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A WELL-POSEDNESS RESULT FOR VISCOUS COMPRESSIBLE FLUIDS WITH ONLY BOUNDED DENSITY

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ABSTRACT. We are concerned with the existence and uniqueness of solutions with only bounded density for the barotropic compressible Navier-Stokes equations. Assuming that the initial velocity has slightly sub-critical regularity and that the initial density is a small perturbation (in the L^∞ norm) of a positive constant, we prove the existence of local-in-time solutions. In the case where the density takes two constant values across a smooth interface (or, more generally, has striated regularity with respect to some nondegenerate family of vector-fields), we get uniqueness. This latter result supplements the work by D. Hoff in [26] with a uniqueness statement, and is valid in any dimension $d \geq 2$ and for general pressure laws.

INTRODUCTION

We are concerned with the multi-dimensional barotropic compressible Navier-Stokes system in the whole space:

$$(0.1) \qquad \begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \,, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mu \Delta u - \lambda \nabla \,\mathrm{div}\, u + \nabla P(\rho) = 0 \,. \end{cases}$$

Here $\rho = \rho(t,x)$ and $u = u(t,x)$, with $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $d \geq 1$, denote the density and velocity of the fluid, respectively. The pressure P is a given function of ρ. We shall take that function locally in $W^{1,\infty}$ in all that follows, and assume (with no loss of generality) that it vanishes at some constant reference density $\bar{\rho} > 0$. The (constant) viscosity coefficients μ and λ satisfy

$$(0.2) \qquad \mu > 0 \quad \text{and} \quad \nu := \lambda + \mu > 0,$$

which ensures ellipticity of the operator

$$(0.3) \qquad \mathcal{L} := -\mu \,\Delta \, - \, \lambda \,\nabla \,\mathrm{div} \,.$$

We supplement System (0.1) with the initial conditions

$$(0.4) \qquad \rho|_{t=0} = \rho_0 \quad \text{and} \quad u|_{t=0} = u_0.$$

A number of recent works have been dedicated to the study of solutions with discontinuous density (so-called "shock-data") for models of viscous compressible fluids. Even though the situation is by now quite well understood for $d = 1$ (see e.g. [24]), the multi-dimensional case is far from being completely elucidated. In this direction, we mention the papers [25, 26] by D. Hoff, who greatly contributed to construct "intermediate solutions" allowing for discontinuity of the density, in between the weak solutions of P.-L. Lions in [32] and the classical ones of e.g. J. Nash in [34].

In [26], in the two-dimensional case, D. Hoff succeeded to get quite an accurate information on the propagation of density discontinuities accros suitably smooth curves (as predicted by the Rankine-Hugoniot condition), under the assumption that the pressure is a linear function of ρ (see Theorem 1.2 therein). In particular, he proved that those curves are convected by the flow

and keep their initial regularity *even though the gradient of the velocity is not continuous.* The result was strongly based on the observation that, for such solutions, the "effective viscous flux" $F := \nu \operatorname{div} u - P(\rho)$ is continuous, although singularities persist in $\operatorname{div} u$ and $P(\rho)$ separately.

The present paper aims at completing the aforementioned works in several directions.

First, we want to supplement them with a uniqueness result. Indeed, in [26], D. Hoff constructed solutions (that are global in time under some smallness assumption) and pointed out very accurate qualitative properties for the geometric structure of singularities, but did not address uniqueness. That latter issue has been considered afterward in [27], but only for *linear* pressure laws. In fact, the main uniqueness theorem therein requires either the pressure law to be linear (as opposed to the standard isentropic assumption $P(\rho) = a\rho^\gamma$ with $\gamma > 1$) or some Lebesgue type information on $\nabla\rho$ (thus precluding our considering jumps across interfaces). To the best of our knowledge, exhibiting an appropriate functional framework for uniqueness *without imposing a special structure for the solutions* has remained an open question for nonlinear pressure laws and discontinuous densities.

Our second goal is to extend Hoff's works concerning discontinuity across interfaces and uniqueness to any dimension and to more general pressure laws and density singularities.

Finding conditions on the initial data that ensure ∇u to belong to $L^1([0, T]; L^\infty)$ for some $T > 0$ is the key to our two goals. That latter property will be achieved by combining parabolic maximal regularity estimates with *tangential* (or striated) *regularity* techniques that are borrowed from the work by J.-Y. Chemin in [4].

In order to introduce the reader with our use of maximal regularity, let us consider the slightly simpler situation of a fluid fulfilling the inhomogeneous incompressible Navier-Stokes equations:

$$(0.5) \qquad \begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u + \nabla \Pi = 0 \\ \operatorname{div} u = 0 \,. \end{cases}$$

We have in mind the case where the initial density is given by

$$(0.6) \qquad \rho_0 = c_1 \mathbb{1}_D + c_2 \mathbb{1}_{D^c}$$

for some positive constants c_1, c_2, with D being a smooth bounded domain of $\mathbb{R}^d$ (above, $\mathbb{1}_A$ designates the characteristic function of a set A).

Several recent works have been devoted to proposing conditions on u_0 allowing for solving (0.5) either locally or globally, and uniquely. The first result in that direction has been obtained by the first author and P.B. Mucha in [13]. It is based on endpoint maximal regularity and requires the jump $c_1 - c_2$ to be small enough. However, that approach requires the use of *multiplier spaces* to handle the low regularity of the density, and is very unlikely to be extendable to the compressible setting, owing to the pressure term that now depends on ρ.

Therefore, we shall rather take advantage of the approach that has been proposed recently in [28] by the third author together with J. Huang and P. Zhang to investigate (0.5) with only bounded density. Indeed, it is based on the standard parabolic maximal regularity and requires only very elementary tools like Hölder inequality and Sobolev embedding. In order to present the main steps, assume for simplicity that the reference density $\bar\rho$ is 1. Then, setting $\varrho := \rho - 1$, system (0.5) rewrites

$$\begin{cases} \partial_t \varrho + u \cdot \nabla \varrho = 0 \\ \partial_t u - \mu \Delta u + \nabla \Pi = -\varrho \partial_t u - (1+\varrho) u \cdot \nabla u \\ \operatorname{div} u = 0 \,. \end{cases}$$

From basic maximal regularity estimates (recalled in Subsection 2.2 below), we have[1] for all $1 < p, r < \infty$,

$$\|u\|_{L_T^\infty(\dot{B}_{p,r}^{2-2/r})} + \|(\partial_t u, \mu\nabla^2 u)\|_{L_T^r(L^p)} \le C\left(\|u_0\|_{\dot{B}_{p,r}^{2-\frac{2}{r}}} + \|\varrho\partial_t u\|_{L_T^r(L^p)} + \|(1+\varrho)u\cdot\nabla u\|_{L_T^r(L^p)}\right).$$

It is obvious that the second term of the r.h.s. may be absorbed by the l.h.s. if the nonhomogeneity ϱ is small enough for the L^∞ norm. As for the last term, it may be absorbed either for short time if the velocity is large, or for all times if the velocity is small, and the norm $L^r(L^p)$ is *scaling invariant* for the incompressible Navier-Stokes equations, that is to say satisfies

(0.7)
$$\frac{2}{r} + \frac{d}{p} = 3.$$

Those simple observations are the keys to the proof of global existence for (0.5) in [28]. As regards uniqueness, owing to the hyperbolic part of the system (viz. the first equation of (0.5)), we need (at least) a $L_T^1(\text{Lip})$ control on the velocity. Note that if one combines the above control in $L_T^r(L^p)$ for $\nabla^2 u$ with the corresponding critical Sobolev embedding, then we miss that information by a little in the critical regularity setting, as having (0.7) and $r > 1$ implies that $p < d$; nonetheless, it turns out that, if working in a slightly subcritical framework (that is $2/r + d/p < 3$), one can get existence and uniqueness altogether for any initial density $\rho_0 \in L^\infty$ that is close enough to some positive constant (see [16, 28] for more details).

In the particular case where ρ_0 is given by (0.6) then, once the Lipschitz control of the transport field is available, it is possible to propagate the Lipschitz regularity of the domain D, as it is just advected by the (Lipschitz continuous) flow of the velocity field. Based on that observation, further developments and more accurate informations on the evolution of the boundary of D have been obtained very recently by X. Liao and P. Zhang in [30, 31], and by the first author with X. Zhang [17] and with P. B. Mucha [15]. In most of those works, a key ingredient is the propagation of tangential regularity, in the spirit of the seminal work by J.-Y Chemin in [4, 5] dedicated to the vortex patch problem for the incompressible Euler equations. We refer also to papers [20], [7], [8], [23] for extensions of the results of [4, 5] to higher dimensions and to viscous homogeneous fluids; the case of non-homogeneous inviscid flows being treated in [19].

The rest of the paper is devoted to obtaining similar results for the compressible Navier-Stokes equations (0.1), and is structured as follows. In the next section, we present our main results and give some insight of the proofs. Then, in Section 2, we recall the definition of Besov spaces and introduce the tools for achieving our results: Littlewood-Paley decomposition, maximal regularity and estimates involving striated regularity. Section 3 is devoted to the proof of our main existence theorem for general discontinuous densities while the next section concerns the propagation of striated regularity and uniqueness. Some technical results that are based on harmonic analysis are postponed in the Appendix.

