Interface effects on the effective behavior of some Kelvin–Voigt viscoelastic heterogeneous bodies

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Abstract. The effective behavior of a solid made from a periodic distribution of inclusions in a matrix is investigated. Inclusions and matrix are linearly elastic or viscoelastic of Kelvin–Voigt type (and possibly rigid for the inclusions) while the link between them can be pure adhesion or viscous friction with bilateral contact or involve a very thin viscoelastic layer. As long as one constituent is viscoelastic, the effective behavior is no longer of Kelvin–Voigt type but with memory.

Keywords: homogenization, Kelvin–Voigt viscoelasticity, viscoelasticity with memory, interface effects, semi-groups of operators, Laplace transform, integro-differential equations

1. Introduction

The effective behavior of a periodically heterogeneous body made of Kelvin–Voigt type viscoelastic materials was long ago investigated by homogenization theory in [4]. It was proven that this effective behavior is *no longer of Kelvin–Voigt type* but rather with fading memory. Here we extend this result to the case of composites with a periodic structure made of two or three phases, with each phase being either viscoelastic of Kelvin–Voigt type or purely elastic, but at least one is viscoelastic. The three-phase case corresponds to a periodic distribution of inclusions in a matrix linked to the matrix through a thin soft layer, or the so called interphase, whereas in the two-phase case, inclusions and matrix are perfectly bonded or a condition involving bilateral contact with friction of Kelvin–Voigt type may occur. The mechanical motivation underlying this study concerns the evident technological difficulty of achieving perfect bonding of inclusions in a matrix when manufacturing composite materials [9].

More precisely, the geometry of the composite can be described as follows. Let $Y := (0,1)^3$ the unit cell of $\mathbb{R}^3$, I a simply connected open set, with a Lipschitz-continuous boundary S, whose closure $\overline{I}$ is included in Y, and $M := Y \setminus \overline{I}$. If h is a small real number (less than $\mathrm{dist}(S, \partial Y)$), let $I_h = \{y \in I; \mathrm{dist}(y, S) > h\}$, $B_h = \{y \in Y; \mathrm{dist}(y, S) < h\}$ and $M_h = Y \setminus (\overline{B}_h \cup I_h)$. Let Ω denote the domain,

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with a Lipschitz-continuous boundary $\partial\Omega$, occupied by the heterogeneous body under consideration. If ε is a small number (the scaling parameter of the periodic structure), let $J_\varepsilon := \{j \in \mathbb{Z}^3;\ \varepsilon(j+Y) \subset \Omega\}$,

$$I_\varepsilon := \bigcup_{j\in J_\varepsilon} \varepsilon(j+I), \qquad S_\varepsilon := \bigcup_{j\in J_\varepsilon} \varepsilon(j+S), \qquad M_\varepsilon := \Omega \setminus \overline{I}_\varepsilon,$$

$$I_{h,\varepsilon} := \bigcup_{j\in J_\varepsilon} \varepsilon(j+I_h), \qquad B_{h,\varepsilon} := \bigcup_{j\in J_\varepsilon} \varepsilon(j+B_h), \qquad M_{h,\varepsilon} := \Omega \setminus (I_{h,\varepsilon} \cup \overline{B}_{h,\varepsilon}).$$

So in the case of two phases, I_ε is the region occupied by the periodic distribution of inclusions, which does not intersect $\partial\Omega$, and M_ε is the region occupied by the matrix. While in the three-phase case, $M_{h,\varepsilon}$, $B_{h,\varepsilon}$, $I_{h,\varepsilon}$ correspond to the regions occupied by the matrix, the interphase and inclusions, respectively.

To describe the constitutive equations of the body, we introduce A_M, B_M in $L^\infty(M; \mathrm{Lin}(\mathbb{S}^3))$ and A_I, B_I in $L^\infty(I; \mathrm{Lin}(\mathbb{S}^3))$, where $\mathrm{Lin}(\mathbb{S}^3)$ denotes the space of symmetric linear mappings from $\mathbb{S}^3$ to $\mathbb{S}^3$, the space of 3×3 symmetric matrices, with the usual inner product and norm denoted by $\cdot$ and $|\ |$ (as in $\mathbb{R}^3$). We assume that there exists $\kappa > 0$ such that

$$\begin{cases} A_M(y)\xi\cdot\xi, B_M(y)\xi\cdot\xi \geqslant \kappa|\xi|^2_{\mathbb{S}^3} & \text{a.e. } y \text{ in } M, \forall\xi \in \mathbb{S}^3, \\ A_I(y)\xi\cdot\xi, B_I(y)\xi\cdot\xi \geqslant \kappa|\xi|^2_{\mathbb{S}^3} & \text{a.e. } y \text{ in } I, \forall\xi \in \mathbb{S}^3 \end{cases} \tag{1.1}$$

and extend these mappings into $\mathbb{R}^3$ by Y-periodicity. We will consider eight examples of composites indexed by J. The first six examples correspond to two phases occupying M_ε and I_ε. In the first four examples, the constituents are assumed to be perfectly bonded along S_ε, whereas, in the last two examples, there are conditions of bilateral contact with Kelvin–Voigt type tangential friction

$$\sigma_T = \mu_e^*[u]_T + \mu_v^*[\dot{u}]_T, \qquad [u]_N = 0.$$

Here μ_e^*, μ_v^* are strictly positive real numbers and $[u]$ denotes the jump of displacement (the relative displacement) along S_ε, defined as the difference of the trace of the displacement u in I_ε with the one in M_ε and n, σn being the inward normal to ∂I_ε and the common stress vector along S_ε, respectively; one has

$$[u]_N = [u]\cdot n, \qquad [u]_T = [u] - [u]_N n, \qquad \sigma_N = \sigma n \cdot n, \qquad \sigma_T = \sigma n - \sigma_N n,$$

while the upper dot $^{\boldsymbol{\cdot}}$ stands for time derivative. In case $J = 1$, the matrix and inclusions are assumed to be viscoelastic of Kelvin–Voigt type, with the elasticity and viscosity tensor equal to $A_M(x/\varepsilon)$, $B_M(x/\varepsilon)$ for all x in M_ε, and $A_I(x/\varepsilon)$, $B_I(x/\varepsilon)$ for all x in I_ε; this is the case treated in [4]. Case $J = 2$ corresponds to a viscoelastic matrix, with elasticity and viscosity tensors $A_M(x/\varepsilon)$ and $B_M(x/\varepsilon)$, surrounding linearly elastic inclusions of elasticity tensor $A_I(x/\varepsilon)$. Case $J = 3$ is the "complement" of the previous one: viscoelastic inclusions with tensors $A_I(x/\varepsilon)$, $B_I(x/\varepsilon)$ are periodically distributed in an elastic matrix with elasticity tensor $A_M(x/\varepsilon)$. Case $J = 4$ deals with rigid inclusions in a viscoelastic matrix with tensors $A_M(x/\varepsilon)$, $B_M(x/\varepsilon)$. Case $J = 5$ corresponds to viscoelastic matrix and inclusions as when $J = 1$, whereas $J = 6$ corresponds to a purely linearly elastic matrix and inclusions with elasticity tensors $A_M(x/\varepsilon)$ and $A_I(x/\varepsilon)$, respectively. This last case with $\mu_e^* = 0$ was treated by the asymptotic expansion method in [5]. Finally, we add two other examples involving three perfectly bonded phases occupying $M_{h,\varepsilon}$, $B_{h,\varepsilon}$ and $I_{h,\varepsilon}$. The interphase is assumed to be isotropically viscoelastic of Kelvin–Voigt type

$$\sigma = \lambda_e \operatorname{tr} e(u) I_{\mathrm{d}} + 2\mu_e e(u) + \lambda_v \operatorname{tr} e(\dot{u}) I_{\mathrm{d}} + 2\mu_v e(\dot{u}), \tag{1.2}$$

where λ_e, μ_e, λ_v, μ_v are strictly positive numbers and σ, $e(u)$, $e(\dot{u})$ denote stress, strain and strain rate tensors, and I_{d} stands for the identity matrix of $\mathbb{S}^3$. When $J = 7$, matrix and inclusions are assumed to be viscoelastic with tensors $A_M(x/\varepsilon)$, $B_M(x/\varepsilon)$, $A_I(x/\varepsilon)$, $B_I(x/\varepsilon)$, while if $J = 8$, matrix and inclusions are purely linearly elastic with elasticity tensors $A_M(x/\varepsilon)$ and $A_I(x/\varepsilon)$, respectively.

