2}{\rho} \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu.$$

*This holds in particular with  $\rho = n$  for the beta Hermite ensemble (1.3) for all  $\beta > 0$ .*

**Theorem 1.3** (Optimality for Poincaré and log-Sobolev inequalities). *Let  $\mu$  be as in (1.1). Assume that  $W$  is convex and that  $V$  is quadratic:  $V(x) = \rho x^2/2$  for some  $\rho > 0$ . Then equality is achieved in the Poincaré inequality of Theorem 1.1 for*

$$f : x \in \mathbb{R}^n \mapsto \lambda(x_1 + \cdots + x_n), \quad \lambda \in \mathbb{R}.$$

*Moreover equality is achieved in the logarithmic Sobolev inequality of Theorem 1.2 for*

$$f : x \in \mathbb{R}^n \mapsto e^{\lambda(x_1 + \cdots + x_n)}, \quad \lambda \in \mathbb{R}.$$

*This is in particular the case for the beta Hermite ensemble (1.3) for all  $\beta > 0$ .*

Theorem 1.1, Theorem 1.2, and Theorem 1.3 are proved in Section 3.

Poincaré and logarithmic Sobolev inequalities for beta ensembles are already known in the literature about random matrix theory, see for instance [1, 29] and references therein. However the optimality that we point out here seems to be new. It is remarkable that both optimal constants and functions match the ones of the underlying Gaussian.

The following corollary of Theorem 1.2 provides concentration of measure around the mean for Lipschitz functions, including linear statistics and maximum.

**Corollary 1.4** (Gaussian concentration inequality for Lipschitz functions). *Let  $\mu$  be as in (1.1). Assume that  $W$  is convex and that  $V$  is  $\rho$ -convex for some positive  $\rho$ . Then for every Lipschitz function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  and for all real parameter  $r > 0$ ,*

$$\mu \left( \left| F - \int F \, d\mu \right| \geq r \right) \leq 2 \exp \left( - \frac{\rho}{\|F\|_{\text{Lip}}^2} \frac{r^2}{2} \right). \quad (1.5)$$

*In particular for any measurable  $f : \mathbb{R} \rightarrow \mathbb{R}$  and all  $r > 0$ , with  $L_n(f)(x) = \frac{1}{n} \sum_{i=1}^n f(x_i)$ ,*

$$\mu \left( \left| L_n(f) - \int L_n(f) \, d\mu \right| \geq r \right) \leq 2 \exp \left( -n \frac{\rho}{\|f\|_{\text{Lip}}^2} \frac{r^2}{2} \right). \quad (1.6)$$

*Additionally, for all  $r > 0$ ,*

$$\mu \left( \left| x_1 - \int x_1 \mu(dx) \right| \geq r \right) \leq 2 \exp \left( -\rho \frac{r^2}{2} \right). \quad (1.7)$$

*This holds in particular with  $\rho = n$  for the beta Hermite ensemble (1.3) for all  $\beta > 0$ .*

The proof of Corollary 1.4 and some additional comments are given in Section 3.

The function  $(x_1, \dots, x_n) \mapsto L_n(f)(x) = \frac{1}{n} \sum_{i=1}^n f(x_i)$  is called a linear statistics. The inequality (1.6) appears for the spectrum of random matrix models in the work of Guionnet and Zeitouni [33] via the logarithmic Sobolev inequality, see also [34]. The exponential speed  $n^2$  in (1.6) is optimal according to the large deviation principle satisfied by  $L_n$  under  $\mu$  established by Ben Arous and Guionnet [8] for the GUE, see [23] and references therein for the general case (1.1). In the GUE case this  $n^2$  scale is also fully compatible with the usage of the Hoffman–Wielandt inequality. Note that in contrast with the concentration inequality obtained by Maïda and Maurel-Segala [42], here the concentration is around the mean, not around the equilibrium measure, and for a single test function, not for a supremum distance such as the Kantorovich–Wasserstein distance. See also Remark 3.3.

**1.2. Symmetric functions.** Let  $\mu$  be the probability measure on  $\mathbb{R}$  defined in (1.1). Let  $\tilde{\mu}$  be the probability measure on  $\mathbb{R}^n$  obtained from  $\mu$  by symmetrizing coordinates: For every test function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  we have

$$\int f \, d\tilde{\mu} = \int f_* \, d\mu$$

where  $f_*$  is the symmetrization of  $f$ , defined by

$$f_*(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

where  $\Sigma_n$  is the symmetric group of permutations of  $\{1, \dots, n\}$ . The measure  $\tilde{\mu}$  is by definition invariant by permutation of the coordinates.

By restricting the Poincaré inequality to symmetric functions we see that the Poincaré constant of  $\tilde{\mu}$  is at least as good as that of  $\mu$  and similarly for log-Sobolev. In particular Theorems 1.1 and 1.2 remain valid if we replace  $\mu$  by  $\tilde{\mu}$ . Moreover since the optimal functions provided by Theorem 1.3 turn out to be symmetric, the measure  $\tilde{\mu}$  also satisfies Theorem 1.3. Notice however that  $\tilde{\mu}$  is not log-concave, even when  $\mu$  is log-concave.

**1.3. Dynamics.** Let us assume in this section that the function  $V$  and  $W$  are smooths on  $\mathbb{R}$  and  $(0, +\infty)$  respectively. Then the energy  $U$  is smooth on its domain  $D$ . Fix  $X_0 \in D$  and consider the Langevin diffusion associated to the potential  $U$  starting from  $X_0$ :

$$X_t = X_0 + \sqrt{2}B_t - \int_0^t \nabla U(X_s) \, ds + \Phi_t, \quad (1.8)$$

where  $(B_t)_{t \geq 0}$  is a standard Brownian Motion of  $\mathbb{R}^n$ , and where  $\Phi_t$  is a reflection at the boundary of  $D$  which constrains the process  $X$  to stay in  $D$ . More precisely

$$\Phi_t = - \int_0^t \mathbf{n}_s L(ds)$$

where  $L$  is a random measure depending on  $X$  and supported on  $\{t \geq 0 : X_t \in \partial D\}$  and where  $\mathbf{n}_t$  is an outer unit normal to the boundary of  $D$  at  $X_t$  for every  $t$  in the support of  $L$ . The process  $L$  is called the “local time” at the boundary of  $D$ . The stochastic differential equation (1.8) writes equivalently

$$dX_t = \sqrt{2} \, dB_t - \nabla U(X_t) \, dt + \mathbf{n}_t L(dt).$$

It is not obvious that equation (1.8) admits a solution. Such diffusions with reflecting boundary conditions were first considered by Tanaka. He proved in [47] that if  $\nabla U$  is globally Lipschitz on  $D$  and grows at most linearly at infinity then (1.8) does admit a unique strong solution.

If it exists, the solution is a Markov process. Its generator is the operator

$$G = \Delta - \langle \nabla U, \nabla \rangle = \sum_{i=1}^n \partial_{x_i}^2 - \sum_{i=1}^n V'(x_i) \partial_{x_i} - \sum_{i \neq j} W'(x_i - x_j) \partial_{x_i} \quad (1.9)$$

with Neumann boundary conditions at the boundary of  $D$ . Stokes formula then shows that  $G$  is symmetric in  $L^2(\mu)$ . As a result the measure  $\mu$  is reversible for the process  $(X_t)$ . By integration by parts the density  $f_t$  of  $X_t$  with respect to the Lebesgue measure satisfies in a sense to the Fokker–Planck equation  $\partial_t f_t = \Delta f_t + \operatorname{div}(f_t \nabla U)$ .