1. Main results

In order to evaluate our chances of getting the same results for (0.1) as for (0.5) after suitable adaptation of the method described above, let us rewrite (0.1) in terms of (ϱ, u). We get, just denoting by P (instead of $P(1 + \varrho)$) the pressure term, the system

(1.1)
$$\begin{cases} \partial_t\varrho + u\cdot\nabla\varrho + (\varrho+1)\operatorname{div} u = 0 \\ \partial_t u - \mu\Delta u - (\lambda+\mu)\nabla\operatorname{div} u = -\varrho\partial_t u - (1+\varrho)u\cdot\nabla u - \nabla P. \end{cases}$$

As the so-called Lamé system (that is, the l.h.s. of the second equation) enjoys the same maximal regularity properties as the heat equation, one can handle the terms $\varrho\partial_t u$ and $(1+\varrho)u\cdot\nabla u$ in a

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[1]Throughout the paper we agree that, whenever E is a Banach space, $r \in [1,\infty]$ and $T > 0$, then $L_T^r(E)$ designates the space $L^r([0,T]; E)$ and $\|\cdot\|_{L_T^r(E)}$ the corresponding norm; when $T = \infty$, we use the notation $L^r(E)$.

suitable $L^r(L^p)$ framework exactly as in the incompressible situation. However, by this method, bounding ∇P requires $\nabla \varrho$ to be in some Lebesgue space, a condition that we want to avoid. In fact, the coupling between the density and the velocity equations is stronger than for (0.5) so that the two equations of (1.1) should not be considered separately. For that reason, it is much more difficult to prove a well-posedness result for rough densities here than in the incompressible case, and the presence of the "out-of-scaling" term ∇P precludes our achieving global existence (even for small data) by simple arguments. A standard way to weaken the coupling between the two equations of (1.1) (that has been used by D. Hoff in [25, 26] and, more recently, by B. Haspot in [22] or by the first author with L. He in [12]) is to reformulate the system in terms of a "modified velocity field", in the same spirit as the effective viscous flux mentioned in the introduction: we set

$$w := u + \nabla(-\nu\Delta)^{-1}P, \quad \text{with} \quad \nu := \lambda + \mu.$$

The modified velocity w absorbs the "dangerous part" of ∇P, as may be observed when writing out the equation for w:

(1.2) $\qquad (1+\varrho)\partial_t w - \mu\Delta w - \lambda\nabla \operatorname{div} w + (1+\varrho)u \cdot \nabla u + (1+\varrho)(-\nu\Delta)^{-1}\nabla\partial_t P = 0.$

As, by virtue of the mass equation, we have

$$\partial_t P = (P - \rho P')\operatorname{div} u - \operatorname{div}(Pu);$$

the last term of (1.2) is indeed of lower order and can be bounded in $L^r_T(L^p)$ whenever ϱ is bounded and belongs to some suitable Lebesgue space.

As we shall see in the present paper, working with w and ϱ rather than with the original unknowns u and ρ proves to be efficient if one is concerned with existence (and possibly uniqueness) results for (0.1) with only bounded density. In fact, we shall implement the maximal regularity estimates on the equation fulfilled by w, and bound ϱ by means of the standard a priori estimates in Lebesgue spaces for the transport equation. For technical reasons however, it will be wise to replace $(-\nu\Delta)^{-1}$ by its non-homogeneous version $(\operatorname{Id} -\nu\Delta)^{-1}$ which is much less singular.

That strategy will enable us to prove the following local-in-time result of existence for (0.1) supplemented with a rough initial density[2].

Theorem 1.1. *Let $d \geq 1$. Let the couple (p, r) satisfy*

(1.3) $\qquad\qquad\qquad d < p < \infty \qquad and \qquad 1 < r < \dfrac{2p}{2p - d},$

and define the couple of indices (r_0, r_1) by the relations

(1.4) $\qquad\qquad\qquad \dfrac{1}{r_0} = \dfrac{1}{r} - 1 + \dfrac{d}{2p} \qquad and \qquad \dfrac{1}{r_1} = \dfrac{1}{r} - \dfrac{1}{2}.$

Let the initial density ρ_0 and velocity u_0 satisfy:

- $\varrho_0 := \rho_0 - 1$ in $\left(L^p \cap L^\infty\right)(\mathbb{R}^d)$;
- $w_0 := u_0 - v_0$ in $\dot{B}^{2-2/r}_{p,r}$, with $v_0 := -\nabla(\operatorname{Id} -\nu\Delta)^{-1}(P(\rho_0))$.

There exist $\varepsilon > 0$ and a time $T > 0$ such that, if

(1.5) $\qquad\qquad\qquad\qquad\qquad \|\varrho_0\|_{L^\infty} \leq \varepsilon,$

then there exists a solution (ρ, u) to System (0.1)-(0.4) on $[0, T] \times \mathbb{R}^d$ with $\varrho := \rho - 1$ satisfying

$$\|\varrho\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq 4\varepsilon \quad and \quad \varrho \in \mathcal{C}([0,T]; L^q) \quad for\ all\ p \leq q < \infty,$$

and $u = v + w$ with $v := -\nabla(\operatorname{Id} -\nu\Delta)^{-1}\left(P(\rho)\right)$ in $\mathcal{C}\left([0,T]; W^{1,q}(\mathbb{R}^d)\right)$ for all $p \leq q < \infty$, and

$$w \in \mathcal{C}\left([0,T]; \dot{B}^{2-2/r}_{p,r}\right) \cap L^{r_0}_T(L^\infty), \quad \nabla w \in L^{r_1}_T(L^p) \quad and \quad \partial_t w,\, \nabla^2 w \in L^r_T(L^p)$$

That solution is unique if $d = 1$.

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[2]The reader is referred to Section 2 below for the definition of the homogeneous Besov spaces $\dot{B}^s_{p,r}$.

If, in addition, u_0 and ϱ_0 are in L^2, and $\inf P' > 0$ on $[1 - 4\varepsilon, 1 + 4\varepsilon]$, then the energy balance

$$(1.6) \quad \frac{1}{2} \left\| \sqrt{\rho(t)} \, u(t) \right\|_{L^2}^2 + \left\| \Pi(\rho(t)) \right\|_{L^1} +$$

$$+ \mu \left\| \nabla u \right\|_{L_t^2(L^2)}^2 + \lambda \left\| \operatorname{div} u \right\|_{L_t^2(L^2)}^2 = \frac{1}{2} \left\| \sqrt{\rho_0} \, u_0 \right\|_{L^2}^2 + \left\| \Pi(\rho_0) \right\|_{L^1}$$

holds true for all $t \in [0, T]$, where the function $\Pi = \Pi(z)$ is defined by the conditions $\Pi(1) = \Pi'(1) = 0$ and $\Pi''(z) = P'(z)/z$.

Combining the above statement with Sobolev embeddings ensures that ∇w and $\operatorname{div} u$ are in $L_T^1(L^\infty)$. However, because operator $\nabla^2 (\operatorname{Id} - \nu \Delta)^{-1}$ does not quite map L^∞ into itself (except if $d = 1$ of course), there is no guarantee that the constructed velocity field u has gradient in $L_T^1(L^\infty)$. This seems to be the minimal requirement in order to get uniqueness of solutions (see e.g. papers [26] and [11]). Keeping the model case (0.6) in mind, the question is whether adding up geometric hypotheses, like interfaces or tangential regularity, ensures that property and, hopefully, uniqueness.

Motivated by the pioneering work by J.-Y. Chemin in [4, 5], we shall assume that the initial density has some "striated regularity" along a nondegenerate family of vector fields. To be more specific, we have to introduce more notation and give some definitions. Before doing that, let us underline that propagating tangential regularity for compressible flows faces us to new difficulties compared to the incompressible case, due to the fact that $\operatorname{div} u$ does not vanish anymore.

First of all, for any p in $]d, \infty]$, we denote by $\mathbb{L}^{\infty, p}$ the space of all continuous and bounded functions with gradient in $L^p(\mathbb{R}^d)$. Now, for a given vector-field Y in $\mathbb{L}^{\infty, p}$, we are interested in the regularity of a function f along Y, i.e. in the quantity

$$\partial_Y f := \sum_{j=1}^d Y^j \, \partial_j f \, .$$

This expression is well-defined if f is smooth enough, in which case we have the identity

$$(1.7) \qquad\qquad \partial_Y f = \operatorname{div}(f \, Y) - f \operatorname{div} Y \, .$$

If f is only bounded (which, typically, will be the case if f is the density given by (0.6)), the above right-hand side makes sense for any vector field Y in $\mathbb{L}^{\infty, p}$, while $\partial_Y f$ has no meaning. Then we take the right-hand side of (1.7) as a definition of $\partial_Y f$.

In order to define striated regularity, fix a family $\mathcal{X} = (X_\lambda)_{1 \le \lambda \le m}$ of m vector-fields with components in $\mathbb{L}^{\infty, p}$ and suppose that it is *non-degenerate*, in the sense that

$$I(\mathcal{X}) := \inf_{x \in \mathbb{R}^d} \sup_{\Lambda \in \Lambda_{d-1}^m} \left| \overset{d-1}{\wedge} X_\Lambda(x) \right|^{\frac{1}{d-1}} > 0 \, .$$

Here $\Lambda \in \Lambda_{d-1}^m$ means that $\Lambda = (\lambda_1, \ldots, \lambda_{d-1})$, with each $\lambda_i \in \{1, \ldots, m\}$ and $\lambda_i < \lambda_j$ for $i < j$, while the symbol $\overset{d-1}{\wedge} X_\Lambda$ stands for the unique element of $\mathbb{R}^d$ such that

$$\forall \, Y \in \mathbb{R}^d, \qquad \left(\overset{d-1}{\wedge} X_\Lambda \right) \cdot Y = \det \left(X_{\lambda_1} \ldots X_{\lambda_{d-1}}, Y \right) \, .$$

Then we set

$$\| X_\lambda \|_{\mathbb{L}^{\infty, p}} := \| X_\lambda \|_{L^\infty} + \| \nabla X_\lambda \|_{L^p} \quad \text{and} \quad \| \mathcal{X} \|_{\mathbb{L}^{\infty, p}} := \sup_{\lambda \in \Lambda} \| X_\lambda \|_{\mathbb{L}^{\infty, p}} \, .$$

More generally, whenever E is a normed space, we use the notation $\| \mathcal{X} \|_E := \sup_{\lambda \in \Lambda} \| X_\lambda \|_E$.

Definition 1.1. *Take a vector-field $Y \in \mathbb{L}^{\infty, p}$, for some $p \in]d, \infty]$. A function $f \in L^\infty$ is said to be of class L^p along Y, and we write $f \in \mathbb{L}_Y^p$, if $\operatorname{div}(f \, Y) \in L^p(\mathbb{R}^d)$.*

If $\mathcal{X} = (X_\lambda)_{1\leq\lambda\leq m}$ is a non-degenerate family of vector-fields in $\mathbb{L}^{\infty,p}$ then we set

$$\mathbb{L}^p_{\mathcal{X}} := \bigcap_{1\leq\lambda\leq m} \mathbb{L}^p_{X_\lambda} \qquad and \qquad \|f\|_{\mathbb{L}^p_{\mathcal{X}}} := \frac{1}{I(\mathcal{X})}\left(\|f\|_{L^\infty}\,\||\mathcal{X}|\|_{\mathbb{L}^{\infty,p}} + \||\mathrm{div}\,(f\,\mathcal{X})|\|_{L^p}\right)\cdot$$

The main motivation for Definition 1.1 is Proposition 2.4 below that states that, if ϱ is bounded and if, in addition, $\varrho \in \mathbb{L}^p_{\mathcal{X}}$ for some non-degenerate family $\mathcal{X}$ of vector-fields in $\mathbb{L}^{\infty,p}$ with $d < p < \infty$, then $(\eta\,\mathrm{Id}\,-\Delta)^{-1}\nabla^2 f(\varrho)$ is in L^∞ for all $\eta > 0$ and smooth enough function f. That fundamental property will enable us to consider data like (0.6) in a functional framework that ensures persistence of interface regularity and uniqueness altogether, as stated just below.