Thus the problem of determining the quasi-static evolution during an interval of time $[0, T]$ of the body in a given initial state, clamped along $\Gamma_0 \subset \partial\Omega$, with strictly positive two-dimensional Hausdorff measure $\mathcal{H}_2(\Gamma_0)$, and subjected to body forces in Ω and surface forces on $\Gamma_1 = \partial\Omega \setminus \overline{\Gamma_0}$ of densities f, g, involves a set s of data which reduces to only ε in the first four examples, is $(\varepsilon, \mu_e^*, \mu_v^*)$ when $J = 5, 6$, and is (ε, s'), $s' = (h, \lambda_e, \mu_e, \lambda_v, \mu_v)$ in the last two examples. In the next section, we show that, under suitable data assumptions, the problem can be formulated as:

$$\begin{cases} \text{Find } u_s \text{ in } C^1\big([0,T]; H_s\big) \quad \text{such that} \\ a_s\big(u_s(t), \varphi\big) + b_s\big(\dot{u}_s(t), \varphi\big) = L(t)(\varphi) \quad \forall (\varphi, t) \in H_s \times [0,T], \\ u_s(0) = u_s^o, \end{cases} \tag{P_s}$$

where H_s is a Hilbert space of possible states with finite total strain energy equipped with the inner product $a_s(\cdot,\cdot)$; b_s is a continuous bilinear form on H_s and $L(t)$ is a continuous linear form representing the work of the loading (f, g) at time t. The existence and uniqueness of a field of displacement u_s solution to (P_s) is easily obtained by transforming (P_s) in an evolution equation in H_s and using the theory of semi-groups of linear operators. In Section 3, to determine the effective behavior of the composites we consider the geometrical and mechanical data s as parameters taking values in a countable set and study the asymptotic behavior of u_s when ε goes to 0, μ_e^* and μ_v^* go to infinity, s' goes to zero according to:

$$\begin{aligned} &\bullet \quad \exists\big(\bar{\mu}_e^*, \bar{\mu}_v^*\big) \in (0, +\infty)^2 \quad \text{such that} \quad \lim\big(\mu_e^*\varepsilon, \mu_v^*\varepsilon\big) = \big(\bar{\mu}_e^*, \bar{\mu}_v^*\big), \\ &\bullet \quad \exists(\bar{\lambda}_e, \bar{\mu}_e, \bar{\lambda}_v, \bar{\mu}_v) \in (0, +\infty)^4 \quad \text{such that} \\ &\qquad \lim(\lambda_e/2h, \mu_e/2h, \lambda_v/2h, \mu_v/2h) = (\bar{\lambda}_e, \bar{\mu}_e, \bar{\lambda}_v, \bar{\mu}_v), \qquad \overline{\lim}\, \varepsilon^2/\mu_e = \overline{\lim}\, \varepsilon^2/h = 0. \end{aligned} \tag{1.3}$$

Note that s' and ε go quite independently to zero under the sole condition $\overline{\lim}\, \varepsilon^2/\mu_e = \overline{\lim}\, \varepsilon^2/h = 0$ which means that the Lamé coefficient μ_e and the thickness of the interphase are not too small. In the following, we will say that s goes to $\bar{s}$ according to (1.3).

By using the Laplace transform and a suitable assumption on the initial state, it can be shown that in *all examples* the effective constitutive equation is of the type

$$\sigma = A^{\mathrm{eff}} e(u) + B^{\mathrm{eff}} e(\dot{u}) + \int_0^t K(t - \tau) e(\dot{u})(\tau)\, \mathrm{d}\tau, \tag{1.4}$$

where A^{eff} and B^{eff} are the standard homogenized tensors corresponding to the heterogeneous elastic and viscous behaviors, while the kernel K is a perfectly identified element of Lin($\mathbb{S}^3$). In some cases B^{eff} may vanish.

The principle of this study is that of [4] but due to the character, say, singular of the cases $J \neq 1$ (purely elastic behavior in some regions and/or soft thin interphase, or viscoelastic friction on the interfaces) *which are relevant in practice*, there are some technical difficulties that must be overcome (H_s may be strongly s-dependent, b_s may have a kernel and consequently (P_s) involves an unbounded evolution operator and convergences do occur in weak-$*$ topologies).

2. A result of existence and uniqueness for (P_s)

We define the space H_s as follows:

$$\begin{cases} 1 \leqslant J \leqslant 3 & H_s := H^1_{\Gamma_0}\big(\Omega;\mathbb{R}^3\big) := \big\{u \in H^1\big(\Omega;\mathbb{R}^3\big); u = 0 \text{ on } \Gamma_0\big\}, \\ J = 4 & H_s := \big\{u \in H^1_{\Gamma_0}\big(\Omega;\mathbb{R}^3\big); e(u) = 0 \text{ in } I_\varepsilon\big\}, \\ & \text{with } e \text{ being the symmetrized gradient in the sense of distributions,} \\ J = 5,6 & H_s := \big\{u \in H^1_{\Gamma_0}\big(\Omega \setminus S_\varepsilon;\mathbb{R}^3\big); [u]_N = 0 \text{ on } S_\varepsilon\big\}, \\ J = 7,8 & H_s := H^1_{\Gamma_0}\big(\Omega;\mathbb{R}^3\big), \end{cases} \tag{2.1}$$

it is equipped with the inner product, equivalent to the classical one, associated with the bilinear form a_s defined by:

$$\begin{cases} J \leqslant 4 & a_s(u,v) := \int_{M_\varepsilon} A_M(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x + \int_{I_\varepsilon} A_I(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x, \\ J = 5,6 & a_s(u,v) := \int_{M_\varepsilon} A_M(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x + \int_{S_\varepsilon} \mu_e^*[u]_T\cdot[v]_T\,\mathrm{d}H_2 \\ & \qquad\qquad + \int_{I_\varepsilon} A_I(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x, \\ & \text{where we still denote the symmetrized gradient in the} \\ & \text{sense of distribution in } \Omega\setminus S_\varepsilon \text{ by } e, \\ J = 7,8 & a_s(u,v) := \int_{M_{h,\varepsilon}} A_M(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x \\ & \qquad\qquad + \int_{B_{h,\varepsilon}} \big(\lambda_e \operatorname{tr} e(u)\operatorname{tr} e(v) + 2\mu_e e(u)\cdot e(v)\big)\,\mathrm{d}x \\ & \qquad\qquad + \int_{I_{h,\varepsilon}} A_I(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x, \\ & \text{where tr denotes the trace operator on } \mathbb{S}^3. \end{cases} \tag{2.2}$$

Let b_s be the continuous bilinear form on H_s deduced from the irreversible part of the behavior of the body

$$\begin{cases} J = 1 & b_s(u,v) := \int_{M_\varepsilon} B_M(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x + \int_{I_\varepsilon} B_I(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x, \\ J = 2,4 & b_s(u,v) := \int_{M_\varepsilon} B_M(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x, \\ J = 3 & b_s(u,v) := \int_{I_\varepsilon} B_I(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x, \\ J = 5 & b_s(u,v) := \int_{M_\varepsilon} B_M(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x + \int_{S_\varepsilon} \mu_v^*[u]_T\cdot[v]_T\,\mathrm{d}H_2 \\ & \qquad\qquad + \int_{I_\varepsilon} B_I(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x, \\ J = 6 & b_s(u,v) := \int_{S_\varepsilon} \mu_v^*[u]_T\cdot[v]_T\,\mathrm{d}H_2, \\ J = 7 & b_s(u,v) := \int_{M_{h,\varepsilon}} B_M(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x \\ & \qquad\qquad + \int_{B_{h,\varepsilon}} \big(\lambda_v \operatorname{tr} e(u)\operatorname{tr} e(v) + 2\mu_v e(u)\cdot e(v)\big)\,\mathrm{d}x \\ & \qquad\qquad + \int_{I_{h,\varepsilon}} B_I(x/\varepsilon)e(u)\cdot e(v)\,\mathrm{d}x, \\ J = 8 & b_s(u,v) := \int_{B_{h,\varepsilon}} \big(\lambda_v \operatorname{tr} e(u)\operatorname{tr} e(v) + 2\mu_v e(u)\cdot e(v)\big)\,\mathrm{d}x. \end{cases} \tag{2.3}$$

If the loading (f, g) is assumed to be in $C^0([0,T]; L^2(\Omega;\mathbb{R}^3) \times L^2(\Gamma_1;\mathbb{R}^3))$, the quasistatic evolution of the body in initial state u_s^o in H_s may be formulated as:

$$\begin{cases} \text{Find } u_s \text{ in } C^1\big([0,T]; H_s\big) \quad \text{such that} \\ a_s\big(u_s(t), v\big) + b_s\big(\dot{u}_s(t), v\big) = L(t)(v) \quad \forall (v,t) \in H_s \times [0,T], \\ u_s(0) = u_s^o, \end{cases} \tag{P_s}$$

where

$$L(t)(v) := \int_\Omega f(x,t) \cdot v(x)\,\mathrm{d}x + \int_{\Gamma_1} g(x,t) \cdot v(x)\,\mathrm{d}H_2 \quad \forall (v,t) \in H_s \times [0,T]. \tag{2.4}$$