It is common to denote  $X_t = (X_t^1, \dots, X_t^n)$  and to interpret  $X_t^1, \dots, X_t^n$  as interacting particles on the real line experiencing confinement and pairwise interactions. Let us discuss now the particular case of the beta ensemble (1.3), for which (1.8) rewrites

$$dX_t^i = \sqrt{2} dB_t^i - nX_t^i dt + \beta \sum_{j: j \neq i} \frac{1}{X_t^i - X_t^j} dt, \quad 1 \leq i \leq n \quad (1.10)$$

as long as the particles have not collided. We call this diffusion the Dyson–Ornstein–Uhlenbeck process. Without the confinement term  $-nX_t^i dt$  this diffusion is known in the literature as the Dyson Brownian motion. Indeed Dyson proved in [27] the following remarkable fact: if  $(M_t)$  is an Ornstein–Uhlenbeck process taking values in the space of complex Hermitian matrices then the eigenvalues of  $(M_t)$  follow the diffusion (1.10) with parameter  $\beta = 2$ , while if  $(M_t)$  is an Ornstein–Uhlenbeck process taking values in the space of real symmetric matrices then the same holds true with  $\beta = 1$ . Dyson also proved an analogue result for the eigenvalues of a Brownian motion on the unitary group. It is natural to ask whether the repulsion term  $1/(X_t^i - X_t^j)$  is strong enough to actually prevent the collision of particles. This was investigated by Rogers and Shi in [46] (see also [1]). They proved that if  $\beta \geq 1$  then there are no collisions: (1.10) admits a unique strong solution and with probability 1, the process  $(X_t)$  stays in the Weyl chamber  $D$  for all time. This means that in that case, Tanaka’s equation (1.8) does admit a unique strong solution, but the reflection at the boundary  $\Phi_t$  is actually identically 0.

Rogers and Shi also proved that when  $\beta < 1$  a collision indeed occurs in finite time, so the reflection  $\Phi_t$  does enter the picture. In that case though, the existence of a process  $(X_t)$  satisfying (1.8) does not follow from Tanaka’s theorem [47], as the potential  $U$  is singular at the boundary of  $D$ . Still (1.8) does admit a unique strong solution. Indeed, this was established by Cépa and Lépingle in [18] using an existence result for multivalued stochastic differential equations due to Cépa [17].

*Long time behavior of the dynamics.* Let us assume that the process (1.8) is well defined. We denote by  $(P_t)$  the associated semigroup: For every test function  $f$

$$P_t f(x) = \mathbb{E}(f(X_t) \mid X_0 = x).$$

Given a probability measure  $\nu$  on  $\mathbb{R}^n$  we denote  $\nu P_t$  the law of the process at time  $t$  when initiated from  $\nu$ . Recall that the measure  $\mu$  is stationary:  $\mu P_t = \mu$  for all time. For all real number  $p \geq 1$ , the  $L^p$  Kantorovich or Wasserstein distance between  $\mu$  and  $\nu$  is

$$W_p(\nu, \mu) = \inf_{\substack{(X, Y) \\ X \sim \nu \\ Y \sim \mu}} \mathbb{E}(|X - Y|^p)^{1/p}. \quad (1.11)$$

Note that  $W_p(\nu, \mu) = +\infty$  if  $|\cdot|^p \notin L^1(\nu) \cap L^1(\mu)$ . It can be shown that the convergence for  $W_p$  is equivalent to weak convergence together with convergence of  $p$ -th moment. If  $\nu$  has density  $f$  with respect to  $\mu$ , the relative entropy of  $\nu$  with respect to  $\mu$  is

$$H(\nu \mid \mu) = \int_{\mathbb{R}^n} \log f d\nu. \quad (1.12)$$

If  $\nu$  is not absolutely continuous we set  $H(\nu \mid \mu) = +\infty$  by convention.

**Theorem 1.5** (Convergence to equilibrium). *If  $W$  is convex and  $V$  is  $\rho$ -convex for some  $\rho > 0$  then for any two probability measures  $\nu_0, \nu_1$  on  $\mathbb{R}^n$  we have, for all  $p \geq 1$  and  $t \geq 0$ ,*

$$W_p(\nu_0 P_t, \nu_1 P_t) \leq e^{-\rho t} W_p(\nu_0, \nu_1).$$

In particular, choosing  $\nu_1 = \mu$  yields

$$W_p(\nu_0 P_t, \mu) \leq e^{-\rho t} W_p(\nu_0, \mu).$$

Moreover we also have, for all  $t \geq 0$ ,

$$H(\nu_0 P_t | \mu) \leq e^{-\rho t} H(\nu_0 P_t | \mu). \quad (1.13)$$

The proof of Theorem 1.5 is given in Section 3.

*Mean field limit of the dynamics.* The interacting particles system above has also a scaling limit which appears as a sort of nonlinear law of large numbers for a mean field particle approximation of a McKean–Vlasov equation. For simplicity, let us consider the case  $\beta = 1$  and let us drop the confinement term. We thus focus on the Dyson Brownian motion  $X$  solution of

$$dX_t^i = \sqrt{2} dB_t^i + \sum_{j: j \neq i} \frac{1}{X_t^i - X_t^j} dt, \quad 1 \leq i \leq n. \quad (1.14)$$

Let us then consider the empirical measure

$$L_t^n = \frac{1}{n} \sum_{i=1}^n \delta_{X_t^i}. \quad (1.15)$$

We refer to [46, 18, 14] for the following theorem, and to [10] for a link with free probability theory. The equation (1.16) below is known as a McKean–Vlasov equation, a granular media equation, and a free Fokker–Planck equation, depending on the context.

**Theorem 1.6** (Mean field limit). *Let  $L_t^n$  be as in (1.15). If the sequence of initial distributions  $(L_0^n)_{n \geq 1}$  converges weakly as  $n \rightarrow \infty$  to a probability measure  $\mu_0$ , then the sequence of measure valued processes  $((L_t^n)_{t \geq 0})_{n \geq 1}$  converges weakly to the unique continuous probability measure valued deterministic process  $(\mu_t)_{t \geq 0}$  satisfying*

$$\langle \mu_t, f \rangle = \langle \mu_0, f \rangle + \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} \mu_s(dx) \mu_s(dy) ds \quad (1.16)$$

for all  $t \geq 0$  and  $f \in \mathcal{C}_b^3(\mathbb{R}, \mathbb{R})$ . The equation (1.16) is a weak formulation of a singular McKean–Vlasov equation without noise. Moreover the Cauchy–Stieltjes transform

$$z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Im z > 0\} \mapsto s_t(z) = \int_{\mathbb{R}} \frac{\mu_t(dx)}{x - z} \quad (1.17)$$

of  $\mu_t$  is the solution of the following noiseless complex Burgers equation

$$\partial_t s_t(z) = s_t \partial_z s_t(z), \quad t \geq 0, z \in \mathbb{C}_+. \quad (1.18)$$

When  $\mu_0 = \delta_0$ , it can be checked that the unique solution of (1.18) is

$$s_t(z) = \frac{\sqrt{z^2 - 4t} - z}{2t}, \quad t \geq 0, z \in \mathbb{C}_+, \quad (1.19)$$

and  $\mu_t$  is the semicircle distribution on  $[-2\sqrt{t}, 2\sqrt{t}]$  of variance  $t$ .

