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► **To cite this version:**

Amina Doumane, Damien Pous. Completeness for Identity-free Kleene Lattices. CONCUR, Sep 2018, Beijing, China. 10.4230/LIPIcs.CONCUR.2018.18 . hal-01780845v2

HAL Id: hal-01780845

<https://hal.science/hal-01780845v2>

Submitted on 2 Jul 2018

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1 Completeness for Identity-free Kleene Lattices*

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8 — Abstract —

9 We provide a finite set of axioms for identity-free Kleene lattices, which we prove sound and
10 complete for the equational theory of their relational models. Our proof builds on the complete-
11 ness theorem for Kleene algebra, and on a novel automata construction that makes it possible to
12 extract axiomatic proofs using a Kleene-like algorithm.

13 **2012 ACM Subject Classification** Theory of computation → Regular languages

14 **Keywords and phrases** Kleene algebra, Graph languages, Petri Automata, Kleene theorem

15 **Funding** This work has been funded by the European Research Council (ERC) under the Eu-
16 ropean Union’s Horizon 2020 programme (CoVeCe, grant agreement No 678157). This work
17 was supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within
18 the program "Investissements d’Avenir" (ANR-11-IDEX-0007) operated by the French National
19 Research Agency (ANR).

20 **1** Introduction

21 Relation algebra is an efficient tool to reason about imperative programs. In this approach,
22 the bigstep semantics of a program P is a binary relation $[P]$ between memory states [20,
23 22, 6, 16, 1]. This relation is built from the elementary relations corresponding to the
24 atomic instructions of P , which are combined using standard operations on relations, for
25 instance composition and transitive closure, that respectively encode sequential composition
26 of programs, and iteration (while loops). Abstracting over the concrete behaviour of atomic
27 instructions, one can compare two programs P, Q by checking whether the expressions $[P]$
28 and $[Q]$ are equivalent in the model of binary relations, which we write as $\mathcal{Rel} \models [P] = [Q]$.

29 To enable such an approach, one should obtain two properties: decidability of the
30 predicate $\mathcal{Rel} \models e = f$, given two expressions e and f as input, and axiomatisability of
31 this relation. Decidability makes it possible to automate the verification process, thus
32 alleviating the burden for the end-user [17, 14, 9, 25, 28]. Axiomatisation offers a better way
33 of understanding the equational theory of relations and provides a certificate for programs
34 verification. Indeed, an axiomatic proof of $e = f$ can be seen as a certificate, which can
35 be exchanged, proofread, and combined in a modular way. Axiomatisations also make it
36 possible to solve hard instances manually, when the existing decision procedures have high
37 complexity and/or when considered instances are large [24, 17, 7].

38 Depending on the class of programs under consideration, several sets of operations
39 on relations can be considered. In this paper we focus on the following set of operations:
40 composition (\cdot), transitive closure ($_+$), union ($+$), intersection (\cap) and the empty relation (0).

* Full version of the extended abstract in Proc. CONCUR 2018 [13].

41 The expressions generated by this signature are called KL^- -expressions. An example of an
 42 inequality in the corresponding theory is $\mathcal{R}el \models (a \cap c) \cdot (b \cap d) \leq (a \cdot b)^+ \cap (c \cdot d)$: when
 43 a, b, c, d are interpreted as arbitrary binary relations, we have $(a \cap c) \cdot (b \cap d) \subseteq (a \cdot b)^+ \cap (c \cdot d)$.
 44 The operations of composition, union and transitive closure arise naturally when defining the
 45 bigstep semantics of sequential programs. In contrast, intersection, which is the operation of
 46 interest in the present paper, is not a standard operation on programs. This operation is
 47 however useful when it comes to specifications: it allows one to express local conjunctions
 48 of specifications. For instance, a specification of the shape $(a \cap b)^+$ expresses the fact that
 49 execution traces must consist of sequences of smaller traces satisfying both a and b .

50 The operations of KL^- contain those of identity-free regular expressions, whose equational
 51 theory inherits the good properties of *Kleene algebra* (KA). We summarise them below.

52 First recall that each regular expression e can be associated with a set of words $\mathcal{L}(e)$ called
 53 its language. Valid inequations between regular expressions inequalities can be characterised
 54 by language inclusions [29]:

$$55 \quad \mathcal{R}el \models e \leq f \quad \text{iff} \quad \mathcal{L}(e) \subseteq \mathcal{L}(f) \quad (1)$$

56 Second, we have the celebrated equivalence between regular expressions and non-deterministic
 57 finite automata (NFA) via a *Kleene theorem*: for every regular expression e , there is an NFA
 58 such that $\mathcal{L}(e)$ is the language of A , and conversely. Decidability follows (in fact, PSPACE-
 59 completeness). Lastly, although every purely equational axiomatisation of this theory must
 60 be infinite [30], Kozen has proved that Conway's finite quasi-equational axiomatisation [12]
 61 is sound and complete [19]. (There is also an independent proof of this result by Boffa [8],
 62 based on the extensive work of Krob [26].)

63 Those three results nicely restrict to identity-free Kleene algebra (KA^-), which form a
 64 proper fragment of Kleene algebra [21]. It suffices to consider languages of non-empty words:
 65 Equation (1) remains, Kleene's theorem still holds, and we have the following characterisation,
 66 where we write $\text{KA}^- \vdash e \leq f$ when $e \leq f$ is derivable from the axioms of KA^- :

$$67 \quad \mathcal{L}(e) \subseteq \mathcal{L}(f) \quad \text{iff} \quad \text{KA}^- \vdash e \leq f \quad (2)$$

68 There are counterparts to the first two points for KL^- -expressions. Each KL^- -expression
 69 e can be associated with a set of graphs $\mathcal{G}(e)$ called its graph language, and valid inequations
 70 of KL^- -expressions can be characterised through these languages of graphs. A subtlety here
 71 is that we have to consider graphs modulo homomorphisms; writing $\triangleleft \mathcal{G}$ for the closure of a
 72 set of graphs \mathcal{G} under graph homomorphisms, we have [10]:

$$73 \quad \mathcal{R}el \models e \leq f \quad \text{iff} \quad \triangleleft \mathcal{G}(e) \subseteq \triangleleft \mathcal{G}(f) \quad (3)$$

74 KL^- -expressions are equivalent to a model of automata over graphs called Petri automata [10].
 75 As for KA^- -expressions, a Kleene-like theorem holds [11]: for every KL^- -expression e , there is
 76 a Petri automaton whose language is $\mathcal{G}(e)$, and conversely. Decidability (in fact, EXPSpace-
 77 completeness) of the equational theory follows [10, 11].

78 What is missing to this picture is an axiomatisation of the corresponding equational theory.
 79 In the present paper, we provide such an axiomatisation, which we call KL^- , and which
 80 comprises the axioms for identity-free Kleene algebra (KA^-) and the axioms of *distributive*
 81 *lattices* for $\{+, \cap\}$. Completeness of this axiomatisation is the difficult result we prove here:

$$82 \quad \triangleleft \mathcal{G}(e) \subseteq \triangleleft \mathcal{G}(f) \quad \text{entails} \quad \text{KL}^- \vdash e \leq f \quad (4)$$

83 We proceed in two main steps. First we show that $\mathcal{G}(e) \subseteq \mathcal{G}(f)$ entails $\text{KL}^- \vdash e \leq f$,
 84 using a technique inspired from [23], this is what we call *completeness for strict language*

85 *inclusion*. The second step is much more involved. There we exploit the Kleene theorem for
 86 Petri automata [11]: starting from expressions e, f such that $\triangleleft \mathcal{G}(e) \subseteq \triangleleft \mathcal{G}(f)$, we build two
 87 Petri automata \mathcal{A}, \mathcal{B} respectively recognising $\mathcal{G}(e)$ and $\mathcal{G}(f)$. Then we design a product
 88 construction to synchronise \mathcal{A} and \mathcal{B} , and a Kleene-like algorithm to extract from this
 89 construction two expressions e', f' such that $\mathcal{G}(e) = \mathcal{G}(e')$, $\text{KL}^- \vdash e' \leq f'$, and $\mathcal{G}(f') \subseteq \mathcal{G}(f)$.
 90 This *synchronised Kleene theorem* suffices to conclude using the first step.

91 To our knowledge, this is the first completeness result for a theory involving Kleene
 92 iteration and intersection. Identity-free Kleene lattices were studied in depth by Andr eka,
 93 Mikulas and Nemeti [3]; they have in particular shown that over this syntax, the equational
 94 theories generated by binary relations and formal languages coincide. But axiomatisability
 95 remained opened. The restriction to the identity-free fragment is important for several
 96 reasons. First of all, it makes it possible to rely on the technique used in [10] to compare
 97 Petri automata, which does not scale in the presence of identity. Second, this is the fragment
 98 for which the Kleene theorem for Petri automata is proved the most naturally [11]. Third,
 99 ‘strange’ laws appear in the presence of 1 [2], *e.g.*, $1 \cap (b \cdot a) \leq a \cdot (1 \cap (b \cdot a)) \cdot b$, and
 100 axiomatisability is still open even in the finitary case where Kleene iteration is absent—see
 101 the erratum about [2].

102 Proofs of completeness for other extensions of Kleene algebra include Kleene algebra with
 103 tests (KAT) [20], nominal Kleene algebra [23], and Concurrent Kleene algebra [27, 18]. The
 104 latter extension is the closest to our work since the parallel operator of concurrent Kleene
 105 algebra shares some properties of the intersection operation considered in the present work
 106 (*e.g.*, it is commutative and it satisfies a weak interchange law with sequential composition).

107 The paper is organised as follows. In Sect. 2, we recall KL^- -expressions, their graph
 108 language and the corresponding model of Petri automata. In Sect. 3 we give our axiomatisation
 109 and state the completeness result. Then we show it following the proof scheme presented
 110 earlier: in Sect. 4 we show completeness for strict language inclusions, we recall in Sect. 5
 111 the Kleene theorem of KL^- expressions, on which we build to show our synchronised Kleene
 112 theorem in Sect. 6.

113 2 Expressions, graph languages and Petri automata

114 2.1 Expressions and their relational semantics

115 We let $a, b \dots$ range over the letters of a fixed alphabet X . We consider the following syntax
 116 of KL^- -expressions, which we simply call expressions if there is no ambiguity:

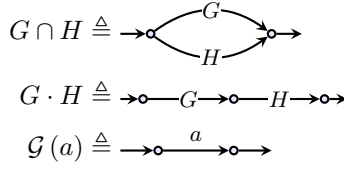
$$117 \quad e, f ::= e \cdot f \mid e + f \mid e \cap f \mid e^+ \mid 0 \mid a \quad (a \in X)$$

119 We denote their set by Exp_X and we often write ef for $e \cdot f$. When we remove intersection
 120 (\cap) from the syntax of KL^- -expressions we get KA^- -expressions, which are the identity-free
 121 regular expressions.

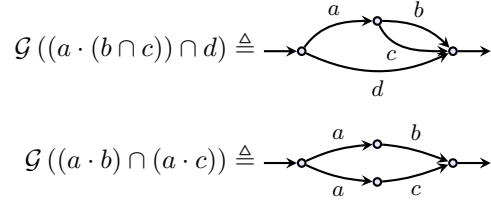
122 If $\sigma : X \rightarrow \mathcal{P}(S \times S)$ is an interpretation of the letters into some space of relations, we
 123 write $\widehat{\sigma}$ for the unique homomorphism extending σ into a function from Exp_X to $\mathcal{P}(S \times S)$.
 124 An inequation between two expressions e and f is *valid*, written $\text{Rel} \models e \leq f$, if for every
 125 such interpretation σ we have $\widehat{\sigma}(e) \subseteq \widehat{\sigma}(f)$.

126 2.2 Terms, graphs, and homomorphisms

127 We let $u, v \dots$ range over expressions built using only letters, \cap and \cdot , which we call *terms*.
 128 (Terms thus form a subset of expressions: they are those expressions not using 0, + and $_+$.)



■ **Figure 1** Operations on graphs.



■ **Figure 2** Graphs associated with some terms.

129 We will use 2-pointed labelled directed graphs, simply called *graphs* in the sequel. Those are
 130 tuples $\langle V, E, s, t, l, \iota, o \rangle$ with V (resp. E) a finite set of vertices (resp. edges), $s, t : E \rightarrow V$ the
 131 *source* and *target* functions, $l : E \rightarrow X$ the *labelling* function, and $\iota, o \in V$ two distinguished
 132 vertices, respectively called *input* and *output*.

133 As depicted in Fig. 1, graphs can be composed in series or in parallel, and a letter can be
 134 seen as a graph with a single edge labelled by that letter. One can thus recursively associate
 135 to every term u a graph $\mathcal{G}(u)$ called the *graph of u* . Two examples are given in Fig. 2; graphs
 136 of terms are *series-parallel* [31].

137 ► **Definition 1** (Graph homomorphism). A *homomorphism* from $G = \langle V, E, s, t, l, \iota, o \rangle$ to
 138 $G' = \langle V', E', s', t', l', \iota', o' \rangle$ is a pair $h = \langle f, g \rangle$ of functions $f : V \rightarrow V'$ and $g : E \rightarrow E'$ that
 139 respect the various components: $s' \circ g = f \circ s$, $t' \circ g = f \circ t$, $l = l' \circ g$, $\iota' = f(\iota)$, and $o' = f(o)$.
 140 We write $G' \triangleleft G$ if there exists a graph homomorphism from G to G' .