To solve (P_s), we make the additional assumption

$$(f,g) \in C^{1,\alpha}\big([0,T]; L^2\big(\Omega;\mathbb{R}^3\big) \times L^2\big(\Gamma_1;\mathbb{R}^3\big)\big), \quad \alpha \in (0,1], \tag{2.5}$$

and seek u_s in the form

$$u_s = u_s^e + u_s^r, \tag{2.6}$$

where u_s^e satisfies

$$\exists! u_s^e(t) \in C^{1,\alpha}\big([0,T]; H_s\big); \quad a_s\big(u_s^e(t), v\big) = L(t)v \quad \forall (v,t) \in H_s \times [0,T]. \tag{2.7}$$

Let

$$\begin{cases} \ker b_s := \big\{u \in H_s; b_s(u,v) = 0 \ \forall v \in H_s\big\}, \\ V_s := \ker b_s^\perp := \big\{u \in H_s; a_s(u,v) = 0 \ \forall v \in \ker b_s\big\}, \end{cases} \tag{2.8}$$

then the remaining part u_s^r will therefore be involved in an evolution equation in V_s governed by the following operator:

$$\begin{cases} D(A_s) = \big\{u \in V_s; \exists! w(u) \in V_s \text{ s.t. } b_s\big(w(u), v\big) + a_s(u,v) = 0 \ \forall v \in V_s\big\}, \\ A_s u = w(u). \end{cases} \tag{2.9}$$

Proposition 2.1. *A_s is an m-dissipative operator.*

Proof.

(i) A_s is dissipative because, for all u in $D(A_s)$, we have

$$a_s(A_s u, u) = a_s\big(w(u), u\big) = -b_s\big(w(u), w(u)\big) \leqslant 0.$$

(ii) $I - A_s$ is onto.
Let F be in V_s, if $u - A_s u = F$, we should have

$$u \in V_s; \quad b_s(u - F, v) + a_s(u,v) = 0 \quad \forall v \in V_s,$$

which has a unique solution $\bar{u}$ by the Lax–Milgram lemma. Hence, $\bar{u} \in D(A_s)$ and $\bar{u} - A_s\bar{u} = F$.

(iii) A_s is self-adjoint: as A_s is m-dissipative, it suffices to prove that A_s is symmetric, which is true because for all u, v in $D(A_s)$ one has

$$a_s(A_s u, v) = a_s\big(w(u), v\big) = -b_s\big(w(u), w(v)\big) = a_s\big(u, w(v)\big) = a_s(u, A_s v). \qquad \square$$

Lastly, it is straight forward to check that (P_s) is equivalent to

$$\begin{cases} \dfrac{\mathrm{d}u_s^r}{\mathrm{d}t} = A_s u_s^r - \dfrac{\mathrm{d}(u_s^e)^{\perp}}{\mathrm{d}t}, \\ u_s^r(0) = u_s^o - u_s^e(0) := u_s^{ro}, \end{cases} \tag{2.10}$$

where $u_s^{e\perp}$ denotes the projection of u_s^e on V_s, which classically has a unique solution if $u_s^{ro} \in u_s^e(0) + D(A_s)$. Hence the following result is established.

Theorem 2.1. *If $(f, g) \in C^{1,\alpha}([0,T]; L^2(\Omega;\mathbb{R}^3) \times L^2(\Gamma_1;\mathbb{R}^3))$ for some α in $(0,1]$, then (P_s) has a unique solution of class $C^1([0,T]; H_s)$ when $u_s^o \in u_s^e(0) + D(A_s)$, of class $C^0([0,T]; H_s) \cap C^1((0,T]; H_s)$ when $u_s^o \in u_s^e(0) + V_s$.*

Remark 2.1. If the kernel of b_s is not reduced to $\{0\}$, the operator A_s is unbounded. In the other cases ($J \in \{1,4,5,7\}$), by equipping H_s with b_s as inner product (like in [4] for $J = 1$), one may formulate (P_s) in terms of an ordinary differential equation, and the conclusions of Theorem 2.1 are reached just with the milder assumption $(f, g) \in C^0([0,T]; L^2(\Omega;\mathbb{R}^3) \times L^2(\Gamma_1;\mathbb{R}^3))$.

3. Effective behavior of the body

As previously stated, the strategy is the one of [4]. By using the Laplace transform and a suitable assumption on the initial state, the problem reduces to two problems of homogenization in elasticity. In the non-standard cases ($J \geqslant 7$), we solve them by a variational convergence method (see [2,3]), and the obtained "homogenized coefficients" expressions permit us to determine the effective behavior of the heterogeneous viscoelastic body. We proceed in five steps, and, as usual, C and C' will denote various constants.

First step (Extension of u_s into $[0,+\infty)$). Let $(\tilde{f}, \tilde{g})$ be a $C^{1,\alpha}([0,+\infty); L^2(\Omega;\mathbb{R}^3) \times L^2(\Gamma_1;\mathbb{R}^3))$ extension of (f, g) with compact support in $[0, T+1)$, then the solution u_s to (P_s) can be viewed as the restriction to $[0,T]$ of the unique solution $\tilde{u}_s$ to

$$\begin{cases} a_s(\tilde{u}_s, v) + b_s(\dot{\tilde{u}}_s, v) = \displaystyle\int_\Omega \tilde{f} \cdot v \,\mathrm{d}x + \int_{\Gamma_1} \tilde{g} \cdot v \,\mathrm{d}H_2 \quad \forall (v,t) \in H_s \times [0,+\infty), \\ \tilde{u}_s(0) = u_s^o \in u_s^e(0) + V_s, \end{cases} \tag{3.1}$$

which definitely exists in $L^\infty(0,+\infty; H_s) \cap C^1([0,+\infty); H_s)$ when $J \in \{1,4,5,7\}$ (see Remark 2.1) and in $L^\infty(0,+\infty; H_s) \cap C^0([0,+\infty); H_s) \cap C^1((0,+\infty); H_s)$ when $J \in \{2,3,6,8\}$. The inequality

$$a_s(u,u) \geqslant \kappa \big|e(u)\big|^2_{L^2(\Omega;\mathbb{S}^3)} \quad \forall u \in H_s, J \in \{1,2,3,4\}, \tag{3.2}$$

due to (1.1), does not hold when $J \in \{5,6,7,8\}$, but one has

$$a_s(u,u) \geqslant C\big(|u|^2_{L^2(\Omega;\mathbb{R}^3)} + |u|^2_{BD(\Omega)}\big) \quad \forall u \in H_s, J \in \{5,6,7,8\}, \tag{3.3}$$

where we recall that

$$\begin{aligned}
&BD(\Omega) := \big\{u \in L^1\big(\Omega;\mathbb{R}^3\big); e(u) \in M_b\big(\Omega;\mathbb{S}^3\big)\big\},\\
&M_b\big(\Omega;\mathbb{S}^3\big) := \big\{\text{bounded } \mathbb{S}^3\text{-valued measures on } \Omega\big\},\\
&LD(\Omega) := \big\{u \in L^1\big(\Omega;\mathbb{R}^3\big); e(u) \in L^1\big(\Omega;\mathbb{S}^3\big)\big\}.
\end{aligned}$$

Indeed, if $J \in \{7,8\}$, one has

$$\begin{aligned}
a_s(u,u) &\geqslant \kappa\big|e(u)\big|^2_{L^2(M_{h,\varepsilon}\cup I_{h,\varepsilon};\mathbb{S}^3)} + \mu_e\big|e(u)\big|^2_{L^2(B_{h,\varepsilon};\mathbb{S}^3)}\\
&\geqslant \big(\kappa/|\Omega|\big)\big|e(u)\big|^2_{L^1(M_{h,\varepsilon}\cup I_{h,\varepsilon};\mathbb{S}^3)} + (C\mu_e/h)\big|e(u)\big|^2_{L^1(B_{h,\varepsilon};\mathbb{S}^3)}\\
&\geqslant C\big|e(u)\big|^2_{L^1(\Omega;\mathbb{S}^3)} \geqslant C|u|^2_{LD(\Omega)} = C|u|^2_{BD(\Omega)}
\end{aligned}$$

because u vanishes on Γ_0, see [12]. To complete the proof of (3.3), we first recall that, using a variant of [11], p. 183, involving rigid displacements, any u in $H^1_{\Gamma_0}(M_{h_o,\varepsilon};\mathbb{R}^3)$ has an extension $\tilde{u}$ into $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ such that $|e(\tilde{u})|_{L^2(\Omega;\mathbb{S}^3)} \leqslant C|e(u)|_{L^2(M_{h_o,\varepsilon};\mathbb{R}^3)}$, where h_o is a fixed positive number less than dist$(S,\partial Y)$. Thus the Korn inequality in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ yields

$$|\tilde{u}|^2_{L^2(\Omega;\mathbb{R}^3)} \leqslant C\big|e(\tilde{u})\big|^2_{L^2(\Omega;\mathbb{S}^3)} \leqslant C'\big|e(u)\big|^2_{L^2(M_{h_o,\varepsilon};\mathbb{S}^3)} \leqslant \big(C'/\kappa\big)a_s(u,u). \tag{3.4}$$

