Completeness of Tree Automata Completion

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Abstract

We consider rewriting of a regular language with a left-linear term rewriting system. We show a completeness theorem on equational tree automata completion stating that, if there exists a regular over-approximation of the set of reachable terms, then equational completion can compute it (or safely under-approximate it). A nice corollary of this theorem is that, if the set of reachable terms is regular, then equational completion can also compute it. This was known to be true for some term rewriting system classes preserving regularity, but was still an open question in the general case. The proof is not constructive because it depends on the regularity of the set of reachable terms, which is undecidable. To carry out those proofs we generalize and improve two results of completion: the Termination and the Upper-Bound theorems. Those theoretical results provide an algorithmic way to safely explore regular approximations with completion. This has been implemented in Timbuk and used to verify safety properties, automatically and efficiently, on first-order and higher-order functional programs.

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1 Introduction

Given a term rewriting system (TRS for short) $\mathcal{R}$ and a tree automaton $\mathcal{A}$ recognizing a regular tree language $\mathcal{L}(\mathcal{A})$, the set of reachable terms is $\mathcal{R}^*(\mathcal{L}(\mathcal{A})) = \{t \mid s \in \mathcal{L}(\mathcal{A}) \text{ and } s \rightarrow_{\mathcal{R}}{}^* t\}$. In this paper, we show that the *equational tree automata completion algorithm* [15] is complete w.r.t. regular approximations. If $\mathcal{R}$ is left-linear and there exists a regular language $\mathscr{L}$ over-approximating $\mathcal{R}^*(\mathcal{L}(\mathcal{A}))$, i.e., $\mathcal{R}^*(\mathcal{L}(\mathcal{A})) \subseteq \mathscr{L}$ then completion can build a tree automaton $\mathcal{A}^*$ such that $\mathcal{R}^*(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathcal{A}^*) \subseteq \mathscr{L}$. We also show that completion is complete w.r.t. TRSs preserving regularity, i.e., if $\mathscr{L} = \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$ then completion can build a tree automaton $\mathcal{A}^*$ such that $\mathcal{R}^*(\mathcal{L}(\mathcal{A})) = \mathcal{L}(\mathcal{A}^*) = \mathscr{L}$. On the one hand, automata built by completion-like algorithms are known to recognize *exactly* the set of reachable terms, for some *restricted* classes of TRSs [17, 24, 8, 10]. On the other hand, automata completion is able to build *over-approximations* for *any* left-linear TRS [9, 23, 15], and even for non-left-linear TRSs [3]. Such approximations are used for program verification [5, 4, 10, 14] as well as to automate termination proofs [16, 20]. To define approximations, completion uses an additional set of equations E and builds a tree automaton $\mathcal{A}^*_{\mathcal{R},E}$ such that $\mathcal{L}(\mathcal{A}^*_{\mathcal{R},E}) \supseteq \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$. Starting from $\mathcal{R}$, $\mathcal{A}$, and E Timbuk[12] is an automatic tool to build $\mathcal{A}^*_{\mathcal{R},E}$. Until now it was an open question whether completion can build *any* regular over-approximation or compute the set of reachable terms if this set is regular. The *first contribution* of this paper is to answer these two questions in the positive, for general left-linear TRSs. The proofs are not constructive but, the *second contribution* is to provide an efficient method to explore regular approximations for TRSs encoding functional programs.

For the approximated case, the proof of completeness is organized as follows. If there exists a regular over-approximation $\mathscr{L}$ such that $\mathcal{R}^*(\mathcal{L}(\mathcal{A})) \subseteq \mathscr{L}$, we know that there exists a tree automaton $\mathcal{B}$ such that $\mathcal{L}(\mathcal{B}) = \mathscr{L}$. From $\mathcal{B}$, using the Myhill-Nerode theorem, we can infer a set of equations E such that the set of E-equivalence classes $\mathcal{T}(\mathcal{F})/_{=_E}$ is finite. Then we prove the following theorems:

(a) If $\mathcal{T}(\mathcal{F})/_{=_E}$ is finite, then it is possible to build from E a set of equations E', equivalent to E, such that completion of any automaton $\mathcal{A}$ by any TRS $\mathcal{R}$ with E' always terminates. This generalizes the termination theorem of [10];

(b) If $\mathcal{T}(\mathcal{F})/_{=_E}$ is finite, then it is possible to build from E and $\mathcal{A}$ a tree automaton $\mathbb{A}$ recognizing the same language as $\mathcal{A}$ such that the completed automaton $\mathbb{A}^*_{\mathcal{R},E}$ has the following precision property: $\mathcal{L}(\mathbb{A}^*_{\mathcal{R},E}) \subseteq \mathcal{R}^*_E(\mathcal{L}(\mathcal{A}))$, where $\mathcal{R}^*_E(\mathcal{L}(\mathcal{A}))$ is the set of reachable terms by rewriting modulo E. It generalizes the Upper Bound theorem of [15].

(c) Then, we show that $\mathcal{R}^*_E(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathcal{B})$, and we get the main completeness theorem: $\mathcal{L}(\mathbb{A}^*_{\mathcal{R},E}) \subseteq \mathcal{R}^*_E(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathcal{B})$.

Besides, we know from [15] that $\mathcal{R}^*(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathbb{A}^*_{\mathcal{R},E})$. Thus, when using the set of equations defined from $\mathcal{B}$ to run completion, (c) implies that we can only get an over-approximation of $\mathcal{R}^*(\mathcal{L}(\mathcal{A}))$ equivalent or better than $\mathscr{L} = \mathcal{L}(\mathcal{B})$. This result has a practical impact for program verification. In particular, for TRSs encoding functional programs, the search space of sets of equations E can be constrained for enumeration to be possible. This has been implemented in the Timbuk [12] tool. Our experiments show that this makes completion automatic enough to carry out safety proofs on first-order and higher-order functional programs. We also get a corollary of (c) when $\mathscr{L}$ is *not* an approximation:

(d) If $\mathscr{L} = \mathcal{L}(\mathcal{B}) = \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$, we can use $\mathcal{R}^*(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathbb{A}^*_{\mathcal{R},E})$ to close-up the $\subseteq$-chain and get that $\mathcal{L}(\mathbb{A}^*_{\mathcal{R},E}) = \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$. Thus if $\mathcal{R}^*(\mathcal{L}(\mathcal{A}))$ is regular, there exists a set of equations E s.t. $\mathcal{L}(\mathbb{A}^*_{\mathcal{R},E}) = \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$

Section 2 defines some basic notions in term rewriting and tree automata and Section 3 recalls the tree automata completion algorithm and the related theorems. Section 4 recalls the Myhill-Nerode theorem for trees and defines the functions to transform a set of equations into a tree automaton and vice versa. Section 5 proves Result (a) and Section 6 shows Result (b). Section 7 assembles (a) and (b) to prove results (c) and (d) using the proof sketched above. Section 8 shows how to take advantage of those results to program verification and presents some experiments. Finally, Section 9 concludes.

2 Preliminaries

In this section we introduce some definitions and concepts that will be used throughout the rest of the paper (see also [2, 6]). Let $\mathcal{F}$ be a finite set of symbols, each associated with an arity function. For brevity, we write $f : n$ if f is a symbol of arity n and $\mathcal{F}^n = \{f \in \mathcal{F} \mid f : n\}$. Let $\mathcal{X}$ be a countable set of *variables*, $\mathcal{T}(\mathcal{F}, \mathcal{X})$ denotes the set of *terms* and $\mathcal{T}(\mathcal{F})$ denotes the set of *ground terms* (terms without variables). The set of variables of a term t is denoted by $\mathscr{V}ar(t)$. A *substitution* is a function σ from $\mathcal{X}$ into $\mathcal{T}(\mathcal{F}, \mathcal{X})$, which can be uniquely extended to an endomorphism of $\mathcal{T}(\mathcal{F}, \mathcal{X})$. A *position* p in a term t is a finite word over $\mathbb{N}$, the set of natural numbers. The empty sequence λ denotes the top-most position. The set $\mathscr{P}os(t)$ of positions of a term t is inductively defined by $\mathscr{P}os(t) = \{\lambda\}$ if $t \in \mathcal{X}$ or t is a constant and $\mathscr{P}os(f(t_1, \ldots, t_n)) = \{\lambda\} \cup \{i.p \mid 1 \leq i \leq n \text{ and } p \in \mathscr{P}os(t_i)\}$ otherwise. If $p \in \mathscr{P}os(t)$, then $t(p)$ denotes the symbol at position p in t, $t|_p$ denotes the subterm of t at position p, and $t[s]_p$ denotes the term obtained by replacing the subterm $t|_p$ at position p by the term s. A ground context $C[\,]$ is a term in $\mathcal{T}(\mathcal{F} \cup \{\Box\})$ containing exactly one occurrence of the symbol $\Box$. If $t \in \mathcal{T}(\mathcal{F})$ then $C[t]$ denotes the term obtained by the replacement of $\Box$ by t in $C[\,]$. A context is empty if it is equal to $\Box$.

A *term rewriting system* (TRS) $\mathcal{R}$ is a set of *rewrite rules* $l \to r$, where $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $l \notin \mathcal{X}$, and $\mathcal{V}ar(l) \supseteq \mathcal{V}ar(r)$. A rewrite rule $l \to r$ is *left-linear* if each variable occurs only once in l. A TRS $\mathcal{R}$ is left-linear if every rewrite rule $l \to r$ of $\mathcal{R}$ is left-linear. The TRS $\mathcal{R}$ induces a rewriting relation $\to_{\mathcal{R}}$ on terms as follows. Let $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and $l \to r \in \mathcal{R}$, $s \to_{\mathcal{R}} t$ denotes that there exists a position $p \in \mathcal{P}os(s)$ and a substitution σ such that $s|_p = l\sigma$ and $t = s[r\sigma]_p$. The set of ground terms irreducible by a TRS $\mathcal{R}$ is denoted by $\textsc{Irr}(\mathcal{R})$. A set $\mathcal{L} \subseteq \mathcal{T}(\mathcal{F})$ is $\mathcal{R}$-closed if for all $s \in \mathcal{L}$ and $s \to_{\mathcal{R}} t$ then $t \in \mathcal{L}$. The reflexive transitive closure of $\to_{\mathcal{R}}$ is denoted by $\to_{\mathcal{R}}^*$, and $s \to_{\mathcal{R}}^! t$ denotes that $s \to_{\mathcal{R}}^* t$ and t is irreducible by $\mathcal{R}$. The set of $\mathcal{R}$-descendants of a set of ground terms I is defined as $\mathcal{R}^*(I) = \{t \in \mathcal{T}(\mathcal{F}) \mid \exists s \in I \text{ s.t. } s \to_{\mathcal{R}}^* t\}$, i.e., the smallest $\mathcal{R}$-closed set containing I.

Let E be a *set of equations* $l = r$, where $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. The relation $=_E$ is the smallest congruence such that for all equations $l = r$ of E and for all substitutions σ we have $l\sigma =_E r\sigma$. The set of equivalence classes defined by $=_E$ on $\mathcal{T}(\mathcal{F})$ is denoted by $\mathcal{T}(\mathcal{F})/_{=_E}$. Given a TRS $\mathcal{R}$ and a set of equations E, a term $s \in \mathcal{T}(\mathcal{F})$ is rewritten modulo E into $t \in \mathcal{T}(\mathcal{F})$, denoted $s \to_{\mathcal{R}/E} t$, if there exists an $s' \in \mathcal{T}(\mathcal{F})$ and a $t' \in \mathcal{T}(\mathcal{F})$ such that $s =_E s' \to_{\mathcal{R}} t' =_E t$. The reflexive transitive closure $\to_{\mathcal{R}/E}^*$ of $\to_{\mathcal{R}/E}$ is defined as usual except that reflexivity is extended to terms equal modulo E, i.e., for all $s, t \in \mathcal{T}(\mathcal{F})$, if $s =_E t$ then $s \to_{\mathcal{R}/E}^* t$. The set of $\mathcal{R}$-descendants modulo E of a set of ground terms I is defined as $\mathcal{R}_E^*(I) = \{t \in \mathcal{T}(\mathcal{F}) \mid \exists s \in I \text{ s.t. } s \to_{\mathcal{R}/E}^* t\}$.

Let $\mathcal{Q}$ be a countably infinite set of symbols with arity 0, called *states*, such that $\mathcal{Q} \cap \mathcal{F} = \emptyset$. Terms in $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ are called *configurations*. A *transition* is a rewrite rule $c \to q$, where c is a configuration and q is a state. A transition is *normalized* when $c = f(q_1, \ldots, q_n)$, $f \in \mathcal{F}$ is of arity n, and $q_1, \ldots, q_n \in \mathcal{Q}$. An *$\epsilon$-transition* is a transition of the form $q \to q'$ where q and q' are states. A bottom-up non-deterministic finite tree automaton (tree automaton for short) over the alphabet $\mathcal{F}$ is a tuple $\mathcal{A} = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$, where $\mathcal{Q}_f \subseteq \mathcal{Q}$ is the set of final states, Δ is a finite set of normalized transitions and ϵ-transitions. An automaton is *epsilon-free* if it is free of ϵ-transitions. The transitive and reflexive *rewriting relation* on $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ induced by the set of transitions Δ (resp. all transitions except ϵ-transitions) is denoted by $\to_{\Delta}^*$ (resp. $\to_{\Delta}^{\not\epsilon *}$). When Δ is attached to a tree automaton $\mathcal{A}$ we also denote those two relations by $\to_{\mathcal{A}}{}^*$ and $\to_{\mathcal{A}}^{\not\epsilon *}$, respectively. A tree automaton $\mathcal{A}$ is complete if for all $s \in \mathcal{T}(\mathcal{F})$ there exists a state q of $\mathcal{A}$ such that $s \to_{\mathcal{A}}{}^* q$. The *language* recognized by $\mathcal{A}$ in a state q is defined by $\mathcal{L}(\mathcal{A}, q) = \{t \in \mathcal{T}(\mathcal{F}) \mid t \to_{\mathcal{A}}^* q\}$. We define $\mathcal{L}(\mathcal{A}) = \bigcup_{q \in \mathcal{Q}_f} \mathcal{L}(\mathcal{A}, q)$. A state q of an automaton $\mathcal{A}$ is *reachable* if $\mathcal{L}(\mathcal{A}, q) \neq \emptyset$. An automaton is *reduced* if all its states are reachable. An automaton $\mathcal{A}$ is $\not\epsilon$*-reduced* if for all states q of $\mathcal{A}$ there exists a ground term $t \in \mathcal{T}(\mathcal{F})$ such that $t \to_{\mathcal{A}}^{\not\epsilon *} q$. An automaton $\mathcal{A}$ is deterministic if for all ground terms $s \in \mathcal{T}(\mathcal{F})$ and all states q, q' of $\mathcal{A}$, if $s \to_{\mathcal{A}}{}^* q$ and $s \to_{\mathcal{A}}{}^* q'$ then $q = q'$. An automaton $\mathcal{A}$ is $\mathcal{R}$-closed if for all terms s, t and all states $q \in \mathcal{Q}$, $s \to_{\mathcal{A}}{}^* q$ and $s \to_{\mathcal{R}} t$ implies $t \to_{\mathcal{A}}{}^* q$.