141 Such a homomorphism is depicted in Fig. 3. A pleasant way to think about graph homomor-
 142 phisms is the following: we have $G \triangleleft H$ if G is obtained from H by merging (or identifying)
 143 some nodes, and by adding some extra nodes and edges. For instance, the graph G in Fig. 3
 144 is obtained from H by merging the nodes 1, 2 and by adding an edge between the input and the
 145 output labelled by d .

146 The starting point of the present work is the following characterisation:

147 ► **Theorem 2** ([5, Thm. 1], [15, p. 208]). *For all terms u, v , $\mathcal{R}el \models u \leq v$ iff $\mathcal{G}(u) \triangleleft \mathcal{G}(v)$.*

148 2.3 Graph language of an expression

149 To generalise the previous characterisation to KL^- -expressions, one interprets expressions by
 150 sets (languages) of graphs: graphs play the role of words for KA -expressions.

151 ► **Definition 3** (Term and graph languages of expressions). The *term language* of an expression
 152 e , written $\llbracket e \rrbracket$, is the set of terms defined recursively as follows:

$$\begin{array}{ll}
 153 & \llbracket e \cdot f \rrbracket \triangleq \{u \cdot v \mid u \in \llbracket e \rrbracket \text{ and } v \in \llbracket f \rrbracket\} & \llbracket 0 \rrbracket \triangleq \emptyset \\
 154 & \llbracket e \cap f \rrbracket \triangleq \{u \cap v \mid u \in \llbracket e \rrbracket \text{ and } v \in \llbracket f \rrbracket\} & \llbracket a \rrbracket \triangleq \{a\} \\
 155 & \llbracket e + f \rrbracket \triangleq \llbracket e \rrbracket \cup \llbracket f \rrbracket & \llbracket e^+ \rrbracket \triangleq \bigcup_{n>0} \{u_1 \cdots u_n \mid \forall i, u_i \in \llbracket e \rrbracket\}
 \end{array}$$

157 The *graph language* of e is the set of graphs $\mathcal{G}(e) \triangleq \{\mathcal{G}(u) \mid u \in \llbracket e \rrbracket\}$.

158 Note that for every term u , $\llbracket u \rrbracket = \{u\}$, so that the graph language of u thus contains just the
 159 graph of u . This justifies the overloaded notation $\mathcal{G}(u)$. Given a set S of graphs, we write
 160 $\triangleleft S$ for its downward closure w.r.t. \triangleleft : $\triangleleft S \triangleq \{G \mid G \triangleleft G', G' \in S\}$. We obtain:

161 ► **Theorem 4** ([10, Thm. 6]). *For all expressions e, f , $\mathcal{R}el \models e \leq f$ iff $\triangleleft \mathcal{G}(e) \subseteq \triangleleft \mathcal{G}(f)$.*

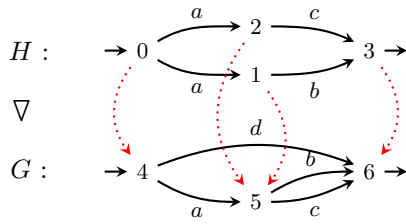


Figure 3 A graph homomorphism.

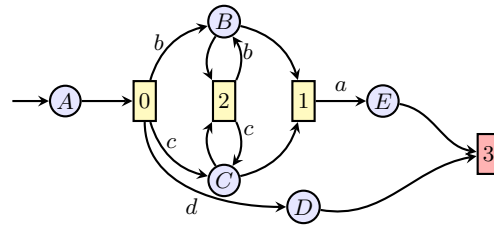


Figure 4 A Petri automaton.

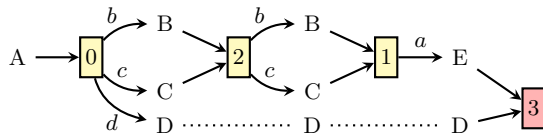


Figure 5 Run of a Petri automaton.

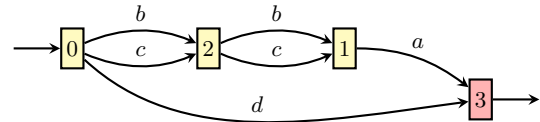


Figure 6 Graph of a run.

2.4 Petri automata

We recall the notion of Petri automata [10, 11], an automata model that recognises precisely the graph languages of our expressions.

► **Definition 5** (Petri Automaton). A *Petri automaton* (PA) over the alphabet X is a tuple $\mathcal{A} = \langle P, \mathcal{T}, \iota \rangle$ where:

- P is a finite set of *places*,
- $\mathcal{T} \subseteq \mathcal{P}(P) \times \mathcal{P}(X \times P)$ is a set of *transitions*,
- $\iota \in P$ is the *initial place* of the automaton.

For each transition $t = \langle {}^a t, t^\flat \rangle \in \mathcal{T}$, ${}^a t$ is assumed to be non-empty; ${}^a t \subseteq P$ is the *input* of t ; and $t^\flat \subseteq X \times P$ is the *output* of t . We write $\pi_2(t^\flat) \triangleq \{p \mid \exists a, \langle a, p \rangle \in t^\flat\}$ for the set of the output places of t . Transitions with empty outputs are called *final*.

A PA is depicted in Fig. 4: places are represented by circles and transitions by squares.

Let us now recall the operational semantics of PA. Fix a PA $\mathcal{A} = \langle P, \mathcal{T}, \iota \rangle$ for the remainder of this section. A *state* of this automaton is a set of places. In a given state $S \subseteq P$, a transition $t = \langle {}^a t, t^\flat \rangle$ is *enabled* if ${}^a t \subseteq S$. In that case, we may fire t , leading to a new state $S' = (S \setminus {}^a t) \cup \pi_2(t^\flat)$. We write $S \xrightarrow{t, \mathcal{A}} S'$ in this case.

► **Definition 6** (Run of a PA). A *run* is a sequence $\langle S_1, t_1, S_2, \dots, t_{n-1}, S_n \rangle$, where S_i are states, t_i are transitions such that $S_i \xrightarrow{t_i, \mathcal{A}} S_{i+1}$ for every $i \in [1, n-1]$, $S_1 = \{\iota\}$ and $S_n = \emptyset$.

A run of the PA from Fig. 4 is depicted in Fig. 5; this run gives rise to a graph, depicted in Fig. 6; see [11, Def. 3] for a formal definition in the general case.

► **Definition 7** (Graph language of a PA). The *graph language* of a PA \mathcal{A} , written $\mathcal{G}(\mathcal{A})$, consists of the graphs of its runs.

PA are assumed to be *safe* (in standard Petri net terminology, places contain at most one *token* at any time—whence the definition of states as sets rather than multisets) and to accept only series-parallel graphs. These two conditions are decidable [11]. Here we moreover assume that all PA have the same set of places P .

PA and KL^- -expressions denote the same class of graph languages:

18:6 Completeness for Identity-free Kleene Lattices

$$\begin{array}{lll}
e \cap (f \cap g) = (e \cap f) \cap g & e \cap f = f \cap e & e \cap e = e \\
e \cap (f + g) = (e \cap f) + (e \cap g) & e \cap (e + f) = e & e + (e \cap f) = e \\
e + (f + g) = (e + f) + g & e + f = f + e & e + e = e \\
e \cdot (f \cdot g) = (e \cdot f) \cdot g & e \cdot (f + g) = e \cdot f + e \cdot g & (e + f) \cdot g = e \cdot g + f \cdot g \\
e + 0 = e & e \cdot 0 = 0 = 0 \cdot e & \\
e + e \cdot e^+ = e^+ = e + e^+ \cdot e & e \cdot f + f = f \Rightarrow e^+ \cdot f + f = f & f \cdot e + f = f \Rightarrow f \cdot e^+ + f = f
\end{array}$$

■ **Figure 7** KL^- : the first three lines correspond to distributive lattices, the last three to KA^- .

189 ► **Theorem 8** (Kleene theorem [11, Thm. 18]).

- 190 (i) For every expression e , there is a Petri automaton \mathcal{A} such that $\mathcal{G}(e) = \mathcal{G}(\mathcal{A})$.
191 (ii) Conversely, for every Petri automaton \mathcal{A} , there is an expression e such that $\mathcal{G}(e) =$
192 $\mathcal{G}(\mathcal{A})$.

193 3 Axiomatisation and structure of completeness proof

194 Let us introduce now our axiomatisation.

- 195 ► **Definition 9.** The axioms of KL^- are the union of
196 ■ the axioms of identity-free Kleene algebra (KA^-) [21], and
197 ■ the axioms of a distributive lattice for $\{+, \cap\}$.

198 It is easy to check that those axioms are valid for binary relations, whence soundness of KL^- :

199 ► **Theorem 10** (Soundness). If $\text{KL}^- \vdash e \leq f$ then $\text{Rel} \models e \leq f$.

200 The rest the paper is devoted the converse implication, which thanks to Thm. 4 amounts to:

201 ► **Theorem 11** (Completeness). If $\triangleleft \mathcal{G}(e) \subseteq \triangleleft \mathcal{G}(f)$ then $\text{KL}^- \vdash e \leq f$.

202 The following very weak form of Thm. 11 is easy to obtain from the results in the literature:

203 ► **Proposition 1.** For all terms u, v , $\mathcal{G}(u) \triangleleft \mathcal{G}(v)$ entails $\text{KL}^- \vdash u \leq v$.

204 **Proof.** Follows from Thm. 4, completeness of semilattice-ordered semigroups [4] for relational
205 models, and the fact the the axioms of KL^- entail those of semilattice-ordered semigroups. ◀

206 As explained in the introduction, our first step consists in proving KL^- completeness w.r.t.
207 strict graph language inclusions, *i.e.*, not modulo homomorphisms:

208 ► **Theorem 12** (Completeness for strict language inclusions). If $\mathcal{G}(e) \subseteq \mathcal{G}(f)$ then $\text{KL}^- \vdash e \leq f$.

209 The proof is given in Sect. 4. Our second step is to get the following theorem (Sect. 6):

210 ► **Theorem 13** (Synchronised Kleene Theorem). If \mathcal{A}, \mathcal{B} are PA such that $\triangleleft \mathcal{G}(\mathcal{A}) \subseteq \triangleleft \mathcal{G}(\mathcal{B})$,
211 then there are expressions e, f such that:

212 $\mathcal{G}(\mathcal{A}) = \mathcal{G}(e), \quad \text{KL}^- \vdash e \leq f, \quad \text{and} \quad \mathcal{G}(f) \subseteq \mathcal{G}(\mathcal{B}).$
213

214 The key observation for the proof is that the state-removal procedure used to transform a
 215 PA into a KL^- expression is highly non-deterministic. When considering two PA at a time,
 216 one can use this flexibility in order to synchronise the computation of the two expressions, so
 217 that they become easier to compare axiomatically. The concrete proof is quite technical and
 218 requires us to first recall many concepts from the proof [11] of Thm. 8(ii) (Sect. 5); it heavily
 219 relies on both Thm. 12 and Prop. 1.

220 Completeness of KL^- follows using Thm. 8(i) and Thm. 12 as explained in the introduction.

221 **4** Completeness for strict language inclusion

222 Recall that the graph language of an expression e , $\mathcal{G}(e)$, is defined as the set of graphs of the
 223 term language of e , $\llbracket e \rrbracket$. We first prove that KL^- is complete for term language inclusions:

224 **► Proposition 2.** *If $\llbracket e \rrbracket \subseteq \llbracket f \rrbracket$ then $\text{KL}^- \vdash e \leq f$.*

225 **Proof.** We follow a technique similar to the one recently used in [23]. We consider the
 226 maximal KA^- -subexpressions, and we compute the atoms of the Boolean algebra of word
 227 languages generated by those expressions. By KA^- completeness [19, 21], we get KA^- (and
 228 thus KL^-) proofs that those are equal to appropriate sums of atoms. We distribute the
 229 surrounding intersections over those sums and replace the resulting intersections of atoms by
 230 fresh letters. This allows us to proceed recursively (on the intersection-depth of the terms),
 231 using substitutivity to recover a KL^- proof of the starting inequality. ◀

232 The difference between the term language and the graph language is that intersection
 233 is interpreted as an associative and commutative operation in the latter. We bury this
 234 difference by defining a ‘saturation’ function s on KL^- -expressions such that for all e ,

$$235 \quad (\dagger) \quad \text{KL}^- \vdash s(e) = e, \quad \text{and} \quad (\ddagger) \quad \llbracket s(e) \rrbracket = \{u \mid \mathcal{G}(u) \in \mathcal{G}(e)\} .$$

237 Intuitively, this function uses distributivity and idempotency of sum to replace all intersections
 238 appearing in the expression by the sum of all their equivalent presentations modulo associativ-
 239 ity and commutativity. For instance, $s(a \cap (b \cap c))$ is a sum of twelve terms (six choices for the
 240 ordering times two choices for the parenthesing). Technically, one should be careful to expand
 241 the expression first by maximally distributing sums, in order to make all potential n-ary
 242 intersections apparent. For instance, $((a \cap b) + d) \cap c$ expands to $((a \cap b) \cap c) + (d \cap c)$ so that
 243 its saturation is a sum of twelve plus two terms. For the same reason, all iterations should be
 244 unfolded once: we unfold and expand $(a \cap b)^+ \cap c$ into $((a \cap b) \cap c) + ((a \cap b) \cdot (a \cap b)^+ \cap c)$
 245 before saturating it. We finally obtain Thm. 12 using (\ddagger) , Prop. 2, and (\dagger) :

$$246 \quad \mathcal{G}(e) \subseteq \mathcal{G}(f) \quad \Rightarrow \quad \llbracket s(e) \rrbracket \subseteq \llbracket s(f) \rrbracket \quad \Rightarrow \quad \text{KL}^- \vdash s(e) \leq s(f) \quad \Rightarrow \quad \text{KL}^- \vdash e \leq f$$

248 **5** Kleene theorem for Petri automata

249 To prove the synchronised Kleene theorem (Thm. 13), we cannot use the Kleene theorem for
 250 PA (Thm. 8) as a black box: we use in a fine way the algorithm underlying the proof of the
 251 second item. We thus explain how it works [11] in details.