Next the periodic structure of $M_{h_o,\varepsilon}$ and the Korn inequality in $H^1_{\Gamma_0}(\Omega \setminus M_{h_o,\varepsilon})$ give

$$\begin{aligned}
|u-\tilde{u}|^2_{L^2(\Omega;\mathbb{R}^3)} &= |u-\tilde{u}|^2_{L^2(\Omega\setminus M_{h_o,\varepsilon};\mathbb{R}^3)} \leqslant C\varepsilon^2\big|e(u-\tilde{u})\big|^2_{L^2(\Omega\setminus M_{h_o,\varepsilon};\mathbb{S}^3)}\\
&\leqslant C\varepsilon^2\big(\big|e(\tilde{u})\big|^2_{L^2(\Omega;\mathbb{S}^3)} + \big|e(u)\big|^2_{L^2(M_{h,\varepsilon};\mathbb{S}^3)} + \big|e(u)\big|^2_{L^2(B_{h,\varepsilon};\mathbb{S}^3)} + \big|e(u)\big|^2_{L^2(I_{h,\varepsilon};\mathbb{S}^3)}\big)\\
&\leqslant C\varepsilon^2\left(1+\frac{1}{\mu_e}\right)a_s(u,u).
\end{aligned} \tag{3.5}$$

Then (3.4) implies

$$|u|^2_{L^2(\Omega;\mathbb{R}^3)} \leqslant C\left(1+\varepsilon^2+\frac{\varepsilon^2}{\mu_e}\right)a_s(u,u) \tag{3.6}$$

which, with assumption (1.3), achieves the proof of (3.3).

Remark 3.1. The weaker assumption $\overline{\lim}\varepsilon^2/\mu_e < +\infty$ is enough to get (3.3). But, as we shall see in the proof of Proposition 3.1, it is crucial to know that a sequence (u_s) such that $a_s(u_s,u_s) \leqslant C$

is strongly relatively compact in $L^2(\Omega;\mathbb{R}^3)$ and that any limit point belongs to $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$. Actually, (3.4) implies that $(\tilde{u}_s)$ is weakly relatively compact in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ and, by Rellich Theorem, strongly relatively compact in $L^2(\Omega;\mathbb{R}^3)$, then (3.5) *with assumption* (1.3) yield the strong relative compactness in $L^2(\Omega;\mathbb{R}^3)$ of (u_s) and that any limit point belongs to $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$.

In the case $J \in \{5,6\}$ the same argument with $h_o = 0$ (see [5]) gives

$$|u|^2_{L^2(\Omega;\mathbb{R}^3)} \leqslant C a_s(u,u) \quad \forall u \in H_s.$$

To conclude, it should be noted, that for all τ in $C^0_0(\overline{\Omega};\mathbb{S}^3) \cap C^1(\Omega;\mathbb{S}^3)$ we have

$$\begin{aligned}
-\int_\Omega u \cdot \operatorname{div}\tau \,\mathrm{d}x &= -\int_{M_\varepsilon} u \cdot \operatorname{div}\tau \,\mathrm{d}x - \int_{I_\varepsilon} u \cdot \operatorname{div}\tau \,\mathrm{d}x \\
&= \int_{M_\varepsilon} e(u) \cdot \tau \,\mathrm{d}x + \int_{I_\varepsilon} e(u) \cdot \tau \,\mathrm{d}x - \int_{S_\varepsilon} \tau n[u] \,\mathrm{d}H_2 \\
&\leqslant C|\tau|_{C^0(\overline{\Omega};\mathbb{S}^3)} a_s(u,u)^{1/2},
\end{aligned}$$

where $C^0_0(\overline{\Omega};\mathbb{S}^3) := \{\tau \in C^0(\overline{\Omega};\mathbb{S}^3); \tau = 0 \text{ on } \partial\Omega\}$. Thus, (3.1)–(3.3) imply

$$\begin{cases}
\text{when } J \in \{1,4\}\ \tilde{u}_s \text{ is bounded in } W^{1,\infty}\big(0,+\infty; H^1_{\Gamma_0}\big(\Omega;\mathbb{R}^3\big)\big), \\
\text{when } J \in \{2,3\}\ \tilde{u}_s \text{ is bounded in } L^{\infty}\big(0,+\infty; H^1_{\Gamma_0}\big(\Omega;\mathbb{R}^3\big)\big), \\
\text{when } J \in \{5,7\}\ \tilde{u}_s \text{ is bounded in } W^{1,\infty}\big(0,+\infty; BD(\Omega) \cap L^2\big(\Omega;S^3\big)\big), \\
\text{when } J \in \{6,8\}\ \tilde{u}_s \text{ is bounded in } L^{\infty}\big(0,+\infty; BD(\Omega) \cap L^2\big(\Omega;S^3\big)\big).
\end{cases} \tag{3.7}$$

Second step (Laplace transform of (P_s)). So, if for any Banach space X, $\mathcal{L}z$ denotes the Laplace transform

$$\mathcal{L}z(p) := \int_0^{+\infty} \exp(-pt) z(t) \,\mathrm{d}t \quad \forall p \in (0,+\infty) \tag{3.8}$$

of any function of $L^\infty(0,+\infty;X)$, one has for all $p \in (0,+\infty)$

$$\begin{cases}
\mathcal{L}u_s(p) \in H_s; \\
(a_s + p b_s)\big(\mathcal{L}u_s(p), v\big) = b_s\big(u^o_s, v\big) + \displaystyle\int_\Omega \mathcal{L}\tilde{f} \cdot v \,\mathrm{d}x + \int_{\Gamma_1} \mathcal{L}\tilde{g} \cdot v \,\mathrm{d}H_2 \quad \forall v \in H_s.
\end{cases} \tag{3.9}$$

Similar to [4], one makes the fundamental assumption of admissibility for the initial state

$$\begin{cases}
\exists \big(f^o, g^o\big) \in L^2\big(\Omega;\mathbb{R}^3\big) \times L^2\big(\Gamma_1;\mathbb{R}^3\big) \quad \text{such that} \\
a_s\big(u^o_s, v\big) = \displaystyle\int_\Omega f^o \cdot v \,\mathrm{d}x + \int_{\Gamma_1} g^o \cdot v \,\mathrm{d}H_2 \quad \forall v \in H_s.
\end{cases} \tag{3.10}$$

Hence (3.9) can be written

$$\begin{cases} \theta_s := p\mathcal{L}u_s(p) - u_s^o \in H_s; \\ (a_s/p + b_s)(\theta_s, v) \\ \quad = \int_\Omega \mathcal{L}\tilde{f} \cdot v \, \mathrm{d}x + \int_{\Gamma_1} \mathcal{L}\tilde{g} \cdot v \, \mathrm{d}H_2 - \frac{1}{p}\Big(\int_\Omega f^o \cdot v \, \mathrm{d}x + \int_{\Gamma_1} g^o \cdot v \, \mathrm{d}H_2\Big) \quad \forall v \in H_s, \end{cases}$$

so that the study of the asymptotic behavior of u_s reduces to two problems of homogenization in linearized elasticity involving the bilinear forms a_s and $c_s(p) := a_s/p + b_s$, respectively.