3 Equational Tree Automata Completion

From a tree automaton $\mathcal{A}_0 = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta_0 \rangle$ and a left-linear TRS $\mathcal{R}$, the completion algorithm computes an automaton $\mathcal{A}^*$ such that $\mathcal{L}(\mathcal{A}^*) = \mathcal{R}^*(\mathcal{L}(\mathcal{A}_0))$ or $\mathcal{L}(\mathcal{A}^*) \supseteq \mathcal{R}^*(\mathcal{L}(\mathcal{A}_0))$.

3.1 Completion General Principles

From $\mathcal{A}^0_{\mathcal{R}} = \mathcal{A}_0$, tree automata completion successively computes tree automata $\mathcal{A}^1_{\mathcal{R}}$, $\mathcal{A}^2_{\mathcal{R}}$, … such that for all $i \geq 0$: $\mathcal{L}(\mathcal{A}^i_{\mathcal{R}}) \subseteq \mathcal{L}(\mathcal{A}^{i+1}_{\mathcal{R}})$ and if $s \in \mathcal{L}(\mathcal{A}^i_{\mathcal{R}})$, and $s \to_{\mathcal{R}} t$ then $t \in \mathcal{L}(\mathcal{A}^{i+1}_{\mathcal{R}})$. For $k \in \mathbb{N}$, if $\mathcal{L}(\mathcal{A}^k_{\mathcal{R}}) = \mathcal{L}(\mathcal{A}^{k+1}_{\mathcal{R}})$ then $\mathcal{A}^k_{\mathcal{R}}$ is a fixpoint and we denote it by $\mathcal{A}^*_{\mathcal{R}}$. To construct $\mathcal{A}^{i+1}_{\mathcal{R}}$ from $\mathcal{A}^i_{\mathcal{R}}$, we perform a *completion step* (denoted by $\mathcal{C}_{\mathcal{R}}$) which consists in finding *critical pairs* between $\to_{\mathcal{R}}$ and $\to_{\mathcal{A}^i_{\mathcal{R}}}$. For a substitution $\sigma : \mathcal{X} \mapsto \mathcal{Q}$ and a rule $l \to r \in \mathcal{R}$, a critical pair is an instance $l\sigma$ of l such that there exists a state $q \in \mathcal{Q}$ satisfying $l\sigma \to^*_{\mathcal{A}^i_{\mathcal{R}}} q$ and $r\sigma \not\to^*_{\mathcal{A}^i_{\mathcal{R}}} q$. For $r\sigma$ to be recognized by the same state and thus model the rewriting of $l\sigma$ into $r\sigma$, it is enough to add the necessary transitions to $\mathcal{A}^i_{\mathcal{R}}$ in order to obtain $\mathcal{A}^{i+1}_{\mathcal{R}}$ such that $r\sigma \to^*_{\mathcal{A}^{i+1}_{\mathcal{R}}} q$. The result of the completion step $\mathcal{C}_{\mathcal{R}}(\mathcal{A}^i_{\mathcal{R}})$ is thus $\mathcal{A}^{i+1}_{\mathcal{R}}$. In [24, 15], critical pairs are joined as in Figure 1.

$$\begin{tikzcd}
l\sigma \arrow[r, "\mathcal{R}"'] \arrow[d, "\mathcal{A}^i_{\mathcal{R}}"'] & r\sigma \arrow[d, "\mathcal{A}^{i+1}_{\mathcal{R}}"] \\
q & q' \arrow[l, "\mathcal{A}^{i+1}_{\mathcal{R}}"]
\end{tikzcd}$$

Figure 1: Completion step

$$\begin{tikzcd}
s\sigma \arrow[r, equal, "E"'] \arrow[d, "\mathcal{A},\not\epsilon"', "*"] & t\sigma \arrow[d, "*"', "\mathcal{A},\not\epsilon"] \\
q_a & q_b
\end{tikzcd}$$

Figure 2: Simplification step

$$\begin{tikzcd}
s(s(q_0)) \arrow[r, equal, "E"'] \arrow[d, "\mathcal{A},\not\epsilon"', "*"] & s(q_0) \arrow[d, "*"', "\mathcal{A},\not\epsilon"] \\
q_2 & q_1
\end{tikzcd}$$

Figure 3: Implication example

From an algorithmic point of view, there remain two problems to solve: find all the critical pairs $(l \to r, \sigma, q)$ and find the transitions to add to $\mathcal{A}^i_{\mathcal{R}}$ to have $r\sigma \to^*_{\mathcal{A}^{i+1}_{\mathcal{R}}} q$. The first problem, called matching, can be efficiently solved using a specific algorithm [8]. The second problem is solved using a normalization algorithm [10]. To have $r\sigma \to^*_{\mathcal{A}^{i+1}_{\mathcal{R}}} q'$ we need a transition of the form $r\sigma \to q'$ in $\mathcal{A}^{i+1}_{\mathcal{R}}$. However, this transition may not be normalized. In this case, it is necessary to introduce new states and new transitions. For instance, to normalize a transition $f(g(a), h(q_1)) \to q'$ w.r.t. a tree automaton $\mathcal{A}^i_{\mathcal{R}}$ with transitions $a \to q_1$, $b \to q_1$, $g(q_1) \to q_1$, we first rewrite $f(g(a), h(q_1))$ with transitions of $\mathcal{A}^i_{\mathcal{R}}$ as far as possible. We obtain $f(q_1, h(q_1))$. Then we introduce the new state q_2 and the new transition $h(q_1) \to q_2$ to recognize the term $h(q_1)$. The new transitions to add to $\mathcal{A}^i_{\mathcal{R}}$ are thus: $h(q_1) \to q_2$, $f(q_1, q_2) \to q'$, and $q' \to q$.

3.2 Simplification of Tree Automata by Equations

Since completion creates new transitions and new states to join critical pairs, it may diverge. Divergence is avoided by *simplifying* the tree automaton with a set of equations E. This operation permits the over-approximation of languages that cannot be recognized *exactly* using tree automata completion, e.g., non-regular languages. Simplification consists in finding E-equivalent terms recognized in $\mathcal{A}$ by different states and then by merging those states.

▶ **Definition 1** (Simplification relation). Let $\mathcal{A} = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$ be a tree automaton and E be a set of equations. For $s = t \in E$, $\sigma : \mathcal{X} \mapsto \mathcal{Q}$, $q_a, q_b \in \mathcal{Q}$ such that $s\sigma \to^{\not\epsilon *}_{\mathcal{A}} q_a$, $t\sigma \to^{\not\epsilon *}_{\mathcal{A}} q_b$ (See Figure 2) and $q_a \neq q_b$ then $\mathcal{A}$ is *simplified* into $\mathcal{A}'$, denoted by $\mathcal{A} \leadsto_E \mathcal{A}'$, where $\mathcal{A}'$ is $\mathcal{A}$ where q_b is replaced by q_a in $\mathcal{Q}$, $\mathcal{Q}_f$ and Δ. ◇

▶ **Example 2.** Let $E = \{s(s(x)) = s(x)\}$ and $\mathcal{A}$ be the tree automaton with $\mathcal{Q}_f = \{q_2\}$ and set of transitions $\Delta = \{a \to q_0, s(q_0) \to q_1, s(q_1) \to q_2\}$. Hence $\mathcal{L}(\mathcal{A}) = \{s(s(a))\}$. We can perform a simplification step using the equation $s(s(x)) = s(x)$ because we found a substitution $\sigma = \{x \mapsto q_0\}$ such that $s(s(q_0)) \to^{\not\epsilon *}_{\mathcal{A}} q_2$, $s(q_0) \to^{\not\epsilon *}_{\mathcal{A}} q_1$ (see Figure 3). Hence, $\mathcal{A} \leadsto_E \mathcal{A}'$ where $\mathcal{A}'$ is $\mathcal{A}$ where q_2 is replaced by q_1 i.e., $\mathcal{A}'$ is the automaton with $\mathcal{Q}'_f = \{q_1\}$, $\Delta = \{a \to q_0, s(q_0) \to q_1, s(q_1) \to q_1\}$. Note that $\mathcal{L}(\mathcal{A}') = \{s^*(s(a))\}$.

The simplification relation $\leadsto_E$ is terminating and confluent (modulo state renaming) [15]. In the following, by $\mathcal{S}_E(\mathcal{A})$ we denote the unique automaton (modulo renaming) $\mathcal{A}'$ such that $\mathcal{A} \leadsto_E^* \mathcal{A}'$ and $\mathcal{A}'$ is irreducible (it cannot be simplified further).

3.3 The full Completion Algorithm

▶ **Definition 3** (Automaton completion)**.** Let $\mathcal{A}$ be a tree automaton, $\mathcal{R}$ a left-linear TRS and E a set of equations.

- $\mathcal{A}^0_{\mathcal{R},E} = \mathcal{A}$,
- $\mathcal{A}^{n+1}_{\mathcal{R},E} = \mathcal{S}_E(\mathcal{C}_\mathcal{R}(\mathcal{A}^n_{\mathcal{R},E}))$, for $n \geq 0$ where $\mathcal{C}_\mathcal{R}(\mathcal{A}^n_{\mathcal{R},E})$ is the tree automaton such that all critical pairs of $\mathcal{A}^n_{\mathcal{R},E}$ are joined.

If there exists $k \in \mathbb{N}$ such that $\mathcal{A}^k_{\mathcal{R},E} = \mathcal{A}^{k+1}_{\mathcal{R},E}$, then we write $\mathcal{A}^*_{\mathcal{R},E}$ for $\mathcal{A}^k_{\mathcal{R},E}$.

▶ **Example 4.** Let $\mathcal{R} = \{f(x,y) \to f(s(x),s(y))\}$, $E = \{s(s(x)) = s(x)\}$ and $\mathcal{A}^0$ be the tree automaton with set of transitions $\Delta = \{f(q_a,q_b) \to q_0, a \to q_a, b \to q_b\}$, i.e., $\mathcal{L}(\mathcal{A}^0) = \{f(a,b)\}$. The completion ends after two completion steps on $\mathcal{A}^2_{\mathcal{R},E}$ which is a fixpoint $\mathcal{A}^*_{\mathcal{R},E}$. Completion steps are summed up in the following table. To simplify the presentation, we do not repeat the common transitions: $\mathcal{A}^i_{\mathcal{R},E}$ and $\mathcal{C}_\mathcal{R}(\mathcal{A}^i)$ columns are supposed to contain all transitions of $\mathcal{A}^0, \ldots, \mathcal{A}^{i-1}_{\mathcal{R},E}$.

$\mathcal{A}^0$	$\mathcal{C}_\mathcal{R}(\mathcal{A}^0)$	$\mathcal{A}^1_{\mathcal{R},E}$	$\mathcal{C}_\mathcal{R}(\mathcal{A}^1_{\mathcal{R},E})$	$\mathcal{A}^2_{\mathcal{R},E}$
$f(q_a,q_b) \to q_0$	$f(q_1,q_2) \to q_3$	$f(q_1,q_2) \to q_3$	$f(q_4,q_5) \to q_6$	$f(q_1,q_2) \to q_6$
$a \to q_a$	$s(q_a) \to q_1$	$s(q_a) \to q_1$	$s(q_1) \to q_4$	$s(q_1) \to q_1$
$b \to q_b$	$s(q_b) \to q_2$	$s(q_b) \to q_2$	$s(q_2) \to q_5$	$s(q_2) \to q_2$
	$q_3 \to q_0$	$q_3 \to q_0$	$q_6 \to q_3$	

On $\mathcal{A}^0$, there is one critical pair $f(q_a,q_b) \to^*_{\mathcal{A}^0} q_0$ and $f(q_a,q_b) \to_\mathcal{R} f(s(q_a),s(q_b))$. The automaton $\mathcal{C}_\mathcal{R}(\mathcal{A}^0)$ contains all the transitions of $\mathcal{A}^0$ with the new transitions (and the new states) necessary to join the critical pair, i.e., to have $f(s(q_a),s(q_b)) \to^*_{\mathcal{C}_\mathcal{R}(\mathcal{A}^0)} q_0$. The automaton $\mathcal{A}^1_{\mathcal{R},E}$ is exactly $\mathcal{C}_\mathcal{R}(\mathcal{A}^0)$ because simplification by E does not apply. Then, $\mathcal{C}_\mathcal{R}(\mathcal{A}^1_{\mathcal{R},E})$ contains all the transitions of $\mathcal{A}^1_{\mathcal{R},E}$ and $\mathcal{A}^0$ plus those obtained by the resolution of the critical pair $f(q_1,q_2) \to^*_{\mathcal{A}^1_{\mathcal{R},E}} q_3$ and $f(q_1,q_2) \to_\mathcal{R} f(s(q_1),s(q_2))$. On $\mathcal{C}_\mathcal{R}(\mathcal{A}^1_{\mathcal{R},E})$ simplification using the equation $s(s(x)) = s(x)$ can be applied on the following instances: $s(s(q_a)) = s(q_a)$ and $s(s(q_b)) = q_b$. Since $s(s(q_a)) \to^*_{\mathcal{C}_\mathcal{R}(\mathcal{A}^1_{\mathcal{R},E})} q_4$ and $s(q_a) \to^*_{\mathcal{C}_\mathcal{R}(\mathcal{A}^1_{\mathcal{R},E})} q_1$, simplification merges q_4 with q_1. Similarly, simplification on $s(s(q_b)) = q_b$ merges q_5 with q_2. Thus, $\mathcal{A}^2_{\mathcal{R},E} = \mathcal{C}_\mathcal{R}(\mathcal{A}^1_{\mathcal{R},E})$ where q_4 is replaced by q_1 and q_5 is replaced by q_2. This automaton is a fixed point because it has no other critical pairs (they are all joined).