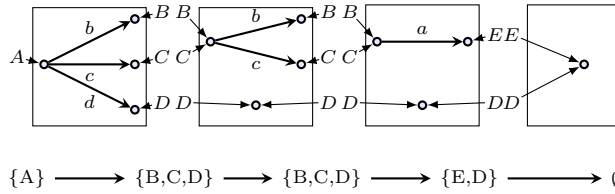
252 Recall that to transform an NFA \mathcal{A} to a regular expression e , one rewrites it using the
 253 rules of Fig. 8 until one reaches an automaton where there is a unique transition from the
 254 initial state to the final one, labelled by an expression e . While doing so, one goes through
 255 generalised NFA, whose transitions are labelled by regular expressions instead of letters.



■ **Figure 8** Rewriting rules for state-removal procedure.

256 We use the same technique for PA: we start by converting the PA into a NFA over a
 257 richer alphabet, which we call a *Template Automaton (TA)*, then we reduce this automaton
 258 using the rules of Fig. 8 until we get a single transition labelled by the desired expression.

259 To get some intuitions about the way we convert a PA into an NFA, consider the run in
 260 Fig. 5 and its graph in Fig. 6. One can decompose the run and the graph as follows:



261

262 The graph can thus be seen as a word over an alphabet of ‘boxes’, and the run as a path in an
 263 NFA whose states are sets of places of the PA. The letters of the alphabet, the above boxes,
 264 can be seen as ‘slices of graphs’; they arise naturally from the transitions of the starting PA
 265 (Fig. 4 in this example).

266 5.1 Template automata

267 In order to make everything work, we need to refine both this notion of states and this notion
 268 of boxes to define template automata:

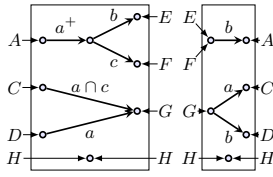
- 269 ■ states (sets of places) are refined into *types*. We let σ, τ range over types. A type is a
 270 tree whose leaves are labelled by places. When we forget the tree structure of a type τ ,
 271 we get a a state $\bar{\tau}$. See [11, Def. 10] for a formal definition of types, which is not needed
 272 here. We call *singleton types* those types whose associated state is a singleton.
- 273 ■ letters will be *templates*: finite sets of boxes like depicted above but with edges labelled
 274 with arbitrary KL⁻-expressions; we define those formally below.

275 Given a directed acyclic graph (DAG) G , we write $\min G$ (resp. $\max G$) for the set of its
 276 sources (resp. sinks). A DAG is non-trivial when it contains at least one edge.

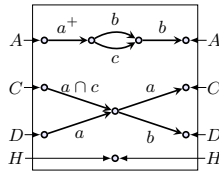
277 ► **Definition 14 (Boxes).** Let σ, τ be types. A *box* from σ to τ is a triple $\langle \vec{p}, G, \overleftarrow{p} \rangle$ where
 278 G is a non-trivial DAG with edges labelled in Exp_X , \vec{p} is a map from $\bar{\sigma}$, the *input ports*, to
 279 the vertices of G , and \overleftarrow{p} is a bijective map from $\bar{\tau}$, the *output ports*, to $\max G$, and where
 280 an additional condition relative to types holds [11, Def. 11]. (This condition can be kept
 281 abstract here.) A *basic* box is a box labelled with letters rather than arbitrary expressions.
 282 A *1-1* box is a box between singleton types.

283 We let α, β range over boxes and we write $\beta : \sigma \rightarrow \tau$ when β is a box from σ to τ .

284 We represent boxes graphically as in Fig. 15. Inside the rectangle is the DAG, with the
 285 input ports on the left-hand side and the output ports on the right-hand side. The maps \vec{p}
 286 and \overleftarrow{p} are represented by the arrows going from the ports to vertices inside the rectangle.



■ **Figure 9** Two boxes and their composition.



■ **Figure 10** An atomic box.

287 Note that unlike $\overleftarrow{\mathfrak{p}}$, the map $\overrightarrow{\mathfrak{p}}$ may reach inner nodes of the DAG. 1-1 boxes are those with
 288 exactly one input port and one output port.

289 Boxes compose like in a category: if $\alpha : \sigma \rightarrow \tau$ and $\beta : \tau \rightarrow \rho$ then we get a box
 290 $\alpha \cdot \beta : \sigma \rightarrow \rho$ by putting the graph of α to the left of the graph of β , and for every port $p \in \overline{\sigma}$,
 291 we identify the node $\overleftarrow{\mathfrak{p}}_1(p)$ with the node $\overrightarrow{\mathfrak{p}}_2(p)$. For instance the third box in Fig. 15 is
 292 obtained by composing the first two.

293 The key property enforced by the condition on types (kept abstract here) is the following:

294 ► **Lemma 15.** *A 1-1 box is just a series-parallel 2-pointed graph labelled in Exp_X .*

295 Accordingly, one can extract a KL^- -expression from any 1-1 box β , which we write $e(\beta)$ and
 296 call its *expression*.

297 ► **Definition 16 (Templates).** A *template* $\Gamma : \sigma \rightarrow \tau$ is a finite set of boxes from σ to τ . A
 298 *1-1 template* is a template of 1-1 boxes. The *expression* of a 1-1 template, written $e(\Gamma)$, is
 299 the sum of the expressions of its boxes.

300 Templates can be composed like boxes, by computing all pairwise box compositions.

301 ► **Definition 17 (Box language of a template).** A basic box is *generated* by a box β if it can
 302 be obtained by replacing each edge $x \xrightarrow{e} y$ of its DAG by a graph $G' \in \mathcal{G}(e)$ with input
 303 vertex x and output vertex y . The *box language* of a template Γ , written $\mathcal{B}(\Gamma)$, is the set of
 304 basic boxes generated by its boxes.

305 As expected, the box language of a template $\Gamma : \sigma \rightarrow \tau$ only contains boxes from σ to τ .
 306 Thanks to Lem. 15, when Γ is a 1-1 template, its box language can actually be seen as a set
 307 of graphs, and we have:

308 ► **Proposition 3.** *For every 1-1 template Γ , we have $\mathcal{B}(\Gamma) = \mathcal{G}(e(\Gamma))$.*

309 We can finally define template automata:

310 ► **Definition 18 (Template automaton (TA)).** A *template automaton* is an NFA whose states
 311 are types, whose alphabet is the set of templates, whose transitions are of the form $\langle \sigma, \Gamma, \tau \rangle$
 312 where $\Gamma : \sigma \rightarrow \tau$, and with a single initial state and a single accepting state which are
 313 singleton types. A *basic TA* is a TA whose all transitions are labelled by basic boxes.

314 By definition, a word accepted by a TA is a sequence of templates that can be composed
 315 into a single 1-1 template Γ , and thus gives rise to a set of graphs $\mathcal{B}(\Gamma)$. The *graph language*
 316 *of a TA \mathcal{E}* , written $\mathcal{G}(\mathcal{E})$, is the union of all those sets of graphs.

317 An important result of [11] is that we can translate every PA into a TA:

318 ► **Proposition 4.** *For every PA \mathcal{A} , there exists a basic TA \mathcal{E} such that $\mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{E})$.*

319 TA were defined so that they can be reduced using the state-removal procedure from Fig. 8.
 320 Templates can be composed sequentially and are closed under unions, so that now we only
 321 miss an operation $_*$ on templates to implement the first rule. Since we work in an identity-
 322 free (and thus star-free) setting, it suffices to define a strict iteration operation $_+$; and to
 323 rely on the following shorthands $\Delta \cdot \Gamma^* = \Delta \cup \Delta \cdot \Gamma^+$ and $\Gamma^* \cdot \Delta = \Delta \cup \Gamma^+ \cdot \Delta$.

324 Such an operation is provided in [11]:

325 ► **Proposition 5.** *There exists a function $_+$ on templates such that if the TA obtained from*
 326 *a PA \mathcal{A} through Prop. 4 reduces to a TA \mathcal{E} by the rules in Fig. 8, then $\mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{E})$.¹*

327 One finally obtains the Kleene theorem for PA by reducing the TA until it consists of a single
 328 transition labelled by a 1-1 template Γ : at this point, $e(\Gamma)$ is the desired KL^- -expression.

329 5.2 Computing the iteration of a template

330 We need to know how the above template iteration can be defined to obtain our synchronised
 331 Kleene theorem, so that we explain it in this section. This section is required only to
 332 understand how we define a synchronised iteration operation in Sect. 6.

333 First notice that templates on which we need to compute $_+$ are of type $\sigma \rightarrow \sigma$. We first
 334 define this operation for a restricted class of templates, which we call *atomic*.

335 ► **Definition 19** (Atomic boxes and templates, Support). A box $\beta = \langle \vec{p}, G, \overleftarrow{p} \rangle : \sigma \rightarrow \sigma$ is
 336 *atomic* if its graph has a single non-trivial connected component C , and if for every vertex v
 337 outside C , there is a unique port $p \in \bar{\sigma}$ such that $\vec{p}(p) = \overleftarrow{p}(p) = v$. An *atomic template* is
 338 a template composed of atomic boxes.

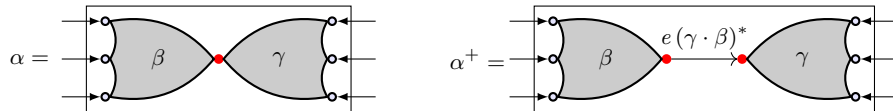
339 The *support* of a box $\beta : \sigma \rightarrow \sigma$ is the set $\text{supp}(\beta) \triangleq \{p \mid \vec{p}(p) \neq \overleftarrow{p}(p)\}$. The support
 340 of a template is the union of the supports of its boxes.

341 The following property of atomic boxes, makes it possible to compute their iteration:

342 ► **Lemma 20** ([11, Lem. 7.18]). *The non-trivial connected component of an atomic box*
 343 *$\beta : \sigma \rightarrow \sigma$ always contains a vertex c , s.t. for every port p mapped inside that component, all*
 344 *paths from $\vec{p}(p)$ to a maximal vertex visit c . We call such a vertex a bowtie for β .*

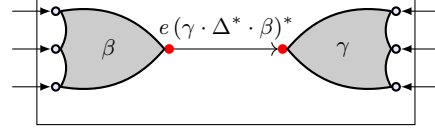
345 Notice that the bowtie of a box is not unique. For instance, the atomic box in Fig. 10
 346 contains two bowties: the blue and the red nodes.

347 We compute the iteration of an atomic box as follows. First choose a bowtie for this box,
 348 then split it at the level of this node into the product $\alpha = \beta \cdot \gamma$. The box $\gamma \cdot \beta$ is 1-1, we can
 349 thus extract from it a term $e(\gamma \cdot \beta)$. We set α^+ to be the template consisting of α and the
 350 box obtained from α by replacing the bowtie by an edge labelled $e(\gamma \cdot \beta)^+$. For the sake of
 351 conciseness, we denote this two-box template as on the right below, with an edge labelled
 352 with a starred expression.



¹ This statement is not simpler because, unfortunately, there is no function $_+$ on templates such that $\mathcal{B}(\Gamma^+) = \mathcal{B}(\Gamma)^+$.

Data: Atomic template Γ
Result: A template Γ^+ s.t. $\mathcal{B}(\Gamma^+) = \mathcal{B}(\Gamma)^+$
if $\Gamma = \emptyset$ **then**
 | Return \emptyset
else
 Write $\Gamma = \Delta \cup \{\alpha\} \cup \Sigma$ such that
 $\text{supp}(\Delta) \subseteq \text{supp}(\alpha)$ and
 $\text{supp}(\Sigma) \cap \text{supp}(\alpha) = \emptyset$;
 Choose a bowtie for α ;
 Split α into $\beta \cdot \gamma$ at the level of this bowtie;
 Return
 $(\Delta^+ \cdot \Sigma^*) \cup (\Delta^* \cdot \Sigma^+) \cup (\Delta^* \cdot \delta \cdot \Delta^* \cdot \Sigma^*)$,
 where δ is the two-box template depicted
 on the right.
end



■ **Figure 11** Iteration of an atomic template.

354 It is not difficult to see that $\mathcal{B}(\alpha^+) = \mathcal{B}(\alpha)^+$. Depending on the bowtie we have chosen, the
355 box α^+ will be different. This is why we will write α_{\bowtie}^+ to say that the bowtie \bowtie has been
356 selected for the computation of the iteration.

357 Now we need to generalise this construction to compute the iteration of an atomic
358 template. For this, we need the following property, saying that the supports of atomic boxes
359 of the same type are either disjoint or comparable:

360 ► **Lemma 21.** *For all atomic boxes $\beta, \gamma : \sigma \rightarrow \sigma$, we have either 1) $\text{supp}(\beta) \subseteq \text{supp}(\gamma)$, or*
361 *2) $\text{supp}(\gamma) \subseteq \text{supp}(\beta)$, or 3) $\text{supp}(\beta) \cap \text{supp}(\gamma) = \emptyset$.*

362 We can compute the iteration of an atomic template by the algorithm in Fig. 11; intuitively,
363 atomic boxes with disjoint support can be iterated in any order: they cannot interfere; in
364 contrast, atomic boxes with small support must be computed before atomic boxes with
365 strictly larger support: the iteration of the latter depends on that of the former. (Also
366 note that since $\text{supp}(\Delta) \subseteq \text{supp}(\alpha)$ we have also $\text{supp}(\Delta^+) \subseteq \text{supp}(\alpha)$ thus the template
367 $\gamma \cdot \Delta^* \cdot \beta$ is 1-1 and it gives rise to an expression $e(\gamma \cdot \Delta^* \cdot \beta)$.)