Third step (Convergence of θ_s and u_s^o). Cases $J \leqslant 3$ are classical (see [10]), while cases $4 \leqslant J \leqslant 6$ were treated in [5]. The other cases seem new and the proof of the homogenization result is given in Section 4. We now need to introduce some tools. If

$$\begin{cases} H^1_{\text{per}}\big(Y;\mathbb{R}^3\big) := \big\{u \in H^1\big(Y;\mathbb{R}^3\big); u \text{ takes the same values on the opposite sides of } \partial Y\big\}, \\ H^1_{\text{per}}\big(Y \setminus S;\mathbb{R}^3\big) \\ \quad := \big\{u \in H^1\big(Y \setminus S;\mathbb{R}^3\big); u \text{ takes the same values on the opposite sides of } \partial Y\big\}, \end{cases} \tag{3.11}$$

on the space

$$H_Y = \big\{u \in H^1_{\text{per}}\big(Y \setminus S, \mathbb{R}^3\big); [u]_N = 0 \text{ on } S\big\}, \quad \text{when } J \in \{5, 6\}, \tag{3.12}$$

are defined continuous bilinear forms a_Y, b_Y whose expressions are derived from those of a_s, b_s by replacing M_ε, I_ε, S_ε and x/ε by M, I, S and y the current point of the unit cell Y which describes the εY-periodic structure of the body. For instance, $a_Y(\varphi, \psi)$ reads as

$$a_Y(\varphi,\psi) := \int_M A_M(y)e(\varphi) \cdot e(\psi) \, \mathrm{d}y + \int_S \mu^*[\varphi]_T \cdot [\psi]_T \, \mathrm{d}H_2 + \int_I A_I(y)e(\varphi) \cdot e(\psi) \, \mathrm{d}y.$$

For $J \in \{7, 8\}$, we have

$$H_Y = H^1_{\text{per}}\big(Y \setminus S;\mathbb{R}^3\big), \tag{3.13}$$

and the bilinear form a_Y is defined by

$$\begin{aligned} a_Y(\varphi,\psi) = &\int_M A_M(y)e(\varphi) \cdot e(\psi) \, \mathrm{d}y \\ &+ \int_S \big(\bar{\lambda}_e[\varphi]_N[\psi]_N + 2\bar{\mu}_e\big([\varphi] \otimes_S n\big) \cdot \big([\psi] \otimes_S n\big)\big) \, \mathrm{d}H_2 \\ &+ \int_I A_I(y)e(\varphi) \cdot e(\psi) \, \mathrm{d}y \quad \forall \varphi, \psi \in H_Y, \end{aligned}$$

while the continuous bilinear form b_Y on H_Y is defined by

$$b_Y(\varphi,\psi) = \int_M B_M(y)e(\varphi) \cdot e(\psi) \, \mathrm{d}y$$

$$+\int_S \big(\bar{\lambda}_v[\varphi]_N[\psi]_N + 2\bar{\mu}_v\big([\varphi]\otimes_S n\big)\cdot\big([\psi]\otimes_S n\big)\big)\,\mathrm{d}H_2$$
$$+\int_I B_I(y)e(\varphi)\cdot e(\psi)\,\mathrm{d}y \quad \forall\varphi,\psi\in H_Y$$

when $J=7$, whilst when $J=8$ only the surface term appears, where $\xi\otimes_S\zeta$ is the symmetrized tensor product of ξ and ζ in $\mathbb{R}^3$.

Let

$$c_Y(p) := \frac{1}{p}a_Y + b_Y, \tag{3.14}$$

as a_Y is coercive on $H_Y/\mathbb{R}^3$, we deduce that for all E in $\mathbb{S}^3$ the problems

$$\min\left\{\frac{1}{2}a_Y(E\cdot+v, E\cdot+v), v\in H_Y\right\}, \tag{P_E^a}$$

$$\min\left\{\frac{1}{2}c_Y(p)(E\cdot+v, E\cdot+v), v\in H_Y\right\}, \tag{$P_E^{c(p)}$}$$

where $E\cdot$ denotes the function $y\in Y\setminus S\mapsto Ey\in\mathbb{R}^3$, have unique solutions denoted by v_E^a, $v_E^{c(p)}$, respectively, and satisfying

$$\begin{cases} a_Y\big(v_E^a,\varphi\big) = -a_Y(E\cdot,\varphi) & \forall\varphi\in H_Y,\\ c_Y(p)\big(v_E^{c(p)},\varphi\big) = -c_Y(p)(E\cdot,\varphi) & \forall\varphi\in H_Y,\\ \displaystyle\int_Y v_E^a(y)\,\mathrm{d}y = \int_Y v_E^{c(p)}(y)\,\mathrm{d}y = 0. \end{cases} \tag{3.15}$$

Let A^{eff} and $C^{\mathrm{eff}}(p)$ be the elements of $\mathrm{Lin}(\mathbb{S}^3)$ defined by

$$\begin{cases} A^{\mathrm{eff}}E\cdot E := a_Y\big(E\cdot+v_E^a, E\cdot+v_E^a\big) & \forall E\in\mathbb{S}^3,\\ C^{\mathrm{eff}}(p)E\cdot E := c_Y(p)\big(E\cdot+v_E^{c(p)}, E\cdot+v_E^{c(p)}\big) & \forall E\in\mathbb{S}^3, \end{cases} \tag{3.16}$$

which classically satisfy:

$$\begin{aligned} &\exists\bar{\kappa}_M; \quad \big|A^{\mathrm{eff}}E\big| \leqslant \bar{\kappa}_M|E|, \qquad \big|C^{\mathrm{eff}}(p)E\big| \leqslant \bar{\kappa}_M|E| \qquad \forall E\in\mathbb{S}^3\\ &\exists\bar{\kappa}_m; \quad A^{\mathrm{eff}}E\cdot E \geqslant \bar{\kappa}_m|E|^2, \qquad C^{\mathrm{eff}}(p)E\cdot E \geqslant \bar{\kappa}_m|E|^2 \quad \forall E\in\mathbb{S}^3. \end{aligned} \tag{3.17}$$

Then the following convergence result, which has its own interest (homogenization and bonding in linearized elasticity) and whose proof is given Section 4 by a method of variational convergence, holds.

Proposition 3.1. *For all p in $(0,+\infty)$, when s goes to $\bar{s}$ according to* (1.3), (u_s^o,θ_s) *converges weakly in* $H^1(\Omega;\mathbb{R}^3)$ *when* $J\leqslant 4$, *weak-$*$ in* $BD(\Omega)$ *and strongly in* $L^2(\Omega;\mathbb{R}^3)$ *when* $J\geqslant 5$, *toward the unique solution* $(\bar{u}^o,\bar{\theta})$ *to*

$$\begin{cases} (\bar{u}^o,\bar{\theta})\in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3);\\ \displaystyle\int_\Omega A^{\mathrm{eff}}e(\bar{u}^o)\cdot e(v)\,\mathrm{d}x=\int_\Omega f^o\cdot v\,\mathrm{d}x+\int_{\Gamma_1}g^o\cdot v\,\mathrm{d}H_2,\\ \displaystyle\int_\Omega C^{\mathrm{eff}}(p)e(\bar{\theta})\cdot e(v)\,\mathrm{d}x\\ \displaystyle\quad=\int_\Omega\mathcal{L}\tilde{f}\cdot\tilde{v}\,\mathrm{d}x+\int_{\Gamma_1}\mathcal{L}\tilde{g}\cdot v\,\mathrm{d}H_2-\frac{1}{p}\Big(\int_\Omega f^o\cdot v\,\mathrm{d}x+\int_{\Gamma_1}g^o\cdot v\,\mathrm{d}H_2\Big)\quad\forall v\in H^1_{\Gamma_0}(\Omega;\mathbb{R}^3).\end{cases}\tag{3.18}$$

Fourth step (A fundamental identity). In order to more thoroughly identify the asymptotic behavior of u_s, we establish a fundamental identity satisfied by $C^{\mathrm{eff}}(p)$ in terms of the effective tensor of elasticity A^{eff} and the "effective tensor of viscosity" B^{eff} defined as follows. Let

$$\begin{aligned}&\ker b_Y:=\{\varphi\in H_Y;b_Y(\varphi,\psi)=0\ \forall\psi\in H_Y\},\\ &V_Y:=\ker b_Y^\perp:=\{\varphi\in H_Y;a_Y(\varphi,\psi)=0\ \forall\psi\in\ker b_Y\}.\end{aligned}\tag{3.19}$$

Actually, the closed affine manifold

$$C_Y:=\{\varphi\in H_Y;b_Y(\varphi,\psi)=-b_Y(E\cdot,\psi)\ \forall\psi\in H_Y\}\tag{3.20}$$

is not empty. This is obvious when $J\in\{1,4,5,7\}$ due to the coercivity of b_Y on H_Y. When $J=3$, any H^1-extension of $-E\cdot$ into M which is Y-periodic belongs to C_Y. When $J=2$, b_Y is coercive on $H^1_{\mathrm{per}}(M;\mathbb{R}^3)=\{\psi\in H^1(M;\mathbb{R}^3);\psi$ takes the same values on opposite sides of $\partial Y\}$, then any H^1-extension into I of a solution φ_E to

$$\begin{cases}\varphi_E\in H^1_{\mathrm{per}}(M;\mathbb{R}^3);\\ b_Y(\varphi_E,\psi)=-b_Y(E\cdot,\psi)\quad\forall\psi\in H^1_{\mathrm{per}}(M;\mathbb{R}^3)\end{cases}\tag{3.21}$$

belongs to C_Y. Lastly, when $J\in\{6,8\}$, as $0=-b_Y(E\cdot,\varphi)\ \forall\varphi\in H_Y$, we deduce that $C_Y=H^1_{\mathrm{per}}(Y;\mathbb{R}^3)$.