Theorem 1.2. *Let $d \geq 1$ and the couple (p,r) fulfill conditions (1.3). Consider initial data (ρ_0, u_0) satisfying the same assumptions as in Theorem 1.1. Assume in addition that there exists a non-degenerate family $\mathcal{X}_0 = (X_{0,\lambda})_{1\leq\lambda\leq m}$ of vector-fields in $\mathbb{L}^{\infty,p}$ such that ρ_0 belongs to $\mathbb{L}^p_{\mathcal{X}_0}$.*

Then, there exists a time $T > 0$ and a unique solution (ρ, u) to System (0.1)-(0.4) on $[0,T] \times \mathbb{R}^d$, such that $\varrho := \rho - 1$, $v := -\nabla(\mathrm{Id}\,-\nu\Delta)^{-1}(P(\rho))$ and $w := u - v$ satisfy the same properties as in Theorem 1.1. Furthermore, ∇u belongs to $L^1([0,T]; L^\infty(\mathbb{R}^d))$ and u has a flow ψ_u with bounded gradient, that is the unique solution of

$$(1.8) \qquad \psi_u(t,x) = x + \int_0^t u(\tau, \psi_u(\tau,x))\,d\tau \qquad for\ all\ \ (t,x) \in [0,T] \times \mathbb{R}^d\,.$$

Finally, if we define $X_{t,\lambda}$ by the formula $X_{t,\lambda}(x) := \partial_{X_{0,\lambda}}\psi_u(t, \psi_u^{-1}(t,x))$ then, for all time $t \in [0,T]$, the family $\mathcal{X}_t := (X_{t,\lambda})_{1\leq\lambda\leq m}$ remains non-degenerate and in the space $\mathbb{L}^{\infty,p}$, and the density $\rho(t)$ belongs to $\mathbb{L}^p_{\mathcal{X}_t}$.

Let us make some comments on the proof of that second main result. As pointed out above, the striated regularity hypothesis ensures that $(\mathrm{Id}\,-\nu\Delta)^{-1}\nabla^2 P$ is bounded. Since we know from Theorem 1.1 that ∇w is in $L^1_T(L^\infty)$, one can conclude that also ∇u belongs to $L^1_T(L^\infty)$. From this property and the remark that, for all $\lambda \in \Lambda$, one has

$$\partial_t X_\lambda + u \cdot \nabla X_\lambda = \partial_{X_\lambda} u \quad and \quad \partial_t\,\mathrm{div}(\rho\,X_\lambda) + \mathrm{div}(\mathrm{div}(\rho\,X_\lambda)\,u) = 0\,,$$

standard estimates for the transport equation will enable us to propagate the tangential regularity.

As regards the proof of uniqueness, we adopt D. Hoff viewpoint in [27]: *solutions with minimal regularity are best if compared in a Lagrangian framework; that is, we compare the instantaneous states of corresponding fluid particles in two different solutions rather than the states of different fluid particles instantaneously occupying the same point of space-time.* However, the proof that is proposed therein relies on stability estimates in the negative Sobolev space $\dot{H}^{-1}$ for the density; at some point, it is crucial that $\rho \in \dot{H}^{-1}$ implies that $P(\rho)$ is in $\dot{H}^{-1}$, too, a property that obviously fails when the pressure law is nonlinear.

In the present paper, adopting the Lagrangian viewpoint will enable us to avoid (for general pressure laws) the loss of one derivative due to the hyperbolic part of System (0.1). As a matter of fact, we shall establish stability estimates for the (Lagrangian) velocity field directly in the energy space, and the presence of variable coefficients owing to the initial density variations, either in front of the time derivative or in the elliptic part of the evolution operator, will be harmless.

Let us finally state an important application of Theorem 1.2.

Corollary 1.1. *Assume that ρ_0 is given by (0.6) for some bounded domain D of class $W^{1,p}$ with $d < p < \infty$.*

Then, there exists $\varepsilon > 0$ such that, if $|c_1 - c_2| \leq \varepsilon$, then for any initial velocity field u_0 satisfying the conditions of Theorem 1.1, there exists $T > 0$ such that System (0.1)-(0.4) admits a unique solution (ρ, u). Furthermore, the density at time t has a jump discontinuity along the interface of the domain D_t transported by the flow of u, and ∂D_t keeps its $W^{1,p}$ regularity.

We end this section with a list of possible extensions/improvements of our paper.

(1) All our results may be readily adapted to the case of periodic boundary conditions; indeed, our techniques rely on Fourier analysis and thus hold true for functions defined on the torus.

(2) We expect a similar existence statement if the fluid domain is a bounded open set Ω with (say) $\mathcal{C}^2$ boundary, and the system is supplemented with homogeneous Dirichlet boundary conditions for the velocity. Indeed, in that setting, the Besov spaces may be defined by real interpolation from the domain of the Lamé (or, equivalently the heat) operator and the maximal regularity estimates remain the same. The reader may refer to [10] for an example of solving (0.1) in that setting, in the case of density in $W^{1,p}$ for some $p > d$.

Concerning the propagation of tangential regularity, the situation where the reference family of vector-fields does not degenerate at the boundary should be tractable with few changes (this is a matter of adapting the work by N. Depauw in [18] to our system). This means that one can consider initial densities like (0.6) provided the boundary of D does not meet that of Ω.

(3) To keep the paper a reasonable size, we refrained from considering the global existence issue for rough densities. We plan to address that interesting question in the near future.

2. Tools

Here we introduce the main tools for our analysis. First of all, we recall basic facts about Littlewood-Paley theory and Besov spaces. The next subsection is devoted to maximal regularity results. Finally, in Subsection 2.3 we present key inequalities involving striated regularity.

2.1. **Littlewood-Paley theory and Besov spaces.** We here briefly present the Littlewood-Paley theory, as it will come into play for proving our main result. We refer e.g. to Chapter 2 of [1] for more details. For simplicity of exposition, we focus on the $\mathbb{R}^d$ case; however, the whole construction can be adapted to the d-dimensional torus $\mathbb{T}^d$.

First of all, let us introduce the so-called "Littlewood-Paley decomposition". It is based on a non-homogeneous dyadic partition of unity with respect to the Fourier variable: fix a smooth radial function χ supported in the ball $B(0,2)$, equal to 1 in a neighborhood of $B(0,1)$ and such that $r \mapsto \chi(r\,e)$ is nonincreasing over $\mathbb{R}_+$ for all unitary vectors $e \in \mathbb{R}^d$. Set $\varphi(\xi) = \chi(\xi) - \chi(2\xi)$ and $\varphi_j(\xi) := \varphi(2^{-j}\xi)$ for all $j \geq 0$. The non-homogeneous dyadic blocks $(\Delta_j)_{j \in \mathbb{Z}}$ are defined by[3]

$$\Delta_j := 0 \quad \text{if } j \leq -2, \qquad \Delta_{-1} := \chi(D) \qquad \text{and} \qquad \Delta_j := \varphi(2^{-j}D) \quad \text{if } j \geq 0.$$

We also introduce the following low frequency cut-off operator:

$$(2.1) \qquad S_j := \chi(2^{-j}D) = \sum_{k \leq j-1} \Delta_k \quad \text{for} \quad j \geq 0, \quad \text{and} \quad S_j = 0 \text{ for } j < 0.$$

It is well known that for any $u \in \mathcal{S}'$, one has the equality

$$u = \sum_{j \geq -1} \Delta_j u \qquad \text{in} \quad \mathcal{S}'.$$

Sometimes, we shall alternately use the following spectral cut-offs $\dot{\Delta}_j$ and $\dot{S}_j$ that are defined by

$$\dot{\Delta}_j := \varphi(2^{-j}D) \quad \text{and} \quad \dot{S}_j = \chi(2^{-j}D) \quad \text{for all } j \in \mathbb{Z}.$$

Note that we have

$$(2.2) \qquad u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u$$

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[3]Throughout we agree that $f(D)$ stands for the pseudo-differential operator $u \mapsto \mathcal{F}^{-1}(f\,\mathcal{F}u)$.

up to polynomials only, which makes decomposition (2.2) unwieldy. A way to have equality in (2.2) in the sense of tempered distributions is to restrict oneself to elements u of the set $\mathcal{S}'_h$ of tempered distributions such that

$$\lim_{j \to -\infty} \big\| \dot{S}_j u \big\|_{L^\infty} = 0 \, .$$

It is now time to introduce Besov spaces.

Definition 2.1. *Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$.*

(i) The non-homogeneous Besov space *$B^s_{p,r}$ is the set of tempered distributions u for which*

$$\|u\|_{B^s_{p,r}} := \left\| \big(2^{js} \, \|\Delta_j u\|_{L^p} \big)_{j \geq -1} \right\|_{\ell^r} < \infty \, .$$

(ii) The homogeneous Besov space *$\dot{B}^s_{p,r}$ is the subset of distributions u in $\mathcal{S}'_h$ such that*

$$\|u\|_{\dot{B}^s_{p,r}} := \left\| \big(2^{js} \, \|\dot{\Delta}_j u\|_{L^p} \big)_{j \in \mathbb{Z}} \right\|_{\ell^r} < \infty \, .$$

It is well known that $B^s_{2,2}$ coincides with H^s (with equivalent norms) and that nonhomogeneous (resp. homogeneous) Besov spaces are interpolation spaces between Sobolev spaces $W^{k,p}$ (resp. $\dot{W}^{k,p}$). Furthermore, for all $p \in]1, \infty[$, one has the following continuous embeddings (see the proof in [1, Chap. 2]):

$$\dot{B}^0_{p,\min(p,2)} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\max(p,2)} \quad \text{and} \quad B^0_{p,\min(p,2)} \hookrightarrow L^p \hookrightarrow B^0_{p,\max(p,2)} .$$

We shall also often use the embeddings that are stated in the following proposition.

Proposition 2.1. *Let $1 \leq p_1 \leq p_2 \leq \infty$. The space $B^{s_1}_{p_1,r_1}(\mathbb{R}^d)$ is continuously embedded in the space $B^{s_2}_{p_2,r_2}(\mathbb{R}^d)$, if*

$$s_2 < s_1 - d \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \qquad or \qquad s_2 = s_1 - d \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \quad and \quad r_1 \leq r_2 \, .$$

The space $\dot{B}^{s_1}_{p_1,r_1}(\mathbb{R}^d)$ is continuously embedded in the space $\dot{B}^{s_2}_{p_2,r_2}(\mathbb{R}^d)$ if

$$s_2 = s_1 - d \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \quad and \quad r_1 \leq r_2 \, .$$

Finally, we shall need the following continuity result.