368 We finally compute the iteration of an arbitrary template $\Gamma : \sigma \rightarrow \sigma$ as follows: from each
369 connected component of the graph of each box in Γ stems an atomic box; let $\text{At}(\Gamma)$ be the
370 set of all these atomic boxes; we set $\Gamma^+ = \text{At}(\Gamma)^+$.

371 The overall algorithm contains two sources of non-determinism. First, one can partially
372 choose in which order to process the atomic boxes. This is reflected by the choice of the box α ,
373 which we will call the *pivot*. For instance if $\Gamma = \{\alpha_1, \alpha_2, \beta\}$ such that $\text{supp}(\alpha_1) = \text{supp}(\alpha_2)$
374 and $\text{supp}(\beta) \cap \text{supp}(\alpha_1) = \emptyset$, then we can choose either α_1 or α_2 as the pivot, and the
375 computation will respectively start with the computation of α_2^+ or that of α_1^+ , yielding two
376 distinct expressions. (In contrast, choices about boxes with disjoint support do not change
377 the final result.) Second, every box of the template is eventually processed, and one must
378 thus choose a bowtie for all of them. We write $\Gamma_{\bowtie, \leq}^+$ to make explicit the choice of the
379 bowties and the computation order.

380 6 Synchronised Kleene theorem for PA

381 We can now prove Thm. 13. To synchronise the computation of two expressions e, f for two
 382 PA \mathcal{A}, \mathcal{B} respectively, we construct a *synchronised product automaton* $\mathcal{E} \times \mathcal{F}$ between a TA
 383 \mathcal{E} for \mathcal{A} and a TA \mathcal{F} for \mathcal{B} .

384 The states of this automaton are triples $\langle \sigma, \eta, \tau \rangle$ where σ and τ are types, *i.e.*, states
 385 from the TA \mathcal{E} and \mathcal{F} , and $\eta : \bar{\tau} \rightarrow \bar{\sigma}$ is a function used to enforce coherence conditions.
 386 Its transitions have the form $\langle \langle \sigma, \eta, \tau \rangle, \langle \Gamma, \Delta \rangle, \langle \sigma', \eta', \tau' \rangle \rangle$ where $\langle \sigma, \Gamma, \sigma' \rangle$ is a transition of
 387 \mathcal{E} , $\langle \tau, \Delta, \tau' \rangle$ is a transition of \mathcal{F} , and Γ and Δ satisfy a certain condition which we call
 388 *refinement*, written $\Gamma \leq \Delta$.

389 The overall strategy is as follows. We reduce $\mathcal{E} \times \mathcal{F}$ using the rules of Fig. 8, where the
 390 operations \cdot and \cup are computed pairwise. The operation $_*$ is also computed pairwise,
 391 but in a careful way, exploiting the non-determinism of this operation to ensure that we
 392 maintain the refinement relation. We eventually get a single transition labelled by a pair of
 393 1-1 templates Γ and Δ such that $\mathcal{B}(\Gamma) = \mathcal{G}(\mathcal{A})$, $\mathcal{B}(\Delta) = \mathcal{G}(\mathcal{B})$, and $\Gamma \leq \Delta$. To conclude, it
 394 suffices to deduce $\text{KL}^- \vdash e(\Gamma) \leq e(\Delta)$ from the latter property. To sum-up, what we need
 395 to do now is:

- 396 ■ **Refinement:** define the refinement relation \leq on templates;
- 397 ■ **Initialisation:** define $\mathcal{E} \times \mathcal{F}$ so that refinement holds;
- 398 ■ **Stability:** show that the refinement relation is maintained during the rewriting process;
- 399 ■ **Finalisation:** show that refinement between 1-1 templates entails KL^- provability.

400 6.1 Refinement relation

401 We first generalise graph homomorphisms to templates; this involves dealing with multiple
 402 ports, with finite sets, and with edge labels which are now arbitrary KL^- -expressions. For
 403 the latter, we do not require strict equality but KL^- -derivable inequalities.

404 ► **Definition 22** (Box and template homomorphisms). Let $\sigma, \tau, \sigma', \tau'$ be four types with two
 405 functions $\eta : \bar{\sigma} \rightarrow \bar{\tau}$ and $\eta' : \bar{\sigma}' \rightarrow \bar{\tau}'$. Let $\beta = \langle \vec{\mathfrak{p}}_\beta, \langle V_\beta, E_\beta, s_\beta, t_\beta, l_\beta \rangle, \overleftarrow{\mathfrak{p}}_\beta \rangle$ be a box
 406 of type $\tau \rightarrow \tau'$ and let $\alpha = \langle \vec{\mathfrak{p}}_\alpha, \langle V_\alpha, E_\alpha, s_\alpha, t_\alpha, l_\alpha \rangle, \overleftarrow{\mathfrak{p}}_\alpha \rangle$ be a box of type $\sigma \rightarrow \sigma'$. A
 407 homomorphism from α to β is a pair $\langle f, g \rangle$ of functions $f : V_\alpha \rightarrow V_\beta$ and $g : E_\alpha \rightarrow E_\beta$ s.t.:

- 408 ■ $s_\beta \circ g = f \circ s_\alpha, t_\beta \circ g = f \circ t_\alpha,$
- 409 ■ $\forall e \in E_\alpha, \quad \text{KL}^- \vdash l_\beta \circ g(e) \leq l_\alpha(e),$
- 410 ■ If $\{v\} \subseteq V_\alpha$ is a trivial connected component, so is $f(v)$.
- 411 ■ $\vec{\mathfrak{p}}_\beta \circ \eta = f \circ \vec{\mathfrak{p}}_\alpha$ and $\overleftarrow{\mathfrak{p}}_\beta \circ \eta' = f \circ \overleftarrow{\mathfrak{p}}_\alpha$. (We call this condition (η, η') -compatibility.)

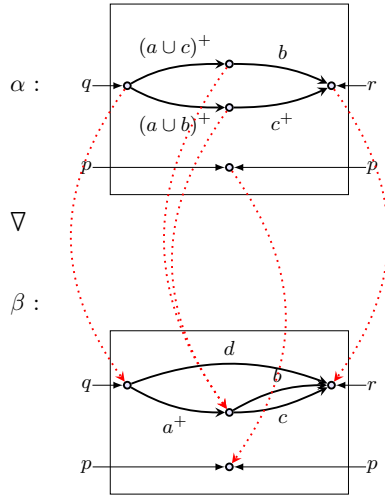
412 We write $\beta \triangleleft_{\eta, \eta'} \alpha$ when there exists such a homomorphism. For two templates $\Gamma : \tau \rightarrow \tau'$
 413 and $\Delta : \sigma \rightarrow \sigma'$, we write $\Gamma \triangleleft_{\eta, \eta'} \Delta$ if for all $\beta \in \Gamma$, there exists $\alpha \in \Delta$ such that $\beta \triangleleft_{\eta, \eta'} \alpha$.

414 We abbreviate $\Gamma \triangleleft_{\eta, \eta'} \Delta$ as $\Gamma \triangleleft \Delta$ when Γ, Δ are 1-1 templates, or when $\sigma = \tau, \sigma' = \tau'$ and
 415 η, η' are the identity function id . A box homomorphism is depicted in Fig. 12.

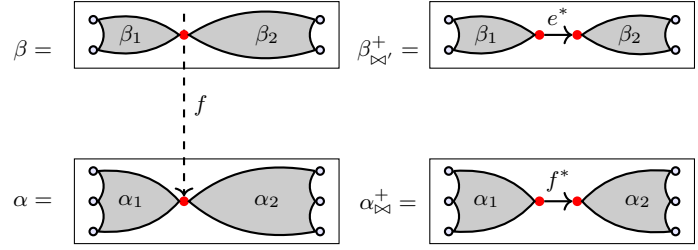
416 The above relation on templates is not enough for our needs; we have to extend it so that
 417 it is preserved during the rewriting process. We first write $\Gamma \sqsubseteq \Delta$ when $\mathcal{B}(\Gamma) \subseteq \mathcal{B}(\Delta)$, for
 418 two templates Γ, Δ of the same type. Refinement is defined as follows:

419 ► **Definition 23** (Refinement). We call *refinement* the relation on templates defined by
 420 $\leq_{\eta, \eta'} \triangleq \triangleleft_{\eta, \eta'} \cdot (\triangleleft_{\text{id}, \text{id}} \cup \sqsubseteq)^*$, where $_*$ is reflexive transitive closure.

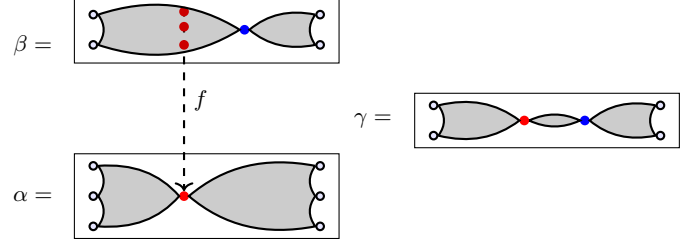
421 The following proposition shows that refinement implies provability of the expressions
 422 extracted from 1-1 templates. This gives us the finalisation step.



■ **Figure 12** A box homomorphism.



■ **Figure 13** Bowtie compatible boxes.



■ **Figure 14** Case of bowtie incompatible boxes.

423 ► **Proposition 6.** *If Δ, Γ are 1-1 templates such that $\Delta \leq \Gamma$, then $\text{KL}^- \vdash e(\Delta) \leq e(\Gamma)$.*

424 **Proof.** When $\Delta \subseteq \Gamma$, it follows from Prop. 3 and Thm. 12; when $\Delta \triangleleft \Gamma$, it follows from
425 Prop. 1. We conclude by transitivity. ◀

426 6.2 Synchronised product automaton (initialisation)

427 ► **Definition 24** (2-Template automata (2-TA)). A 2-template automaton is an NFA whose
428 states are tuples of the form $\langle \tau, \eta, \sigma \rangle$ where τ, σ are types and $\eta : \bar{\sigma} \rightarrow \bar{\tau}$, whose alphabet is
429 the set of pairs of templates, whose transitions are of the form $\langle \langle \sigma, \eta, \tau \rangle, \langle \Gamma, \Delta \rangle, \langle \sigma', \eta', \tau' \rangle \rangle$
430 where $\Gamma : \sigma \rightarrow \sigma'$, $\Delta : \tau \rightarrow \tau'$, and $\Gamma \leq_{\eta, \eta'} \Delta$, and with a single initial state and a single
431 accepting state which consist of singleton types.

432 If \mathcal{T} is a 2-TA, we denote by $\pi_1(\mathcal{T})$ (resp. $\pi_2(\mathcal{T})$) the automaton obtained by projecting the
433 alphabet, the states and the transitions of \mathcal{T} on the first (resp. last) component. Note that
434 $\pi_1(\mathcal{T})$ and $\pi_2(\mathcal{T})$ are TA.

435 ► **Definition 25** (Synchronised product of TA). Let \mathcal{E}, \mathcal{F} be two TA. The *synchronised product*
436 of \mathcal{E} and \mathcal{F} , written $\mathcal{E} \times \mathcal{F}$ is the 2-TA where $\langle \langle \tau, \eta, \sigma \rangle, \langle \Gamma, \Delta \rangle, \langle \tau', \eta', \sigma' \rangle \rangle$ is a transition of
437 $\mathcal{E} \times \mathcal{F}$ iff $\langle \tau, \Gamma, \tau' \rangle$ is a transition of \mathcal{E} , $\langle \sigma, \Delta, \sigma' \rangle$ is a transition of \mathcal{F} and $\Gamma \leq_{\eta, \eta'} \Delta$. (And
438 with initial and accepting states defined from those of \mathcal{E} and \mathcal{F} .)