3.4 Lower Bound, Upper Bound and Termination of Completion

▶ **Theorem 5** (Lower Bound [15])**.** *Let $\mathcal{R}$ be a left-linear TRS, $\mathcal{A}$ be a tree automaton and E be a set of equations. If completion terminates on $\mathcal{A}^*_{\mathcal{R},E}$ then $\mathcal{L}(\mathcal{A}^*_{\mathcal{R},E}) \supseteq \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$.*

To state the upper bound theorem, we need the notion of $\mathcal{R}/E$-coherence we now define.

▶ **Definition 6** (Coherent automaton)**.** Let $\mathcal{A} = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$ be a tree automaton, $\mathcal{R}$ a TRS and E a set of equations. The automaton $\mathcal{A}$ is said to be $\mathcal{R}/E$-*coherent* if $\forall q \in \mathcal{Q} : \exists s \in \mathcal{T}(\mathcal{F}) :$

$$s \to_{\mathcal{A}}^{\cancel{\epsilon}*} q \wedge [\forall t \in \mathcal{T}(\mathcal{F}) : (t \to_{\mathcal{A}}^{\cancel{\epsilon}*} q \implies s =_E t) \wedge (t \to_{\mathcal{A}}{}^* q \implies s \to^*_{\mathcal{R}/E} t)].$$

Here is the intuition behind $\mathcal{R}/E$-coherence. An $\mathcal{R}/E$-coherent automaton is $\not\epsilon$-reduced, its ϵ-transitions represent rewriting steps and normalized ($\not\epsilon$-transitions) transitions recognize E-equivalence classes. More precisely, in an $\mathcal{R}/E$-coherent tree automaton, if two terms s, t are recognized in the same state q using only normalized transitions then they belong to the same E-equivalence class. Otherwise, if at least one ϵ-transition is necessary to recognize, say, t in q then at least one step of rewriting with $\mathcal{R}$ was necessary to obtain t from s.

▶ **Example 7.** Let $\mathcal{R} = \{a \to b\}$, $E = \{c = d\}$ and $\mathcal{A} = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta\rangle$ with $\Delta = \{a \to q_0, b \to q_1, c \to q_2, d \to q_2, q_1 \to q_0\}$. The automaton $\mathcal{A}$ is $\mathcal{R}/E$-coherent because it is $\not\epsilon$-reduced and the state q_2 recognizes with $\to^{\not\epsilon}_{\Delta}$ two terms c and d but they satisfy $c =_E d$. Finally, $a \to^*_\Delta q_0$ and $b \to^*_\Delta q_0$ but $a \to^{\not\epsilon}_\Delta q_0$, $b \to^{\not\epsilon}_\Delta q_1 \to q_0$ and $a \to_\mathcal{R} b$.

▶ **Theorem 8** (Upper Bound [15])**.** *Let $\mathcal{R}$ be a left-linear TRS, E a set of equations and $\mathcal{A}$ an $\mathcal{R}/E$-coherent automaton. For any $i \in \mathbb{N}$: $\mathcal{L}(\mathcal{A}^i_{\mathcal{R},E}) \subseteq \mathcal{R}^*_E(\mathcal{L}(\mathcal{A}))$ and $\mathcal{A}^i_{\mathcal{R},E}$ is $\mathcal{R}/E$-coherent.*

Finally, we state the termination theorem which relies on E-compatibility. Roughly speaking, E-compatibility is the symmetric of E-coherence. An automaton $\mathcal{A}$ is E-compatible if for all states $q_1, q_2 \in \mathcal{A}$ and all terms $s, t \in \mathcal{T}(\mathcal{F})$ such that $s \to^{\not\epsilon *}_{\mathcal{A}} q_1$, $t \to^{\not\epsilon *}_{\mathcal{A}} q_2$ and $s =_E t$ then we have $q_1 = q_2$.

▶ **Theorem 9** (Termination of completion [10])**.** *Let $\mathcal{A}$ be a $\not\epsilon$-reduced tree automaton, $\mathcal{R}$ a left-linear TRS, and E a set of equations such that $\mathcal{T}(\mathcal{F})/_{=_E}$ is finite. If for all $i \in \mathbb{N}$, $\mathcal{A}^i_{\mathcal{R},E}$ is E-compatible then there exists a natural number $k \in \mathbb{N}$ such that $\mathcal{A}^k_{\mathcal{R},E}$ is a fixpoint.*

To prove our final result, we first have to generalize Theorems 8 and 9 to discard the technical $\mathcal{R}/E$-coherence and E-compatibility assumptions. This is the objective of the next sections.

4 From automata to equations and vice versa

Theorem 9 uses the assumption that the automata $\mathcal{A}^i_{\mathcal{R},E}$ are all E-compatible. This is not true in general. Unlike $\mathcal{R}/E$-coherence, E-compatibility is not preserved by tree automaton completion: $\mathcal{A}^{i+1}_{\mathcal{R},E}$ may not be E-compatible even if $\mathcal{A}^i_{\mathcal{R},E}$ is. Proofs can be found in [11].

▶ **Example 10.** Let $\mathcal{F} = \{f : 1, a : 0, b : 0, c : 0\}$, $\mathcal{R} = \{f(x) \to f(f(x)), f(f(x)) \to a\}$, $\mathcal{A}$ be the automaton such that $\Delta = \{a \to q_1, c \to q_1, f(q_1) \to q_f\}$ and $E = \{f(a) = f(b), f(b) = b, f(c) = f(b)\}$. Note that $\mathcal{T}(\mathcal{F})/_{=_E}$ has 3 equivalence classes: the class of $\{a\}$, the class of $\{b, f(a), f(b), f(c), \ldots\}$ and the class of $\{c\}$. However, completion does not terminate on this example. Automaton $\mathcal{A}$ is E-compatible ($f(a) =_E f(c)$ and both terms are recognized with $\to^{\not\epsilon}_{\mathcal{A}}$ by the same state: q_f) but $\mathcal{A}^1_{\mathcal{R},E}$ is not: it has one new state q_2 and contains additional transitions $\{f(q_f) \to q_2, q_2 \to q_f\}$. We thus have $f(f(a)) \to^{\not\epsilon *}_{\mathcal{A}^1_{\mathcal{R},E}} q_2$ and $f(a) \to^{\not\epsilon *}_{\mathcal{A}^1_{\mathcal{R},E}} q_f$ and $f(f(a)) =_E f(f(b)) =_E f(b) =_E f(a)$ but $q_2 \neq q_f$. Since b is not recognized by $\mathcal{A}^n_{\mathcal{R},E}$ for any n, the equation $f(b) = b$ never applies and completion diverges.

E-compatibility can be ensured for particular cases of $\mathcal{R}$ and E, e.g., for typed functional programs [10]. Here, we show how to transform the set E into a set $E_\mathcal{B}$ for which completed automata are $E_\mathcal{B}$-compatible, and completion is thus terminating. We also build $E_\mathcal{B}$ so that its precision is similar to E, i.e., $=_E \equiv =_{E_\mathcal{B}}$. This transformation is based on the Myhill-Nerode theorem for trees [18, 6]. We first produce a tree automaton $\mathcal{B}$ whose states recognize the equivalence classes of E. Then, from $\mathcal{B}$, we perform the inverse operation and obtain a set $E_\mathcal{B}$ whose set of equivalence classes is similar to the classes of E, but whose equations avoid the problem shown in Example 10. In this paper, we mainly consider sets E of ground equations because they are sufficient to prove our completeness results and for the practical applications of Section 7. However, this can be extended to general equations if E can be oriented into a weakly terminating TRS $\mathcal{R}$ s.t. $\textsc{Irr}(\mathcal{R})$ is finite [11].

4.1 From equations to automata

Provided that $\mathcal{T}(\mathcal{F})/_{=_E}$ is finite, the Myhill-Nerode theorem for trees [18, 6] relates $\mathcal{T}(\mathcal{F})/_{=_E}$ with tree automata. This theorem is constructive and provides an algorithm to switch from one form to the other, provided that $=_E$ is decidable. In the following we denote by **MN** the function that builds a tree automaton from a set of equations E [18].

▶ **Definition 11** (Function **MN**). Let E be a set of equations such that $\mathcal{T}(\mathcal{F})/_{=_E}$ is finite and $=_E$ is decidable. Let $\mathcal{Q}$ be a set of states and $state : \mathcal{T}(\mathcal{F})/_{=_E} \mapsto \mathcal{Q}$ be an injective function. $\mathbf{MN}(E) = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}, \Delta\rangle$ where $\Delta = \{f(state(u_1), \ldots, state(u_n)) \to state(u) \mid f \in \mathcal{F},\ u_1, \ldots, u_n, u \in \mathcal{T}(\mathcal{F})/_{=_E}$ and $f(u_1, \ldots, u_n) =_E u\}$

▶ **Theorem 12** (Myhill-Nerode theorem for trees [18]). *If $\mathcal{T}(\mathcal{F})/_{=_E}$ is finite and $=_E$ decidable, $\mathcal{B} = \mathbf{MN}(E)$ is a reduced, deterministic, epsilon-free and complete tree automaton such that for all $s, t \in \mathcal{T}(\mathcal{F})$, $s =_E t \iff (\exists q : \{s, t\} \subseteq \mathcal{L}(\mathcal{B}, q))$.*

When all equations of E are ground, E can be oriented into a complete TRS (confluent and terminating) $\overrightarrow{E}$, using for instance [22]. Then $=_E$ is decidable using $\overrightarrow{E}$ and finiteness of $\mathcal{T}(\mathcal{F})/_{=_E}$ is equivalent to finiteness of $\textsc{Irr}(\overrightarrow{E})$, which is decidable [6].

▶ **Example 13.** Consider the set E of Example 10. We can orient E into a complete TRS $\overrightarrow{E} = \{f(a) \to f(b), f(b) \to b, f(c) \to f(b)\}$. The set $\textsc{Irr}(\overrightarrow{E})$ is $\{a, b, c\}$. The automaton $\mathbf{MN}(E)$ has 3 states q_0, q_1, q_2 such that $state(a) = q_0$, $state(b) = q_1$ and $state(c) = q_2$. It has six transitions $a \to q_0$ (because $a \to^!_{\overrightarrow{E}} a$), $b \to q_1$ (because $b \to^!_{\overrightarrow{E}} b$), $c \to q_2$ (because $c \to^!_{\overrightarrow{E}} c$), $f(q_0) \to q_1$ (because $f(a) \to^!_{\overrightarrow{E}} b$), $f(q_1) \to q_1$ (because $f(b) \to^!_{\overrightarrow{E}} b$), $f(q_2) \to q_1$ (because $f(c) \to^!_{\overrightarrow{E}} b$).

4.2 From automata to equations

In the other direction, starting from a tree automaton $\mathcal{B}$ it is possible to build a set of equations $E_{\mathcal{B}}$ such that languages recognized by states of $\mathcal{B}$ and equivalence classes of $\mathcal{T}(\mathcal{F})/_{=_{E_{\mathcal{B}}}}$ coincide [18]. We reformulate the original algorithm into a function called **A2E** because we need some additional properties on the generated set of equations for completion to terminate. For simplicity we assume that $\mathcal{B}$ is **R**educed and epsilon-**F**ree. Some properties of $E_{\mathcal{B}}$ will hold only if $\mathcal{B}$ is also **C**omplete and **D**eterministic. In the following, we use the **RF** and **RDFC** short-hands for automata having the related properties. Recall that for any tree automaton, there exists an equivalent **RF** or **RDFC** automaton [6].

For an **RF** automaton $\mathcal{B}$, the construction of $E_{\mathcal{B}} = \mathbf{A2E}(\mathcal{B})$ is straightforward and follows [18]: for all states q we identify a ground term recognized by q, a representative, and for all transitions $f(q_1, \ldots, q_n) \to q$ we generate an equation $f(t_1, \ldots, t_n) = t$ where t_i, $1 \leq i \leq n$ are representatives for q_i and t is a representative for q. However, for this set of equations to guarantee termination of completion it needs some redundancy: for each state we generate a *set* of *state representatives* and the equations are defined for each representative of the set. As shown in Example 10, the equation $f(b) = b$ cannot be applied during completion because b does not occur in the tree automaton. However, a logical consequence of this equation is that $f(f(a)) =_E f(a)$ and terms $f(f(a))$ and $f(a)$

that occur in the tree automaton could be merged. In our setting the term $f(a)$ will be a state representative and the equation $f(f(a)) = f(a)$ will appear in the set of generated equations. Roughly speaking, every constant symbol a appearing in a transition $a \to q$ is a state representative for q. Every term of the form $f(u_1, \ldots, u_n)$ is a state representative for q if (1) u_i's are not state representatives of q, (2) $f(q_1, \ldots, q_n) \to q$ is a transition of $\mathcal{B}$ and (3) u_i's are state representatives for the q_i's. The property (1) ensures finiteness of the set of representatives.