**1.4. Hermite–Lassalle orthogonal polynomials.** Recall that for all  $n \geq 1$ , the classical Hermite polynomials  $(H_{k_1, \dots, k_n})$  are the orthogonal polynomials for the standard Gaussian distribution  $\gamma_n$  on  $\mathbb{R}^n$ . Among several remarkable properties, these polynomials satisfy a differential equation which writes  $LH_{k_1, \dots, k_n} = -(k_1 + \dots + k_n)H_{k_1, \dots, k_n}$  where  $L = \Delta - \langle x, \nabla \rangle$  is the infinitesimal generator of the Ornstein–Uhlenbeck process, which admits  $\gamma_n$  as a reversible invariant measure. In other words these orthogonal polynomials form a complete set of eigenfunctions of this operator. Such a structure is relatively rare, see [44] for a complete classification when  $n = 1$ .

Michel Lassalle discovered in the 1990s that a very similar phenomenon takes place for beta Hermite ensembles and the Dyson–Ornstein–Uhlenbeck process, provided that we

restrict to symmetric polynomials. Observe first that this cannot hold for all polynomials, simply because the Dyson–Ornstein–Uhlenbeck operator  $G$  does not preserve polynomials, for instance  $Gx_1 = -nx_1 + \beta \sum_{j \neq 1} \frac{1}{x_1 - x_j}$ . However, rewriting the operator  $G$  by symmetrization as

$$G = \sum_{i=1}^n \partial_{x_i}^2 - n \sum_{i=1}^n x_i \partial_{x_i} + \frac{\beta}{2} \sum_{i \neq j} \frac{1}{x_i - x_j} (\partial_{x_i} - \partial_{x_j}), \quad (1.20)$$

it is easily seen that the set of symmetric polynomials in  $n$  variables is left invariant by  $G$ .

Let  $\mu$  be the beta Hermite ensemble defined in (1.3). Lassalle studied in [38] multivariate symmetric polynomials  $(P_{k_1, \dots, k_n})_{k_1 \geq \dots \geq k_n \geq 0}$  which are orthogonal with respect to  $\mu$ . He called them “generalized Hermite” but we decide to call them “Hermite–Lassalle”. For all  $k_1 \geq \dots \geq k_n \geq 0$  and  $k'_1 \geq \dots \geq k'_n \geq 0$ ,

$$\int P_{k_1, \dots, k_n}(x_1, \dots, x_n) P_{k'_1, \dots, k'_n}(x_1, \dots, x_n) \mu(dx) = \mathbb{1}_{(k_1, \dots, k_n) = (k'_1, \dots, k'_n)}. \quad (1.21)$$

They can be obtained from the standard basis of symmetric polynomials by using the Gram–Schmidt algorithm in the Hilbert space  $L^2_{\text{sym}}(\mu)$  of square integrable symmetric functions. The total degree of  $P_{k_1, \dots, k_n}$  is  $k_1 + \dots + k_n$ . The numbering in terms of  $k_1, \dots, k_n$  used in [38] is related to Jack polynomials. Beware that [38] comes without proofs. We refer to [3] for proofs, and to [26] for symbolic computation via Jack polynomials.

The Hermite–Lassalle symmetric polynomials form an orthogonal basis in  $L^2_{\text{sym}}(\mu)$  of eigenfunctions of the Dyson–Ornstein–Uhlenbeck operator  $G$ . Restricted to symmetric functions, this operator is thus exactly solvable, just like the classical Ornstein–Uhlenbeck operator. Here is the result of Lassalle in [38], see [3] for a proof.

**Theorem 1.7** (Eigenfunctions). *For all  $n \geq 2$  and  $k_1 \geq \dots \geq k_n \geq 0$ ,*

$$GP_{k_1, \dots, k_n} = -n(k_1 + \dots + k_n)P_{k_1, \dots, k_n}. \quad (1.22)$$

where  $G$  is the operator (1.9) namely

$$G = \Delta - \nabla U \cdot \nabla = \sum_{i=1}^n \partial_{x_i}^2 - n \sum_{i=1}^n x_i \partial_{x_i} + \beta \sum_{i \neq j} \frac{1}{x_i - x_j} \partial_{x_i}. \quad (1.23)$$

In particular the spectral gap of  $G$  in the space  $L^2_{\text{sym}}(\mu)$  is  $n$ , as reflected in the Poincaré inequality provided by Theorem 1.1. The right hand side is compatible since

$$\int fGf \, d\mu = - \int |\nabla f|^2 \, d\mu.$$

Beware that the Poincaré inequality of Theorem 1.1 concerns non-necessarily symmetric functions, but an optimal function is symmetric from Theorem 1.3!

Hermite–Lassalle polynomials can be decomposed in terms of Jack polynomials, and this decomposition generalizes the hypergeometric expansion of classical Hermite polynomials.

**Remark 1.8** (Examples and formulas). *It is not difficult to check that up to normalization*

$$x_1 + \dots + x_n \quad \text{and} \quad x_1^2 + \dots + x_n^2 - 1 - \beta \frac{n-1}{2}$$

are Hermite–Lassalle polynomials. In the GUE case,  $\beta = 2$ , Lassalle gave in [38], using Jack polynomials and Schur functions, a formula for  $P_{k_1, \dots, k_n}$  in terms of a ratio of a determinant involving classical Hermite polynomials and a Vandermonde determinant.



**1.5. Comments and open questions.** Regarding functional inequalities, one can probably extend the results to the class of Gaussian  $\varphi$ -Sobolev inequalities such as the Beckner inequality [7], see also [20]. Lassalle has studied not only the beta Hermite ensemble in [38], but also the beta Laguerre ensemble in [40] with density proportional to

$$x \in D \mapsto \prod_{k=1}^n x_k^a e^{-bnx_k} \prod_{i<j} (x_i - x_j)^\beta \mathbb{1}_{x_1 \geq \dots \geq x_n \geq 0},$$

and the beta Jacobi ensemble in [39] with density proportional to

$$x \in D \mapsto \prod_{k=1}^n x_k^{a-1} (1-x_k)^{b-1} \prod_{i<j} (x_i - x_j)^\beta \mathbb{1}_{1 \geq x_1 \geq \dots \geq x_n \geq 0}.$$

It is tempting to study functional inequalities and concentration of measure for these ensembles. The proofs of Lassalle, based on Jack polynomials, are not in [40, 39, 38] but can be found in [3]. It is natural (maybe naive) to ask about direct proofs of these results without using Jack polynomials. The study of beta ensembles can be connected to  $H$ -transforms and to the work [32] on Brownian motion in a Weyl chamber, see also [24]. The analogue of the Dyson Brownian motion for the Laguerre ensemble is studied in [13], see also [24]. Tridiagonal matrix models for Dyson Brownian motion are studied in [36]. The factorization phenomenon captured by Lemma 2.6, which is behind the optimality provided by Theorem 1.3, reminds some kind of concentration-compactness related to continuous spins systems as in [19] and [43] for instance. Regarding exact solvability, we suspect that despite the lack of convexity, the complex Ginibre ensemble and its natural gradient flow dynamics studied in [22] admit a rigid polynomial eigenfunction structure similar to the one of the Gaussian unitary ensemble and that its spectral gap over symmetric functions is also of order  $n$ .

## 2. USEFUL OR BEAUTIFUL FACTS

**2.1. Random matrices, GUE, and beta Hermite ensemble.** The following result from random matrix theory goes back to Dyson, see [28, 45, 1, 31].