439 Note that we enforce refinement in the definition of this product, so that $\pi_1(\mathcal{E} \times \mathcal{F})$ is
440 a sub-automaton of \mathcal{E} and $\pi_2(\mathcal{E} \times \mathcal{F})$ is a sub-automaton of \mathcal{F} . Thus $\mathcal{G}(\pi_1(\mathcal{E} \times \mathcal{F})) \subseteq$
441 $\mathcal{G}(\mathcal{E})$ and $\mathcal{G}(\pi_2(\mathcal{E} \times \mathcal{F})) \subseteq \mathcal{G}(\mathcal{F})$. When \mathcal{E}, \mathcal{F} are TA coming from PA \mathcal{A}, \mathcal{B} such that
442 $\triangleleft \mathcal{G}(\mathcal{A}) \subseteq \triangleleft \mathcal{G}(\mathcal{B})$, we can use the results from [11] about simulations to strengthen the first
443 inclusion into an equality:

444 ► **Theorem 26.** *Let \mathcal{A}, \mathcal{B} be two PA, \mathcal{E}, \mathcal{F} be basic TA such that $\mathcal{G}(\mathcal{A}) = \mathcal{L}(\mathcal{E})$ and
445 $\mathcal{G}(\mathcal{B}) = \mathcal{L}(\mathcal{F})$ (given by Prop. 4). If $\triangleleft \mathcal{G}(\mathcal{A}) \subseteq \triangleleft \mathcal{G}(\mathcal{B})$ then:*

- 446 ■ $\mathcal{G}(\pi_1(\mathcal{E} \times \mathcal{F})) = \mathcal{G}(\mathcal{A});$
- 447 ■ $\mathcal{G}(\pi_2(\mathcal{E} \times \mathcal{F})) \subseteq \mathcal{G}(\mathcal{B}).$

448 **Proof.** The second point follows from the observation above. The first one comes from the sim-
 449 ulation result ([11, Prop. 9.10]) for PA. Indeed, if $\triangleleft \mathcal{G}(\mathcal{A}) \subseteq \triangleleft \mathcal{G}(\mathcal{B})$, then there is a simulation
 450 ([11, Def. 9.2]) between \mathcal{A} and \mathcal{B} . This implies that for every run $\langle \tau_1, \Gamma_1, \tau_2, \dots, \Gamma_{n-1}, \tau_n \rangle$ of
 451 \mathcal{E} , there is a run $\langle \sigma_1, \Delta_1, \sigma_2, \dots, \Delta_{n-1}, \sigma_n \rangle$ of \mathcal{F} and a set of mapping $\eta_i : \bar{\sigma}_i \rightarrow \bar{\tau}_i$, $i \in [1, n]$
 452 such that $\Gamma_i \triangleleft_{\eta_i, \eta_{i+1}} \Delta_i$ for every $i \in [1, n-1]$. \blacktriangleleft

453 6.3 Maintaining refinement during reductions

454 Let us finally show that refinement is stable by composition, union, and iteration.

455 **► Theorem 27** (Stability of refinement by \cdot and \cup).

456 **■** If $\Gamma_1 \leq_{\eta, \eta'} \Gamma_2$ and $\Delta_1 \leq_{\eta', \eta''} \Delta_2$ then $\Gamma_1 \cdot \Delta_1 \leq_{\eta, \eta''} \Gamma_2 \cdot \Delta_2$.

457 **■** If $\Gamma_1 \leq_{\eta, \eta'} \Gamma_2$ and $\Delta_1 \leq_{\eta, \eta'} \Delta_2$ then $\Gamma_1 \cup \Delta_1 \leq_{\eta, \eta'} \Gamma_2 \cup \Delta_2$.

Proof. To show the first property it suffices to show the following results:

$$\text{If } \Gamma_1 \triangleleft_{\eta, \eta'} \Gamma_2 \quad \text{and} \quad \Delta_1 \triangleleft_{\eta', \eta''} \Delta_2 \quad \text{then} \quad \Gamma_1 \cdot \Delta_1 \triangleleft_{\eta', \eta''} \Gamma_2 \cdot \Delta_2. \quad (L_1)$$

$$\text{If } \Gamma_1 \sqsubseteq \Gamma_2 \quad \text{and} \quad \Delta_1 \sqsubseteq \Delta_2 \quad \text{then} \quad \Gamma_1 \cdot \Delta_1 \sqsubseteq \Gamma_2 \cdot \Delta_2. \quad (L_2)$$

$$\text{If } \Gamma_1 \triangleleft \Gamma_2 \quad \text{and} \quad \Delta_1 \sqsubseteq \Delta_2 \quad \text{then} \quad \Gamma_1 \cdot \Delta_1 (\triangleleft \sqsubseteq)^* \Gamma_2 \cup \Delta_2. \quad (L_3)$$

458 To show (L_1) , consider a box $\alpha_1 \in \Gamma_1$ and $\beta_1 \in \Delta_1$. By hypothesis, there is a box $\alpha_2 \in \Gamma_2$
 459 and an (η, η') -compatible homomorphism $h = \langle f, g \rangle$ from α_2 to α_1 and a box $\beta_2 \in \Delta_2$ and
 460 an (η', η'') -compatible homomorphism $h' = \langle f', g' \rangle$ from β_2 to β_1 . Let $h'' = \langle f'', g'' \rangle$, where
 461 f'' equals f in $\text{dom}(f)$ and f' in $\text{dom}(f')$, and g'' equals g in $\text{dom}(g)$ and g' in $\text{dom}(g')$.
 462 Using (η, η') -compatibility of h and (η', η'') -compatibility of h' , it is easy to show that h'' is
 463 an (η, η'') -compatible homomorphism from $\alpha_2 \cdot \beta_2$ to $\alpha_1 \cdot \beta_1$, which concludes the proof of
 464 (L_1) . (L_2) follows easily from the definition of \sqsubseteq . For (L_3) , note that $\Delta_1 \triangleleft \Delta_1$ (we choose
 465 the identity homomorphism), thus by (L_1) , we have that $\Gamma_1 \cdot \Delta_1 \triangleleft \Gamma_2 \cdot \Delta_1$. By (L_2) , we have
 466 that $\Gamma_2 \cdot \Delta_1 \sqsubseteq \Gamma_2 \cdot \Delta_2$, which concludes the proof.

467 To show the first property, we proceed by induction on the length of the sequences
 468 justifying that $\Gamma_1 \leq_{\eta, \eta'} \Gamma_2$ and $\Delta_1 \leq_{\eta', \eta''} \Delta_2$, using (L_1) , (L_2) and (L_3) for the base cases.

469 To show the second property, we follow the same proof schema, showing results similar
 470 to $(L_1) - (L_3)$ where \cdot is replaced by \cup . \blacktriangleleft

471 **► Remark.** Thm. 27 justifies our definition of $\leq_{\eta, \eta'}$. Indeed, a more permissive definition
 472 would seem natural, but the first property of Thm 27 would fail. For instance, if $\Gamma_1 \sqsubseteq \Gamma_2$
 473 and $\Delta_1 \triangleleft_{\eta, \eta'} \Delta_2$, we do not have in general that $\Gamma_1 \cdot \Delta_1 \leq_{\eta, \eta'} \Gamma_2 \cdot \Delta_2$.

474 The main theorem of this section is Thm 28, stating that the refinement relation is stable
 475 under iteration.

476 **► Theorem 28** (Stability of refinement by $_+$). If $\Gamma \leq_{\eta, \eta} \Delta$ then there are bowtie choices
 477 \bowtie, \bowtie' and computation orders \preceq, \preceq' , for Γ and Δ respectively, such that: $\Gamma_{\bowtie, \preceq}^+ \leq_{\eta, \eta} \Delta_{\bowtie', \preceq'}^+$.

478 **Proof.** To prove Thm. 28, it is enough to show the following properties.

479 **■** If $\Gamma \sqsubseteq \Delta$ then, for every bowtie choices \bowtie, \bowtie' , and every computation orders \preceq, \preceq' for Γ
 480 and Δ respectively, we have that $\Gamma_{\bowtie, \preceq}^+ \sqsubseteq \Delta_{\bowtie', \preceq'}^+$.

481 **■** If $\Gamma \triangleleft_{\eta, \eta} \Delta$ then there are two bowtie choices \bowtie, \bowtie' and two computation orders \preceq, \preceq' ,
 482 for Γ and Δ respectively, such that $\Gamma_{\bowtie, \preceq}^+ \leq_{\eta, \eta} \Delta_{\bowtie', \preceq'}^+$.

483 The first property follows from $\mathcal{B}(\Gamma_{\bowtie, \preceq}^+) = \mathcal{B}(\Gamma)^+$ for every bowtie choice \bowtie and order \preceq .

484 For the sake of clarity, we give here the proof of the second proposition in the case where
 485 Γ and Δ are singletons of atomic boxes $\{\alpha\}$ and $\{\beta\}$ respectively. The general case is treated
 486 in App. B. Let \bowtie, \bowtie' be bowtie choices for α and β respectively, and let $h = \langle f, g \rangle$ be a
 487 homomorphism from β to α .

488 Let us first treat the case where $f^{-1}(\bowtie) = \{\bowtie'\}$ (we say that α, β are bowtie compatible).
 489 This is illustrated by the boxes α, β of Fig. 13, where the bowties are the red nodes. If we
 490 decompose α and β at the level of their bowties, we get $\alpha = \alpha_1 \cdot \alpha_2$ and $\beta = \beta_1 \cdot \beta_2$, where
 491 $\alpha_2 \cdot \alpha_1$ and $\beta_2 \cdot \beta_1$ are 1-1 boxes. We write $e = e(\alpha_2 \cdot \alpha_1)$ and $f = e(\beta_2 \cdot \beta_1)$. The boxes α_{\bowtie}^+
 492 and $\beta_{\bowtie'}^+$ are depicted in Fig. 13. Let us show that there is a homomorphism from $\beta_{\bowtie'}^+$ to α_{\bowtie}^+ .
 493 The homomorphism h induces a homomorphism h_1 from β_1 to α_1 and a homomorphism h_2
 494 from β_2 to α_2 (Lem. 42 in App. B). Combining h_1 and h_2 , we get almost a homomorphism
 495 from $\beta_{\bowtie'}^+$ to α_{\bowtie}^+ (See Fig. 13), we need only to show that $\text{KL}^- \vdash e \leq f$. But this follows from
 496 Prop. 6: indeed, we can combine h_1 and h_2 to get a homomorphism from $\beta_2 \cdot \beta_1$ to $\alpha_2 \cdot \alpha_1$.
 497 We have thus that $\alpha_{\bowtie}^+ \triangleleft_{\eta, \eta} \beta_{\bowtie'}^+$ ((η, η) -compatibility is easy).

498 Let us now treat the case where $N := f^{-1}(\bowtie)$ is not necessarily $\{\bowtie'\}$ (N is illustrated
 499 by the red node of β in Fig. 14). Let γ be the box obtained from β by merging the nodes
 500 N (see Fig. 14). There are two bowtie choices for γ : a bowtie \bowtie_b inherited from β (blue in
 501 Fig. 14) and a bowtie \bowtie_r coming from the nodes of N (red in Fig. 14).

502 Let h' be the homomorphism from β to γ that maps each node (and each edge) to itself,
 503 except for the nodes of N which are mapped to \bowtie_r . If we consider the bowtie \bowtie_b for γ , then
 504 β and γ are bowtie compatible w.r.t. to h' , thus $\gamma_{\bowtie_b}^+ \triangleleft \beta_{\bowtie'}^+$ using the previous case.

505 Let h'' be the homomorphism from γ to α , which is exactly h except that it maps the
 506 node \bowtie_r to the bowtie \bowtie of α . If we consider the bowtie \bowtie_r for γ , then γ and α are bowtie
 507 compatible w.r.t. h'' , thus $\alpha_{\bowtie}^+ \triangleleft_{\eta, \eta} \gamma_{\bowtie_r}^+$ using the previous case again.

508 Notice finally that $\gamma_{\bowtie_r}^+ \sqsubseteq \gamma_{\bowtie_b}^+$. To sum up, we have: $\alpha_{\bowtie}^+ \triangleleft_{\eta, \eta} \gamma_{\bowtie_r}^+ \sqsubseteq \gamma_{\bowtie_b}^+ \triangleleft \beta_{\bowtie'}^+$. ◀

509 The last case in this proof explains the need to work with refinement (\leq) rather than just
 510 homomorphisms (\triangleleft): when starting from templates that are related by homomorphism and
 511 iterating them, the templates we obtain are not necessarily related by a single homomorphism,
 512 only by a sequence of homomorphisms and inclusions.

513 7 Future work

514 We have proven that KL^- axioms are sound and complete w.r.t. the relational models of
 515 identity-free Kleene lattices, and thus also w.r.t. their language theoretic models, by the
 516 results from [3].

517 Whether one can obtain a finite axiomatisation in presence of identity remains open.
 518 This question is important since handling the identity relation is the very first step towards
 519 handling *tests*, which are crucial in order to model the control flow of sequential programs
 520 precisely (*e.g.*, as in Kleene algebra with tests [20]).

521 An intermediate problem, which is still open to the best of our knowledge, consists in
 522 finding an axiomatisation for the fragment with composition, intersection and identity (not
 523 including transitive closure) [2, see errata available online].

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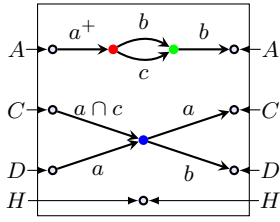


Figure 15 Example of a box.

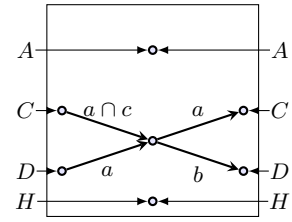
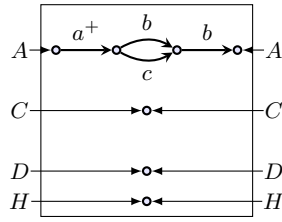


Figure 16 Atomic boxes stemming from the box of Fig. 15.

A Iteration of a template

In this section, we address in detail the definitions relative to the construction of the iteration of a template.

We have seen that there are two sources of non-determinism when computing the iteration of a template (Algorithm 11). The first is the bowtie choice and the second is the computation order. Let us introduce them more precisely.

A.1 Bowtie choice for a template

We have seen in Sec. 5.2 that the non-trivial connected component of an atomic box can be associated with a specific node called its bowtie (Lem.20). We do the same for non atomic boxes.

► **Definition 29** (Connected component of a box). If $\beta = \langle \vec{p}, G, \overleftarrow{p} \rangle$ is a box, we denote by $\mathcal{C}(\beta)$ the set of non-trivial connected components of G , which we call simply connected component of β .