▶ **Definition 14** (State representatives)**.** Let $\mathcal{B} = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$ be an **RF** tree automaton and $q \in \mathcal{Q}$. The set of state representatives of q of height lesser or equal to $k \in \mathbb{N}$, denoted by $[\![q]\!]^k_{\mathcal{B}}$, is inductively defined by:

- $[\![q]\!]^1_{\mathcal{B}} = \{a \mid a \to q \in \Delta\}$
- $[\![q]\!]^k_{\mathcal{B}} = [\![q]\!]^{k-1}_{\mathcal{B}} \cup \{f(u_1, \ldots, u_n) \mid f(q_1, \ldots, q_n) \to q \in \Delta$ and $\forall i \in \{1, \ldots, n\} : u_i \in [\![q_i]\!]^{k-1}_{\mathcal{B}}$, and $\forall p \in \mathscr{P}os(u_i) : u_i|_p \notin [\![q]\!]^{k-1}_{\mathcal{B}}\}$

In the above definition, the fact that $\mathcal{B}$ is reduced and epsilon-free ensures that there exists at least one (non-epsilon) transition for every state and that each state has at least one state representative.

▶ **Example 15.** Let $\mathcal{B}$ be the **RF** automaton that we obtained in Example 13 and whose set of transitions is $a \to q_0$, $b \to q_1$, $c \to q_2$, $f(q_0) \to q_1$, $f(q_1) \to q_1$, $f(q_2) \to q_1$.

- $[\![q_0]\!]^1_{\mathcal{B}} = \{a\}$, $[\![q_1]\!]^1_{\mathcal{B}} = \{b\}$, and $[\![q_2]\!]^1_{\mathcal{B}} = \{c\}$.
- $[\![q_0]\!]^2_{\mathcal{B}} = [\![q_0]\!]^1_{\mathcal{B}}$, $[\![q_1]\!]^2_{\mathcal{B}} = \{b, f(a), f(c)\}$, and $[\![q_2]\!]^2_{\mathcal{B}} = [\![q_2]\!]^1_{\mathcal{B}}$. The term $f(b)$ of height 2 and recognized by q_1 is not added to $[\![q_1]\!]^2_{\mathcal{B}}$ because its subterm b belongs to $[\![q_1]\!]^1_{\mathcal{B}}$.
- The fixpoint is reached because terms $f(f(a))$ and $f(f(c))$ recognized by q_1 are not added to $[\![q_1]\!]^3_{\mathcal{B}}$ because $f(a)$ and $f(c)$ belong to $[\![q_1]\!]^2_{\mathcal{B}}$.

We denote by $[\![q]\!]_{\mathcal{B}}$ the set of all state representatives for the state q i.e., the fixpoint of the above equations. We know that such a fixpoint exists and is always a finite set. Omitted proofs can be found in [11].

▶ **Lemma 16** (The set of state representatives is finite)**.** *For all* **RF** *tree automata $\mathcal{B}$, for all states $q \in \mathcal{B}$ there exists a natural number $k \in \mathbb{N}$ for which the set $[\![q]\!]^k_{\mathcal{B}}$ is a fixpoint.*

▶ **Definition 17** (Function **A2E**: set of equations $E_{\mathcal{B}}$ from a tree automaton $\mathcal{B}$)**.** Let $\mathcal{B} = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$ be an **RF** tree automaton. The set of equations $E_{\mathcal{B}}$ inferred from $\mathcal{B}$ is $\mathbf{A2E}(\mathcal{B}) = \{f(u_1, \ldots, u_n) = u \mid f(q_1, \ldots, q_n) \to q \in \mathcal{B}$, $u \in [\![q]\!]_{\mathcal{B}}$ and $u_i \in [\![q_i]\!]_{\mathcal{B}}$ for $1 \leq i \leq n\}$.

▶ **Example 18.** Starting from the automaton $\mathcal{B}$ and the state representatives of Example 15, the set $\mathbf{A2E}(\mathcal{B})$ contains the following equations: $a = a$ (because of transition $a \to q_0$), $c = c$ (because of transition $c \to q_2$), $b = b$, $b = f(a)$, $b = f(c)$ (because of transition $b \to q_1$), $f(a) = f(a)$, $f(a) = b$, $f(a) = f(c)$ (because of transition $f(q_0) \to q_1$), $f(f(a)) = f(a)$, $f(f(a)) = b$, $f(f(a)) = f(c)$, $f(b) = f(a)$, $f(b) = b$, $f(b) = f(c)$, $f(f(c)) = f(a)$, $f(f(c)) = b$, $f(f(c)) = f(c)$ (because of transition $f(q_1) \to q_1$), $f(c) = f(a)$, $f(c) = b$, and $f(c) = f(c)$ (because of transition $f(q_2) \to q_1$).

Since $\mathcal{B}$ is finite and the set of state representatives is finite then so is $E_{\mathcal{B}}$. Note that many equations of $E_{\mathcal{B}}$ are useless w.r.t. the underlying equational theory. This is the case, in the above example, for equations of the form $a = a$ as well as the equation $f(a) = f(c)$ which is redundant w.r.t. $b = f(a)$ and $b = f(c)$. However, as shown in Example 10 those equations are necessary for equational simplification to produce $E_{\mathcal{B}}$-compatible automata and completion to terminate. With the above $E_{\mathcal{B}}$, completion of Example 10 terminates. Below, Theorem 23 shows that, if $\mathcal{B}$ is **RDFC** then completion with **A2E**$(\mathcal{B})$ always terminates. Unsurprisingly, if $\mathcal{B}$ is deterministic then equivalence classes of $E_{\mathcal{B}}$ coincide with languages recognized by states of $\mathcal{B}$. This is the purpose of the next two lemmas.

▶ **Lemma 19.** *Let $\mathcal{B} = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$ be an* **RDFC** *tree automaton and $E_{\mathcal{B}} =$* **A2E**$(\mathcal{B})$*. For all $s \in \mathcal{T}(\mathcal{F})$, there exists a unique state $q \in \mathcal{Q}$ such that $s \rightarrow^*_{\mathcal{B}} q$ and for all state representatives $u \in [\![q]\!]_{\mathcal{B}}$, $s =_{E_{\mathcal{B}}} u$.*

Now we can relate equivalence classes of $E_{\mathcal{B}}$ and languages recognized by states of $\mathcal{B}$.

▶ **Lemma 20** (Equivalence classes of $E_{\mathcal{B}}$ coincide with languages recognized by states of $\mathcal{B}$)**.** *Let $\mathcal{B} = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$ be an* **RDFC** *tree automaton and $E_{\mathcal{B}} =$* **A2E**$(\mathcal{B})$*. For all $s, t \in \mathcal{T}(\mathcal{F})$, $s =_{E_{\mathcal{B}}} t \iff (\exists q : \{s, t\} \subseteq \mathcal{L}(\mathcal{B}, q))$.*

▶ **Corollary 21** ($\mathcal{T}(\mathcal{F})/_{=_{E_{\mathcal{B}}}}$ is finite)**.** *Let $\mathcal{B} = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$ be an* **RDFC** *tree automaton. If $E_{\mathcal{B}}$ is the set of equations inferred from $\mathcal{B}$ then $\mathcal{T}(\mathcal{F})/_{=_{E_{\mathcal{B}}}}$ is finite.*

5 Generalizing the termination theorem

Now, we prove that using $E_{\mathcal{B}}$ built from an **RDFC** tree automaton $\mathcal{B}$, completion terminates.

5.1 Proving termination of completion with $E_{\mathcal{B}}$

In the following, the automaton $\mathcal{A}^*$ is the limit of the (possibly) infinite completion of an initial $\not\epsilon$-reduced tree automaton $\mathcal{A}$ with $\mathcal{R}$ and $E_{\mathcal{B}}$. If the initial automaton is not $\not\epsilon$-reduced then completion may diverge. For instance, completion of the automaton whose set of transitions is $\{f(q_0) \rightarrow q_1\}$, with $\mathcal{R} = \{f(x) \rightarrow f(f(x))\}$ and $E = \{f(a) = a\}$ diverges (simplification never happens because q_0 does not recognize any term). Now we show that all state representatives are recognized by epsilon-free derivations in $\mathcal{A}^*$.

▶ **Lemma 22** (All states of $\mathcal{A}^*$ recognize at least one state representative)**.** *Let $\mathcal{R}$ be a TRS, $\mathcal{A}$ a $\not\epsilon$-reduced tree automaton, $\mathcal{B}$ an* **RDFC** *tree automaton and $E_{\mathcal{B}} =$* **A2E**$(\mathcal{B})$*. Let $\mathcal{A}^*$ be the limit of the completion of $\mathcal{A}$ by $\mathcal{R}$ and $E_{\mathcal{B}}$. For all states $q \in \mathcal{A}^*$, for all terms $s \in \mathcal{T}(\mathcal{F})$ such that $s \rightarrow^{\not\epsilon *}_{\mathcal{A}^*} q$, there exists a state $q'_{\mathcal{B}} \in \mathcal{B}$, a term $u \in [\![q'_{\mathcal{B}}]\!]_{\mathcal{B}}$ such that $u =_{E_{\mathcal{B}}} s$ and $u \rightarrow^{\not\epsilon *}_{\mathcal{A}^*} q$.*

Now, we can state the termination theorem with $E_{\mathcal{B}}$.

▶ **Theorem 23** (Completion with $E_{\mathcal{B}}$ terminates)**.** *Let $\mathcal{R}$ be a TRS, $\mathcal{A}$ a $\not\epsilon$-reduced tree automaton, $\mathcal{B}$ be an* **RDFC** *tree automaton and $E_{\mathcal{B}} =$* **A2E**$(\mathcal{B})$*. Let n be the number of all states representatives of $\mathcal{B}$. The automaton $\mathcal{A}^*$, limit of the completion of $\mathcal{A}$ with $\mathcal{R}$ and $E_{\mathcal{B}}$, has n states or less.*

Proof. Recall that the number n of state representatives is finite (cf. Lemma 16). Assume that $\mathcal{A}^*$ has m distinct states with $m > n$. From Lemma 22 we know that for all states $q \in \mathcal{A}^*$, there exists a state representative u such that $u \rightarrow^{\not\epsilon *}_{\mathcal{A}^*} q$. Since there are only n state representatives, by pigeon hole principle, we know that there is necessarily one state representative u recognized by two distinct states q_1 and q_2 of $\mathcal{A}^*$. Thus, $u \rightarrow^{\not\epsilon *}_{\mathcal{A}^*} q_1$ and $u \rightarrow^{\not\epsilon *}_{\mathcal{A}^*} q_2$. Besides, by construction of $E_{\mathcal{B}}$, we know that the equation $u = u$ is part of $E_{\mathcal{B}}$. This contradicts the fact that $\mathcal{A}^*$ is simplified w.r.t. $E_{\mathcal{B}}$. ◀

5.2 Building $E_{\mathcal{B}}$ from any set of equations E

Now, we combine the transformations **A2E** and **MN** to produce a set of equations $E_{\mathcal{B}}$ that ensures termination of completion. Unsurprisingly, $E_{\mathcal{B}}$ is equivalent to E.

▶ **Lemma 24.** *Let E be a set of equations. If $\mathcal{T}(\mathcal{F})/_{=_E}$ is finite and $=_E$ is decidable then $E_{\mathcal{B}} = \mathbf{A2E}(\mathbf{MN}(E))$ and $=_E \equiv =_{E_{\mathcal{B}}}$.*

▶ **Theorem 25** (Generalized termination theorem for completion)**.** *Let E be a set of ground equations such that $\mathcal{T}(\mathcal{F})/_{=_E}$ is finite. For all $\not\epsilon$-reduced tree automata $\mathcal{A}$ and TRSs $\mathcal{R}$, completion of $\mathcal{A}$ with $\mathcal{R}$ and $\mathbf{A2E}(\mathbf{MN}(E))$ terminates.*

Proof. As mentioned in Section 4.1, since E is ground $=_E$ is decidable. By Theorem 12, we know that $\mathcal{B} = \mathbf{MN}(E)$ exists and is **RDFC**. Let $E_{\mathcal{B}}$ be the set of equations $\mathbf{A2E}(\mathcal{B})$. Using Theorem 23, we know that completion of $\mathcal{A}$ with $\mathcal{R}$ and $E_{\mathcal{B}}$ is terminating. ◀

The above theorem shows how to tune a set of equations E into $E_{\mathcal{B}}$ to guarantee termination of completion. Note that tuning E into $E_{\mathcal{B}}$ does not jeopardize the precision of the completion since Lemma 24 guarantees that $=_E \equiv =_{E_{\mathcal{B}}}$. Combining this lemma with Theorem 8 (the Upper Bound Theorem) yields that completion of $\mathcal{R}$ with $E_{\mathcal{B}}$ is upper-bounded by $\mathcal{R}_E^*$.

6 Improving the Precision of Equational completion

Looking at our overall goal, we are half way there. If $\mathscr{L}$ is regular and $\mathscr{L} \supseteq \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$ (or $\mathscr{L} = \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$) then it can be recognized by an automaton $\mathcal{B}$. Using the results of the last section, we can build a set of equations $E_{\mathcal{B}}$ guaranteeing termination of completion. What remains to be proved is that completion with $E_{\mathcal{B}}$ ends on a tree automaton under-approximating $\mathscr{L}$ (or recognizing exactly $\mathscr{L} = \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$). As it is, Theorem 8 (the Upper Bound Theorem) fails to tackle this goal because it needs $\mathcal{R}/E$-coherence of $\mathcal{A}$. However, if $\mathcal{A}$ is not $\mathcal{R}/E$-coherent the full precision, granted by this theorem, may not be obtained.