**Theorem 2.1** (Gaussian random matrices and GUE). *The Gaussian unitary ensemble  $\mu$  defined by (1.4) is the law of the ordered eigenvalues of a random  $n \times n$  Hermitian matrix  $H$  with density proportional to  $h \mapsto e^{-\frac{n}{2} \text{Trace}(h^2)} = e^{-\frac{n}{2} \sum_{i=1}^n h_{ii}^2 - n \sum_{i<j} |h_{ij}|^2}$  in other words the  $n^2$  real random variables  $\{H_{ii}, \Re H_{ij}, \Im H_{ij}\}_{1 \leq i < j \leq n}$  are independent, with  $\Re H_{ij}$  and  $\Im H_{ij} \sim \mathcal{N}(0, 1/(2n))$  for any  $i < j$  and  $H_{ii} \sim \mathcal{N}(0, 1/n)$  for any  $1 \leq i \leq n$ .*

There is an analogue theorem for the GOE case  $\beta = 1$  with random Gaussian real symmetric matrices, and for the GSE case  $\beta = 4$  with random Gaussian quaternion selfdual matrices. The following result holds for all beta Hermite ensemble (1.3), see [25].

**Theorem 2.2** (Tridiagonal random matrix model for beta Hermite ensemble). *The beta Hermite ensemble  $\mu$  defined by (1.3) is the distribution of the ordered eigenvalues of the random tridiagonal symmetric  $n \times n$  matrix*

$$H = \frac{1}{\sqrt{2n}} \begin{pmatrix} \mathcal{N}(0, 2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & \mathcal{N}(0, 2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & \mathcal{N}(0, 2) & \chi_\beta \\ & & & \chi_\beta & \mathcal{N}(0, 2) \end{pmatrix}$$

where, up to the scaling prefactor  $1/\sqrt{2n}$ , the entries in the upper triangle including the diagonal are independent, follow a Gaussian law  $\mathcal{N}(0, 2)$  on the diagonal, and  $\chi$ -distributions just above the diagonal with a decreasing parameter with step  $\beta$  from  $(n-1)\beta$  to  $\beta$ .

In particular the trace follows the Gaussian law  $\mathcal{N}(0, 1)$ . Such random matrix models with independent entries allow notably to compute moments of (1.3) via traces of powers.

**2.2. Isotropy of beta Hermite ensembles.** This helps to understand the structure.

**Lemma 2.3** (Isotropy of beta Hermite ensemble). *Let  $\mu$  be the beta Hermite ensemble with density (1.3), and let  $\tilde{\mu}$  be its symmetrized version. For every  $1 \leq i \neq j \leq n$ ,*

$$\int x_i d\tilde{\mu} = 0, \quad \int x_i^2 d\tilde{\mu} = \frac{\beta}{2} + \frac{2-\beta}{2n}, \quad \int x_i x_j d\tilde{\mu} = -\frac{\beta}{2n}.$$

*In particular, the law  $\tilde{\mu}$  is asymptotically isotropic.*

Recall that isotropy means zero mean and covariance matrix multiple of the identity.

*Proof of Lemma 2.3.* Observe first that if  $X \sim \mu$  then  $\sum X_i$  is a standard Gaussian. This can be seen using Theorem 2.2, and observing that  $\sum X_i$  coincides with the trace of the matrix  $H$ . Actually this is true as soon as the confinement potential  $V$  is quadratic, regardless of the interaction potential  $W$ , see Lemma 2.6 below. In particular

$$\int (x_1 + \cdots + x_n) \mu(dx) = 0,$$

hence, by definition  $\tilde{\mu}$ ,

$$\int x_i \tilde{\mu}(dx) = \frac{1}{n} \int (x_1 + \cdots + x_n) \mu(dx) = 0,$$

for every  $i \leq n$ . Since  $\sum X_i$  is a standard Gaussian we also have

$$\int (x_1 + \cdots + x_n)^2 \mu(dx) = 1. \quad (2.1)$$

Next we compute  $\int |x|^2 d\mu$ . This can be done using Theorem 2.2, namely

$$\int |x|^2 \mu(dx) = \mathbb{E}(\text{Trace}(H^2)) = 1 + \frac{\beta}{n} \sum_{k=1}^{n-1} k = 1 + \frac{(n-1)\beta}{2}. \quad (2.2)$$

Alternatively one can use the fact that the square of the norm  $|\cdot|^2$  is, up to an additive constant, an eigenvector of  $G$ , see Remark 1.8. Namely, recall the definition (1.23) of the operator  $G$  and note that

$$G(|\cdot|^2)(x) = 2n - 2n|x|^2 + 2\beta \sum_{i \neq j} \frac{x_i}{x_i - x_j} = 2n - 2n|x|^2 + n(n-1)\beta.$$

In particular  $G(|\cdot|^2) \in L^2(\mu)$ . Since  $\mu$  is stationary, we then have  $\int G|x|^2 d\mu = 0$ , and we thus recover (2.2). Combining (2.1) and (2.2) we get

$$\int x_i^2 \tilde{\mu}(dx) = \frac{1}{n} \int |x|^2 \mu(dx) = \frac{\beta}{2} + \frac{2-\beta}{2n},$$

and

$$\int x_i x_j \tilde{\mu}(dx) = \frac{1}{n(n-1)} \int (x_1 + \cdots + x_n)^2 - (x_1^2 + \cdots + x_n^2) \tilde{\mu}(dx) = -\frac{\beta}{2n}.$$

□

**Remark 2.4** (Mean and covariance of beta Hermite ensembles). *Let  $\mu$  and  $\tilde{\mu}$  be as in Lemma 2.3. In contrast with the probability measure  $\tilde{\mu}$ , the probability measure  $\mu$  is log-concave but is not centered, even asymptotically as  $n \rightarrow \infty$ , and this is easily seen from  $0 \notin D$ . Moreover, if  $X_n = (X_{n,1}, \dots, X_{n,n}) \sim \mu$  then the famous Wigner theorem for the*

beta Hermite ensemble, see for instance [33], states that almost surely and in  $L^1$ , regardless of the way we choose the common probability space,

$$X_{n,1} \xrightarrow[n \rightarrow \infty]{} -\sqrt{2\beta}, \quad X_{n,n} \xrightarrow[n \rightarrow \infty]{} \sqrt{2\beta}, \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^n \delta_{X_{n,i}} \xrightarrow[n \rightarrow \infty]{\text{weak}} \nu_\beta$$

where

$$\nu_\beta = \arg \inf_{\mu} \left( \int \frac{x^2}{2} d\mu(x) - \beta \iint \log(x-y) d\mu(x) d\mu(y) \right) = \frac{\sqrt{2\beta-x^2}}{\beta\pi} \mathbb{1}_{[-\sqrt{2\beta}, \sqrt{2\beta}]}(x) dx.$$

This follows for instance from a large deviation principle. This suggests in a sense that asymptotically, as  $n \rightarrow \infty$ , the mean is supported by the whole interval  $[-\sqrt{2\beta}, \sqrt{2\beta}]$ . It is quite natural to ask about the asymptotic shape of the covariance matrix of  $\mu$ .

**2.3. Log-concavity and curvature.** The following Lemma is essentially the key of the proof of Theorem 1.1 and Theorem 1.2.

**Lemma 2.5** (Log-concavity and curvature). *Let  $\mu$  be as in (1.1) and let  $\rho \in \mathbb{R}$ . If the interaction potential is convex and if the confinement potential is  $\rho$ -convex then  $U$  is  $\rho$ -convex. In particular, in the case of the beta Hermite ensemble (1.3), the potential  $U$  is  $n$ -convex, for every  $\beta$ .*

*Proof of Lemma 2.5.* Recall from (1) that  $U(x) = U_V(x) + U_W(x)$ . Observe that  $U_W$  is convex as a sum of linear maps composed with the convex function  $W$ . Similarly, under the hypothesis that  $V(r) - \rho r^2/2$  is convex the function

$$U_V(x) - \frac{\rho}{2}|x|^2 = \sum_{i=1}^n \left( V(x_i) - \frac{\rho}{2}x_i^2 \right)$$

is convex. Hence the result.  $\square$

**2.4. Factorization by projection.** The following factorization lemma is the key of the proof of Theorem 1.3. Let  $u$  be the unit vector of  $\mathbb{R}^n$  given by

$$u = \frac{1}{\sqrt{n}}(1, \dots, 1)$$

and let  $\pi$  be the orthogonal projection onto  $u^\perp$ .