► **Lemma 30** (Bowtie lemma [11, Lem. 7.1]). Let $B = \langle \vec{p}, G, \overleftarrow{p} \rangle$ be a box of type $\tau \rightarrow \tau$. For every $C \in \mathcal{C}(\beta)$ there is a vertex c such that for every port p where $\vec{p}(p) \in C$, all paths from $\vec{p}(p)$ to a maximal vertex of C visit c . We call such a vertex a bowtie for C .

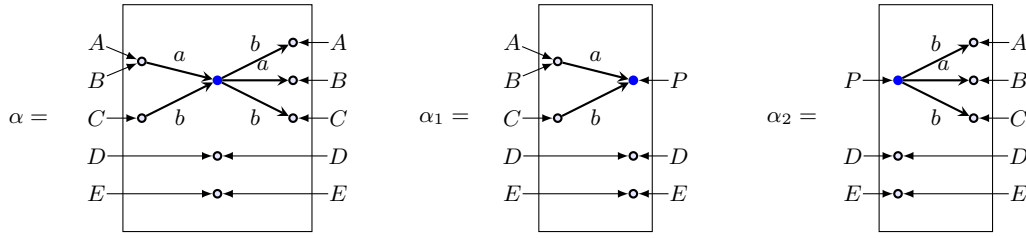
► **Definition 31** (Bowtie choice for a template). A bowtie choice for a box is a function mapping a bowtie to every connected component.

A bowtie choice for a template is a function mapping a bowtie to every connected component of every box.

► **Remark.** When β is atomic, it has only one connected component, so we may identify the bowtie choice that maps this component to a node, with the node itself.

► **Example 32.** Consider the box of Figure 15. It has two connected components. The first has two bowtie choices: the red and the green node. The second has only one bowtie choice, the blue node.

► **Notation 1.** If α is an atomic box and \bowtie is a bowtie choice for α , then we can decompose α at the level of this bowtie to get two boxes such that $\alpha = \alpha_1 \cdot \alpha_2$. We write $\alpha \stackrel{\bowtie}{=} \alpha_1 \cdot \alpha_2$ for this decomposition. In Fig. 17, the box α can be decomposed at the level of its bowtie (the blue node) into α_1 and α_2 .



■ **Figure 17** Decomposition of an atomic box.

A.2 Computation order

Let us analyse computation order of algorithm 11 in the simple case where $\Gamma = \{\alpha, \beta\}$. If $\text{supp}(\alpha) \subsetneq \text{supp}(\beta)$ then the algorithm starts necessarily by processing α . If $\text{supp}(\alpha) \cap \text{supp}(\beta) = \emptyset$, then the order in which the computation proceeds does not matter, and we will get the same result no matter if we start with processing α or β . The only case where we have a freedom to choose the computation order, and in which this order may affect the result is when $\text{supp}(\alpha) = \text{supp}(\beta)$. In general, to specify the computation order, it is enough to order the elements of Γ having the same support.

► **Definition 33** (Computation order). A *computation order* for an atomic template Γ is a partial order \preceq on its elements such that if $\alpha \preceq \beta$ then $\text{supp}(\alpha) = \text{supp}(\beta)$.

A.3 Atomic template of a template

To compute the iteration of a template, we start by decomposing its boxes into atomic ones.

► **Definition 34** ($At(\Gamma)$). Let $\beta : \sigma \rightarrow \sigma$ be a box. From each $C \in \mathcal{C}(\beta)$ stems an atomic box of the same type having C as a connected component. We set $At(\beta)$ to be the set of atomic boxes stemming from its connected components.

If $\Gamma : \sigma \rightarrow \sigma$ is a template we write $At(\Gamma)$ for the set of boxes stemming from the connected components of the boxes of Γ .

For instance, the boxes of Figure 16 are the boxes stemming from the connected components of the box of Figure 15.

► **Remark.** Note that every bowtie choice for Γ induces a bowtie choice for $At(\Gamma)$.

► **Definition 35.** A computation order for a template Γ is a computation order for $At(\Gamma)$.

A.4 The iteration algorithm

Fig. 18 shows the algorithm computing the iteration of an atomic template, parameterised by a bowtie choice and a computation order.

If \bowtie is a bowtie choice and \preceq is a computation order for Γ , we set $\Gamma_{\bowtie, \preceq}^+ := At(\Gamma)_{\bowtie, \preceq}^+$.

B Stability of \leq under iteration

In the whole section, we will work under the following proviso:

► **Proviso 1.** We suppose that all templates are of type $\tau \rightarrow \tau$ and that all the box and template homomorphisms are (η, η) -compatible, where τ is a fixed type and $\eta : \tau \rightarrow \tau$ a fixed mapping. We will not write explicitly $\triangleleft_{\eta, \eta}$ for (η, η) -compatible homomorphisms but simply

Data: Atomic template Γ , a bowtie choice \bowtie
and a computation order \preceq for Γ

Result: A template $\Gamma_{\bowtie, \preceq}^+$ such that
 $\mathcal{B}(\Gamma_{\bowtie, \preceq}^+) = \mathcal{B}(\Gamma)^+$

if $\Gamma = \emptyset$ **then**

| Return \emptyset

else

Write $\Gamma = \Delta \cup \{\alpha\} \cup \Sigma$ such that

$\text{supp}(\Delta) \subseteq \text{supp}(\alpha)$, $\forall \alpha' \in \Delta$ if

$\text{supp}(\alpha)' = \text{supp}(\alpha)$ then $\alpha' \preceq \alpha$, and

$\text{supp}(\Sigma) \cap \text{supp}(\alpha) = \emptyset$;

Set $\bowtie' := \bowtie(C)$, where $\mathcal{C}(\alpha) = \{C\}$.

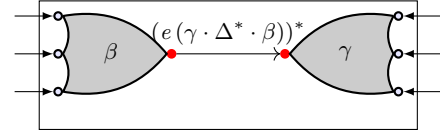
Split α into $\alpha \stackrel{\bowtie'}{=} \beta \cdot \gamma$;

Return

$(\Delta^+ \cdot \Sigma^*) \cup (\Delta^* \cdot \Sigma^+) \cup (\Delta^* \cdot \delta \cdot \Delta^* \cdot \Sigma^*)$,

where δ is the two-box template depicted
on the right.

end



■ **Figure 18** Algorithm computing the iteration of an atomic template

650 \triangleleft . All the theorems, propositions, lemmas of this section hold under this proviso, which will
651 not be mentioned explicitly in their statements.

652 In this section, we will show the following theorem:

653 ► **Theorem 36.** If $\Delta \triangleleft \Gamma$ then there are two bowtie choices \bowtie, \bowtie' for Δ and Γ respectively,
654 and two computation orders \preceq, \preceq' for Δ and Γ respectively such that: $\Delta_{\bowtie, \preceq}^+ \leq \Gamma_{\bowtie', \preceq'}^+$.

655 To prove theorem 36, we will show that template homomorphisms can be decomposed into
656 simpler template homomorphisms \triangleleft_1 and \triangleleft_2 (Def. 37, Def. 39, Prop. 7). It is thus enough
657 to show Thm. 36 in the case where $\Delta \triangleleft_1 \Gamma$ and $\Delta \triangleleft_2 \Gamma$, these results are precisely Prop. 8
658 and Prop. 9.

659 B.1 Decomposing \triangleleft into \triangleleft_1 and \triangleleft_2

660 Let us first define the template homomorphisms \triangleleft_1 and \triangleleft_2 .

661 ► **Definition 37** (\triangleleft_1). Let α, β be two boxes. We set $\alpha \triangleleft_1 \beta$ if there are bowtie choices
662 \bowtie, \bowtie' for α and β respectively, and a box homomorphism h from β to α such that:

663 ■ If $C \in \mathcal{C}(\beta)$ then $h(C) \in \mathcal{C}(\alpha)$.

664 ■ If $C, D \in \mathcal{C}(\beta)$ and $C \neq D$ then $h(C) \neq h(D)$.

665 ■ If $C \in \mathcal{C}(\beta)$ then $h(\bowtie'(C)) = \bowtie(h(C))$.

666 If Γ, Δ are templates, we set $\Gamma \triangleleft_1 \Delta$ if for every $\alpha \in \Gamma$, there is $\beta \in \Delta$ such that $\alpha \triangleleft_1 \beta$.

667 Figure 19 shows two boxes α, β such that $\alpha \triangleleft_1 \beta$. Indeed, the blue connected component
668 of β and its bowtie are mapped to the blue connected component of α and its bowtie. The
669 same holds for the red connected component.

670 To define the homomorphism \triangleleft_2 , we need to define formally the operation of "merging"
671 (or "identifying") nodes in a graph.

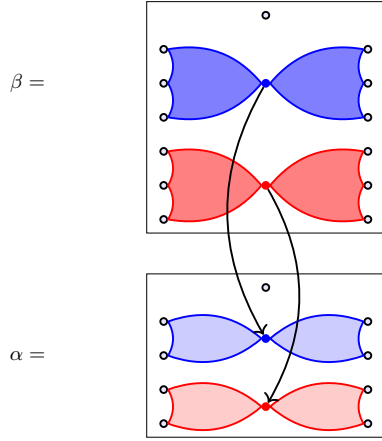


Figure 19 Boxes α, β such that $\alpha \triangleleft_1 \beta$.

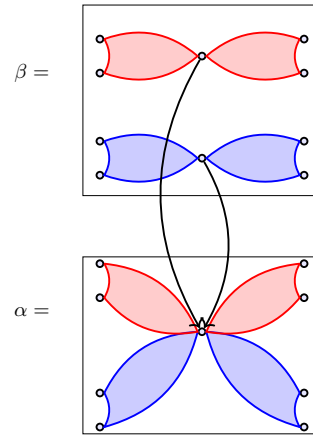


Figure 20 Boxes α, β such that $\alpha \triangleleft_2 \beta$.

672 ▶ **Definition 38** (Identification of nodes in a graph). Let $G = \langle V, E \rangle$ be a graph and
 673 $N_1, \dots, N_k \subseteq V$ be pairwise disjoint sets of nodes. Let \equiv be the smallest equivalence relation
 674 on V containing all the pairs $\langle n, m \rangle$, such that $\exists i \in [1, k], n, m \in N_i$. We write $G|_{\equiv\{N_1, \dots, N_k\}}$
 675 for the graph $\langle \{[n] \mid n \in V\}, E' \rangle$ where $[n] = \{m \mid m \equiv n\}$ and $\langle [n], x, [m] \rangle \in E'$ if and only
 676 if $\langle n, x, m \rangle \in E$.

677 Let $\beta = \langle \vec{p}, G, \overleftarrow{p} \rangle$ is a box, and N_1, \dots, N_k be pairwise disjoint sets of the nodes of G .
 678 We write $\beta|_{\equiv\{N_1, \dots, N_k\}}$ for the box $\langle \vec{p}', G|_{\equiv\{N_1, \dots, N_k\}}, \overleftarrow{p}' \rangle$ where \vec{p}' and \overleftarrow{p}' are defined by:
 679 $\vec{p}'(x) = [n]$ if $\vec{p}(x) = n$ and $\overleftarrow{p}'(x) = [n]$ if $\overleftarrow{p}(x) = n$.

680 ▶ **Definition 39** (\triangleleft_2). Let α, β be two boxes. We set $\alpha \triangleleft_2 \beta$ if there is a bowtie choice \bowtie
 681 for β and $C, D \in \mathcal{C}(\beta)$ such that when we set $N = \{\bowtie(C), \bowtie(D)\}$ we have $\alpha = \beta|_{\equiv\{N\}}$.

682 If Γ, Δ are templates, we set $\Gamma \triangleleft_2 \Delta$ if $\Gamma = \Sigma \cup \{\alpha\}$ and $\Delta = \Sigma \cup \{\beta\}$ such that $\alpha \triangleleft_2 \beta$.

683 In other words, $\alpha \triangleleft_2 \beta$ if α is obtained by "merging" the bowties of two connected components
 684 of β . Figure 20 show two boxes α, β such that $\alpha \triangleleft_2 \beta$.

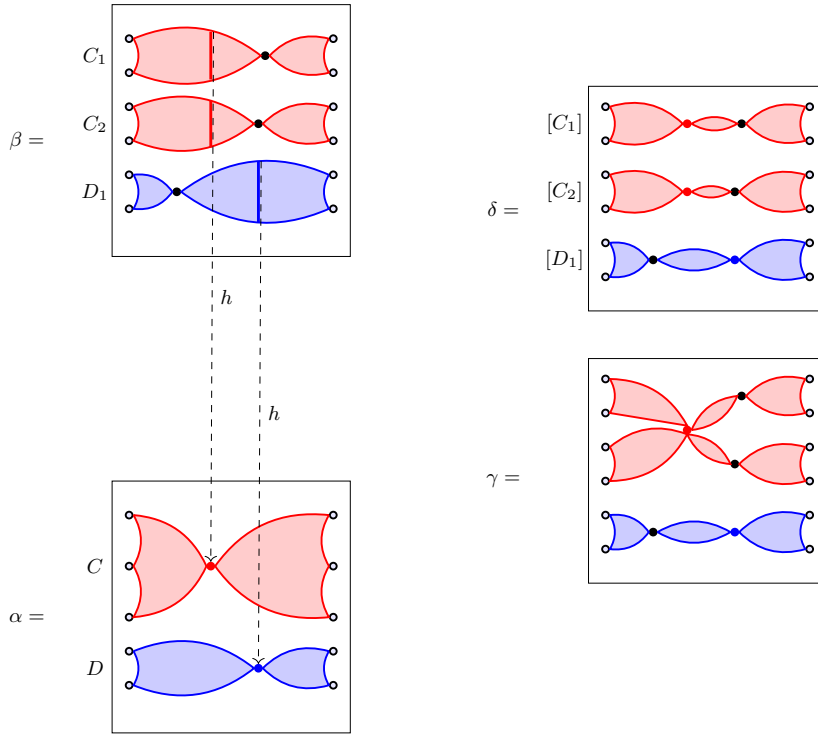
685 To show that \triangleleft can be decomposed into \triangleleft_1 and \triangleleft_2 (Prop. 7), we need the following
 686 lemma, which says that the converse image of a connected component by a homomorphism
 687 is a collection of connected components.