▶ **Example 26.** Starting from Example 10, together with the set of equations $E_{\mathcal{B}}$ of Example 18, the initial tree automaton is not $\mathcal{R}/E_{\mathcal{B}}$-coherent (nor $\mathcal{R}/E$-coherent): $a \rightarrow_{\mathcal{A}}^{\not\epsilon *} q_1$ and $c \rightarrow_{\mathcal{A}}^{\not\epsilon *} q_1$ though $a \neq_{E_{\mathcal{B}}} c$. As a consequence, if we complete $\mathcal{A}$ with $\mathcal{R}$ and $E_{\mathcal{B}}$, we obtain an automaton that roughly approximates $\mathcal{R}^*(\mathcal{L}(\mathcal{A}))$. This can be done using the Timbuk tool [12]:

States `q0 q1` **Final States** `q0` **Transitions** `c->q1 a->q1 c->q0 f(q0)->q0 f(q1)->q0 a->q0`

This automaton recognizes the term c that is not reachable by rewriting the initial language $\mathcal{L}(\mathcal{A}) = \{f(a), f(c)\}$ with $\mathcal{R}$ (nor by rewriting with $\mathcal{R}/E_{\mathcal{B}}$). We propose to transform $\mathcal{A}$ so that it becomes $\mathcal{R}/E$-coherent: we build the product between $\mathcal{A}$ and $\mathbf{MN}(E)$. We recall the definition of a product automaton and we show that the product is $\mathcal{R}/E$-coherent.

▶ **Definition 27** (Product automaton [6])**.** Let $\mathcal{A} = (\mathcal{F}, Q, Q_F, \Delta_{\mathcal{A}})$ and $\mathcal{B} = (\mathcal{F}, P, P_F, \Delta_{\mathcal{B}})$ be automata. The product of $\mathcal{A}$ and $\mathcal{B}$ is $\mathcal{A} \times \mathcal{B} = (\mathcal{F}, Q \times P, Q_F \times P_F, \Delta)$ where $\Delta = \{f((q_1, p_1), \ldots, (q_k, p_k)) \rightarrow (q', p') \mid f(q_1, \ldots, q_k) \rightarrow q' \in \Delta_{\mathcal{A}}$ and $f(p_1, \ldots, p_k) \rightarrow p' \in \Delta_{\mathcal{B}}\}$.

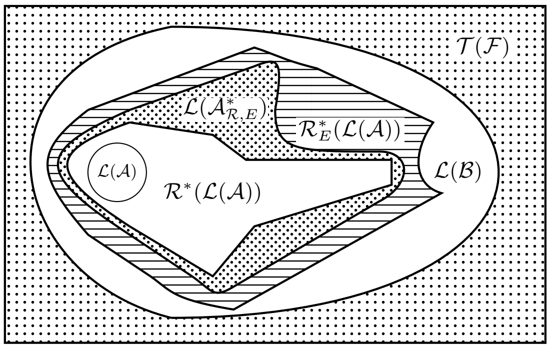
▶ **Theorem 28** (Generalized Upper Bound)**.** *Let $\mathcal{R}$ be a left-linear TRS, $\mathcal{A}$ an epsilon-free automaton, and E a set of ground equations such that $\mathcal{T}(\mathcal{F})/_{=_E}$ is finite. If $\mathcal{B} = \mathbf{MN}(E)$ and $\mathbb{A} = \mathcal{A} \times \mathcal{B}$ then for any $i \in \mathbb{N}$: $\mathcal{L}(\mathbb{A}^i_{\mathcal{R},E}) \subseteq \mathcal{R}_E^*(\mathcal{L}(\mathcal{A}))$.*

Proof. Since $\mathcal{L}(\mathbb{A}) = \mathcal{L}(\mathcal{A} \times \mathcal{B}) = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B})$ and $\mathcal{L}(\mathcal{B}) = \mathcal{T}(\mathcal{F})$, we get that $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathbb{A})$. Since both $\mathcal{A}$ and $\mathcal{B}$ are epsilon-free, so is $\mathcal{B}$. Thus, to prove $\mathcal{R}/E$-coherence of $\mathbb{A}$, we only have to prove that for all states q of $\mathbb{A}$ and for all two terms $s, t \in \mathcal{T}(\mathcal{F})$ such that (1) $s \rightarrow_{\mathbb{A}}^{\cancel{\epsilon}*} q$ and (2) $t \rightarrow_{\mathbb{A}}^{\cancel{\epsilon}*} q$ then $s =_E t$. Since $\mathbb{A}$ is a product automaton, q is a pair of the form (q_1, q_2) where $q_1 \in \mathcal{A}$ and $q_2 \in \mathcal{B}$. From (1) and (2) we can deduce that $s \rightarrow_{\mathcal{B}}^{\cancel{\epsilon}*} q_2$ and $t \rightarrow_{\mathcal{B}}^{\cancel{\epsilon}*} q_2$. Then, using Lemma 12, we get $s =_E t$. Thus $\mathbb{A}$ is $\mathcal{R}/E$-coherent and from Theorem 8, we get that $\mathcal{L}(\mathbb{A}^i_{\mathcal{R},E}) \subseteq \mathcal{R}^*_E(\mathcal{L}(\mathbb{A}))$ and $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathbb{A})$ ends the proof. ◀

▶ **Example 29.** Starting from Example 26, we can build the product between $\mathcal{A}$ and the automaton $\mathcal{B}$ found in Example 13. In $\mathcal{A} \times \mathcal{B}$, a and c are recognized by two different states, avoiding the $\mathcal{R}/E$-coherence problem of Example 26. The $\cancel{\epsilon}$-reduced product $\mathbb{A} = \mathcal{A} \times \mathcal{B}$ (where product states are renamed) is the automaton with $\mathcal{Q}_f = \{q_2\}$ and $\Delta = \{c \rightarrow q_0, a \rightarrow q_1, f(q_0) \rightarrow q_2, f(q_1) \rightarrow q_2\}$. Running Timbuk on $\mathbb{A}$, $\mathcal{R}$, and $E_\mathcal{B}$, we obtain $\mathbb{A}^*_{\mathcal{R},E}$ whose precision is now bounded by $\mathcal{R}^*_{E_\mathcal{B}}(\mathcal{L}(\mathcal{A}))$ and does not recognize c in a final state:

```
States q0 q1 q2 Final States q0 Transitions a->q1 f(q0)->q0 f(q1)->q0 f(q2)->q0
a->q0 c->q2
```

This provides hints to define equations for completion: we can start from an automaton $\mathcal{B}$ defining a rough approximation of the target language and build $E = \mathbf{A2E}(\mathcal{B})$. Then, we complete $\mathbb{A} = \mathcal{A} \times \mathcal{B}$ with $\mathcal{R}$ and E and obtain a tree automaton $\mathbb{A}^*_{\mathcal{R},E}$ whose precision is better or equal to $\mathcal{B}$. The set $\mathcal{R}^*_E(\mathcal{L}(\mathcal{A}))$ acts as a safeguard for completion (see Figure 4). In particular, terms of $\mathcal{R}^*_E(\mathcal{L}(\mathcal{A}))$ may not belong to $\mathcal{L}(\mathbb{A}^*_{\mathcal{R},E})$. This is the case in Example 29, where the term b belongs to $\mathcal{R}^*_{E_\mathcal{B}}(\mathcal{L}(\mathcal{A}))$ but not to $\mathcal{L}(\mathbb{A}^*_{\mathcal{R},E})$. In practice, we still need to know if E always exists (Section 7) and to generate a satisfactory E (Section 8).


$\mathcal{T}(\mathcal{F})$
$\mathcal{L}(\mathcal{A}^*_{\mathcal{R},E})$
$\mathcal{R}^*_E(\mathcal{L}(\mathcal{A}))$
$\mathcal{L}(\mathcal{B})$
$\mathcal{L}(\mathcal{A})$
$\mathcal{R}^*(\mathcal{L}(\mathcal{A}))$
Figure 4: The Generalized Upper Bound theorem (precision of completion)

7 Completeness Theorems

In this section, we prove two completeness theorems on completion. The first theorem states that if the set of reachable terms can be over-approximated by a regular language $\mathscr{L}$, then we can find a language containing reachable terms and under-approximating $\mathscr{L}$ using equational completion. The second theorem states that if the set of reachable terms is regular then completion can build it. Since the upper-bound of completion depends on $\mathcal{R}^*_E$, we first need a lemma showing that if E is built from $\mathscr{L}$ then $\mathcal{R}^*_E$ is upper-bounded by $\mathscr{L}$.

▶ **Lemma 30.** *Let $\mathcal{R}$ be a TRS over $\mathcal{F}$, $S \subseteq \mathcal{T}(\mathcal{F})$, and $\mathcal{B}$ an* **RDFC** *automaton such that $\mathcal{L}(\mathcal{B}) \supseteq \mathcal{R}^*(S)$ and $\mathcal{L}(\mathcal{B})$ is $\mathcal{R}$-closed. If $E_\mathcal{B} = \mathbf{A2E}(\mathcal{B})$ then $\mathcal{R}^*_{E_\mathcal{B}}(S) \subseteq \mathcal{L}(\mathcal{B})$.*

Example 31 shows that the $\mathcal{R}$-closed assumption on $\mathscr{L}$ is necessary for the lemma to hold.

▶ **Example 31.** Let $\mathcal{F} = \{a : 0, b : 0, c : 0, d : 0\}$, $S = \{a\}$, $\mathcal{R} = \{a \to b, c \to d\}$, and $\mathscr{L} = \{a, b, c\}$ where $\mathscr{L} \supseteq \mathcal{R}^*(S)$ but $\mathscr{L}$ is not $\mathcal{R}$-closed. A possible **RDFC** automaton $\mathcal{B}$, s.t. $\mathcal{L}(\mathcal{B}) = \mathscr{L}$, has a unique final state q and transitions $\{a \to q, b \to q, c \to q\}$. Thus $E_{\mathcal{B}} = \mathbf{A2E}(\mathcal{B})$ includes the equation $b = c$. Finally $\mathcal{R}^*_{E_{\mathcal{B}}}(S) = \{a, b, c, d\} \not\subseteq \mathscr{L}$.

▶ **Theorem 32** (Completeness). *Let $\mathcal{A}$ be a reduced epsilon-free tree automaton and $\mathcal{R}$ a left-linear TRS. Let $\mathcal{T}(\mathcal{F}) \supseteq \mathscr{L} \supseteq \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$. If $\mathscr{L}$ is regular and $\mathcal{R}$-closed then there exists a set of ground equations E such that $\mathbb{A} = \mathcal{A} \times \mathbf{MN}(E)$, $\mathbb{A}^*_{\mathcal{R},E}$ exists and $\mathcal{R}^*(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathbb{A}^*_{\mathcal{R},E}) \subseteq \mathscr{L}$.*

Proof. Since $\mathscr{L}$ is regular, we know that there exists an **RDFC** tree automaton, say $\mathcal{B}$, recognizing $\mathscr{L}$. From $\mathcal{B}$ we can infer $E_{\mathcal{B}} = \mathbf{A2E}(\mathcal{B})$ and then use completion to compute reachable terms. From Theorem 23, we know that completion of the automaton $\mathcal{A}$ with $\mathcal{R}$ and the set of equations $E_{\mathcal{B}}$ always terminates on a tree automaton $\mathcal{A}^*_{\mathcal{R},E_{\mathcal{B}}}$. From Theorem 8, we know that $\mathcal{L}(\mathcal{A}^*_{\mathcal{R},E_{\mathcal{B}}}) \subseteq \mathcal{R}^*_{E_{\mathcal{B}}}(\mathcal{L}(\mathcal{A}))$ provided that $\mathcal{A}$ is $\mathcal{R}/E_{\mathcal{B}}$-coherent. To enforce $\mathcal{R}/E_{\mathcal{B}}$-coherence of $\mathcal{A}$, we apply the transformation presented in Section 6. Let $\mathbb{A} = \mathcal{A} \times \mathbf{MN}(E_{\mathcal{B}})$. Note that since $E_{\mathcal{B}}$ is obtained by using the **A2E** transformation, $\mathcal{T}(\mathcal{F})/_{=_E}$ is finite (Corollary 21) and since equations of $E_{\mathcal{B}}$ are ground, $=_{E_{\mathcal{B}}}$ is decidable. The resulting automaton $\mathbb{A}$ is $\mathcal{R}/E_{\mathcal{B}}$-coherent. Besides, Theorem 23 also applies to $\mathbb{A}$. Thus, completion of $\mathbb{A}$ with $\mathcal{R}$ and $E_{\mathcal{B}}$ always ends on an automaton $\mathbb{A}^*_{\mathcal{R},E_{\mathcal{B}}}$. The automaton $\mathbb{A}^*_{\mathcal{R},E_{\mathcal{B}}}$ satisfies $\mathcal{R}^*(\mathcal{L}(\mathbb{A})) \subseteq \mathcal{L}(\mathbb{A}^*_{\mathcal{R},E_{\mathcal{B}}})$ (by Theorem 5) and $\mathcal{L}(\mathbb{A}^*_{\mathcal{R},E_{\mathcal{B}}}) \subseteq \mathcal{R}^*_{E_{\mathcal{B}}}(\mathcal{L}(\mathbb{A}))$ (by Theorem 28). Since $\mathcal{L}(\mathcal{A}) = \mathcal{L}(\mathbb{A})$, we have $\mathcal{R}^*(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathbb{A}^*_{\mathcal{R},E_{\mathcal{B}}})$ and $\mathcal{L}(\mathbb{A}^*_{\mathcal{R},E_{\mathcal{B}}}) \subseteq \mathcal{R}^*_{E_{\mathcal{B}}}(\mathcal{L}(\mathcal{A}))$. With Lemma 30, we get that $\mathcal{R}^*_{E_{\mathcal{B}}}(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathcal{B}) = \mathscr{L}$. ◀

In general we do not have $\mathcal{L}(\mathcal{A}^*_{\mathcal{R},E}) \supseteq \mathscr{L}$ because $\mathcal{L}(\mathcal{A}^*_{\mathcal{R},E})$ can be more precise than $\mathscr{L}$ (See Example 29). However, this is true when $\mathscr{L} = \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$, as we show in the next theorem.