Lemma 2.1. *There exists a constant C, depending only on d, such that for all $p \in [1, \infty]$, we have*

$$\|\Delta (\mathrm{Id} - \Delta)^{-1} f\|_{L^p} \leq C \|f\|_{L^p} .$$

Proof. It suffices to notice that

$$\Delta (\mathrm{Id} - \Delta)^{-1} f = (\mathrm{Id} - \Delta)^{-1} f - f$$

and that $(\mathrm{Id} - \Delta)^{-1}$ maps L^p to $B^2_{p,\infty}$ (see Proposition 2.78 of [1]) hence to L^p, with a constant independent of p. $\square$

Corollary 2.1. *Let u solve the elliptic equation $(\mathrm{Id} - \Delta) u = f$ in $\mathbb{R}^d$, with $f \in L^p$ for some $p \in [1, \infty]$. Then $u \in W^{1,p}(\mathbb{R}^d)$, with $\Delta u \in L^p$, and one has the estimate*

$$\|u\|_{W^{1,p}} + \|\Delta u\|_{L^p} \leq C \|f\|_{L^p} ,$$

for some positive constant C depending just on d.

If $1 < p < \infty$, then $u \in W^{2,p}$ and we have

$$\|u\|_{W^{2,p}} \leq C \|f\|_{L^p}.$$

Proof. From Lemma 2.1, we gather that $\Delta u \in L^p$, hence $u = \Delta u + f$ belongs to L^p, too. This relation in particular implies the control $\|u\|_{L^p} \leq C\|f\|_{L^p}$. At this point, the control of the gradient of u in L^p follows e.g. from Gagliardo-Nirenberg inequalities (or a decomposition into low and high frequencies). Finally, in the case $1 < p < \infty$, the fact that $\Delta u \in L^p$ implies that $\nabla^2 u \in L^p$ by Calderón-Zygmund theory. $\qquad\square$

2.2. Maximal regularity and propagation of L^p norms.

In this subsection we recall some results about maximal regularity for the heat equation, then extend them to the elliptic operator $\mathcal{L}$ defined in (0.3). Those results will be essentially the key to Theorem 1.1, namely existence of solutions in an L^p setting.

2.2.1. *The case of the heat kernel.* Here we focus on maximal regularity results for the heat semi-group, as they will entail similar ones for the Lamé semi-group generated by $-\mathcal{L}$ (see Paragraph 2.2.2 below). We first look at the propagation of regularity for the initial datum. Our starting point is the proposition below, that corresponds to Theorem 2.34 of [1].

Proposition 2.2. *Let $s > 0$ and $(p, r) \in [1, \infty]^2$. A constant C exists such that*

$$C^{-1} \|z\|_{\dot{B}_{p,r}^{-s}} \leq \left\| \|t^{s/2} e^{t\Delta} z\|_{L^p} \right\|_{L^r(\mathbb{R}_+; \frac{dt}{t})} \leq C \|z\|_{\dot{B}_{p,r}^{-s}}.$$

Thus we deduce that, for all $r \in [1, \infty[$, having z in $\dot{B}_{p,r}^{-2/r}(\mathbb{R}^d)$ is equivalent to the condition $e^{t\Delta} z \in L^r(\mathbb{R}_+; L^p(\mathbb{R}^d))$. In particular, taking $z = \Delta u_0$ and assuming that u_0 has *critical regularity* $\dot{B}_{p,r}^{-1+d/p}(\mathbb{R}^d)$ for some $p \in (1, \infty)$ and r fulfilling

$$2 - \frac{2}{r} = \frac{d}{p} - 1 \quad \text{with} \ \ 1 < r < \infty, \tag{2.3}$$

Proposition 2.2 combined with the classical L^p theory for the Laplace operator imply that $\nabla^2 e^{t\Delta} u_0$ is in $L^r(\mathbb{R}_+; L^p(\mathbb{R}^d))$.

Note however that (2.3) gives the constraint $d/3 < p \leq d$, which is too restrictive for our scopes: we will need $p > d$ in order to guarantee that ∇u is in $L^1([0, T]; L^\infty(\mathbb{R}^d))$ for some $T > 0$ (see Section 3 for more details). This fact will preclude us from working in the critical regularity setting.

Before going on, let us introduce more notations: throughout this section, we will use an index j to designate the regularity of Lebesgue exponents $(p_j, r_j) \in [1, \infty]^2$ pertaining to the term $\nabla^j h$.

According to Proposition 2.2, if u_0 is in $\dot{B}_{p_2,r_2}^{s_2}$ with $s_2 = 2 - 2/r_2$ and $1 < p_2, r_2 < \infty$, then

$$\nabla^2 e^{t\Delta} u_0 \in L^{r_2}(\mathbb{R}_+; L^{p_2}(\mathbb{R}^d)). \tag{2.4}$$

Furthermore, we have

$$e^{t\Delta} u_0 \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{p_2,r_2}^{s_2}) \tag{2.5}$$

since

$$\|e^{t\Delta} u_0\|_{\dot{B}_{p_2,r_2}^{2-\frac{2}{r_2}}} \sim \|\nabla^2 e^{t\Delta} u_0\|_{\dot{B}_{p_2,r_2}^{-\frac{2}{r_2}}} \sim \left(\int_{\mathbb{R}_+} \|e^{\tau\Delta}(\Delta e^{t\Delta} u_0)\|_{L^{p_2}}^{r_2} \, d\tau \right)^{\frac{1}{r_2}}$$

and, using the fact that the heat semi-group is contractive on L^{p_2},

$$\int_{\mathbb{R}_+} \|e^{\tau\Delta}(\Delta e^{t\Delta} u_0)\|_{L^{p_2}}^{r_2} \, d\tau \leq \int_{\mathbb{R}_+} \|e^{\tau\Delta} \Delta u_0\|_{L^{p_2}}^{r_2} \, d\tau \leq C \|\Delta u_0\|_{\dot{B}_{p_2,r_2}^{2-\frac{2}{r_2}}}^{r_2}.$$

Time continuity in (2.5) just follows from the fact that $\mathcal{S}$ is densely embedded in L^{p_2}.

Next, by the embedding properties of Proposition 2.1, we have, if $p_1 \geq p_2$ and $r_1 \geq r_2$,

$$\nabla u_0 \in \dot{B}_{p_1,r_1}^{s_1} \qquad \text{with} \qquad s_1 = 1 - \frac{2}{r_2} - d\left(\frac{1}{p_2} - \frac{1}{p_1}\right).$$

In order to be in position of applying Proposition 2.2 so as to get that $\nabla e^{t\Delta} u_0 \in L^{r_1}\left(\mathbb{R}_+; L^{p_1}(\mathbb{R}^d)\right)$, we need to have in addition $s_1 < 0$, that is to say

$$(2.6) \qquad \frac{2}{r_2} + \frac{d}{p_2} - \frac{d}{p_1} > 1.$$

Then, defining r_1 by

$$(2.7) \qquad \frac{2}{r_2} + \frac{d}{p_2} = 1 + \frac{2}{r_1} + \frac{d}{p_1},$$

we get

$$(2.8) \qquad \nabla e^{t\Delta} u_0 \in L^{r_1}\left(\mathbb{R}_+; L^{p_1}(\mathbb{R}^d)\right).$$

Finally, let us consider $e^{t\Delta} u_0$. By critical embedding, we have, if $p_0 \geq p_2$ and $r_0 \geq r_2$,

$$u_0 \in \dot{B}_{p_0,r_0}^{s_0} \qquad \text{with} \qquad s_0 = 2 - \frac{2}{r_2} - d\left(\frac{1}{p_2} - \frac{1}{p_0}\right).$$

If we want to resort again to Proposition 2.2, we need to have in addition that

$$(2.9) \qquad s_0 = 2 - \frac{2}{r_2} - \frac{d}{p_1} < 0.$$

Under that condition, choosing r_0 so that $s_0 = -2/r_0$, that is to say such that

$$(2.10) \qquad \frac{2}{r_2} + \frac{d}{p_2} = 2 + \frac{2}{r_0} + \frac{d}{p_0},$$

we end up with

$$(2.11) \qquad e^{t\Delta} u_0 \in L^{r_0}\left(\mathbb{R}_+; L^{p_0}(\mathbb{R}^d)\right).$$

Let us next consider the propagation of regularity for the forcing term in the heat equation. We start by presenting the standard maximal $L^r(L^p)$ regularity for the heat semi-group (see the proof in [29, Lem. 7.3] for instance).

Lemma 2.2. *Let us define the operator $\mathcal{A}_2$ by the formula*

$$\mathcal{A}_2 : \quad f \longmapsto \int_0^t \nabla^2 e^{(t-s)\Delta} f(s, \cdot) \, ds.$$

Then $\mathcal{A}_2$ is bounded from $L^{r_2}\left(\,]0, T[\,; L^{p_2}(\mathbb{R}^d)\right)$ to $L^{r_2}\left(\,]0, T[\,; L^{p_2}(\mathbb{R}^d)\right)$ for every $T \in \,]0, \infty]$ and $1 < p_2, r_2 < \infty$. Moreover, there holds:

$$\|\mathcal{A}_2 f\|_{L_T^{r_2}(L^{p_2})} \leq C \|f\|_{L_T^{r_2}(L^{p_2})}.$$

As for the propagation of regularity for the first derivatives, we have the following statement.

Lemma 2.3. *Assume that the Lebesgue exponents p_1 and p_2 fulfill $0 \leq 1/p_2 - 1/p_1 < 1/d$, and that $1 < r_2 < r_1 < \infty$ are interrelated by the relation (2.7). Let us define the operator $\mathcal{A}_1$ by*

$$\mathcal{A}_1 : \quad f \mapsto \int_0^t \nabla e^{(t-s)\Delta} f(s, \cdot) \, ds.$$

Then $\mathcal{A}_1$ is bounded from $L^{r_2}\left(\,]0, T[\,; L^{p_2}(\mathbb{R}^d)\right)$ to $L^{r_1}\left(\,]0, T[\,; L^{p_1}(\mathbb{R}^d)\right)$ for every $T \in \,]0, \infty]$, and there holds

$$\|\mathcal{A}_1 f\|_{L_T^{r_1}(L^{p_1})} \leq C \|f\|_{L_T^{r_2}(L^{p_2})},$$

for a suitable constant $C > 0$ depending only on the space dimension $d \geq 1$ and on p_1, r_1, p_2, r_2.