688 ▶ **Lemma 40.** Let α, β be two boxes and h be a box homomorphism from β to α . For every
 689 $C \in \mathcal{C}(\alpha)$, there is a set $\{C_1, \dots, C_n\} \subseteq \mathcal{C}(\beta)$ such that $h^{-1}(C) = C_1 \cup \dots \cup C_n$.

690 **Proof.** Let $C \in \mathcal{C}(\alpha)$. By contradiction suppose that there is a connected component
 691 $C' \in \mathcal{C}(\beta)$ and two nodes $x, y \in C'$ such that $\langle x, a, y \rangle$ is a vertex of the graph of α , $h(x) \in C$
 692 and $h(y) \notin C$. Since h is a homomorphism, we have that there is a vertex in the graph of β
 693 between $h(x)$ and $h(y)$, thus $h(y) \in C$. This gives us a contradiction. ◀

694 Let us show now that we can indeed decompose \triangleleft into \triangleleft_1 and \triangleleft_2 .

695 ▶ **Proposition 7.** We have that $\triangleleft \subseteq (\triangleleft_1 \cup \triangleleft_2)^+$, where the operation $_+^+$ is the transitive
 696 closure on relations.



■ **Figure 21** Decomposing $\alpha \triangleleft \beta$ into $\alpha \triangleleft_1 \gamma$, $\gamma \triangleleft_2^+ \delta$ and $\delta \triangleleft_1 \beta$.

697 **Proof.** Let us show that if $\Gamma \triangleleft \Delta$ then there is $\Sigma_1, \dots, \Sigma_n$ such that $\Sigma_1 = \Gamma$, $\Sigma_n = \Delta$ and
 698 for every $i \in [1, n - 1]$ either $\Sigma_i \triangleleft_1 \Sigma_{i+1}$ or $\Sigma_i \triangleleft_2 \Sigma_{i+1}$. For that, we proceed by induction
 699 on the size of Γ .

700 Let $\alpha \in \Gamma$ and set $\Sigma = \Gamma \setminus \{\alpha\}$. Since $\Gamma \triangleleft \Delta$, there is a box $\beta \in \Delta$ such that $\alpha \triangleleft \beta$.
 701 Let h be a homomorphism from β to α and let \bowtie, \bowtie' be two bowtie choices for α and β
 702 respectively.

703 Let us show first that there are two boxes γ and δ such that $\alpha \triangleleft_1 \gamma$, $\gamma \triangleleft_2^+ \delta$ and $\delta \triangleleft_1 \beta$. We
 704 will illustrate the construction of γ and δ by Figure 21. In this figure, α has two connected
 705 components C and D , and β has three connected components C_1, C_2 and D_1 such that
 706 $h(C_1 \cup C_2) = C$ and $h(D_1) = D$. The bowtie choices for α and β are illustrated by the nodes
 707 in the middle of each connected component.

Let us construct δ . By Lem. 40, we know that for every connected component C of α ,
 $h^{-1}(C) = C_1 \cup \dots \cup C_n$ where $C_i \in \mathcal{C}(\beta)$. We set $C^{-1} = \{C_1, \dots, C_n\}$. For every $C' \in C^{-1}$
 we set:

$$b(C, C') = h^{-1}(\bowtie(C)) \cap C'$$

708 Let $\delta = \beta|_{\equiv \{b(C, C') \mid C \in \mathcal{C}(\alpha), C' \in C^{-1}\}}$. As illustrated by Figure 21, δ is obtained from β by
 709 merging in every connected component the nodes that are mapped to a bowtie of α by h .

710 The box δ has two possible bowtie choices: one inherited from the bowtie \bowtie' of β (the
 711 black bowties of δ in Figure 21) and another coming from the nodes $b(C, C')$ that have being
 712 merged (the red and the blue bowties for δ in Figure 21). We call the former \bowtie_1 and the
 713 later \bowtie_2 .

714 If we take \bowtie_1 as a bowtie choice for δ , then we have easily that $\delta \triangleleft_1 \beta$.

Let us construct γ now. We set $\bowtie^{-1}(C) = h^{-1}(\bowtie(C))$. Note that we have $\bowtie^{-1}(C) = \bigcup_{C' \in C^{-1}} b(C, C')$. We let

$$\gamma = \beta|_{\{\bowtie^{-1}(C) \mid C \in \mathcal{C}(\alpha)\}}$$

715 In other words, if we denote by $[C]$ the connected component of δ coming from the connected
716 component C of β , then γ is obtained by identifying every two nodes $\bowtie_2([C_1])$ and $\bowtie_2([C_2])$,
717 where $C_1, C_2 \in C^{-1}$ and $C \in \mathcal{C}(\alpha)$. If we call $\gamma_1, \dots, \gamma_k$ these intermediate boxes where we
718 merged only two nodes, we have that $\gamma \triangleleft_2 \delta_1 \triangleleft_2 \dots \triangleleft_2 \delta_k \triangleleft_2 \delta$. Figure 21 illustrates the
719 construction of γ .

720 If we consider the bowtie choice \bowtie of α and the bowtie choice \bowtie_3 of γ induced by merging
721 the nodes $\bowtie^{-1}(C)$ of β (The red node of γ in Figure 21), it is easy to see that $\alpha \triangleleft_1 \gamma$.

722 Let us make a final observation before showing the general result. Notice that if B, B'
723 are two boxes, and Θ is a template, then $B \triangleleft_1 B'$ entails $B \cup \Theta \triangleleft_1 B' \cup \Theta$ and $B \triangleleft_2 B'$
724 entails $B \cup \Theta \triangleleft_2 B' \cup \Theta$. Thus if $B(\triangleleft_1 \cup \triangleleft_2)^+ B'$ then $(B \cup \Theta)(\triangleleft_1 \cup \triangleleft_2)^+(B' \cup \Theta)$.

725 Let us go back to the proof of our result. Recall that $\Gamma = \Sigma \uplus \{\alpha\}$, that $\beta \in \Delta$, and that
726 $\alpha(\triangleleft_1 \cup \triangleleft_2)^+ \beta$. By the remark above, we have that $\Gamma(\triangleleft_1 \cup \triangleleft_2)^+(\{\beta\} \cup \Sigma)$. Since $\Gamma \triangleleft \Delta$ we
727 have also that $\Sigma \triangleleft \Delta$, thus by induction hypothesis we have $\Sigma(\triangleleft_1 \cup \triangleleft_2)^+ \Delta$, and again by
728 the remark above, we have that $(\Sigma \cup \{\beta\})(\triangleleft_1 \cup \triangleleft_2)^+ \Delta$, which concludes the proof. ◀

729 B.2 \triangleleft_1 is stable under iteration

730 Let us show now that \triangleleft_1 is stable under iteration:

731 ▶ **Proposition 8.** *If $\Gamma \triangleleft_1 \Delta$ then there are two bowtie choices \bowtie, \bowtie' and two template orders*
732 *\preceq, \preceq' for Γ and Δ respectively such that: $\Gamma_{\bowtie, \preceq}^+ \leq \Delta_{\bowtie', \preceq'}^+$.*

733 To show Prop. 8, we need the following lemma.

734 ▶ **Lemma 41.** *If $\alpha_1, \beta_1, \alpha_2, \beta_2$ are atomic boxes such that $\alpha_1 \triangleleft \beta_1$ and $\alpha_2 \triangleleft \beta_2$ then:*

- 735 ■ $\text{supp}(\alpha_1) \subseteq \text{supp}(\alpha_2) \Rightarrow \text{supp}(\beta_1) \subseteq \text{supp}(\beta_2)$.
- 736 ■ $\text{supp}(\alpha_1) \cap \text{supp}(\alpha_2) = \emptyset \Rightarrow \text{supp}(\beta_1) \cap \text{supp}(\beta_2) = \emptyset$.

Proof. To show this result, let us make first the following observation. If α, β are atomic boxes such that $\alpha \triangleleft \beta$ then:

$$p \in \text{supp}(\beta) \text{ if and only if } \eta(p) \in \text{supp}(\alpha).$$

737 Let us see why this observation holds. We set $\alpha = \langle \vec{\mathfrak{p}}_\alpha, G, \overleftarrow{\mathfrak{p}}_\alpha \rangle$ and $\beta = \langle \vec{\mathfrak{p}}_\beta, H, \overleftarrow{\mathfrak{p}}_\beta \rangle$, and
738 let h be a homomorphism from β to α .

739 Suppose by contradiction that there is $p \in \text{supp}(\beta)$ such that $\eta(p) \notin \text{supp}(\alpha)$. We
740 set $v = \vec{\mathfrak{p}}_\beta(p)$ and $w = \vec{\mathfrak{p}}_\alpha(\eta(p))$. By (η, η) -compatibility, we have $h(v) = w$. Since
741 $p \in \text{supp}(\beta)$, $\vec{\mathfrak{p}}_\beta(p)$ is a node of a non-trivial component of G , thus there is an edge $\langle v, a, u \rangle$
742 in G . Since h is a homomorphism from β to α we should have an edge $\langle w, b, h(u) \rangle$ in H . But
743 since $\eta(p) \notin \text{supp}(\alpha)$, we have that w is an isolated node of H , this gives us a contradiction.

744 Conversely, if $p \notin \text{supp}(\beta)$ then $v := \vec{\mathfrak{p}}_\alpha(p)$ is an isolated node of β , thus $h(v)$ is an
745 isolated node by definition of a box homomorphism. By $(\eta - \eta)$ -compatibility, we have that
746 $\vec{\mathfrak{p}}_\alpha(\eta(p)) = h(v)$, thus $\eta(p) \notin \text{supp}(\alpha)$.

747 Let us go back to the proof of our lemma. Suppose that $\text{supp}(\alpha_1) \subseteq \text{supp}(\alpha_2)$ and let
748 $p \in \text{supp}(\beta_1)$. By the observation above, we have that $\eta(p) \in \text{supp}(\alpha_1)$ thus $\eta(p) \in \text{supp}(\alpha_2)$.
749 By the above observation again, we have $\eta(p) \in \text{supp}(\alpha_2)$.

750 Suppose that $p \in \text{supp}(\beta_1) \cap \text{supp}(\beta_2)$. By the above observation, we have that $\eta(p) \in$
 751 $\text{supp}(\alpha_1) \cap \text{supp}(\alpha_2)$. ◀

752 ► **Lemma 42.** Let α, β be two atomic boxes and h be a homomorphism from β to α . Let
 753 \bowtie, \bowtie' be bowtie choices for α, β , and let $\alpha \stackrel{\bowtie}{=} \alpha_1 \cdot \alpha_2$ and $\beta \stackrel{\bowtie'}{=} \beta_1 \cdot \beta_2$. If $h(\bowtie') = \bowtie$ then
 754 $\alpha_1 \triangleleft \beta_1$ and $\alpha_2 \triangleleft \beta_2$.

755 **Proof.** We show that the homomorphism h induces a homomorphism from β_i to α_i , for
 756 $i = 1, 2$. For that we only need to show that h maps the graph of β_1 to the graph of α_1 and
 757 maps the graph of β_2 to the graph of α_2 . In other words, for $i = 1, 2$:

758 n is a node of β_i if and only if $h(n)$ is a node of α_i

759 Suppose (by symmetry) that there is a node n of β_1 such that $h(n) \in \alpha_2$. There is a path
 760 from n to \bowtie in the graph of β . This path can be mapped by h to a path from $h(n)$ to \bowtie' in
 761 the graph of α . This is not possible by well-typedness of the α . ◀

762 Let us show now Prop. 8.

763 **Proof of Prop. 8.** It is not difficult to see that if $\Gamma \triangleleft_1 \Delta$ then $\text{At}(\Gamma) \triangleleft_1 \text{At}(\Delta)$, thus we
 764 suppose *w.l.o.g.* that Γ and Δ are atomic.

765 Let \bowtie, \bowtie' be the bowtie choices for Γ and Δ respectively, witnessing that $\Gamma \triangleleft_1 \Delta$. We set
 766 $\Gamma = \{\alpha_1, \dots, \alpha_n\}$. Since $\Gamma \triangleleft \Delta$, we have that for every $i \in [1, n]$, there is $\beta_i \in \Delta$ such that
 767 $\alpha_i \triangleleft_1 \beta_i$. We set $\Sigma = \{\beta_1, \dots, \beta_n\}$. Since $\Sigma \subseteq \Delta$, it is enough to show that there are \preceq, \preceq'
 768 such that $\Gamma_{\bowtie, \preceq}^+ \leq \Sigma_{\bowtie', \preceq'}^+$.