**Lemma 2.6** (Factorization by projection). *Let  $\mu$  be as in (1.1) and let  $X$  be a random vector distributed according to  $\mu$ . Assume that the confinement potential  $V$  is quadratic:  $V(r) = \rho r^2/2$  for some  $\rho > 0$ . Then  $\langle X, u \rangle$  and  $\pi(X)$  are independent. Moreover*

$$\langle X, u \rangle \sim \mathcal{N}\left(0, \frac{1}{\rho}\right).$$

and  $\pi(X)$  has density proportional to  $e^{-U}$  with respect to the Lebesgue measure on  $u^\perp$ .

In the special case of the beta Hermite ensemble, the law of  $\langle X, u \rangle = \text{Trace}(H)/\sqrt{n}$  is easily seen on the random matrix model  $H$  provided by theorems 2.1 and 2.2.

*Proof of Lemma 2.6.* Since  $\pi$  is the orthogonal projection on  $u^\perp$  we have

$$|x|^2 = \langle x, u \rangle^2 + |\pi(x)|^2.$$

Besides it is easily seen that  $U_W(x) = U_W(\pi(x))$  for all  $x$ , a property which comes from the shift invariance of the interaction energy  $U_W$  along  $\mathbb{R}u$ . Therefore

$$e^{-U(x)} = e^{-\rho \langle x, u \rangle^2 / 2} \times e^{-\rho |\pi(x)|^2 / 2 - U_W(\pi(x))} = e^{-\rho \langle x, u \rangle^2 / 2} \times e^{-U(\pi(x))}.$$

So the density of  $X$  is the product of a function of  $\langle x, u \rangle$  by a function of  $\pi(x)$ . The result follows immediately.  $\square$

## 3. PROOFS

*Proof of Theorems 1.1 and 1.2.* Let us first mention that Theorem 1.2 actually implies Theorem 1.1. Indeed it is well-known that applying log-Sobolev to a function  $f$  of the form  $f = 1 + \epsilon h$  and letting  $\epsilon$  tend to 0 yields the Poincaré inequality for  $h$ , with half the constant if the log-Sobolev inequality. See for instance [2] or [5] for details.

In the discussion below, we call *potential* of a probability measure  $\mu$  the function  $-\log \rho$ , where  $\rho$  is the density of  $\mu$  with respect to the Lebesgue measure. In view of Lemma 2.5 it is enough to prove that a probability measure  $\mu$  on  $\mathbb{R}^n$  whose potential  $U$  is  $\rho$ -convex for some positive  $\rho$  satisfies the logarithmic Sobolev inequality with constant  $2/\rho$ . This is actually a well-known fact. It can be seen in various ways which we briefly spell out now. Some of these arguments require extra assumptions on  $U$ , namely that the domain of  $U$  equals  $\mathbb{R}^n$  (equivalently  $\mu$  has full support) and that  $U$  is  $\mathcal{C}^2$ -smooth on  $\mathbb{R}^n$ . For this reason we first explain a regularization procedure showing that these hypothesis can be added without loss of generality.

*Regularization procedure.* Let  $\gamma$  be the Gaussian measure whose density is proportional to  $e^{-\rho|x|^2/2}$  and let  $f$  be the density of  $\mu$  with respect to  $\gamma$ . Clearly  $U$  is  $\rho$ -convex if and only if  $\log f$  is concave. Next let  $(Q_t)$  be the Ornstein–Uhlenbeck semigroup having  $\gamma$  as a stationary measure, namely for every test function  $g$

$$Q_t g(x) = \mathbb{E} \left[ g \left( e^{-t} x + \sqrt{1 - e^{-2t}} G \right) \right]$$

where  $G \sim \gamma$ . Since  $\gamma$  is reversible for  $(Q_t)$  the measure  $\mu Q_t$  has density  $Q_t f$  with respect to  $\gamma$ . Moreover the semigroup  $(Q_t)$  satisfies the following property

$$f \text{ log-concave} \quad \Rightarrow \quad Q_t f \text{ log-concave.}$$

This is indeed an easy consequence of the Prékopa–Leindler inequality, see (3.1) below. As a result the potential  $U_t$  of  $\mu Q_t$  is also  $\rho$ -convex. Besides  $U_t$  is clearly  $\mathcal{C}^\infty$  smooth on the whole  $\mathbb{R}^n$ . Lastly since  $\lim_{t \rightarrow 0} Q_t f(x) = f(x)$  for almost every  $x$ , we have  $\mu P_t \rightarrow \mu$  weakly as  $t$  tends to 0. As a result, if  $\mu P_t$  satisfies log-Sobolev with constant  $2/\rho$  for every  $t$ , then so does  $\mu$ .

*First proof: The Brascamp–Lieb inequality.* A theorem due to Brascamp and Lieb [12] states that if the potential of  $\mu$  is smooth and satisfies  $\text{Hess}(U)(x) > 0$  for all  $x \in \mathbb{R}^n$ , then for any  $\mathcal{C}^\infty$  compactly supported test function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , we have the inequality

$$\text{var}_\mu(f) \leq \int_{\mathbb{R}^n} \langle \text{Hess}^{-1} U(\nabla f), \nabla f \rangle d\mu.$$

If  $U$  is  $\rho$ -convex then  $\text{Hess}^{-1} U \leq (1/\rho)I_n$  and we obtain

$$\text{var}_\mu(f) \leq \frac{1}{\rho} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu.$$

The extension of this inequality to all  $f \in H^1(\mu)$  follows by truncation and regularization. Note that this method only works for Poincaré. The Brascamp–Lieb inequality does not seem to admit a logarithmic Sobolev inequality counterpart, see [11] for a discussion on this.

*Second proof: The Bakry–Émery criterion.* Assume that  $U$  is finite and smooth on the whole  $\mathbb{R}^n$  and consider the Langevin diffusion

$$dX_t = \sqrt{2} dB_t - \nabla U(X_t) dt.$$

The generator of the diffusion is the operator  $G = \Delta - \langle \nabla U, \nabla \rangle$ . The carré du champs  $\Gamma$  and its iterated version  $\Gamma_2$  are easily computed:

$$\begin{aligned}\Gamma(f, g) &= \frac{1}{2}(G(fg) - fG(g) - gG(f)) = \langle \nabla f, \nabla g \rangle \\ \Gamma_2(f) &= \frac{1}{2}(G\Gamma(f) - 2\Gamma(f, Gf)) = \|\text{Hess}(f)\|_{\text{HS}}^2 + \langle \text{Hess}(U)(\nabla f), \nabla f \rangle.\end{aligned}$$

Under the hypothesis that  $U$  is  $\rho$ -convex we thus obtain

$$\Gamma_2(f) \geq \rho\Gamma(f).$$

In the language of Bakry–Émery, see [4, 2, 5], the diffusion satisfies the curvature dimension criterion  $\text{CD}(\rho, \infty)$ . This criterion implies that the stationary measure  $\mu$  satisfies the following logarithmic Sobolev inequality

$$\text{ent}_\mu(f^2) \leq \frac{2}{\rho} \int_{\mathbb{R}^n} \Gamma(f, f) d\mu,$$

see [5, Proposition 5.7.1]. Formally this proof also work if  $\mu$  does not have full support by adding a reflection at the boundary, just as in section 1.3. However this poses some technical issues which are not always easy to overcome. As a matter of fact, diffusions with reflecting boundary conditions are not treated in the book [5].