▶ **Theorem 33** (Completeness for regularity preserving TRSs). *Let $\mathcal{A}$ be a reduced epsilon-free tree automaton and $\mathcal{R}$ a left-linear TRS. If $\mathcal{R}^*(\mathcal{L}(\mathcal{A}))$ is regular then it is possible to compute a tree automaton recognizing $\mathcal{R}^*(\mathcal{L}(\mathcal{A}))$ by equational tree automata completion.*

Proof. Let $\mathscr{L} = \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$. It is $\mathcal{R}$-closed. By assumption, it is also regular. Thus, we can apply Theorem 32 to get that there exists a set of equations E and a tree automaton $\mathbb{A} = \mathcal{A} \times \mathbf{MN}(E)$ such that $\mathbb{A}^*_{\mathcal{R},E}$ exists and $\mathcal{R}^*(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathbb{A}^*_{\mathcal{R},E}) \subseteq \mathscr{L}$. Since $\mathscr{L} = \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$, we get $\mathcal{L}(\mathbb{A}^*_{\mathcal{R},E}) = \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$. ◀

Thus, completion is complete w.r.t. *all left-linear TRS classes preserving regularity.*

8 Application of the Completeness Theorem

Let us show how to take advantage of Theorem 32 to automatically verify safety properties on programs. Given an initial regular language S and a program represented by a TRS $\mathcal{R}$, we can prove that the program never reaches terms in a set *Bad* by checking that there exists a regular over-approximation $\mathscr{L} \supseteq \mathcal{R}^*(S)$ such that $\mathscr{L} \cap Bad = \emptyset$. This technique has been used to verify cryptographic protocols [1], Java programs [4] and Functional Programs [10, 14]. Theorem 32 ensures that, if there exists an $\mathcal{R}$-closed regular approximation $\mathscr{L}$ such that $\mathscr{L} \cap Bad = \emptyset$, then we can build it (or under-approximate it) using completion and an appropriate set of ground equations E. To explore all the possible E, it is enough to explore $G_{\mathcal{F}}(k)$ with $k \in \mathbb{N}^*$.

▶ **Definition 34** (Generated Equations for $\mathcal{F}$ and $k \in \mathbb{N}^*$). Let $\mathbb{B}(k)$ be the set of all possible **RDFC** tree automata on $\mathcal{F}$ with exactly k states. The set of *generated equations* of size k is $G_{\mathcal{F}}(k) = \{E \mid \mathcal{B} \in \mathbb{B}(k) \text{ and } E = \mathbf{A2E}(\mathcal{B})\}$.

The semi-algorithm to prove that $\mathcal{R}^*(\mathcal{L}(\mathcal{A})) \cap Bad = \emptyset$ works as follows: (a) We start from $k = 1$, (b) we generate $G_{\mathcal{F}}(k)$, (c) we try completion with $\mathcal{A}$, $\mathcal{R}$ and all $E \in G_{\mathcal{F}}(k)$ (completion terminates with all those E, Theorem 23). If $\mathcal{L}(\mathcal{A}^*_{\mathcal{R},E}) \cap Bad = \emptyset$ for one E, we are done. Otherwise if $\mathcal{L}(\mathcal{A}^*_{\mathcal{R},E}) \cap Bad \neq \emptyset$ for all $E \in G_{\mathcal{F}}(k)$, we increase k and go back to step (b). If there exists a regular over-approximation $\mathscr{L} \supseteq \mathcal{R}^*(S)$ such that $\mathscr{L} \cap Bad = \emptyset$, then this algorithm eventually reaches a tree automaton $\mathcal{B}$ such that $\mathcal{L}(\mathcal{B}) = \mathscr{L}$, $E = \mathbf{A2E}(\mathcal{B})$, and by Theorem 32, we know that $\mathcal{L}(\mathcal{A}^*_{\mathcal{R},E}) \subseteq \mathscr{L}$. Finally, since $\mathscr{L} \cap Bad = \emptyset$, we have $\mathcal{L}(\mathcal{A}^*_{\mathcal{R},E}) \cap Bad = \emptyset$.

For general TRSs, we can enumerate all equation sets of $G_{\mathcal{F}}(k)$ but the search space is huge. When the TRS $\mathcal{R}$ encodes a functional program, we can restrict the search space to equation sets of the form $E = E_{\mathcal{R}} \cup E_r \cup E_{\mathcal{C}}$ [10], where $E_{\mathcal{R}}$ and E_r are fixed and $E_{\mathcal{C}}$ only ranges over $\textsc{Irr}(\mathcal{R})$. If program's functions are complete and terminating, $\textsc{Irr}(\mathcal{R})$ is the set of constructor terms. The set $\mathcal{F}$ can be separated into a set of *defined symbols* $\mathcal{D} = \{f \mid \exists l \to r \in \mathcal{R} \text{ s.t. } \mathscr{R}oot(l) = f\}$ and *constructor symbols* $\mathcal{C} = \mathcal{F} \setminus \mathcal{D}$.

▶ **Definition 35** (E_r). For a given set of symbols $\mathcal{F}$, $E_r = \{f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) \mid f \in \mathcal{F}, \text{ and arity of } f \text{ is } n\}$, where $x_1 \ldots x_n$ are pairwise distinct variables.

▶ **Definition 36** ($E_{\mathcal{R}}$). Let $\mathcal{R}$ be a TRS, the set of $\mathcal{R}$-equations is $E_{\mathcal{R}} = \{l = r \mid l \to r \in \mathcal{R}\}$.

▶ **Definition 37** ($E_{\mathcal{C}}$ contracting equations for $\mathcal{T}(\mathcal{C})$). A set of equations is contracting for $\mathcal{T}(\mathcal{C})$, denoted by $E_{\mathcal{C}}$, if all equations of $E_{\mathcal{C}}$ are of the form $u = u|_p$ with $u \in \mathcal{T}(\mathcal{C})$, $p \neq \lambda$, $\overrightarrow{E_{\mathcal{C}}} = \{u \to u|_p \mid u = u|_p \in E_{\mathcal{C}}\}$, and $\textsc{Irr}(\overrightarrow{E_{\mathcal{C}}})$ (terms of $\mathcal{T}(\mathcal{C})$ irreducible by $\overrightarrow{E_{\mathcal{C}}}$) is finite.

Completion is terminating if $E = E_{\mathcal{R}} \cup E_r \cup E_{\mathcal{C}}$ and $\mathcal{R}$ encodes a functional program which is terminating, complete, and is either first order [10] or higher-order [14]. Now, our objective is to define a completeness theorem for TRSs encoding those programs. Since E contains $E_{\mathcal{R}}$, all completed automata $\mathcal{A}^*_{\mathcal{R},E}$ will be $\mathcal{R}$-closed because $s \to^*_{\mathcal{A}^*_{\mathcal{R},E}} q$, $s \to_{\mathcal{R}} t$, $t \to^*_{\mathcal{A}^*_{\mathcal{R},E}} q'$ implies that $s =_{E_{\mathcal{R}}} t$ and $q = q'$ ($\mathcal{A}^*_{\mathcal{R},E}$ is simplified w.r.t. $E \supseteq E_{\mathcal{R}}$). Thus, the completeness theorem says that if there exists an $\mathcal{R}$-closed automaton $\mathcal{B}$ s.t. $\mathcal{L}(\mathcal{B}) \supseteq \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$ then there exists $E_{\mathcal{C}}$ such that $E = E_{\mathcal{R}} \cup E_r \cup E_{\mathcal{C}}$ and $\mathcal{L}(\mathcal{A}^*_{\mathcal{R},E}) \subseteq \mathcal{L}(\mathcal{B})$. To prove such a theorem, we need to explain how to construct a satisfying $E_{\mathcal{C}}$ from $\mathcal{B}$. We propose to project $\mathcal{B}$ on $\mathcal{C}$ (denoted by $\mathcal{B}/\mathcal{C}$), produce equations from $\mathcal{B}/\mathcal{C}$ with **A2E**, and finally filter out all equations that are not of the form $u = u|_p$ (this is function *ct*).

▶ **Definition 38** (Automaton projection on $\mathcal{C}$). Let $\mathcal{B} = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$ be an epsilon free tree automaton. The automaton $\mathcal{B}/\mathcal{C}$ is the tree automaton $\langle \mathcal{C}, \mathcal{Q}_{\mathcal{C}}, \mathcal{Q}_f \cap \mathcal{Q}_{\mathcal{C}}, \Delta_{\mathcal{C}} \rangle$ where $\Delta_{\mathcal{C}} = \{s \to q \mid s \to q \in \Delta \wedge \mathscr{R}oot(s) \in \mathcal{C}\}$ and $\mathcal{Q}_{\mathcal{C}}$ is the set of states occurring in the right-hand side of transitions of $\Delta_{\mathcal{C}}$.

Note that $\mathcal{L}(\mathcal{B}/\mathcal{C}) = \mathcal{L}(\mathcal{B}) \cap \mathcal{T}(\mathcal{C})$ and if $\mathcal{B}$ is **RDFC** so is $\mathcal{B}/\mathcal{C}$. In particular, if $\mathcal{B}$ is complete for $\mathcal{F}$, $\mathcal{B}/\mathcal{C}$ is complete for $\mathcal{C}$.

▶ **Definition 39.** Given a set of equations E, $ct(E) = \{l = r \in E \mid r = l|_p \text{ and } p \neq \lambda\}$.

In the following, we show that $E = ct(\mathbf{A2E}(\mathcal{B}))$ is a contracting set of equations, provided that $\mathcal{B}$ is **RDFC**. In particular, we show that $\textsc{Irr}(\overrightarrow{E})$ is finite.

▶ **Lemma 40.** *If $\mathcal{B}$ is an* **RDFC** *automaton on $\mathcal{C}$ and $E = ct(\mathbf{A2E}(\mathcal{B}))$, then* $\textsc{Irr}(\overrightarrow{E})$ *is finite and E is contracting for $\mathcal{T}(\mathcal{C})$.*

The above lemma states that $ct(\mathbf{A2E}(\mathcal{B}))$ is contracting for $\mathcal{T}(\mathcal{C})$. To have a finite set of equivalence classes on $\mathcal{T}(\mathcal{F})$ (and a terminating completion) we use $E = E_{\mathcal{R}} \cup E_r \cup E_{\mathcal{C}}$ where $E_{\mathcal{C}} = ct(\mathbf{A2E}(\mathcal{B}/\mathcal{C}))$. Now we prove that, w.r.t. approximations, E is as precise as $E_{\mathcal{B}}$.

▶ **Lemma 41.** *For a TRS $\mathcal{R}$ and an automaton $\mathcal{B}$ on $\mathcal{F}$, if $\mathcal{B}$ is* **RDFC** *and $\mathcal{R}$-closed and $E_{\mathcal{B}} = \mathbf{A2E}(\mathcal{B})$, $E_{\mathcal{C}} = ct(\mathbf{A2E}(\mathcal{B}/\mathcal{C}))$, and $E = E_{\mathcal{R}} \cup E_r \cup E_{\mathcal{C}}$ then $=_E \subseteq =_{E_{\mathcal{B}}}$.*

▶ **Theorem 42** **($E_{\mathcal{R}} \cup E_r \cup E_{\mathcal{C}}$ covers all $\mathcal{R}$-closed approximation automata).** *Let $\mathcal{R}$ be a left-linear TRS and $\mathcal{A}$ a reduced and epsilon-free tree automaton on $\mathcal{F}$. Let $\mathcal{B}$ be an $\mathcal{R}$-closed* **RDFC** *tree automaton such that $\mathcal{L}(\mathcal{B}) \supseteq \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$. Let $E_{\mathcal{C}} = ct(\mathbf{A2E}(\mathcal{B}/\mathcal{C}))$, $E = E_{\mathcal{C}} \cup E_{\mathcal{R}} \cup E_r$, and $\mathbb{A} = \mathcal{A} \times \mathbf{MN}(E)$. If $\mathbb{A}^*_{\mathcal{R},E}$ exists then $\mathcal{R}^*(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathbb{A}^*_{\mathcal{R},E}) \subseteq \mathcal{L}(\mathcal{B})$.*

Proof. The fact that $\mathcal{R}^*(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathbb{A}^*_{\mathcal{R},E})$ is ensured by Theorem 5. Using the Generalized Upper Bound theorem (Theorem 28), we deduce that (1) $\mathcal{L}(\mathbb{A}^*_{\mathcal{R},E}) \subseteq \mathcal{R}^*_E(\mathcal{L}(\mathcal{A}))$. From Lemma 41, we know that $=_E \subseteq =_{E_{\mathcal{B}}}$ and thus that (2) $\mathcal{R}^*_E(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{R}^*_{E_{\mathcal{B}}}(\mathcal{L}(\mathcal{A}))$. Besides, since $\mathcal{B}$ is $\mathcal{R}$-closed, $\mathcal{L}(\mathcal{B})$ is $\mathcal{R}$-closed and we can use Lemma 30 to get that (3) $\mathcal{R}^*_{E_{\mathcal{B}}}(\mathcal{L}(\mathcal{A})) \subseteq \mathcal{L}(\mathcal{B})$. Finally, using transitivity of $\subseteq$ on (1), (2) and (3) we get $\mathcal{L}(\mathbb{A}^*_{\mathcal{R},E}) \subseteq \mathcal{L}(\mathcal{B})$. ◀

Note that, for functional programs classes of [10] and [14], since $E_{\mathcal{C}} = ct(\mathbf{A2E}(\mathcal{B}/\mathcal{C}))$ is contracting (Lemma 40), $\mathbb{A}^*_{\mathcal{R},E}$ always exists. Thus, if there exists an $\mathcal{R}$-closed tree automaton $\mathcal{B}$ such that $\mathcal{L}(\mathcal{B}) \supseteq \mathcal{R}^*(\mathcal{L}(\mathcal{A}))$ and $\mathcal{L}(\mathcal{B}) \cap \mathit{Bad} = \emptyset$, it is enough to enumerate all possible $E = E_{\mathcal{R}} \cup E_r \cup E_{\mathcal{C}}$ to find it. Since $E_{\mathcal{R}}$ and E_r are fixed, it is enough to enumerate all possible $E_{\mathcal{C}}$ on $\mathcal{C}$ using Definition 37 and the algorithm of Definition 34 (generating on $\mathcal{C}$).