Proof. We use the fact that for all $0 \leq s \leq t \leq T$,

$$\nabla e^{(t-s)\Delta} f(s, x) = \frac{\sqrt{\pi}}{(4\pi(t-s))^{\frac{d+1}{2}}} \int_{\mathbb{R}^d} \frac{(x-y)}{2\sqrt{(t-s)}} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) f(s, y)\, dy$$

$$\stackrel{\text{def}}{=} \frac{\sqrt{\pi}}{(4\pi(t-s))^{\frac{d+1}{2}}} K_1\left(\frac{\cdot}{\sqrt{4(t-s)}}\right) *_x f(s, \cdot).$$

Applying Young's inequality in the space variables yields,

$$\left\|\nabla e^{(t-s)\Delta}\left(\mathbb{1}_{[0,T]} f\right)(s, \cdot)\right\|_{L^{p_1}} \leq C((t-s))^{-(d+1)/2} \left\|K_1\left(\frac{\cdot}{\sqrt{4\pi(t-s)}}\right)\right\|_{L^{m_1}} \|\mathbb{1}_{[0,T]} f\|_{L^{p_2}}$$

$$\leq C((t-s))^{-\beta} \|\mathbb{1}_{[0,T]} f\|_{L^{p_2}},$$

where we have defined m_1 and β by

$$\frac{1}{p_1} + 1 = \frac{1}{m_1} + \frac{1}{p_2} \quad \text{and} \quad \beta = \frac{d+1}{2} - \frac{d}{2m_1} = \frac{1}{2} + \frac{d}{2p_2} - \frac{d}{2p_1}.$$

Note that the conditions in the lemma ensure that $\beta \in [1/2, 1[$. At this point, we apply Hardy-Littlewood-Sobolev inequality (see e.g. Theorem 1.7 of [1]) with respect to time: since r_1 and r_2 verify $1/r_1 + 1 = 1/r_2 + \beta$ by hypothesis (2.7), we immediately get the claimed inequality. The lemma is thus proved. $\qquad\square$

We now state integrability properties concerning f itself, without taking any derivative.

Lemma 2.4. *Assume that the Lebesgue exponents p_0 and p_2 fulfill $0 \leq 1/p_2 - 1/p_0 < 2/d$, and that $1 < r_2 < r_0 < \infty$ are interrelated by the relation (2.10). Define the operator $\mathcal{A}_0$ by the formula*

$$\mathcal{A}_0 : \quad f \longmapsto \int_0^t e^{(t-s)\Delta} f(s, \cdot)\, ds.$$

Let $s_2 := 2 - 2/r_2$. Then $\mathcal{A}_0$ is bounded from $L^{r_2}\left(]0, T[\, ; L^{p_2}\right)$ to $L^{r_0}\left(]0, T[\, ; L^{p_0}\right) \times \mathcal{C}([0, T]; \dot{B}^{s_2}_{p_2, r_2})$ for every $T \in]0, \infty]$, and there holds

$$\|\mathcal{A}_0 f\|_{L^\infty_T(\dot{B}^{s_2}_{p_2, r_2})} + \|\mathcal{A}_0 f\|_{L^{r_0}_T(L^{p_0})} \leq C \|f\|_{L^{r_2}_T(L^{p_2})},$$

for a suitable constant $C > 0$ depending on the space dimension $d \geq 1$ and on p_0, r_0, p_2, r_2.

Proof. The proof of the continuity in $L^{r_0}(]0, T[\, ; L^{p_0})$ goes as in Lemma 2.3: we start by writing

$$e^{(t-s)\Delta} f(s, x) = \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) f(s, y)\, dy$$

$$\stackrel{\text{def}}{=} \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} K_0\left(\frac{\cdot}{\sqrt{4(t-s)}}\right) *_x f(s, \cdot).$$

Then, we apply Young's inequality in the space variables and get, for every $s > 0$ fixed,

$$\left\|e^{(t-s)\Delta}\left(\mathbb{1}_{[0,T]} f\right)(s, \cdot)\right\|_{L^{p_0}} \leq C((t-s))^{-d/2} \left\|K_0\left(\frac{\cdot}{\sqrt{4\pi(t-s)}}\right)\right\|_{L^{m_0}} \|\mathbb{1}_{[0,T]} f\|_{L^{p_2}}$$

$$\leq C((t-s))^{-\gamma} \|\mathbb{1}_{[0,T]} f\|_{L^{p_2}},$$

where, exactly as before, we have defined m_0 and γ by the relations

$$1 + \frac{1}{p_0} = \frac{1}{m_0} + \frac{1}{p_2} \quad \text{and} \quad \gamma = \frac{d}{2p_2} - \frac{d}{2p_0}.$$

Our assumptions ensure that $\gamma \in]0, 1[$ and one may apply Hardy-Littlewood-Sobolev inequality with respect to time. Since r_0 and r_2 verify $1/r_0 + 1 = 1/r_2 + \gamma$ by hypothesis (2.10), we immediately get that $\mathcal{A}_0$ is bounded from $L^{r_2}_T(L^{p_2})$ to $L^{r_0}_T(L^{p_0})$.

The second part of the statement is classical. Arguing by density, it suffices to establish that

$$\|\mathcal{A}_0 f(T)\|_{\dot{B}^{s_2}_{p_2,r_2}} \leq C \|f\|_{L^{r_2}_T(L^{p_2})}.$$

To this end, we write, using Proposition 2.2 and an obvious change of variable, that

$$\|\mathcal{A}_0 f(T)\|_{\dot{B}^{s_2}_{p_2,r_2}} \sim \left(\int_0^\infty \left\| \Delta e^{t\Delta} \mathcal{A}_0 f(T) \right\|_{L^{p_2}}^{r_2} dt \right)^{\frac{1}{r_2}}$$

$$\sim \left(\int_0^\infty \left\| \int_0^T \Delta e^{(t+T-\tau)\Delta} f(\tau) d\tau \right\|_{L^{p_2}}^{r_2} dt \right)^{\frac{1}{r_2}}$$

$$\leq C \left(\int_T^\infty \left\| \int_0^{t'} \Delta e^{(t'-\tau)\Delta} (f \mathbb{1}_{[0,T]})(\tau) d\tau \right\|_{L^{p_2}}^{r_2} dt \right)^{\frac{1}{r_2}}.$$

Then using Lemma 2.2 to bound the right-hand side yields the claimed inequality. $\qquad\square$

2.2.2. *Maximal regularity results for operator $\mathcal{L}$.* Here we want to extend the results of the previous paragraph to the elliptic operator $\mathcal{L} = -\mu\Delta - \lambda\nabla\operatorname{div}$, under Condition (0.2): for suitable initial datum h_0 and external force f, let us consider the equation

$$(2.12) \qquad \begin{cases} \partial_t h + \mathcal{L}h = f \\ h_{|t=0} = h_0. \end{cases}$$

The following statement will be a key ingredient in the proof of our existence result.

Proposition 2.3. *Let $\big((p_j, r_j)\big)_{j=0,1,2}$ satisfy $1 < p_2, r_2 < \infty$, $r_2 < r_0$, $r_2 < r_1$, $p_0 \geq p_2$, $p_1 \geq p_2$, and the relations (2.7) and (2.10). Let h_0 be in $\dot{B}^{s_2}_{p_2,r_2}$ with $s_2 := 2 - 2/r_2$, and let f be in $L^{r_2}_{\mathrm{loc}}\big(\mathbb{R}_+; L^{p_2}(\mathbb{R}^d)\big)$. Let $(\mu, \lambda) \in \mathbb{R}^2$ satisfy condition (0.2).*

Then, for all $T > 0$, System (2.12) has a unique solution h in $\mathcal{C}([0,T]; \dot{B}^{s_2}_{p_2,r_2}) \cap L^{r_0}([0,T]; L^{p_0})$, with $\nabla h \in L^{r_1}([0,T]; L^{p_1})$ and $\partial_t h, \nabla^2 h \in L^{r_2}([0,T]; L^{p_2})$. Moreover, there exists a constant $C_0 > 0$ (depending just on μ, λ, d, p_0, p_1, p_2 and r_2) such that the following estimate holds true:

$$(2.13) \quad \|h\|_{L^\infty_T(\dot{B}^{s_2}_{p_2,r_2})} + \|h\|_{L^{r_0}_T(L^{p_0})} + \|\nabla h\|_{L^{r_1}_T(L^{p_1})} + \left\|(\partial_t h, \nabla^2 h)\right\|_{L^{r_2}_T(L^{p_2})}$$

$$\leq C_0 \left(\|h_0\|_{\dot{B}^{s_2}_{p_2,r_2}} + \|f\|_{L^{r_2}_T(L^{p_2})} \right).$$

Proof. Let us write down the Helmholtz decomposition of the vector field h: denoting by $\mathbb{P}$ the Leray projector onto the space of divergence-free vector fields and by $\mathbb{Q}$ the projector onto the space of irrotational vector fields, we have $h = \mathbb{P}h + \mathbb{Q}h$. Recall that we have in Fourier variables

$$\mathcal{F}(\mathbb{Q}h)(\xi) = \frac{1}{|\xi|^2} \big(\xi \cdot \widehat{h}(\xi)\big) \xi.$$

Hence $\mathbb{P}$ and $\mathbb{Q}$ are linear combinations of composition of Riesz transforms and thus act continuously on L^p for all $1 < p < \infty$.

Now, applying those two operators to System (2.12), we discover that $\mathbb{P}h$ and $\mathbb{Q}h$ satisfy the heat equations

$$\big(\partial_t - \mu\Delta\big)\mathbb{P}h = \mathbb{P}f \quad \text{and} \quad \big(\partial_t - \nu\Delta\big)\mathbb{Q}h = \mathbb{Q}f, \quad \text{with } \nu := \mu + \lambda.$$

with initial data $\mathbb{P}h_0$ and $\mathbb{Q}h_0$, respectively. Therefore, denoting by $\mathbb{A}$ the operator $\mathbb{P}$ or $\mathbb{Q}$, and by α either μ or ν, Duhamel's formula gives us

$$(2.14) \qquad \mathbb{A}h(t) = e^{\alpha t\Delta}\mathbb{A}h_0 + \int_0^t e^{\alpha(t-s)\Delta}\mathbb{A}f(s) ds.$$

For the term containing the initial datum, we apply (2.4), (2.5), (2.8) and (2.11), while for the source term we apply Lemmas 2.2, 2.3 and 2.4. Therefore, we conclude by continuity of operators $\mathbb{P}$ and $\mathbb{Q}$ over L^p, for $1 < p < \infty$, and over $\dot{B}^s_{q,r}$, for all $s \in \mathbb{R}$ and all $(q,r) \in]1, \infty[^2$. □

2.3. Tangential regularity.
We establish here fundamental stationary estimates about propagation of *striated* (or *tangential*) *regularity* in the L^p setting.