*Third proof: Caffarelli's contraction theorem.* Again let  $\gamma$  be the Gaussian measure on  $\mathbb{R}^n$  whose density is proportional to  $e^{-\rho|x|^2/2}$ . The theorem of Caffarelli [16, 15] states that if the potential of  $\mu$  is  $\rho$ -convex then the Brenier map from  $\gamma$  to  $\mu$  is 1-Lipschitz. This easily implies that the the Poincaré constant of  $\mu$  is at least as good as that of  $\gamma$ , namely  $1/\rho$ . Let us sketch the argument briefly. Let  $F$  be the Brenier map from  $\mu$  to  $\gamma$  and let  $f$  be a smooth function on  $\mathbb{R}^n$ . Using  $F_{\#}\mu = \gamma$ , the Poincaré inequality for  $\gamma$  and the fact that  $F$  is 1-Lipschitz we get

$$\begin{aligned}\text{var}_\mu(f) = \text{var}_\gamma(f \circ F) &\leq \frac{1}{\rho} \int_{\mathbb{R}^n} |\nabla(f \circ F)|^2 d\gamma \\ &\leq \frac{1}{\rho} \int_{\mathbb{R}^n} |\nabla f|^2 \circ F d\gamma \\ &= \frac{1}{\rho} \int_{\mathbb{R}^n} |\nabla f|^2 d\mu.\end{aligned}$$

This contraction principle works just the same for log-Sobolev.

*Fourth proof: An argument of Bobkov and Ledoux.* This fourth proof is the one that requires the less background. Another nice feature is that the regularization procedure is not needed for this proof. It is based on the Prékopa–Leindler inequality. The latter, which is a functional form of the Brunn–Minkowski inequality, states that if  $f, g, h$  are functions on  $\mathbb{R}^n$  satisfying

$$(1-t)f(x) + tg(y) \leq h((1-t)x + ty)$$

for every  $x, y \in \mathbb{R}^n$  and for some  $t \in [0, 1]$ , then

$$\left( \int e^f dx \right)^{1-t} \left( \int e^g dx \right)^t \leq \int e^h dx. \quad (3.1)$$

We refer to [6] for a nice presentation of this inequality. Let  $F: \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth function with compact support, and for  $s > 0$  let  $R_s f$  be the infimum convolution

$$R_s f(x) = \inf_{y \in \mathbb{R}^n} \left\{ f(x+y) + \frac{1}{2s}|y|^2 \right\}.$$

Fix  $t \in (0, 1)$ . Using the  $\rho$ -convexity of  $U$ :

$$(1-t)U(x) + tU(y) \leq U((1-t)x + ty) - \frac{\rho t(1-t)}{2}|x-y|^2,$$

it is easily seen that the functions  $f = R_{t/\rho}F - U$ ,  $g = -U$  and  $h = (1-t)F - U$  satisfy the hypothesis of the Prékopa-Leindler inequality. The conclusion (3.1) rewrites in this case

$$\left( \int_{\mathbb{R}^n} e^{R_{t/\rho}F} d\mu \right)^{1-t} \leq \int_{\mathbb{R}^n} e^{(1-t)F} d\mu. \quad (3.2)$$

It is well-known that  $(R_s)$  solves the Hamilton–Jacobi equation

$$\partial R_s f + \frac{1}{2} |\nabla R_s f|^2 = 0,$$

see for instance [30]. Using this and differentiating the inequality (3.2) at  $t = 0$  yields

$$\text{ent}_\mu(e^F) \leq \frac{1}{2\rho} \int_{\mathbb{R}^n} |\nabla F|^2 e^F d\mu$$

which is equivalent to the desired log-Sobolev inequality. We refer to Bobkov and Ledoux article [11] for more details.  $\square$

**Remark 3.1** (Proof for GUE/GOE via Hoffmann–Wielandt inequality). *For the GUE and the GOE one can give a fifth proof, based on the contraction principle, like the proof using Caffarelli’s theorem above. The Hoffman–Wielandt inequality [35, 37, 9], states that for all  $n \times n$  Hermitian matrices  $A$  and  $B$  with ordered eigenvalues  $x_1(A) \geq \dots \geq x_n(A)$  and  $x_1(B) \geq \dots \geq x_n(B)$  respectively, we have*

$$\sum_{i=1}^n (x_i(A) - x_i(B))^2 \leq \sum_{i,j=1}^n |A_{ij} - B_{ij}|^2.$$

*In other words the map which associates to an  $n \times n$  Hermitian matrix  $A$  its vector of eigenvalues  $(x_1(A), \dots, x_n(A)) \in \mathbb{R}^n$  is 1-Lipschitz for the Euclidean structure on  $n \times n$  Hermitian matrices, given by  $\langle A, B \rangle = \text{Trace}(AB)$ . On the other hand, as we saw in section 2.1, the Gaussian unitary ensemble is the image by this map of the Gaussian measure on  $\mathbb{H}_n$  whose density is proportional to  $e^{-n\text{Tr}(H^2)/2}$ . The Poincaré constant of this Gaussian measure is  $1/n$  so by the contraction principle the Poincaré constant of the GUE is  $1/n$  at most. The argument works just the same for log-Sobolev and for the GOE.*

*Proof of Theorem 1.3.* According to Lemma 2.6, if  $V$  is quadratic:  $V(u) = \rho u^2/2$  then the image of  $\mu$  by the map  $x \mapsto \sum_{i \leq n} x_i / \sqrt{n}$  is the centered Gaussian measure on  $\mathbb{R}$  with variance  $1/\rho$ . This Gaussian satisfies Poincaré with constant  $1/\rho$  and log-Sobolev with constant  $2/\rho$ . Moreover, linear functions are optimal in Poincaré and log-linear functions, i.e. functions of the form  $f(u) = e^{\lambda u}$ , are optimal in log-Sobolev, see for instance [2, 5]. Hence the result.  $\square$

**Remark 3.2** (Transport). *Many proofs involve the following simple transportation facts:*

$$\mathcal{N}(0, n^{-1}I_n) \xrightarrow{\text{Caffarelli}} \mu \xrightarrow{x_1 + \dots + x_n} \mathcal{N}(0, 1)$$

and

$$\mathcal{N}(0, n^{-1}I_n) \xrightarrow{x_1 + \dots + x_n} \mathcal{N}(0, 1)$$

and

$$\text{Law}(H) \xrightarrow{\text{Spectrum}} \mu \xrightarrow{x_1 + \dots + x_n} \mathcal{N}(0, 1)$$

and

$$\text{Law}(H) \xrightarrow{\text{Trace}} \mathcal{N}(0, 1)$$

where  $H$  is a random Hermitian matrix as in Theorem 2.1 or Theorem 2.2.

*Proof of Corollary 1.4.* The Gaussian concentration can be deduced from the log-Sobolev inequality via an argument due to Herbst, see for instance [41], which consists in using log-Sobolev with  $f = e^F$  to get the Gaussian upper bound on the Laplace transform

$$\int e^F d\mu \leq \exp\left(\int F d\mu + \frac{\|F\|_{\text{Lip}}^2}{2\rho}\right), \quad (3.3)$$

which leads in turn to the concentration inequality (1.5) via the Markov inequality. Alternatively we can use the intermediate inequality (3.2) obtained in the course of the fourth proof of Theorem 1.2. Indeed applying Jensen's inequality to the right-hand side of (3.2) and letting  $t \rightarrow 1$ , we obtain

$$\int e^{R_{1/\rho}F} d\mu \leq \exp\left(\int F d\mu\right). \quad (3.4)$$

Moreover, if  $F$  is Lipschitz it is easily seen that

$$R_{1/\rho}F \geq F - \frac{1}{2\rho}\|F\|_{\text{Lip}}^2.$$

Plugging this into the previous inequality yields (3.3). Note that a result due to Bobkov and Götze states that (3.4) is equivalent to a Talagrand  $W_2$  transportation inequality for  $\mu$ , see for instance [41] and references therein.