▶ **Example 43.** Let $\mathcal{C} = \{0:0, s:1\}$. For $k = 1$, there is only one **RDFC** automaton with 1 state. Its transitions are $\{s(q_0) \rightarrow q_0, 0 \rightarrow q_0\}$. Thus, $G_{\mathcal{C}}(1) = \{\{s(0) = 0\}\}$. For $k = 2$ there are 2 **RDFC** automata : one with transitions $\{0 \rightarrow q_0, s(q_0) \rightarrow q_1, s(q_1) \rightarrow q_1\}$ and the other with transitions $\{0 \rightarrow q_0, s(q_0) \rightarrow q_1, s(q_1) \rightarrow q_0\}$. Thus, $G_{\mathcal{C}}(2) = \{\{s(s(0)) = s(0)\}, \{s(s(0)) = 0, s(s(s(0))) = s(0)\}\}$.

We implemented this in Timbuk and used it to verify more than 20 safety properties of several first-order and higher-order functions on lists, ordered lists, trees and ordered trees. Higher-order properties include state-of-the-art examples from [21, 19, 14]. In [14], contracting equations of $E_{\mathcal{C}}$ contain variables and are generated from test sets. Here, we generate ground contracting equations $E_{\mathcal{C}}$ as shown above and use $E = E_{\mathcal{R}} \cup E_r \cup E_{\mathcal{C}}$ for completion. We transform the initial automaton $\mathcal{A}$ into $\mathbb{A}$ as in Theorem 28. The approximation is, thus, upper-bounded by $\mathcal{R}^*_E$ and we can benefit from the coverage guarantee of Theorem 42. On examples taken from [21, 19], we managed to do the same proofs with comparable execution times. On all the examples of [14], we do the same proofs (or find the counter-examples, see [14]), but in a much faster way. Appendix A presents a summary of those experiments and full details are here: `http://people.irisa.fr/Thomas.Genet/timbuk/funExperiments/`. For each example, we provide the specifications, Timbuk output, and the full result with completed automaton and generated equations in a Coq checkable file `comp.res`.

9 Conclusion and perspectives

Tree automata completion is known to cover many TRS classes preserving regularity [8, 10]. For some other classes, the question was still open. We established that, for all those classes (including those not known yet), given $\mathcal{A}$ and $\mathcal{R}$, there exists a set of equations E such that $\mathcal{A}^*_{\mathcal{R},E}$ recognizes $\mathcal{R}^*(\mathcal{L}(\mathcal{A}))$. We proved a similar theorem for the approximated case. The proofs are not constructive but give hints to enumerate sets of equations E. Finally, we showed that if a regular approximation satisfying a given property exists, we can find it by enumerating the sets E and running completion. From an algorithmic point of view and *in the general case* (where $\mathcal{T}(\mathcal{F})/_{=E}$ is finite), since we enumerate tree automata $\mathcal{B}$ on $\mathcal{T}(\mathcal{F})$ to generate sets of equations E, we could directly take advantage of $\mathcal{B}$ to perform automata simplification and thus replace equations.

However, equations are strictly more powerful than tree automata to define approximations. This can be observed on functional programs (Section 8) where $\mathcal{T}(\mathcal{C})/_{=E_\mathcal{C}}$ is finite (and $E_\mathcal{C}$ is generated using a tree automaton) but $\mathcal{T}(\mathcal{F})/_{=E}$ is not [14] and E cannot be defined with an automaton. On functional programs, Theorem 42 shows that enumeration can be restricted to sets of ground contracting equations on constructor symbols. This makes enumeration efficient enough to automatically verify properties on first-order and higher programs. Experiments shows that this approach tackles state-of-the-art automatic verification problems for first-order and higher-order programs. The completeness Theorem for functional programs ranges over $\mathcal{R}$-closed **RDFC** approximation automata. However, there exist $\mathcal{R}$-closed approximations that are not recognized by $\mathcal{R}$-closed **RDFC** tree automata.

▶ **Example 44.** Let $\mathcal{F} = \{f : 1, a : 0, b : 0\}$, $\mathcal{R} = \{a \to b\}$ and $\mathscr{L} = \{f(b), a, b\}$. The language $\mathscr{L}$ is $\mathcal{R}$-closed and regular. There exists no $\mathcal{R}$-closed **RDFC** tree automaton recognizing $\mathscr{L}$. In any $\mathcal{R}$-closed **RDFC** tree automaton, a and b need to be recognized by the same state, say q, and thus $f(b)$ needs to be recognized using a transition $f(q) \to q_f$ where q_f is final. Thus, this automaton recognizes $f(a)$ which does not belong to $\mathscr{L}$.

Such approximations are thus out of the scope of Theorem 42, and cannot be found by enumerating $E_\mathcal{C}$, because E contains $E_\mathcal{R}$ and the completed automata are $\mathcal{R}$-closed. However, the above approximation is in the scope of Theorem 32. We think that it is possible to explore the set of all possible equation sets using $E = E_r \cup E_\mathcal{F}$ where $E_\mathcal{F}$ is contracting on $\mathcal{T}(\mathcal{F})$ and to prune the search space using Counter Example Guided Abstraction Refinement like [19]. This would permit to have an efficient equation generation for general TRSs and widen its applicability to non-terminating functional programs, cryptographic protocols, etc.

A last perspective is to extend those results to non-left-linear TRSs. Dealing with regular languages and non-left-linear rules is known to be more challenging than the left-linear case [24, 3, 7]. Nevertheless, there could be a nice surprise here. For non-left-linear TRSs, completion is known to be sound and precise as long as the completed tree automaton is kept deterministic [8]. Completion itself does not preserve determinism but, in Section 8, all the completed automata of the experiments are deterministic. This is a consequence of the fact that E contains E_r (makes the automaton $\not\epsilon$-deterministic) and $E_\mathcal{R}$ (merges all states related by an ϵ-transition). Thus, when using $E = E_\mathcal{R} \cup E_r \cup E_\mathcal{C}$, completion may build precise over-approximations for non-left-linear TRSs as it does for left-linear ones.

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A Experiments

Timbuk Spec.	Description	P/C	Comp. Time	Eq. Gen. Time
delete	not (member A (delete A A_and_B_list))	P	0.01s	0.01s
delete2	(member B (delete A A_and_B_list))	P	0.01s	0.01s
deleteBasic	(delete A A_and_B_list) removes all occurrences of A	P	0.01s	0.01s
reverseFirstOrder	reverse [A,...A,B,...,B] does not produce lists with a A before a B	P	0.01s	0.03s
reverseFirstOrder2	invsorted (reverse [A,...A,B,...,B])	P	0.02s	0.13s
incTree	not (member 0 (increment nat_tree))	P	0.08s	1.05s
replaceTree	not (member A (replace A C A_and_B_tree))	P	0.44s	6.13s
orderedTree	ordered ordered_A_and_B_tree	P	0.16s	6.73s
insertTree	ordered (insert A_and_B_list emptyTree)	-	-	Timeout
orderedTreeTraversal	sorted (infix-traversal ordered_A_and_B_tree)	P	0.13s	1.71s
orderTreeTraversalBug	sorted (prefix-traversal ordered_A_and_B_tree)	C	0.2s	-
mapPlus	no 0 in (map (plus 1) nat_list)	P	0.02s	0.08s
filterEven	not (exists even (filter odd nat_list))	P	0.12s	1.16s
filterEvenBug	not (exists odd (filter odd nat_list))	C	0.09s	-
insertionSort	(sorted leq (sort leq A_and_B_list))	P	0.04s	0.11s
insertionSortBug	(sorted geq (sort leq A_and_B_list))	C	0.59s	-
filterNz	(forAll nz (filter nz nat_list))	P	0.01s	0.11s
mapTree	no 0 in (map (plus 1) nat_tree)	P	0.03s	16.15s
mapTree2	not (member 0 (map (plus 1) nat_tree)	-	-	Timeout
reverse	(sorted geq (reverse ordered_A_B_list))	P	0.04s	0.47s
mapSquare	(filter (eq 2) (map square nat_list)) is empty	P	0.31s	4.25s
foldRightMult	(foldRight mult nonzero_nat_list 1) is not 0	P	0.01s	0.01s
foldRightMult2	(foldRight mult nonzero_nat_list 3) is not 2	P	0.05s	0.29s
foldLeftPlus	even (foldLeft plus 0 even_nat_list)	P	0.01s	0.21s

The above table gives a summary of the experiments carried out with Timbuk. The source of the programs, trace of execution, Coq certificates, etc. can be found here: `http://people.irisa.fr/Thomas.Genet/timbuk/funExperiments/`. The 'Timbuk Spec.' column gives the name of the Timbuk specification file that was used (it is also available in Timbuk's distribution). The first 11 examples are first order programs and the 13 remaining are higher-order programs. The 'Description' column gives a short description of the property we want to prove. In the corresponding Timbuk specification this is the initial language and encoded by either a tree automaton or a *simplified regular expression* [13]. The 'P/C' column says if Timbuk has done a (P)roof of the property or found a (C)ounter example. 'Comp. Time' stands for completion time and 'Eq. Gen. Time' for equation generation time. On some examples, the equation generation algorithm times out and completion cannot be performed.

B Additional proofs

This section contains some the proofs of [11].

▶ Lemma (16). For all **RF** tree automata $\mathcal{B}$, for all states $q \in \mathcal{B}$ there exists a natural number $k \in \mathbb{N}$ for which the set $[\![q]\!]_{\mathcal{B}}^k$ is a fixpoint.

Proof. We make a proof by contradiction. Assume that one set of state representatives $[\![q]\!]_{\mathcal{B}}$ is infinite. Let $\mathcal{Q}$ be the set of states of $\mathcal{B}$ and $t \in [\![q]\!]_{\mathcal{B}}$ be a term s.t. $|t| > \mathcal{C}\mathrm{ard}(\mathcal{Q})$. Assume that we label each subterm of t by the state recognizing it in $\mathcal{B}$. Since height of t is greater than $\mathcal{C}\mathrm{ard}(\mathcal{Q})$, by the pigeonhole principle we know that there exists $q' \in \mathcal{B}$ and a path in the tree t such that q' appears at least two times. Let $p, r \in \mathscr{P}os(t)$ be the positions of the two subterms recognized by q'. By definition of state representatives, we know that $t|_p \in [\![q']\!]_{\mathcal{B}}$ and $t|_r \in [\![q']\!]_{\mathcal{B}}$. Since p and r are on the same path, we know that $t|_p$ is a strict subterm of $t|_r$ (or the opposite). This contradicts Definition 14 that forbids a term and a strict subterm to belong to the same set of representatives. ◀

▶ Lemma (19). Let $\mathcal{B} = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$ be an **RDFC** tree automaton and $E_{\mathcal{B}} = \mathbf{A2E}(\mathcal{B})$. For all $s \in \mathcal{T}(\mathcal{F})$, there exists a unique state $q \in \mathcal{Q}$ such that $s \to_{\mathcal{B}}^* q$ and for all state representatives $u \in [\![q]\!]_{\mathcal{B}}$, $s =_{E_{\mathcal{B}}} u$.

Proof. We make a proof by induction on the height of s. If s is a constant, since $\mathcal{B}$ is complete and deterministic there exists a unique transition $s \to q \in \Delta$. By construction of $E_{\mathcal{B}}$, we know that there are equations with s on the left-hand side and all state representatives of $[\![q]\!]_{\mathcal{B}}$ on the right-hand side. For all equations $s = u$ with $u \in [\![q]\!]_{\mathcal{B}}$ we thus trivially have $s =_{E_{\mathcal{B}}} u$. This concludes the base case.

Now, we assume that the property is true for terms of height lesser or equal to n. Let $s = f(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n$ are terms of height lesser or equal to n. Since $\mathcal{B}$ is complete, we know that there exists a state q such that $f(t_1, \ldots, t_n) \to_{\mathcal{B}}^* q$, i.e., there exists states $q_1, \ldots, q_n$ such that $f(q_1, \ldots, q_n) \to q \in \Delta$ and $t_i \to_{\mathcal{B}}^* q_i$ for $1 \leq i \leq n$. Using the induction hypothesis we get that there exist states q_i' in $\mathcal{B}$ and terms $[\![q_i']\!]_{\mathcal{B}}$ such that $t_i \to_{\mathcal{B}}^* q_i$ and $t_i =_{E_{\mathcal{B}}} u_i$ for $u_i \in [\![q_i']\!]_{\mathcal{B}}$ and for $1 \leq i \leq n$. Since $\mathcal{B}$ is deterministic, from $t_i \to_{\mathcal{B}}^* q_i$ and $t_i \to_{\mathcal{B}}^* q_i'$ we get that $q_i = q_i'$ and thus $t_i =_{E_{\mathcal{B}}} u_i$ for $u_i \in [\![q_i]\!]_{\mathcal{B}}$, with $1 \leq i \leq n$. Besides, since $f(q_1, \ldots, q_n) \to q \in \Delta$, we know that $E_{\mathcal{B}}$ contains the equations $f(u_1, \ldots, u_n) = u$ for all $u_i \in [\![q_i]\!]_{\mathcal{B}}$, for all $1 \leq i \leq n$ and for all $u \in [\![q]\!]_{\mathcal{B}}$. Thus q is the unique state such that $f(t_1, \ldots, t_n) \to_{\mathcal{B}}^* q$. Furthermore, $f(t_1, \ldots, t_n) =_{E_{\mathcal{B}}} f(u_1, \ldots, u_n) =_{E_{\mathcal{B}}} u$ for all $u_i \in [\![q_i]\!]_{\mathcal{B}}$, for all $1 \leq i \leq n$ and for all $u \in [\![q]\!]_{\mathcal{B}}$. ◀

▶ Lemma (20). Let $\mathcal{B} = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$ be an **RDFC** tree automaton and $E_{\mathcal{B}} = \mathbf{A2E}(\mathcal{B})$. For all $s, t \in \mathcal{T}(\mathcal{F})$, $s =_{E_{\mathcal{B}}} t \iff (\exists q : \{s, t\} \subseteq \mathcal{L}(\mathcal{B}, q))$.