Before starting the presentation, let us recall that the classical result on zero order pseudo-differential operators ensures that, if $g \in L^\infty$, then $\Delta^{-1}\partial_i\partial_j g \in BMO$. The main result of this paragraph states that, if g has suitable tangential regularity properties (similar to those exhibited by J.-Y. Chemin in [4] for the vortex patches problem), then $\Delta^{-1}\partial_i\partial_j g$ is in L^∞. Our starting point is an adaptation of Lemma 5.1 in [35] enabling us to handle operator $\nabla(\eta\,\mathrm{Id} - \Delta)^{-1}$, valid in any dimension and for non-zero divergence vector fields.

Before stating that lemma, the proof of which is postponed in Appendix, let us introduce, for $m \in \mathbb{R}$, the class S^m of symbols of order m, that is the space of $\mathcal{C}^\infty(\mathbb{R}^d)$ functions σ such that for all $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ satisfying for all $\xi \in \mathbb{R}^d$,

$$|\partial^\alpha \sigma(\xi)| \leq C_\alpha \, (1 + |\xi|)^{m - |\alpha|}.$$

Lemma 2.5. *Let $1 < p < \infty$. Consider a vector field X in $\mathbb{L}^{\infty,p}$ and a function $g \in L^\infty$ such that $\partial_X g \in L^p$. Let σ be a smooth Fourier multiplier in the class S^{-1}. Then, for any fixed $0 < s < 1$, the following estimate holds true:*

$$\|\partial_X \sigma(D)g\|_{B^s_{p,\infty}} \leq C \left(\|\partial_X g\|_{L^p} + \|\nabla X\|_{L^p} \|g\|_{L^\infty} \right).$$

Since the nonhomogeneous Besov space $B^s_{p,\infty}$ is embedded in L^∞ whenever $s > d/p$, that lemma implies the following fundamental result.

Corollary 2.2. *Assume that $d < p < \infty$, and consider a vector field X in $\mathbb{L}^{\infty,p}$ and a function L^∞ such that $\partial_X g \in L^p$. Then, for any Fourier multiplier σ in the class S^{-1}, there exists a constant $C > 0$ such that*

$$\|\partial_X \sigma(D)g\|_{L^\infty} \leq C \left(\|\partial_X g\|_{L^p} + \|\nabla X\|_{L^p} \|g\|_{L^\infty} \right).$$

From the previous results, we immediately obtain the following fundamental stationary estimate.

Proposition 2.4. *Fix $p \in]d, \infty[$ and an integer $m \geq d - 1$, and take a non-degenerate family $\mathcal{X} = (X_\lambda)_{1 \leq \lambda \leq m}$ of vector-fields belonging to $\mathbb{L}^{\infty,p}$. Let $g \in L^\infty(\mathbb{R}^d)$ be such that $g \in \mathbb{L}^p_\mathcal{X}$.*

Then for all $\eta > 0$, one has the property $(\eta\,\mathrm{Id} - \Delta)^{-1}\nabla^2 g \in L^\infty(\mathbb{R}^d)$. Moreover, there exists a positive constant C such that the following estimates hold true:

$$\left\|\nabla^2(\eta\,\mathrm{Id} - \Delta)^{-1}g\right\|_{L^\infty} \leq C\left(\left(1 + \frac{\||\mathcal{X}\||_{L^\infty}^{4d-5}\||\nabla\mathcal{X}\||_{L^p}}{\left(I(\mathcal{X})\right)^{4d-4}}\right)\|g\|_{L^\infty} + \frac{\||\mathcal{X}\||_{L^\infty}^{4d-5}}{\left(I(\mathcal{X})\right)^{4d-4}}\||\partial_\mathcal{X} g\||_{L^p}\right).$$

Proof. Fix some $\Lambda := (\lambda_1, \cdots, \lambda_{d-1}) \in \Lambda^m_{d-1}$, and consider the set U_Λ of those $x \in \mathbb{R}^d$ such that one has $\left(\overset{d-1}{\wedge} X_\Lambda\right)(x) \geq \left(I(\mathcal{X})\right)^{d-1}$. By Lemma 3.2 of [8], there exists a family of functions $b^{k\ell}_{ij}$, where $(i,j,k) \in \{1, \cdots, d\}^3$ and $\ell \in \{1 \ldots d-1\}$, which are homogeneous of degree $4d - 5$ with respect to the coefficients of $\left(X_{\lambda_1} \ldots X_{\lambda_{d-1}}\right)$ and such that the following identity holds true on U_Λ, for all $\xi \in \mathbb{R}^d$:

$$\xi_i \xi_j = \frac{\left(\overset{d-1}{\wedge} X_\Lambda(x)\right)^i \left(\overset{d-1}{\wedge} X_\Lambda(x)\right)^j}{\left|\overset{d-1}{\wedge} X_\Lambda(x)\right|^2} |\xi|^2 + \frac{1}{\left|\overset{d-1}{\wedge} X_\Lambda(x)\right|^4} \sum_{k,\ell} b^{k\ell}_{ij} \xi_k \left(X_{\lambda_\ell}(x) \cdot \xi\right).$$

Then, we multiply both sides by $(\eta + |\xi|^2)^{-1}\, \widehat{g}(\xi)$ and take the inverse Fourier transform at x:

$$(\eta \operatorname{Id} - \Delta)^{-1} \partial_i \partial_j g = \frac{\left(\overset{d-1}{\wedge} X_\Lambda \right)^i \left(\overset{d-1}{\wedge} X_\Lambda \right)^j}{\left| \overset{d-1}{\wedge} X_\Lambda \right|^2} \Delta (\eta \operatorname{Id} - \Delta)^{-1} g$$

$$+ \frac{1}{\left| \overset{d-1}{\wedge} X_\Lambda \right|^4} \sum_{k,\ell} b_{ij}^{k\ell}\, \partial_{X_{\lambda_\ell}} \left(\partial_k (\eta \operatorname{Id} - \Delta)^{-1} g \right) \cdot$$

Hence, thanks also to Lemma 2.1, for all $\Lambda \in \Lambda^{d-1}$, we deduce the following bound on the set U_Λ:

$$(2.15) \quad \left\| (\eta \operatorname{Id} - \Delta)^{-1} \partial_i \partial_j\, g \right\|_{L^\infty(U_\Lambda)} \leq \|g\|_{L^\infty}$$

$$+ \frac{C}{\left(I(\mathcal{X}) \right)^{4d-4}} \sum_\ell \|X_\Lambda\|_{L^\infty}^{4d-5} \left\| \partial_{X_{\lambda_\ell}} \left(\nabla (\eta \operatorname{Id} - \Delta)^{-1} g \right) \right\|_{L^\infty}.$$

In order to bound the last term in the right-hand side, we apply Corollary 2.2: we get, for some constant $C > 0$ also depending on d and on p, the estimate

$$\left\| \partial_{X_{\lambda_\ell}} \left(\nabla (\eta \operatorname{Id} - \Delta)^{-1} g \right) \right\|_{L^\infty} \leq C \left(\left\| \partial_{X_{\lambda_\ell}} g \right\|_{L^p} + \|g\|_{L^\infty} \|\nabla X_{\lambda_\ell}\|_{L^p} \right).$$

Inserting this bound into (2.15) immediately gives us the result. $\square$

We conclude this part by presenting a new estimate concerning tangential regularity. This statement that will be proved in the appendix, turns out to be of tremendous importance to exhibit the Lipschitz regularity of the velocity field u, see Subsection 4.1 below.

Proposition 2.5. *Let the hypotheses of Proposition 2.4 be in force. Then there exists a constant $C > 0$ such that, for all $\eta > 0$, the following estimate holds true:*

$$\|\partial_{\mathcal{X}} \nabla^2 (\eta \operatorname{Id} - \Delta)^{-1} g\|_{L^p} \leq C \left(\||\nabla \mathcal{X}\||_{L^p} \left(1 + \frac{\||\mathcal{X}\||_{L^\infty}^{4d-5} \||\nabla \mathcal{X}\||_{L^p}}{\left(I(\mathcal{X}) \right)^{4d-4}} \right) \|g\|_{L^\infty} \right.$$

$$\left. + \left(1 + \frac{\||\mathcal{X}\||_{L^\infty}^{4d-5} \||\nabla \mathcal{X}\||_{L^p}}{\left(I(\mathcal{X}) \right)^{4d-4}} \right) \||\partial_{\mathcal{X}} g\||_{L^p} + \frac{\||\mathcal{X}\||_{L^\infty}^{4d-4}}{\left(I(\mathcal{X}) \right)^{4d-4}} \||\nabla \mathcal{X}\||_{L^p} \|g\|_{L^\infty} \right).$$

3. An existence statement for almost critical data and only bounded density

The goal of the present section is to prove Theorem 1.1. After reformulating the original system (0.1), we establish *a priori* estimates on smooth solutions, then provide the reader with the construction of a family of approximate smooth solutions to our (new) system. As a last step, we show the convergence of the sequence to a true solution.

3.1. Reformulation of the system.

In all that follows, we assume for notational simplicity that $\nu = 1$. This is of course not restrictive, owing to the change of unknowns

$$(3.1) \qquad\qquad (\widetilde{\rho}, \widetilde{u})(t, x) := (\rho, u)(\nu t, \nu x).$$

First of all, we want to reformulate our system in terms of new unknowns, to which maximal regularity results of Subsection 2.2 apply. As already explained at the beginning of the paper, in order to handle the pressure term, it is convenient to introduce the auxiliary vector-field

$$(3.2) \qquad\qquad v := -\nabla (\operatorname{Id} - \Delta)^{-1} P,$$

and the following *modified velocity field*:

$$(3.3) \qquad\qquad w := u - v = u + \nabla (\operatorname{Id} - \Delta)^{-1} P.$$

For future uses, we observe that

$$(3.4) \qquad \operatorname{div} u \,=\, \operatorname{div} w \,-\, \Delta(\operatorname{Id}-\Delta)^{-1}P \,.$$

We want to reformulate System (0.1) in terms of the new unknowns (ϱ, w), keeping in mind that estimates for v may be deduced from those for ϱ, and that combining with information on w enables us to bound the original velocity field u. From the first equation of (1.1) and relation (3.4), we immediately deduce that

$$(3.5) \qquad \partial_t \varrho \,+\, u \cdot \nabla \varrho \,=\, -\rho \operatorname{div} w \,+\, \rho \Delta(\operatorname{Id}-\Delta)^{-1}P \,,$$

with $\rho := 1 + \varrho$ and $u := w - \nabla(\operatorname{Id}-\Delta)^{-1}P$.