In the case  $F(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^n f(x_i) = L_n(f)(x)$  we have

$$\|F\|_{\text{Lip}} \leq \frac{\|f\|_{\text{Lip}}}{\sqrt{n}}$$

so that (1.6) follows from (1.5).

Finally taking  $F(x_1, \dots, x_n) = \max(x_1, \dots, x_n)$  ( $= x_1$  on  $D$ ) in (1.5) gives (1.7).  $\square$

**Remark 3.3** (Concentration in transportation distance). *Let  $\mu$  be the Gaussian unitary ensemble (1.4). Let us give a nice argument that we have learned from Nathaël Gozlan giving concentration in Kantorovich–Wasserstein distance  $W_2$ , from the Hoffman–Wielandt inequality and the fact that  $\mu$  is a Lipschitz image of  $\mathcal{N}(0, \frac{1}{n}I_n)$ . Now if  $A$  and  $B$  are  $n \times n$  Hermitian matrices with respective ordered eigenvalues  $x_1(A) \geq \dots \geq x_n(A)$  and  $x_1(B) \geq \dots \geq x_n(B)$ , and if we define the empirical measures  $L_A = \frac{1}{n} \sum_{i=1}^n \delta_{x_i(A)}$  and  $L_B = \frac{1}{n} \sum_{i=1}^n \delta_{x_i(B)}$ , then the Hoffman–Wielandt inequality reads*

$$nW_2(L_A, L_B)^2 = \sum_{i=1}^n (x_i(A) - x_i(B))^2 \leq \text{Trace}((A - B)^2). \quad (3.5)$$

*Thanks to the triangle inequality for  $W_2$ , this implies that for all probability measure  $\mu$  with finite second moment, the map  $A \mapsto W_2(L_A, \mu)$  is Lipschitz with Lipschitz norm smaller than or equal to  $1/\sqrt{n}$ . Therefore, denoting  $L_n = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ , we get, for all  $r > 0$ ,*

$$\mu(|W_2(L_n, \mu) - \mathbb{E}W_2(L_n, \mu)| > r) \leq 2 \exp\left(-n^2 \frac{r^2}{2}\right). \quad (3.6)$$

*Note that  $\mu$  is arbitrary. This simple argument remains essentially valid as soon as  $\mu$  is the distribution of the eigenvalues of a random Hermitian matrix with independent entries satisfying a logarithmic Sobolev inequality with constant of order  $1/n$ .*

*Proof of Theorem 1.5.* The exponential decay of relative entropy (1.13) is a well-known consequence of the logarithmic Sobolev inequality, see for instance [5, Theorem 5.2.1]. The decay in Wasserstein distance follows from the Bakry–Émery machinery, see [5, Theorem 9.7.2]. Alternatively it can be seen using parallel coupling. We explain this argument

briefly. Let  $X$  and  $Y$  be two solutions of the SDE (1.8) driven by the same Brownian motion:

$$\begin{aligned} dX_t &= \sqrt{2} dB_t - \nabla U(X_t) dt + d\Phi_t \\ dY_t &= \sqrt{2} dB_t - \nabla U(Y_t) dt + d\Psi_t. \end{aligned}$$

Assume additionally that  $X_0 \sim \nu_0$ ,  $Y_0 \sim \nu_1$  and that

$$\mathbb{E}(|X_0 - Y_0|^p) = W_p(\nu_0, \nu_1)^p.$$

By Itô's formula we get

$$d|X_t - Y_t|^2 = -2\langle X_t - Y_t, \nabla U(X_t) - \nabla U(Y_t) \rangle dt + 2\langle X_t - Y_t, d\Phi_t \rangle + 2\langle Y_t - X_t, d\Psi_t \rangle.$$

Since  $U$  is  $\rho$ -convex  $\langle X_t - Y_t, \nabla U(X_t) - \nabla U(Y_t) \rangle \geq \rho|X_t - Y_t|^2$ . Besides  $d\Phi_t = -n_t dL_t$  where  $L$  is the local time of  $X$  at the boundary of  $D$  and  $n_t$  is an outer unit normal at  $X_t$ . Since  $Y_t \in K$  we get in particular  $\langle X_t - Y_t, d\Phi_t \rangle \leq 0$  for all  $t$ , and similarly  $\langle Y_t - X_t, d\Psi_t \rangle \leq 0$ . Therefore  $d|X_t - Y_t|^2 \leq -2\rho|X_t - Y_t|^2 dt$ , hence

$$|X_t - Y_t| \leq e^{-\rho t} |X_0 - Y_0|.$$

Taking the  $p$ -th power and expectation we get

$$\mathbb{E}[|X_t - Y_t|^p]^{1/p} \leq e^{-\rho t} \mathbb{E}(|X_0 - Y_0|^p)^{1/p} = e^{-\rho t} W_p(\nu_0, \nu_1).$$

Moreover since  $X_t \sim \nu_0 P_t$  and  $Y_t \sim \nu_1 P_t$  we have by definition of  $W_p$

$$W_p(\nu_0 P_t, \nu_1 P_t) \leq \mathbb{E}[|X_t - Y_t|^p]^{1/p}.$$

Hence the result. □

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## REFERENCES

- [1] G. W. ANDERSON, A. GUIONNET & O. ZEITOUNI – *An introduction to random matrices*, Cambridge Studies in Advanced Mathematics, vol. 118, Cambridge Univ. Press, Cambridge, 2010. 3, 5, 8
- [2] C. ANÉ, S. BLACHÈRE, D. CHAFAÏ, P. FOUGÈRES, I. GENTIL, F. MALRIEU, C. ROBERTO & G. SCHEFFER – *Sur les inégalités de Sobolev logarithmiques*, Panoramas et Synthèses, vol. 10, Soc. Math. France, Paris, 2000. 11, 12, 13
- [3] T. H. BAKER & P. J. FORRESTER – “The Calogero-Sutherland model and generalized classical polynomials”, *Comm. Math. Phys.* **188** (1997), no. 1, p. 175–216. 7, 8, 15
- [4] D. BAKRY & M. ÉMERY – “Diffusions hypercontractives”, in *Séminaire de probabilités, XIX, 1983/84*, Lecture Notes in Math., vol. 1123, Springer, Berlin, 1985, p. 177–206. 12
- [5] D. BAKRY, I. GENTIL & M. LEDOUX – *Analysis and geometry of Markov diffusion operators*, Grundlehren der Math. Wiss., vol. 348, Springer, Cham, 2014. 11, 12, 13, 14
- [6] K. BALL – “An elementary introduction to modern convex geometry”, in *Flavors of geometry*, Math. Sci. Res. Inst. Publ., vol. 31, Cambridge Univ. Press, Cambridge, 1997, p. 1–58. 12
- [7] W. BECKNER – “A generalized Poincaré inequality for Gaussian measures”, *Proc. Amer. Math. Soc.* **105** (1989), no. 2, p. 397–400. 8
- [8] G. BEN AROUS & A. GUIONNET – “Large deviations for Wigner’s law and Voiculescu’s non-commutative entropy”, *Probab. Theory Related Fields* **108** (1997), no. 4, p. 517–542. 4
- [9] R. BHATIA – *Matrix analysis*, Graduate Texts in Mathematics, vol. 169, Springer-Verlag, New York, 1997. 13
- [10] P. BIANE & R. SPEICHER – “Free diffusions, free entropy and free Fisher information”, *Ann. Inst. H. Poincaré Probab. Statist.* **37** (2001), no. 5, p. 581–606. 6