Then we may define v_E^b by

$$v_E^b\in C_Y;\quad a_Y(v_E^b,v_E^b-\psi)=-a_Y(E\cdot,v_E^b-\psi)\quad\forall\psi\in C_Y\tag{3.22}$$

which does exist and is unique up to an element of $\mathbb{R}^3$ by the Stampacchia theorem and also satisfies

$$a_Y(v_E^b,\varphi)=-a_Y(E\cdot,\varphi)\quad\forall\varphi\in\ker b_Y.$$

Thus $v_E^b-v_E^a$ belongs to V_Y because

$$\forall\varphi\in\ker b_Y,\quad a_Y(v_E^b-v_E^a,\varphi)=-a_Y(E\cdot,\varphi)+a_Y(E\cdot,\varphi)=0.$$

Hence we define B^{eff} in $\mathrm{Lin}(\mathbb{S}^3)$ by

$$B^{\text{eff}}E \cdot E = b_Y\bigl(E\cdot + v_E^b, E\cdot + v_E^b\bigr) \quad \forall E \in \mathbb{S}^3; \tag{3.23}$$

note that B^{eff} vanishes when $J \in \{3, 6, 8\}$, that is to say when the viscoelastic phase occupies a nonconnected region.

Then, proceeding as for A_s, it is straight forward to check that the operator A_Y defined by

$$\begin{aligned} &D(A_Y) = \bigl\{\varphi \in V_Y; \exists! w(\varphi) \in V_Y; b_Y\bigl(w(\varphi), \psi\bigr) + a_Y(\varphi, \psi) = 0 \; \forall \psi \in V_Y\bigr\},\\ &A_Y(\varphi) = w(\varphi) \end{aligned} \tag{3.24}$$

is m-dissipative, so that the evolution equation

$$\begin{cases} \dfrac{\mathrm{d}z_E}{\mathrm{d}t} = A_Y z_E, \\ z_E(0) = v_E^b - v_E^a \end{cases} \tag{3.25}$$

has a unique generalized solution in $C^0([0, +\infty); V_Y) \cap C^\infty((0, +\infty); V_Y) \cap L^\infty(0, +\infty; V_Y)$ which, for all p in $(0, +\infty)$, does have a Laplace transform satisfying

$$a_Y(\mathcal{L}z_E, \varphi) + b_Y\bigl(p\mathcal{L}z_E + v_E^a - v_E^b, \varphi\bigr) = 0 \quad \forall \varphi \in H_Y,$$

because $\mathcal{L}z_E$ belongs to V_Y. As v_E^b belongs to C_Y and $c_Y(p)$ is coercive on $H_Y/\mathbb{R}$, (3.15) implies that up to an element of $\mathbb{R}^3$

$$v_E^{c(p)} = pz_E + v_E^a. \tag{3.26}$$

Let $K(t) \in \mathrm{Lin}(\mathbb{S}^3)$ be defined by

$$K(t)E \cdot E = a_Y(z_E, E\cdot) + b_Y(\dot{z}_E, E\cdot) \quad \forall E \in \mathbb{S}^3, \tag{3.27}$$

then (3.25), (3.26) yield

$$C^{\text{eff}}(p) = \mathcal{L}K(p) + B^{\text{eff}} + \frac{1}{p}A^{\text{eff}} \quad \forall p \in (0, +\infty). \tag{3.28}$$

Last step (Asymptotic behavior of u_s). The fact that $C^{\text{eff}}(p)$ differs from $A^{\text{eff}}/p + B^{\text{eff}}$ implies that the effective behavior of the media is no longer of Kelvin–Voigt type. More precisely, Proposition 3.1 and (3.25) infer that for all p in $(0, +\infty)$ $\mathcal{L}\tilde{u}_s(p)$ converges weakly in $H^1(\Omega; \mathbb{R}^3)$ when $J \leqslant 4$, weak-$*$ in $BD(\Omega)$ and strongly in $L^2(\Omega; \mathbb{R}^3)$ when $J \geqslant 5$, toward the unique solution $\check{u}(p)$ to

$$\begin{cases} \check{u}(p) \in H^1_{\Gamma_0}\bigl(\Omega; \mathbb{R}^3\bigr); \\ \displaystyle\int_\Omega A^{\text{eff}} e\bigl(\check{u}(p)\bigr) \cdot e(v)\,\mathrm{d}x + \int_\Omega \bigl[B^{\text{eff}} + \mathcal{L}K(p)\bigr] e\bigl(p\check{u}(p) - \bar{u}_0\bigr) \cdot e(v)\,\mathrm{d}x \\ \quad \displaystyle = \int_\Omega \mathcal{L}\tilde{f} \cdot v\,\mathrm{d}x + \int_{\Gamma_1} \mathcal{L}\tilde{g} \cdot v\,\mathrm{d}H_2 \quad \forall v \in H^1_{\Gamma_0}\bigl(\Omega; \mathbb{R}^3\bigr). \end{cases} \tag{3.29}$$

And as the Laplace transform is one-to-one, (3.7) yields the convergence results.

Theorem 3.1. *If the initial state of the body and the loading satisfy* (3.10) *and*

$$\begin{cases} (f,g) \in C^0\big([0,T]; L^2\big(\Omega;\mathbb{R}^3\big) \times L^2\big(\Gamma_1;\mathbb{R}^3\big)\big) & \text{when } J \in \{1,4,5,7\}, \\ (f,g) \in C^{1,\alpha}\big([0,T]; L^2\big(\Omega;\mathbb{R}^3\big) \times L^2\big(\Gamma_1;\mathbb{R}^3\big)\big) & \text{when } J \in \{2,3,6,8\} \text{ for } \alpha \text{ in } (0,1), \end{cases}$$

respectively, then, when s goes to $\bar{s}$, the solution u_s of (P_s) converges weak-$*$ in $W^{1,\infty}(0,T; H^1_{\Gamma_0}(\Omega;\mathbb{R}^3))$ if $J \in \{1,4\}$, in $L^\infty(0,T; H^1_{\Gamma_0}(\Omega;\mathbb{R}^3))$ if $J \in \{2,3\}$, in $W^{1,\infty}(0,T; BD(\Omega) \cap L^2(\Omega;\mathbb{R}^3))$ if $J \in \{5,7\}$, in $L^\infty(0,T; BD(\Omega) \cap L^2(\Omega;\mathbb{R}^3))$ if $J \in \{6,8\}$ toward the unique solution $\bar{u}$ to

$$\begin{cases} \bar{u} \in C^1\big((0,T]; H^1_{\Gamma_0}\big(\Omega;\mathbb{R}^3\big)\big); \quad \bar{u}(0) = u^o, \\ \displaystyle\int_\Omega \left(A^{\mathrm{eff}} e(\bar{u}) + B^{\mathrm{eff}} e(\dot{\bar{u}}) + \int_0^t K(t-\tau) e(\dot{\bar{u}})(\tau)\,\mathrm{d}\tau \right) \cdot e(v)\,\mathrm{d}x \\ \quad = \displaystyle\int_\Omega f \cdot v\,\mathrm{d}x + \int_{\Gamma_1} g \cdot v\,\mathrm{d}H_2 \quad \forall (v,t) \in H^1_{\Gamma_0}\big(\Omega;\mathbb{R}^3\big) \times (0,T]. \end{cases} \tag{3.30}$$

Hence, as in the case $J = 1$ studied in [4], the effective behavior of the body is no longer of Kelvin–Voigt type but rather of viscoelastic type with memory. When $J \in \{1,4,5,7\}$, $\ker b_Y = \{0\}$ so that A_Y is bounded and consequently K decreases exponentially fast with time, hence the effective behavior is viscoelastic with fading memory. When $J \in \{3,6,8\}$ B^{eff} vanishes.

The case of bilateral contact with purely linear viscous friction ($J = 6$ with $\mu_e^* = 0$) considered in [5] by the asymptotic expansion method can be rigorously handled by our method through the standard change of unknown $u_s \mapsto v_s$, $v_s(t) = \exp(-t)u_s(t)\ \forall t \in [0,T]$.