Let \preceq be a template order for Γ . Let us define a template order \preceq' for Σ . Note that if
 $\text{supp}(\beta_i) = \text{supp}(\beta_j)$, by Lem. 41 we cannot have $\text{supp}(\alpha_i) \cap \text{supp}(\alpha_j) = \emptyset$, thus by Lem. 21,
 either $\text{supp}(\alpha_i) \subseteq \text{supp}(\alpha_j)$ or $\text{supp}(\alpha_j) \subseteq \text{supp}(\alpha_i)$. We set define \preceq' as follows:

$$\beta_i \preceq' \beta_j \quad \text{iff} \quad \text{supp}(\alpha_i) \subsetneq \text{supp}(\alpha_j) \quad \text{or} \quad \text{supp}(\alpha_i) = \text{supp}(\alpha_j) \quad \text{and} \quad \alpha_i \preceq \alpha_j$$

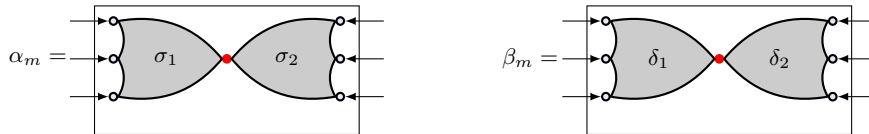
769 Let us show now, by induction on Γ , that $\Gamma_{\bowtie, \preceq}^+ \leq \Sigma_{\bowtie', \preceq'}^+$. We decompose Γ into $\Gamma_1 \cup \{\alpha_m\} \cup \Gamma_2$
 770 such that:

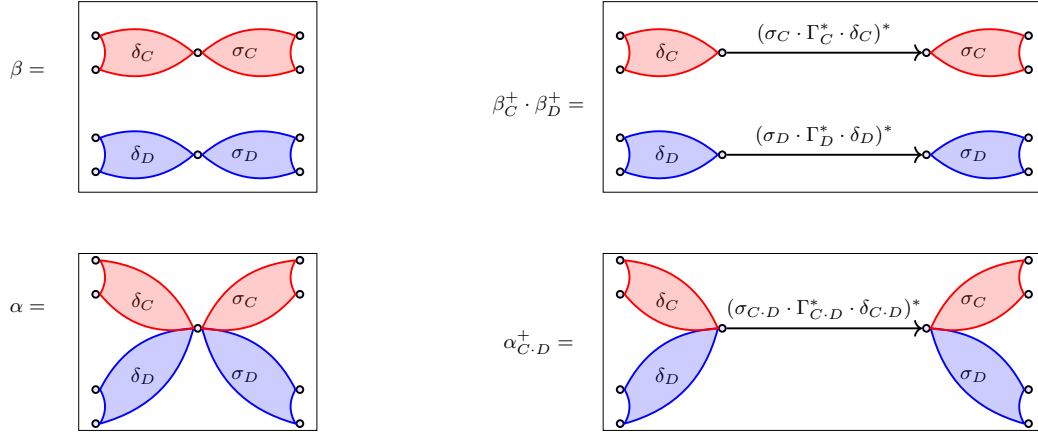
- 771 1. $\forall \alpha \in \Gamma_1, \text{supp}(\alpha) \subseteq \text{supp}(\alpha_i)$.
- 772 2. If $\alpha \preceq \alpha_m$ then $\alpha \in \Gamma_1$.
- 773 3. $\forall \alpha \in \Gamma_2, \text{supp}(\alpha) \cap \text{supp}(\alpha_m) = \emptyset$.

774 We set $\Gamma_1 = \{\alpha_k\}_{k \in I}, \Gamma_2 = \{\alpha_k\}_{k \in J}$ and $\Sigma_1 = \{\beta_k\}_{k \in I}, \Sigma_2 = \{\beta_k\}_{k \in J}$. We have that
 775 $\Sigma = \Sigma_1 \cup \{\beta_m\} \cup \Sigma_2$. Let us show that this decomposition of Σ is relevant for the computation
 776 of its iteration, in particular that β_m can be chosen as a pivot.

- 777 ■ $\forall \beta \in \Sigma_1, \text{supp}(\beta) \subseteq \text{supp}(\beta_m)$. (By item 1 above and Prop. 41)
- 778 ■ If $\beta \preceq' \beta_m$ then $\beta \in \Sigma_1$. (Indeed, by definition of \preceq' , if $\beta_j \preceq' \beta_m$ then $\text{supp}(\alpha_j) \subseteq$
 779 $\text{supp}(\alpha_m)$ thus $\alpha_j \in \Gamma_1$ and then $\beta_j \in \Sigma_1$.)
- 780 ■ $\forall \beta \in \Sigma_2, \text{supp}(\beta) \cap \text{supp}(\beta_i) = \emptyset$. (By item 3 above and Prop. 41).

781 To compute the iteration of Γ and Σ , we decompose the pivots α_m and β_m at the level of
 782 their bowtie choices: $\alpha_m \stackrel{\bowtie}{=} \sigma_1 \cdot \sigma_2$ and $\beta_m \stackrel{\bowtie'}{=} \delta_1 \cdot \delta_2$.





■ **Figure 22** Boxes α, β in the proof of Prop. 9

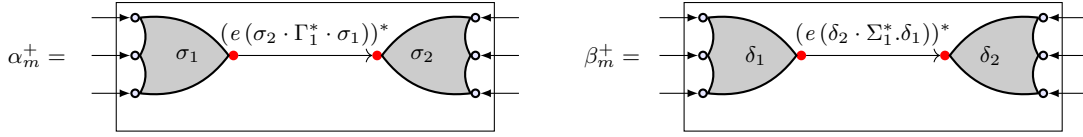
■ **Figure 23** Boxes $\alpha_{C,D}^+$ and $\beta_C^+ \cdot \beta_D^+$ of the proof of Prop. 9

If we set $\Gamma_i^+ = (\Gamma_i)_{\bowtie, \gamma}^+$ and $\Sigma_i^+ = (\Sigma_i)_{\bowtie, \gamma}^+$, for $i = 1, 2$ then we have:

$$\Gamma_{\bowtie, \gamma}^+ = (\Gamma_1^+ \cdot \Gamma_2^*) \cup (\Gamma_1^* \cdot \Gamma_2^+) \cup (\Gamma_1^* \cdot \alpha_m^+ \cdot \Gamma_1^* \cdot \Gamma_2^*)$$

$$\Sigma_{\bowtie, \gamma}^+ = (\Sigma_1^+ \cdot \Sigma_2^*) \cup (\Sigma_1^* \cdot \Sigma_2^+) \cup (\Sigma_1^* \cdot \beta_m^+ \cdot \Sigma_1^* \cdot \Sigma_2^*)$$

Where α_m^+ are and β_m^+ are the following boxes.



784 By induction hypothesis, we have that $\Gamma_1^+ \leq \Sigma_1^+$ and $\Gamma_2^+ \leq \Sigma_2^+$. Since \leq is stable by set
 785 union and composition, it is enough to show that $\alpha_m^+ \leq \beta_m^+$ to conclude. More precisely, we
 786 will show that $\alpha_m^+ \triangleleft_1 \beta_m^+$.

787 Since $\alpha_m \triangleleft_1 \beta_m$, we know by Lem. 42 that $\sigma_1 \triangleleft \delta_1$ and $\sigma_2 \triangleleft \delta_2$. To show that $\alpha_m^+ \triangleleft_1 \beta_m^+$,
 788 it is enough to show that $\text{KL}^- \vdash (e(\sigma_2 \cdot \Gamma_1^* \cdot \sigma_1))^+ \leq (e(\delta_2 \cdot \Sigma_1^* \cdot \delta_1))^+$ or simply that
 789 $\text{KL}^- \vdash e(\sigma_2 \cdot \Gamma_1^* \cdot \sigma_1) \leq e(\delta_2 \cdot \Sigma_1^* \cdot \delta_1)$. For that observe that $\sigma_2 \cdot \Gamma_1^* \cdot \sigma_1 \leq \delta_2 \cdot \Sigma_1^* \cdot \delta_1$ (because
 790 $\sigma_2 \triangleleft \delta_2$, $\Gamma_1^* \leq \Sigma_1^*$ and $\sigma_1 \triangleleft \delta_1$). We can thus conclude by Prop. 6. ◀

791 B.3 \triangleleft_2 is stable under iteration

792 ► **Proposition 9.** *If $\Gamma \triangleleft_2 \Delta$ then there are two bowtie choices \bowtie, \bowtie' and two computation*
 793 *orders \preceq, \preceq' for Γ and Δ respectively such that: $\Gamma_{\bowtie, \preceq}^+ \leq \Delta_{\bowtie', \preceq'}^+$.*

794 **Proof.** Since $\Gamma \triangleleft_2 \Delta$, we can write $\Gamma = \Sigma \cup \{\alpha\}$ and $\Delta = \Sigma \cup \{\beta\}$ such that $\alpha \triangleleft_2 \beta$. This
 795 means that there is a bowtie choice \bowtie' for β , and two connected components C and D of the
 796 graph of β , such that α is obtained by merging $\bowtie(C)$ and $\bowtie(D)$. We denote by $C \cdot D$ the
 797 connected component of α obtained by merging C and D at the level of $\bowtie(C)$ and $\bowtie(D)$.
 798 (see Figure 22)

799 Let us define a bowtie choice \bowtie for α . For the connected component $C \cdot D$, we set
 800 $\bowtie(C \cdot D)$ to be the node resulting from the merge of $\bowtie'(C)$ and $\bowtie'(D)$. For the other

18:26 Completeness for Identity-free Kleene Lattices

801 connected components, \bowtie and \bowtie' coincide. We extend \bowtie and \bowtie' to bowtie choices for Γ and
802 Δ .

803 Let β_C, β_D and $\alpha_{C \cdot D}$ be the atomic boxes stemming respectively from the connected
804 component C, D and $C \cdot D$. Observe that we can write $At(\Gamma)$ and $At(\Delta)$ as $At(\Gamma) = \Theta \cup \{\alpha_{C \cdot D}\}$
805 and $At(\Delta) = \Theta \cup \{\beta_C, \beta_D\}$.

806 Let \preceq be a template order on $At(\Gamma)$ for which $\alpha_{C \cdot D}$ is minimal. Let \preceq' be a template
807 order on $At(\Delta)$ for which β_C and β_D are maximal elements.

We set $\Gamma_C = \{\delta \mid \delta \in \Theta, \text{supp}(\delta) \subseteq \text{supp}(\alpha_C)\}$ and $\Gamma_D = \{\delta \mid \delta \in \Theta, \text{supp}(\delta) \subseteq \text{supp}(\alpha_D)\}$.
The computation of $At(\Gamma)_{\bowtie, \preceq}^+$ starts with the computation of $(\Gamma_C \cup \Gamma_D \cup \{\alpha_{C \cdot D}\})_{\bowtie, \preceq}^+$ and that
of $At(\Delta)_{\bowtie', \preceq'}^+$ starts with the computation of $(\Gamma_C \cup \Gamma_D \cup \{\beta_C, \beta_D\})_{\bowtie', \preceq'}^+$. Both carry on in
exactly the same way, using respectively $(\Gamma_C \cup \Gamma_D \cup \{\alpha_{C \cdot D}\})_{\bowtie, \preceq}^+$ and $(\Gamma_C \cup \Gamma_D \cup \{\beta_C, \beta_D\})_{\bowtie', \preceq'}^+$
as black-boxes. It is thus enough to show that:

$$(\Gamma_C \cup \Gamma_D \cup \{\alpha_{C \cdot D}\})_{\bowtie, \preceq}^+ \leq (\Gamma_C \cup \Gamma_D \cup \{\beta_C, \beta_D\})_{\bowtie', \preceq'}^+$$

We decompose β_C, β_D and $\alpha_{C \cdot D}$ as follows (See Figure 22):

$$\begin{array}{lcl} \beta_C & \stackrel{\bowtie'}{\cong} & \delta_C \cdot \sigma_C \\ \beta_D & \stackrel{\bowtie'}{\cong} & \delta_D \cdot \sigma_D \\ \alpha_{C \cdot D} & \stackrel{\bowtie}{\cong} & \delta_{C \cdot D} \cdot \sigma_{C \cdot D} \end{array}$$

808 Since \preceq and \preceq' (resp. \bowtie and \bowtie') coincide on the elements of Θ , we will write Γ_C^+ (resp. Γ_D^+)
809 for the iteration of Γ_C (resp. Γ_D) under the bowtie choice \bowtie and the template order \preceq or
810 under the bowtie choice \bowtie' and the template order \preceq' .

Since $\text{supp}(\Gamma_C) \cap \text{supp}(\Gamma_D) = \emptyset$, we have that $\Gamma_C^+ \cdot \Gamma_D^+ = \Gamma_D^+ \cdot \Gamma_C^+$, we denote this product
simply by $\Gamma_{C \cdot D}^+$. We have also that:

$$\begin{aligned} (\Gamma_C \cup \Gamma_D \cup \{\alpha_{C \cdot D}\})_{\bowtie, \preceq}^+ &= \Gamma_{C \cdot D}^+ \cup (\Gamma_{C \cdot D}^* \cdot \alpha_{C \cdot D}^+ \cdot \Gamma_{C \cdot D}^*) \\ (\Gamma_C \cup \Gamma_D \cup \{\beta_C, \beta_D\})_{\bowtie, \preceq}^+ &\supseteq \Gamma_{C \cdot D}^+ \cup (\Gamma_{C \cdot D}^* \cdot \beta_C^+ \cdot \beta_D^+ \cdot \Gamma_{C \cdot D}^*) \end{aligned}$$

811 Where $\alpha_{C \cdot D}^+$ and $\beta_C^+ \cdot \beta_D^+$ are the boxes depicted in Figure 23. It is not difficult to see that
812 $\alpha_{C \cdot D}^+ \triangleleft \beta_C^+ \cdot \beta_D^+$, whence the result. \blacktriangleleft