Proof. For s and t, using Lemma 19, we know that there exist unique states $q, q' \in \mathcal{Q}$ such that $s \to_{\mathcal{B}}^* q$, $t \to_{\mathcal{B}}^* q'$ and for all state representatives $u \in [\![q]\!]_{\mathcal{B}}$ and $v \in [\![q']\!]_{\mathcal{B}}$, we have $s =_{E_{\mathcal{B}}} u$ and $t =_{E_{\mathcal{B}}} v$. We first prove the left to right implication. From $s =_{E_{\mathcal{B}}} t$

we obtain that $u =_{E_\mathcal{B}} v$, where u and v are state representatives. By construction of term representatives, for all states q we know that $[\![q]\!]_\mathcal{B}$ only contains terms recognized by q in $\mathcal{B}$. Since $\mathcal{B}$ is deterministic, if $q \neq q'$ then we can conclude that $[\![q]\!]_\mathcal{B} \cap [\![q']\!]_\mathcal{B} = \emptyset$. Thus, the only possibility to have $u =_{E_\mathcal{B}} v$ is to have an equation $u = v$ in $E_\mathcal{B}$. This entails that u and v belong to the same set of representatives: $[\![q]\!]_\mathcal{B} = [\![q']\!]_\mathcal{B}$, which entails that $q = q'$. Then $s \to_\mathcal{B}^* q$ and $t \to_\mathcal{B}^* q$ entails that $\{s, t\} \subseteq \mathcal{L}(\mathcal{B}, q)$. To prove the right to left implication, it is enough to point out that because of the determinism of $\mathcal{B}$ having $t \to_\mathcal{B}^* q'$ (the initial assumption) and having $t \to_\mathcal{B}^* q$ (the fact that $t \in \mathcal{L}(\mathcal{B}, q)$) is possible only if $q = q'$. This entails that u and v have a common set of representatives and thus for all representatives u of this set $s =_{E_\mathcal{B}} u =_{E_\mathcal{B}} t$. ◀

▶ Lemma (22). Let $\mathcal{R}$ be a TRS, $\mathcal{A}$ a $\cancel{\epsilon}$-reduced tree automaton, $\mathcal{B}$ an **RDFC** tree automaton and $E_\mathcal{B} = \mathbf{A2E}(\mathcal{B})$. Let $\mathcal{A}^*$ be the limit of the completion of $\mathcal{A}$ by $\mathcal{R}$ and $E_\mathcal{B}$. For all states $q \in \mathcal{A}^*$, for all terms $s \in \mathcal{T}(\mathcal{F})$ such that $s \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q$, there exists a state $q'_\mathcal{B} \in \mathcal{B}$, a term $u \in [\![q'_\mathcal{B}]\!]_\mathcal{B}$ such that $u =_{E_\mathcal{B}} s$ and $u \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q$.

Proof. Note that if $\mathcal{A}$ is $\cancel{\epsilon}$-reduced, then so is $\mathcal{A}^*$ (cf. Lemma 44 of [10]). This is easy to figure out since all states added during completion recognize at least one term with $\to_{\mathcal{A}}^{\cancel{\epsilon}*}$, and this is trivially preserved by simplification. By induction on the height of s we show that the representative u exists and is recognized by q. If s is of height 1 (it is a constant) then, by construction of state representatives, we know that s is a representative. Thus $s = u \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q$.

For the inductive case, assume that the property is true for all terms of height lesser or equal to n. Let $s = f(s_1, \ldots, s_n)$ be a term of height $n+1$. By assumption, we know that $f(s_1, \ldots, s_n) \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q$. From $f(s_1, \ldots, s_n) \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q$, we obtain that there exists states $q_1, \ldots, q_n$ of $\mathcal{A}^*$ such that $s_i \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q_i$ for $i = 1, \ldots, n$ and a transition $f(q_1, \ldots, q_n) \to q$ in $\mathcal{A}^*$. Using the induction hypothesis on q_i, $i = 1, \ldots, n$ we get that there exist state representatives u_i such that $s_i =_{E_\mathcal{B}} u_i$ and $u_i \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q_i$ for $i = 1, \ldots, n$. Then, since $f(q_1, \ldots, q_n) \to q$ in $\mathcal{A}^*$ we know that $f(u_1, \ldots, u_n) \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q$. If $f(u_1, \ldots, u_n)$ is a state representative we are done since $f(s_1, \ldots, s_n) =_{E_\mathcal{B}} f(u_1, \ldots, u_n)$ and $f(u_1, \ldots, u_n) \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q$. Otherwise, by definition of state representatives, for $u = f(u_1, \ldots, u_n)$ not to belong to the representatives there is a position p in u, different from the root position such that the subterm $u|_p$ is itself a state representative and it belongs to the same class as u, i.e., $u =_{E_\mathcal{B}} u|_p$. Since $u_1, \ldots, u_n$ are state representatives and $f(u_1, \ldots, u_n)$ is in the same equivalence class as $u|_p$ which is a state representative, we know that the equation $f(u_1, \ldots, u_n) = u|_p$ necessarily belongs to $E_\mathcal{B}$. Besides, for $u \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q$ to hold, we know that there exists a state q' such that $u[u|_p]_p \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} u[q']_p \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q$. Thus, $f(u_1, \ldots, u_n) \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q$ and $u|_p \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q'$. Then, since $E_\mathcal{B}$ contains the equation $f(u_1, \ldots, u_n) = u|_p$, and since $\mathcal{A}^*$ is simplified w.r.t. $E_\mathcal{B}$, we necessarily have $q = q'$ in $\mathcal{A}^*$. Finally, we have $f(s_1, \ldots, s_n) =_{E_\mathcal{B}} f(u_1, \ldots, u_n) =_{E_\mathcal{B}} u|_p$ and $u|_p \to_{\mathcal{A}^*}^{\cancel{\epsilon}*} q$ where $u|_p$ is a state representative. ◀

▶ Lemma (21). Let $\mathcal{B} = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$ be an **RDFC** tree automaton. If $E_\mathcal{B}$ is the set of equations inferred from $\mathcal{B}$ then $\mathcal{T}(\mathcal{F})/_{=_{E_\mathcal{B}}}$ is finite.

Proof. Using Lemma 19, we know that for all terms $t \in \mathcal{T}(\mathcal{F})$ there exists a state $q \in \mathcal{Q}$ and a state representative $u \in [\![q]\!]_\mathcal{B}$ such that $t \to_\mathcal{B}^* q$ and $t =_{E_\mathcal{B}} u$. Since the number of states of $\mathcal{B}$ is finite, and since the set of state representatives u is finite for all states of $\mathcal{B}$ (Lemma 16), so is the number of equivalence classes of $\mathcal{T}(\mathcal{F})/_{=_{E_\mathcal{B}}}$. ◀

▶ Lemma (22). Let $\mathcal{R}$ be a TRS, $\mathcal{A}$ a $\not\epsilon$-reduced tree automaton, $\mathcal{B}$ an **RDFC** tree automaton and $E_{\mathcal{B}} = \mathbf{A2E}(\mathcal{B})$. Let $\mathcal{A}^*$ be the limit of the completion of $\mathcal{A}$ by $\mathcal{R}$ and $E_{\mathcal{B}}$. For all states $q \in \mathcal{A}^*$, for all terms $s \in \mathcal{T}(\mathcal{F})$ such that $s \to^{\not\epsilon *}_{\mathcal{A}^*} q$, there exists a state $q'_{\mathcal{B}} \in \mathcal{B}$, a term $u \in [\![q'_{\mathcal{B}}]\!]_{\mathcal{B}}$ such that $u =_{E_{\mathcal{B}}} s$ and $u \to^{\not\epsilon *}_{\mathcal{A}^*} q$.

Proof. Note that if $\mathcal{A}$ is $\not\epsilon$-reduced, then so is $\mathcal{A}^*$ (cf. Lemma 44 of [10]). This is easy to figure out since all states added during completion recognize at least one term with $\to^{\not\epsilon *}_{\mathcal{A}}$, and this is trivially preserved by simplification. By induction on the height of s we show that the representative u exists and is recognized by q. If s is of height 1 (it is a constant) then, by construction of state representatives, we know that s is a representative. Thus $s = u \to^{\not\epsilon *}_{\mathcal{A}^*} q$.

For the inductive case, assume that the property is true for all terms of height lesser or equal to n. Let $s = f(s_1, \dots, s_n)$ be a term of height $n+1$. By assumption, we know that $f(s_1, \dots, s_n) \to^{\not\epsilon *}_{\mathcal{A}^*} q$. From $f(s_1, \dots, s_n) \to^{\not\epsilon *}_{\mathcal{A}^*} q$, we obtain that there exists states $q_1, \dots, q_n$ of $\mathcal{A}^*$ such that $s_i \to^{\not\epsilon *}_{\mathcal{A}^*} q_i$ for $i = 1, \dots, n$ and a transition $f(q_1, \dots, q_n) \to q$ in $\mathcal{A}^*$. Using the induction hypothesis on q_i, $i = 1, \dots, n$ we get that there exist state representatives u_i such that $s_i =_{E_{\mathcal{B}}} u_i$ and $u_i \to^{\not\epsilon *}_{\mathcal{A}^*} q_i$ for $i = 1, \dots, n$. Then, since $f(q_1, \dots, q_n) \to q$ in $\mathcal{A}^*$ we know that $f(u_1, \dots, u_n) \to^{\not\epsilon *}_{\mathcal{A}^*} q$. If $f(u_1, \dots, u_n)$ is a state representative we are done since $f(s_1, \dots, s_n) =_{E_{\mathcal{B}}} f(u_1, \dots, u_n)$ and $f(u_1, \dots, u_n) \to^{\not\epsilon *}_{\mathcal{A}^*} q$. Otherwise, by definition of state representatives, for $u = f(u_1, \dots, u_n)$ not to belong to the representatives there is a position p in u, different from the root position such that the subterm $u|_p$ is itself a state representative and it belongs to the same class as u, i.e., $u =_{E_{\mathcal{B}}} u|_p$. Since $u_1, \dots, u_n$ are state representatives and $f(u_1, \dots, u_n)$ is in the same equivalence class as $u|_p$ which is a state representative, we know that the equation $f(u_1, \dots, u_n) = u|_p$ necessarily belongs to $E_{\mathcal{B}}$. Besides, for $u \to^{\not\epsilon *}_{\mathcal{A}^*} q$ to hold, we know that there exists a state q' such that $u[u|_p]_p \to^{\not\epsilon *}_{\mathcal{A}^*} u[q']_p \to^{\not\epsilon *}_{\mathcal{A}^*} q$. Thus, $f(u_1, \dots, u_n) \to^{\not\epsilon *}_{\mathcal{A}^*} q$ and $u|_p \to^{\not\epsilon *}_{\mathcal{A}^*} q'$. Then, since $E_{\mathcal{B}}$ contains the equation $f(u_1, \dots, u_n) = u|_p$, and since $\mathcal{A}^*$ is simplified w.r.t. $E_{\mathcal{B}}$, we necessarily have $q = q'$ in $\mathcal{A}^*$. Finally, we have $f(s_1, \dots, s_n) =_{E_{\mathcal{B}}} f(u_1, \dots, u_n) =_{E_{\mathcal{B}}} u|_p$ and $u|_p \to^{\not\epsilon *}_{\mathcal{A}^*} q$ where $u|_p$ is a state representative. ◀

▶ Lemma (30). Let $\mathcal{R}$ be a TRS over $\mathcal{F}$, $S \subseteq \mathcal{T}(\mathcal{F})$, and $\mathcal{B}$ an **RDFC** automaton such that $\mathcal{L}(\mathcal{B}) \supseteq \mathcal{R}^*(S)$ and $\mathcal{L}(\mathcal{B})$ is $\mathcal{R}$-closed. If $E_{\mathcal{B}} = \mathbf{A2E}(\mathcal{B})$ then $\mathcal{R}^*_{E_{\mathcal{B}}}(S) \subseteq \mathcal{L}(\mathcal{B})$.

Proof. We prove that for all natural number $k >= 0$, if $s \in S$ and $s \to^{k}_{\mathcal{R}/E_{\mathcal{B}}} t$ then $t \in \mathcal{L}(\mathcal{B})$ where $\to^{k}_{\mathcal{R}/E_{\mathcal{B}}}$ denotes k steps of rewriting by $\mathcal{R}$ modulo $E_{\mathcal{B}}$. By induction on k. If $k = 0$ then $s =_{E_{\mathcal{B}}} t$. Using Lemma 20 on $s =_{E_{\mathcal{B}}} t$, we get that there exists a state q of $\mathcal{B}$ such that $s \to^*_{\mathcal{B}} q$ and $t \to^*_{\mathcal{B}} q$. Since $s \in S$ and $S \subseteq \mathcal{L}(\mathcal{B})$ there exists a final state q_f of $\mathcal{B}$ such that $s \to^*_{\mathcal{B}} q_f$. Since $\mathcal{B}$ is deterministic we obtain that $q = q_f$. Thus t is recognized by $\mathcal{B}$. For the inductive case, we assume that the property is true for a given k and we show that it is true for $k+1$. Let $s \to^{k+1}_{\mathcal{R}/E} t$, i.e., we have terms s', s'', and t' such that $s \to^{k}_{\mathcal{R}/E} s' =_{E_{\mathcal{B}}} s'' \to_{\mathcal{R}} t' =_{E_{\mathcal{B}}} t$. Using the induction hypothesis, we get that s' is recognized by $\mathcal{B}$. Since $\mathcal{L}(\mathcal{B})$ is $\mathcal{R}$-closed, we know that t' is also recognized by $\mathcal{B}$. Thus, there exists a final state q_f such that $t' \to^*_{\mathcal{B}} q_f$. Finally, as above, applying Lemma 20 on the fact that $t' \to^*_{\mathcal{B}} q_f$ and $t' =_{E_{\mathcal{B}}} t$ gives us that $t \to^*_{\mathcal{B}} q_f$. ◀