Regarding w, we see from the second equation of (1.1) and (3.3) that

$$\rho \, \partial_t w \,-\, \mu \, \Delta w \,-\, \lambda \nabla \operatorname{div} w \,=\, -(\operatorname{Id}-\Delta)^{-1}\nabla P - \rho u \cdot \nabla u \,-\, \rho(\operatorname{Id}-\Delta)^{-1}\nabla \partial_t P.$$

Using once again the mass equation in (0.1), we find

$$\partial_t P \,+\, \operatorname{div}\big(P\,u\big) \,=\, g(\rho) \operatorname{div} u \quad \text{with} \quad g(\rho) := P(\rho) - \rho P'(\rho),$$

so that the equation for w can be recast in

$$(3.6) \qquad \rho \partial_t w \,+\, \mathcal{L}w \,=\, -\rho F \,,$$

with

$$(3.7) \quad F \,=\, \frac{1}{\rho}(\operatorname{Id}-\Delta)^{-1}\nabla P \,+\, w \cdot \nabla w \,+\, w \cdot \nabla v \,+\, v \cdot \nabla w \,+\, v \cdot \nabla v$$
$$+\, (\operatorname{Id}-\Delta)^{-1}\nabla \Big(g(\rho) \operatorname{div} u \,-\, \operatorname{div}\big(P\,u\big)\Big).$$

The existence part of Theorem 1.1 will be a consequence of the following statement.

Proposition 3.1. *Let $\varrho_0 \in L^p \cap L^\infty(\mathbb{R}^d)$ and $w_0 \in \dot{B}_{p,r}^{2-2/r}$, with*

$$(3.8) \qquad d < p < \infty \qquad and \qquad 1 < r < \frac{2p}{2p-d}.$$

Define r_0 and r_1 by

$$(3.9) \qquad \frac{1}{r_0} := \frac{1}{r} + \frac{d}{2p} - 1 \quad and \quad \frac{1}{r_1} := \frac{1}{r} - \frac{1}{2}.$$

There exists some small enough constant $\varepsilon > 0$ depending only on d, p and r, such that, if in addition ϱ_0 fulfills (1.5), then there exist a time $T > 0$ and a weak solution (ϱ, w) to system (3.5)-(3.6) on $[0, T[\,\times\mathbb{R}^d$ such that $\varrho \in \mathcal{C}([0, T]; L^p)$ and

$$(3.10) \qquad \|\varrho\|_{L_T^\infty(L^\infty)} \,\leq\, 4\varepsilon \,,$$

and $w \in \mathcal{C}\big([0, T]; \dot{B}_{p,r}^{2-2/r}\big) \cap L^{r_0}\big([0, T]; L^\infty(\mathbb{R}^d)\big)$ with $\nabla w \in L^{r_1}\big([0, T]; L^p(\mathbb{R}^d)\big)$ and $\partial_t w$, $\nabla^2 w \in L^r\big([0, T]; L^p(\mathbb{R}^d)\big)$.
If in addition to the above hypotheses, we have $\inf P' > 0$ on $[1 - 4\varepsilon, 1 + 4\varepsilon]$ and $\varrho_0, u_0 \in L^2(\mathbb{R}^d)$, where $u_0 := w_0 + v_0$ and v_0 is defined as in the statement of Theorem 1.1, then $u := w - \nabla(\operatorname{Id}-\Delta)^{-1}P$ fulfills $u \in L_T^\infty(L^2) \cap L_T^2(H^1)$ and the energy equality (1.6) holds true.

3.2. **A priori bounds for smooth solutions.** We start by establishing a priori estimates for smooth solutions to the new system (3.5)-(3.6). Our goal is to "close the estimates" in the space

$$E_T := \left\{ (\varrho, w) \in L_T^\infty(L^p \cap L^\infty) \times \left(\mathcal{C}([0,T]; \dot{B}_{p,r}^{2-2/r}) \cap L_T^{r_0}(L^\infty) \right) \mid \nabla w \in L_T^{r_1}(L^p), \ \nabla^2 w \in L_T^{r_2}(L^p) \right\}$$

for some small enough $T > 0$.

We denote

$$(3.11) \quad \mathcal{N}(T) := \|\varrho\|_{L_T^\infty(L^p \cap L^\infty)} + \|w\|_{L_T^\infty(\dot{B}_{p,r}^{2-2/r}) \cap L_T^{r_0}(L^\infty)} + \|\nabla w\|_{L_T^{r_1}(L^p)} + \left\| (\partial_t w, \nabla^2 w) \right\|_{L_T^r(L^p)}.$$

As it fulfills a transport equation, it is easy to propagate any Lebesgue norm L^q for ϱ once we know that $\operatorname{div} u$ is in $L_T^1(L^\infty)$ and that the right-hand side of (3.5) is in $L_T^1(L^q)$. Given the expected properties on w, this will give us the constraint $q \geq p$. As for equation (3.6), we want to apply the maximal regularity estimates given by Proposition 2.3.

3.2.1. *Bounds for the density.* Throughout, we fix some $\varepsilon > 0$ and constant $C > 0$ so that (recall that $P(1) = 0$)

$$(3.12) \quad |P(z)| \leq C|z - 1| \quad \text{for all } z \in [1 - 4\varepsilon, 1 + 4\varepsilon].$$

As a first step, let us establish estimates for the density term. Let us take some $q \in [p, \infty[$. By multiplying equation (3.5) by $|\varrho|^{q-2}\varrho$ and integrating in space, we easily get

$$\frac{1}{q}\frac{d}{dt}\|\varrho\|_{L^q}^q - \frac{1}{q}\int |\varrho|^q \operatorname{div} u + \int |\varrho|^q \operatorname{div} w + \int \varrho|\varrho|^{q-2} \operatorname{div} w = \int (\varrho + 1)\varrho|\varrho|^{q-2}\Delta(\operatorname{Id} - \Delta)^{-1}P.$$

By (3.4), one can rewrite the previous relation as

$$\frac{1}{q}\frac{d}{dt}\|\varrho\|_{L^q}^q + \left(1 - \frac{1}{q}\right)\int |\varrho|^q \left(\operatorname{div} w - \Delta(\operatorname{Id} - \Delta)^{-1}P\right) = \int \varrho|\varrho|^{q-2}\left(\Delta(\operatorname{Id} - \Delta)^{-1}P - \operatorname{div} w\right),$$

which immediately implies

$$\|\varrho(t)\|_{L^q} \leq \|\varrho_0\|_{L^q} + \left(1 - \frac{1}{q}\right)\int_0^t \|\varrho\|_{L^q}\|\Delta(\operatorname{Id} - \Delta)^{-1}P - \operatorname{div} w\|_{L^\infty}\, d\tau$$
$$+ \int_0^t \|\Delta(\operatorname{Id} - \Delta)^{-1}P - \operatorname{div} w\|_{L^q}\, d\tau.$$

Of course, passing to the limit $q \to \infty$, we see that the same inequality holds true for $q = \infty$. Then, using Lemma 2.1 and the fact that, under (3.12), we have

$$(3.13) \quad \|P\|_{L^r} \leq C\|\varrho\|_{L^r} \quad \text{for all } r \in [1, \infty],$$

an application of Gronwall lemma implies that, for all $q \in [p, \infty]$, there holds

$$(3.14) \quad \|\varrho(t)\|_{L^q} \leq e^{Ct + \int_0^t \|\operatorname{div} w\|_{L^\infty}\, d\tau}\left(\|\varrho_0\|_{L^q} + \int_0^t \|\operatorname{div} w\|_{L^q}\, d\tau\right).$$

Define now the time $T > 0$ as

$$(3.15) \quad T := \sup\left\{ t > 0 \mid Ct + \int_0^t \|\operatorname{div} w\|_{L^\infty}\, d\tau \leq \log 2 \quad \text{and} \quad \int_0^t \|\operatorname{div} w\|_{L^\infty} \leq \varepsilon \right\};$$

then, on $[0, T]$ one has, owing to (3.14),

$$\|\varrho(t)\|_{L^\infty} \leq 2\|\varrho_0\|_{L^\infty} + 2\varepsilon.$$

Hence, if we take the initial density satisfying (1.5), for ε fixed in (3.12) above, then we get (3.10).

3.2.2. *Bounds for w.* Throughout we fix the time $T > 0$ as defined in (3.15) and assume that (3.10) is fulfilled. Then, applying Proposition 2.3 to equation (3.6) with $(p_0, p_1, p_2) = (\infty, p, p)$, $r_2 = r$ and (r_0, r_1) according to (3.9), we get, treating the term $\varrho\, \partial_t w$ as a perturbation,

$$\|w\|_{L^\infty_T(\dot{B}^{2-2/r}_{p,r}) \cap L^{r_0}_T(L^\infty)} + \|\nabla w\|_{L^{r_1}_T(L^p)} + \left\|(\partial_t w,\, \nabla^2 w)\right\|_{L^r_T(L^p)} \le$$

$$\le C_0 \left(\|w_0\|_{\dot{B}^{2-2/r}_{p,r}} + \|\varrho\|_{L^\infty_T(L^\infty)} \|\partial_t w\|_{L^r_T(L^p)} + \|F\|_{L^r_T(L^p)} \right),$$

for a suitable constant $C_0 > 0$. Now, assuming that ε in (1.5) has been fixed so small that

$$(3.16) \qquad\qquad\qquad\qquad 8\, C_0\, \varepsilon \le 1,$$

thanks to (3.10) we gather the estimate

$$(3.17) \quad \|w\|_{L^\infty_T(\dot{B}^{2-2/r}_{p,r}) \cap L^{r_0}_T(L^\infty)} + \|\nabla w\|_{L^{r_1}_T(L^p)} + \left\|(\partial_t w,\, \nabla^2 w)\right\|_{L^{r_2}_T(L^p)}$$

$$\le 2\, C_0 \left(\|w_0\|_{\dot{B}^{2-2/r}_{p,r}} + \|F\|_{L^r_T(L^p)} \right).$$

Our next goal is to bound F, defined by (3.7). First of all, as operator $\nabla(\mathrm{Id} - \Delta)^{-1}$ maps continuously L^q into $W^{1,q}$ for any $1 < q < \infty$, one deduces that

$$(3.18) \qquad\qquad\qquad\qquad \|v\|_{W^{1,q}} \le C\|\varrho\|_{L^q},$$

where v is the vector-field defined in (3.2). Hence, the first term in (3.7) can be bounded, thanks to (3.13), in the following way:

$$(3.19) \qquad\qquad \|(\mathrm{Id} - \Delta)^{-1}\nabla P\|_{L^r_T(L^p)} \le C\, T^{1/r} \|\varrho\|_{L^\infty_T(L^p)}.$$