Finally to keep the analysis of reasonable length, we did not consider "pathological cases" like $(\bar{\lambda}_e, \bar{\mu}_e, \bar{\lambda}_v, \bar{\mu}_v) \in \{0,+\infty\}^4$ when $J \in \{7,8\}$. Based on the discussion of [6] and [8] about the nature of the mechanical constraint equivalent to a soft viscoelastic layer, the effective behavior will be of viscoelastic type with memory except when $J = 8$, and $\bar{\mu}_v = \bar{\lambda}_v = 0$ or $\bar{\mu}_e = +\infty$, or $\bar{\mu}_v = +\infty$, where it should be elastic.

4. Proof of Proposition 3.1

It suffices to only consider the convergence of u_s^o and case $J = 7$. Clearly u_s^o is the only minimizer of the functional

$$v \in H^1_{\Gamma_0}\big(\Omega;\mathbb{R}^3\big) \mapsto F_s(v) := \frac{1}{2} a_s(v,v) - \int_\Omega f^o \cdot v\,\mathrm{d}x - \int_{\Gamma_1} g^o \cdot v\,\mathrm{d}H_2. \tag{4.1}$$

One proceeds in four steps:

First step (A compactness property).

Lemma 4.1. *Let (v_s) be a sequence in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ such that $F_s(v_s) \leqslant C$, then there exists v in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ and a nonrelabeled subsequence such that v_s weak$*$ converges in $BD(\Omega)$ and strongly in $L^2(\Omega;\mathbb{R}^3)$ toward some v in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$.*

Proof. The very definition of the extension $\tilde{v}_s$ of v_s, (3.4) and (3.6) (see Section 3, First step) imply

$$\frac{1}{2}a_s(v_s,v_s) \leqslant C + |f^o|_{L^2(\Omega;\mathbb{R}^3)}|v_s|_{L^2(\Omega;\mathbb{R}^3)} + \int_{\Gamma_1} g^o\cdot\tilde{v}_s\,\mathrm{d}H_2$$
$$\leqslant C\big(1 + |f^o|_{L^2(\Omega;\mathbb{R}^3)} + |g^o|_{L^2(\Gamma_1;\mathbb{R}^3)}\big)a_s(v_s,v_s)^{1/2}.$$

Thus $a_s(v_s,v_s)$ is bounded and the sought after result stems from Remark 3.1. $\square$

Second step (More about (P_E^a)). If v_E^a still denotes the extension into $\mathbb{R}^3$ by Y-periodicity of the solution v_E^a to (P_E^a), let σ_E^a be the Y-periodic field such that

$$\sigma_E^a(y) = \begin{cases} A_M(y)\big(E + e\big(v_E^a\big)(y)\big) & \text{a.e. } y\in M,\\ A_I(y)\big(E + e\big(v_E^a\big)(y)\big) & \text{a.e. } y\in I.\end{cases} \tag{4.2}$$

It is easy to check that σ_E^a satisfies $\operatorname{div}\sigma_E^a = 0$ in the sense of distribution in $\mathbb{R}^3$ and that $\sigma_E^a(\cdot/\varepsilon)$ weakly converges in $L^2(\Omega;\mathbb{S}^3)$ toward $\int_Y \sigma_E^a(y)\,\mathrm{d}y$. For almost y in Y, let the strictly convex quadratic form $W_{s'}$ be defined on $\mathbb{S}^3$ by

$$W_{s'}(y,e) := \begin{cases} \dfrac{1}{2}A_M(y)e\cdot e & \text{if } y\in M_h,\\ \dfrac{\lambda_e}{2}(\mathrm{tr}\,e)^2 + \mu_e|e|^2 & \text{if } y\in B_h,\\ \dfrac{1}{2}A_I(y)e\cdot e & \text{if } y\in I_h.\end{cases}$$

Then the problem

$$\mathrm{Min}\left\{\int_Y W_{s'}\big(y, E + e(\varphi)(y)\big)\,\mathrm{d}y;\ \varphi\in H^1_{\mathrm{per}}\big(Y;\mathbb{R}^3\big)\right\} \tag{$P_E^{a,s'}$}$$

has a unique solution $v_E^{a,s'}$ satisfying $\int_Y v_E^{a,s'}(y)\,\mathrm{d}y = 0$ and whose extension into $\mathbb{R}^3$ by Y-periodicity is still denoted by $v_E^{a,s'}$. Let $\sigma_E^{a,s'}$ defined by

$$\sigma_E^{a,s'}(y) = DW_{s'}\big(y, E + e\big(v_E^{a,s'}\big)(y)\big) \quad \text{a.e. } y\in\mathbb{R}^3, \tag{4.3}$$

which clearly satisfies $\operatorname{div}\sigma_E^{a,s'} = 0$ in the sense of distributions in $\mathbb{R}^3$. Actually, $(P_E^{a,s'})$ is a kind of approximation of (P_E^a) in the sense that, by arguing as in [1,7], one has

$$\begin{cases} \lim\limits_{s'\to 0}\big|v_E^{a,s'} - v_E^a\big|_{L^2(Y;\mathbb{R}^3)} = 0,\\ \lim\limits_{s'\to 0}\displaystyle\int_Y W_{s'}\big(E + e\big(v_E^{a,s'}\big)(y)\big)\,\mathrm{d}y = \frac{1}{2}a_Y\big(E\cdot + v_E^a, E\cdot + v_E^a\big) = \frac{1}{2}A^{\mathrm{eff}}E\cdot E,\\ \lim\limits_{s'\to 0}\big|\sigma_E^{a,s'} - \sigma_E^a\big|_{L^2(Y;\mathbb{R}^3)} = 0\end{cases} \tag{4.4}$$

and consequently

$$\lim_{s\to s'}\int_\Omega\left|\sigma_E^{a,s'}(x/\varepsilon)-\sigma_E^a(x/\varepsilon)\right|^2\mathrm{d}x=0. \tag{4.5}$$

Now we add some ingredients of the mathematical theory of bonded joints ([1,7]) to the classical proof by [2] for homogenization in elasticity.

Third step (Upper bound for $F_s(v_s)$).

Lemma 4.2. *For all v in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ there exists a sequence (v_s) in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ such that v_s weak-$*$ converges in $BD(\Omega)$ and strongly in $L^2(\Omega;\mathbb{R}^3)$ toward v and*

$$\begin{aligned}F^{\mathrm{eff}}(v)&:=\frac{1}{2}\int_\Omega A^{\mathrm{eff}}e(v)\cdot e(v)\,\mathrm{d}x-\int_\Omega f^o\cdot v\,\mathrm{d}x-\int_{\Gamma_1}g^o\cdot v\,\mathrm{d}H_2\\&=\lim_{s\to\bar{s}}F_s(v_s).\end{aligned} \tag{4.6}$$

Proof. First we assume that v is affine: $v(x)=Ex+d$, $E\in\mathbb{S}^3$, $d\in\mathbb{R}^3$. Let $w_{E,s}$ such that

$$w_{E,s}(x)=\varepsilon v_E^{a,s'}(x/\varepsilon)\quad\text{a.e. }x\in\Omega, \tag{4.7}$$

then the field $v_s:=v+w_{E,s}$ belongs to $H^1(\Omega;\mathbb{R}^3)$, converges weak-$*$ in $BD(\Omega)$ and strongly in $L^2(\Omega;\mathbb{R}^3)$ toward v by due account of (4.4) and the definition of $v_E^{a,s'}$. Moreover, as

$$\frac{1}{2}a_s(v_s,v_s)=|\Omega|\int_Y W_{s'}\big(E+e\big(v_E^{a,s'}\big)(y)\big)\,\mathrm{d}y+\mathrm{o}(\varepsilon),$$

(4.4) gives $\lim_{s\to\bar{s}}\frac{1}{2}a_s(v_s,v_s)=\frac{1}{2}\int_\Omega A^{\mathrm{eff}}e(v)\cdot e(v)\,\mathrm{d}x$. Next we take v as a continuous piecewise-affine function $v(x)=E^ix+d^i$ on Ω^i, i belonging to a finite set $\mathcal{I}$ of indices, where $\{\Omega^i\}_{i\in\mathcal{I}}$ forms a partition by polyhedral sets. As in the first step, we define v_s by $v_s^i=v+w_{E^i,s}$ on each Ω^i, but by due account of possible discontinuities on the interface $\sum^{jk}$ between Ω^j and Ω^k we need to introduce ϕ_δ in $W^{1,\infty}(\Omega)$, $0\leqslant\phi_\delta\leqslant1$, $\phi_\delta=1$ on $\sum_\delta^{jk}:=\{x\in\Omega;\mathrm{dist}(x,\sum_i^{jk})<\delta\},\delta>0$, $\phi_\delta=0$ on $\Omega\setminus\sum_{2\delta}^{jk}$ and $v_{\delta_s}=\phi_\delta v+(1-\phi_\delta)v_s^i$ on Ω^i. Hence we can repeat the end of the proof by [2], pp. 47–48, because v_s^i converges strongly toward v in $L^2(\Omega;\mathbb{R}^3)$ and A_I, A_M, λ_e, μ_e are bounded.