- [11] S. G. BOBKOV & M. LEDOUX – “From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities”, *Geom. Funct. Anal.* **10** (2000), no. 5, p. 1028–1052. 11, 13
- [12] H. J. BRASCAMP & E. H. LIEB – “On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation”, *J. Functional Analysis* **22** (1976), no. 4, p. 366–389. 11
- [13] M.-F. BRU – “Wishart processes”, *J. Theoret. Probab.* **4** (1991), no. 4, p. 725–751. 8
- [14] T. CABANAL DUVILLARD & A. GUIONNET – “Large deviations upper bounds for the laws of matrix-valued processes and non-communicative entropies”, *Ann. Probab.* **29** (2001), no. 3, p. 1205–1261. 6
- [15] L. A. CAFFARELLI – “Monotonicity properties of optimal transportation and the FKG and related inequalities”, *Comm. Math. Phys.* **214** (2000), no. 3, p. 547–563. 12
- [16] ———, “Erratum: “Monotonicity of optimal transportation and the FKG and related inequalities” [Comm. Math. Phys. **214** (2000), no. 3, 547–563; MR1800860 (2002c:60029)]”, *Comm. Math. Phys.* **225** (2002), no. 2, p. 449–450. 12
- [17] E. CÉPA – “Équations différentielles stochastiques multivoques”, in *Séminaire de Probabilités, XXIX*, Lecture Notes in Math., vol. 1613, Springer, Berlin, 1995, p. 86–107. 5
- [18] E. CÉPA & D. LÉPINGLE – “Diffusing particles with electrostatic repulsion”, *Probab. Theory Related Fields* **107** (1997), no. 4, p. 429–449. 5, 6
- [19] D. CHAFAÏ – “Glauber versus Kawasaki for spectral gap and logarithmic Sobolev inequalities of some unbounded conservative spin systems”, *Markov Process. Related Fields* **9** (2003), no. 3, p. 341–362. 8
- [20] D. CHAFAÏ – “Entropies, convexity, and functional inequalities: on  $\Phi$ -entropies and  $\Phi$ -Sobolev inequalities”, *J. Math. Kyoto Univ.* **44** (2004), no. 2, p. 325–363. 8
- [21] D. CHAFAÏ – “<http://djalil.chafai.net/blog/2016/12/27/mind-the-gap/>”, blogpost, 2016. 15
- [22] D. CHAFAÏ, F. BOLLEY & J. FONTBONA – “Dynamics of a planar coulomb gas”, preprint, 2017. 8
- [23] D. CHAFAÏ, N. GOZLAN & P.-A. ZITT – “First-order global asymptotics for confined particles with singular pair repulsion”, *Ann. Appl. Probab.* **24** (2014), no. 6, p. 2371–2413. 4
- [24] Y. DOUMERC – “Matrices aléatoires, processus stochastiques et groupes de réflexions”, Thèse, Université de Toulouse, 2005. 8
- [25] I. DUMITRIU & A. EDELMAN – “Matrix models for beta ensembles”, *J. Math. Phys.* **43** (2002), no. 11, p. 5830–5847. 2, 8
- [26] I. DUMITRIU, A. EDELMAN & G. SHUMAN – “MOPS: multivariate orthogonal polynomials (symbolically)”, *J. Symbolic Comput.* **42** (2007), no. 6, p. 587–620. 7
- [27] F. J. DYSON – “A Brownian-motion model for the eigenvalues of a random matrix”, *J. Mathematical Phys.* **3** (1962), p. 1191–1198. 5
- [28] ———, “The threefold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics”, *J. Mathematical Phys.* **3** (1962), p. 1199–1215. 2, 8
- [29] L. ERDŐS & H.-T. YAU – “Dynamical approach to random matrix theory”, book in preparation, 2017. 3
- [30] L. C. EVANS – *Partial differential equations*, second éd., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010. 13
- [31] P. J. FORRESTER – *Log-gases and random matrices*, London Math. Soc. Monographs Series, vol. 34, Princeton Univ. Press, Princeton, 2010. 8
- [32] D. J. GRABINER – “Brownian motion in a Weyl chamber, non-colliding particles, and random matrices”, *Ann. Inst. H. Poincaré Probab. Statist.* **35** (1999), no. 2, p. 177–204. 8
- [33] A. GUIONNET & O. ZEITOUNI – “Concentration of the spectral measure for large matrices”, *Electron. Comm. Probab.* **5** (2000), p. 119–136. 4, 10
- [34] A. GUIONNET – *Large random matrices: lectures on macroscopic asymptotics*, Lecture Notes in Mathematics, vol. 1957, Springer-Verlag, Berlin, 2009, Lectures from the 36th Probability Summer School held in Saint-Flour, 2006. 4
- [35] A. J. HOFFMAN & H. W. WIELANDT – “The variation of the spectrum of a normal matrix”, *Duke Math. J.* **20** (1953), p. 37–39. 13
- [36] D. HOLCOMB & E. PAQUETTE – “Tridiagonal models for Dyson Brownian motion”, preprint arXiv:1707.02700, 2017. 8
- [37] R. A. HORN & C. R. JOHNSON – *Matrix analysis*, second éd., Cambridge University Press, Cambridge, 2013. 13
- [38] M. LASSALLE – “Polynômes de Hermite généralisés”, *C. R. Acad. Sci. Paris Sér. I Math.* **313** (1991), no. 9, p. 579–582. 7, 8, 15
- [39] ———, “Polynômes de Jacobi généralisés”, *C. R. Acad. Sci. Paris Sér. I Math.* **312** (1991), no. 6, p. 425–428. 8
- [40] ———, “Polynômes de Laguerre généralisés”, *C. R. Acad. Sci. Paris Sér. I Math.* **312** (1991), no. 10, p. 725–728. 8

- [41] M. LEDOUX – *The concentration of measure phenomenon*, Mathematical Surveys and Monographs, vol. 89, American Mathematical Society, Providence, RI, 2001. 14
- [42] M. MAÏDA & E. MAUREL-SEGALA – “Free transport-entropy inequalities for non-convex potentials and application to concentration for random matrices”, *Probab. Theory Related Fields* **159** (2014), no. 1-2, p. 329–356. 4
- [43] F. MALRIEU – “Logarithmic Sobolev inequalities for some nonlinear PDE’s”, *Stochastic Process. Appl.* **95** (2001), no. 1, p. 109–132. 8
- [44] O. MAZET – “Classification des semi-groupes de diffusion sur  $\mathbf{R}$  associés à une famille de polynômes orthogonaux”, in *Séminaire de Probabilités, XXXI*, Lecture Notes in Math., vol. 1655, Springer, Berlin, 1997, p. 40–53. 6
- [45] M. L. MEHTA – *Random matrices*, Pure and Applied Math. (Amsterdam), vol. 142, Elsevier/Academic Press, Amsterdam, 2004. 8
- [46] L. C. G. ROGERS & Z. SHI – “Interacting Brownian particles and the Wigner law”, *Probab. Theory Related Fields* **95** (1993), no. 4, p. 555–570. 5, 6
- [47] H. TANAKA – “Stochastic differential equations with reflecting boundary condition in convex regions”, *Hiroshima Math. J.* **9** (1979), no. 1, p. 163–177. 4, 5

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