Eventually the proof of the convergence of $\frac{1}{2}a_s(v_s,v_s)$ toward $\frac{1}{2}\int_\Omega A^{\mathrm{eff}}e(v)\cdot e(v)\,\mathrm{d}x$ is completed by diagonalization and density arguments.

To tackle loading term g^o into account, it suffices to introduce $w_{E,s}$ only on $\bigcup_{j\in J_\varepsilon}\varepsilon(j+Y)$, while the convergence of the term involving f^o stems from the strong convergence of v_s toward v in $L^2(\Omega;\mathbb{R}^3)$. □

Fourth step (Lower bound for $F_s(v_s)$).

Lemma 4.3. *For all v in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ and all sequences (v_s) in $H^1_{\Gamma_0}(\Omega;\mathbb{R}^3)$ which weak-$*$ converges in $BD(\Omega)$ and strongly in $L^2(\Omega;\mathbb{R}^3)$ toward v, we have*

$$F^{\mathrm{eff}}(v)\leqslant\liminf_{s\to\bar{s}}F_s(v_s).$$

Proof. Once more we proceed by introducing a continuous piecewise-affine function $w = E^i \cdot + d^i$ as *approximation in* H^1 of v on Ω^i. For each Ω^i, let $\phi^i \in \mathcal{D}(\Omega^i)$ such that $0 \leqslant \phi^i \leqslant 1$.

By using functions $w_{E^i,s}$ defined like (4.7), the subdifferential inequality yields

$$\frac{1}{2}a_s(v_s, v_s) \geqslant \sum_{i\in\mathcal{I}} \left[\int_{\Omega^i} \phi^i(x) W_{s'}\big(x/\varepsilon, E^i + e(w_{E^i,s})(x)\big)\,\mathrm{d}x + \int_{\Omega^i} \phi^i(x) DW_{s'}\big(x/\varepsilon, E^i + e(w_{E^i,s})(x)\big) \cdot e(v_s - w - w_{E^i,s})\,\mathrm{d}x \right].$$

A slight and obvious modification of the argument used at the beginning of the proof of Lemma 4.2 gives

$$\lim_{s\to\bar{s}} \int_{\Omega^i} \phi^i(x) W_{s'}\big(x/\varepsilon, E^i + e(w_{E^i,s})\big)(x)\,\mathrm{d}x = \frac{1}{2}\int_{\Omega^i} \phi^i(x) A^{\mathrm{eff}} e(w) \cdot e(w)\,\mathrm{d}x.$$

Taking into account (4.3), $\operatorname{div} \sigma^{a,s'}_{E^i} = 0$ and (4.5), one has

$$\begin{aligned}
&\lim_{s\to\bar{s}} \int_{\Omega^i} \phi^i(x) DW_{s'}\big(x/\varepsilon, E^i + e(w_{E^i,s})\big) \cdot e(v_s - w - w_{E^i,s})\,\mathrm{d}x \\
&\quad = \lim_{s\to\bar{s}} \int_{\Omega^i} \phi^i(x) \sigma^{a,s'}_{E^i}(x/\varepsilon) \cdot e(v_s - w - w_{E^i,s})\,\mathrm{d}x \\
&\quad = \lim_{s\to\bar{s}} \bigg(\int_{\Omega^i} \sigma^{a,s'}_{E^i}(x/\varepsilon) \cdot e\big(\phi^i(v_s - w - w_{E^i,s})\big)\,\mathrm{d}x \\
&\qquad - \int_{\Omega^i} \sigma^{a,s'}_{E^i}(x/\varepsilon) \cdot \nabla\phi^i \otimes_S (v_s - w - w_{E^i,s})\,\mathrm{d}x \bigg) \\
&\quad = -\lim_{s\to\bar{s}} \int_{\Omega^i} \sigma^{a,s'}_{E^i}(x/\varepsilon) \cdot \nabla\phi^i \otimes_S (v_s - w - w_{E^i,s})\,\mathrm{d}x \\
&\quad = -\lim_{s\to\bar{s}} \int_{\Omega^i} \sigma^{a}_{E^i}(x/\varepsilon) \cdot \nabla\phi^i \otimes_S (v_s - w - w_{E^i,s})\,\mathrm{d}x \\
&\quad = -\int_{\Omega^i} \left(\int_Y \sigma^a_{E^i}(y)\,\mathrm{d}y \right) \cdot \nabla\phi^i \otimes_S (v - w)\,\mathrm{d}x
\end{aligned}$$

because of the strong convergence in $L^2(\Omega;\mathbb{R}^3)$ of $v_s - w - w_{E^i,s}$ toward $v - w$. Hence

$$\liminf_{s\to\bar{s}} \frac{1}{2}a_s(v_s, v_s) \geqslant \frac{1}{2}\sum_{i\in\mathcal{I}} \int_{\Omega^i} \phi^i A^{\mathrm{eff}} e(w) \cdot e(w)\,\mathrm{d}x + \sum_{i\in\mathcal{I}} \int_{\Omega^i} \phi^i A^{\mathrm{eff}} e(w) \cdot e(v - w)\,\mathrm{d}x.$$

As in [2], by letting ϕ^i converge increasingly to 1 on Ω^i for the first term, and by using (3.17) and the density of piecewise-affine functions in $H^1(\Omega;\mathbb{R}^3)$ for the second term, we conclude that

$$\liminf_{s\to\bar{s}} \frac{1}{2}a_s(v_s, v_s) \geqslant \frac{1}{2}\int_\Omega A^{\mathrm{eff}} e(v) \cdot e(v)\,\mathrm{d}x.$$

Lastly, the arguments used in the proof of Lemma 4.1 (Remark 3.1) yield

$$\begin{aligned}\lim_{s\to\bar{s}}\int_{\Omega} f^{o}\cdot v_s\,\mathrm{d}x+\int_{\Gamma_1} g^{o}\cdot v_s\,\mathrm{d}H_2 &= \lim_{s\to\bar{s}}\int_{\Omega} f^{o}\cdot v_s\,\mathrm{d}x+\int_{\Gamma_1} g^{o}\cdot \tilde{v}_s\,\mathrm{d}H_2\\ &= \int_{\Omega} f^{o}\cdot v\,\mathrm{d}x+\int_{\Gamma_1} g^{o}\cdot v\,\mathrm{d}H_2.\end{aligned}$$

Thus Proposition 3.1 stems classically (see [2,3]) from a combination of Lemmas 4.1–4.3. □

References

[1] A. Ait-Moussa, A modelling of elastic bonded joints, Modélisation et étude des singularités de contraintes d'un joint collé très mince, PhD thesis, Université Montpellier II, France, 1989.

[2] H. Attouch, *Variational Convergence for Functions and Operators*, Applicable Mathematics Series, Pitman Advanced Publishing Program, 1984.

[3] H. Attouch, G. Buttazzo and G. Michaille, *Variational Analysis in Sobolev and BV Spaces: Application to PDEs and Optimization*, MPS–SIAM Book Series on Optimization, 2005.

[4] G. Francfort and P. Suquet, Homogenization and mechanical dissipation in thermoviscoelasticity, *Arch. Rational Mech. Anal.* **96**(3) (1986), 265–293.

[5] F. Lénée, Contribution à l' étude des matériaux composites et de leur endommagement, Thèse doctorat d' état, Université Paris VI, 1984.

[6] C. Licht, A. Léger, S. Orankitjaroen and A.O. Khaoua, Dynamics of elastic bodies connected by a thin soft viscoelastic layer, *J. Math. Pures Appl.* **99**(6) (2013), 685–703.

[7] C. Licht and G. Michaille, A modelling of elastic bonded joints, *Advances in Mathematical Sciences and Applications* **7** (1997), 711–740.

[8] C. Licht and S. Orankitjaroen, Dynamics of elastic bodies connected by a thin soft inelastic layer, *C. R. Mecanique* **341**(3) (2013), 323–332.

[9] P.K. Mallick, *Fiber-Reinforced Composites: Materials, Manufacturing, and Design*, 3rd edn, CRC Press, Taylor & Francis Group, 2007.

[10] E. Sanchez-Palencia, *Non-homogeneous Media and Vibration Theory*, Lecture Notes in Physics, Vol. 127, Springer, 2009.

[11] L. Tartar, *The General Theory of Homogenization, A Personalized Introduction*, Lecture Notes of the Unione Mathematica Italiana, Vol. 7, Springer, 2009.

[12] R. Temam, *Mathematical Problem in Plasticity*, Gauthier-Villars, 1985.