Next, we estimate the transport terms of F by means of Hölder inequality, using that

$$\frac{1}{r} - \frac{1}{r_0} - \frac{1}{r_1} = \frac{3}{2} - \frac{1}{r} - \frac{d}{2p} = \frac{1}{2} - \frac{1}{r_0} > 0.$$

We get the following inequality:

$$(3.20) \quad \|w \cdot \nabla w + w \cdot \nabla v + v \cdot \nabla w + v \cdot \nabla v\|_{L^r_T(L^p)} \le$$

$$\le C\, T^{\frac{1}{2} - \frac{1}{r_0}} \left(\|v\|_{L^{r_0}_T(L^\infty)} + \|w\|_{L^{r_0}_T(L^\infty)} \right) \left(\|\nabla v\|_{L^{r_1}_T(L^p)} + \|\nabla w\|_{L^{r_1}_T(L^p)} \right).$$

Hence, using the definition of $\mathcal{N}(T)$ and (3.18), inequality (3.20) becomes

$$(3.21) \qquad \|w \cdot \nabla w + w \cdot \nabla v + v \cdot \nabla w + v \cdot \nabla v\|_{L^r_T(L^p)} \le C\, T^{\frac{1}{2} - \frac{1}{r_0}}(1 + T^{\frac{1}{r_0}})(1 + T^{\frac{1}{r_1}})\mathcal{N}^2(T).$$

Let us now consider the term $\nabla(\mathrm{Id} - \Delta)^{-1} \mathrm{div}(Pu)$ occurring in the definition of F. From Corollary 2.1 and (3.10), combined with the continuity of the function P, we deduce that

$$\left\|\nabla(\mathrm{Id} - \Delta)^{-1} \mathrm{div}(P\,u)\right\|_{L^r_T(L^p)} \le C\, \|P\,u\|_{L^r_T(L^p)}$$

$$\le C\, T^{\frac{1}{r} - \frac{1}{r_0} - \frac{1}{r_1}} \|P\|_{L^{r_1}_T(L^p)} \left(\|v\|_{L^{r_0}_T(L^\infty)} + \|w\|_{L^{r_0}_T(L^\infty)} \right)$$

$$\le C\, T^{\frac{1}{r} - \frac{1}{r_0}} \|\varrho\|_{L^\infty_T(L^p)} \left(T^{\frac{1}{r_0}} \|\varrho\|_{L^\infty_T(L^\infty)} + \|w\|_{L^{r_0}_T(L^\infty)} \right).$$

Hence, using the definition of $\mathcal{N}(T)$, we get

$$(3.22) \qquad\qquad \left\|\nabla(\mathrm{Id} - \Delta)^{-1} \mathrm{div}(P\,u)\right\|_{L^r_T(L^p)} \le C\, T^{\frac{1}{r} - \frac{1}{r_0}} (1 + T^{\frac{1}{r_0}}) \mathcal{N}^2(T).$$

To handle the last term of F, we write that

$$\nabla(\mathrm{Id} - \Delta)^{-1}\big(g(\rho)\, \mathrm{div}\, u\big) = g(1)\, \nabla(\mathrm{Id} - \Delta)^{-1} \mathrm{div}\, u + \nabla(\mathrm{Id} - \Delta)^{-1}\Big(\big(g(\rho) - g(1)\big)\, \mathrm{div}\, u\Big).$$

To bound the first term, we use that, thanks to (3.4),

$$\nabla(\mathrm{Id} - \Delta)^{-1} \mathrm{div}\, u = \nabla(\mathrm{Id} - \Delta)^{-1} \mathrm{div}\, w + \nabla(\mathrm{Id} - \Delta)^{-2}\Delta P.$$

Because both $\nabla(\mathrm{Id} - \Delta)^{-1}$ and $\nabla(\mathrm{Id} - \Delta)^{-2}\Delta$ map L^p to itself, we get

$$\left\|\nabla(\mathrm{Id} - \Delta^{-1})\operatorname{div} u\right\|_{L_T^r(L^p)} \le C\left(T^{\frac{1}{2}}\|\nabla w\|_{L_T^{r_1}(L^p)} + T^{\frac{1}{r}}\|P\|_{L_T^\infty(L^p)}\right) \le C\left(T^{\frac{1}{2}} + T^{\frac{1}{r}}\right)\mathcal{N}(T).$$

Similarly, we have

$$\left\|\nabla(\mathrm{Id} - \Delta)^{-1}\Big(\big(g(\rho) - g(1)\big)\operatorname{div} u\Big)\right\|_{L_T^r(L^p)} \le C\left\|\big(g(\rho) - g(1)\big)\operatorname{div} u\right\|_{L_T^r(L^p)}$$

$$\le C\,T^{\frac{1}{r} - \frac{1}{r_0} - \frac{1}{r_1}}\|\varrho\|_{L_T^{r_0}(L^\infty)}\|\operatorname{div} u\|_{L_T^{r_1}(L^p)};$$

so, keeping in mind (3.4), one can conclude that

$$(3.23) \qquad \left\|\nabla(\mathrm{Id} - \Delta)^{-1}\Big(\big(g(\rho) - g(1)\big)\operatorname{div} u\Big)\right\|_{L_T^r(L^p)} \le C\,T^{\frac{1}{2}}\big(1 + T^{\frac{1}{r_1}}\big)\mathcal{N}^2(T).$$

In the end, plugging the inequalities (3.14) and (3.19) to (3.23) in (3.17), whenever relation (3.10) is fulfilled, we get

$$\mathcal{N}(T) \le C\left(\|\varrho_0\|_{L^p \cap L^\infty} + \|w_0\|_{\dot{B}_{p,r}^{2-2/r}} + \big(T^{\frac{1}{2}} + T^{\frac{1}{r}}\big)\mathcal{N}(T) + \big(T^{\frac{1}{2} - \frac{1}{r_0}} + T^{\frac{1}{r}}\big)\mathcal{N}^2(T)\right),$$

for some constant C depending only on the pressure function and on the regularity parameters. From that inequality and a standard bootstrap argument, one may conclude that there exists a time $T > 0$, depending only on the norm of the initial data, such that

$$(3.24) \qquad\qquad \mathcal{N}(T) \le 2C\left(\|\varrho_0\|_{L^p \cap L^\infty} + \|w_0\|_{\dot{B}_{p,r}^{2-2/r}}\right).$$

3.2.3. *Classical energy estimates.* We establish here energy estimates for solutions to system (0.1), under the additional assumption that $u_0 \in L^2$. The computations being quite standard, we only sketch the arguments, and refer the reader to e.g. Chapter 5 of [32] for the details: we take the L^2 scalar product of the momentum equation in (0.1) with u, integrate by parts and make use of the mass equation. Defining Π as in the statement of Theorem 1.1, we end up with the relation

$$\frac{1}{2}\frac{d}{dt}\int \rho|u|^2\,dx + \frac{d}{dt}\int \Pi(\rho)\,dx + \mu\int|\nabla u|^2\,dx + \lambda\int|\operatorname{div} u|^2\,dx = 0.$$

The previous relation, after integration in time, leads to the classical energy balance (1.6).

Now, keeping in mind the smallness assumption (1.5), we gather that if $\inf P' > 0$ on $[1-4\varepsilon, 1+4\varepsilon]$ then it holds on $[0, T]$:

$$C^{-1}\|\varrho(t)\|_{L^2}^2 \le \left\|\Pi\big(\rho(t)\big)\right\|_{L^1} \le C\,\|\varrho(t)\|_{L^2}^2.$$

Hence, by the hypotheses on the initial data, we get that the right-hand side of (1.6) is finite, and then, for all $t \in [0, T]$, one has that $\sqrt{\rho}\,u$ belongs to $L_t^\infty(L^2)$, $\varrho \in L_t^\infty(L^2)$ and $\nabla u \in L_t^2(L^2)$.

To make a long story short, one can eventually assert that for all $t \in [0, T]$, we have

$$(3.25) \qquad\qquad \|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(\tau)\|_{L^2}^2\,d\tau + \|\varrho(t)\|_{L^2}^2 \le C,$$

for some $C > 0$ just depending on the initial energy, on P and on ε.

3.3. **The proof of existence.** In this subsection, we derive, from the estimates of the previous part, the existence of a weak solution to system (3.5)-(3.6).

We start by smoothing out the initial data (ϱ_0, u_0) by convolution with a family of nonnegative mollifiers:

$$\varrho_0^n := \chi^n * \varrho_0 \quad \text{and} \quad u_0^n := \chi^n * u_0.$$

Then ϱ_0^n still satisfies (1.5), and both ϱ_0^n and u_0^n belong to all Sobolev spaces $W^{k,p}$, with $k \in \mathbb{N}$. Note that one can in addition multiply the regularized data by a family of cut-off functions, to have

ϱ_0^n and u_0^n in L^2, which enables us to apply Theorem A of [33][4]. We get a sequence of solutions (ϱ^n, u^n) on $[0, T^n]$ (with $T^n > 0$) to (0.1), supplemented with initial data $(1 + \varrho_0^n, u_0^n)$, fulfilling (1.5), the energy balance (1.6), and the following properties:

$$\varrho^n \in \mathcal{C}([0, T^n]; W^{1,p}), \qquad u^n \in \mathcal{C}([0, T^n]; L^2), \qquad \partial_t w^n, \nabla^2 w^n \in L^r([0, T^n]; L^p).$$

Furthermore, by taking advantage of Inequality (3.24), one can exhibit some $T > 0$, depending only on norms of (ϱ_0, u_0), such that $T^n \geq T$ for all $n \in \mathbb{N}$, and eventually get, for some constant C depending only on p, r, d and ε, the bound

$$\|\varrho^n\|_{L_T^\infty(L^p \cap L^\infty)} + \|w^n\|_{L_T^\infty(\dot{B}_{p,r}^{2-2/r}) \cap L_T^{r_0}(L^\infty)} + \|\nabla w^n\|_{L_T^{r_1}(L^p)}$$
$$+ \left\|\left(\partial_t w^n, \nabla^2 w^n\right)\right\|_{L_T^r(L^p)} \leq C\left(\|\varrho_0\|_{L^p \cap L^\infty} + \|w_0\|_{\dot{B}_{p,r}^{2-2/r}}\right).$$

The previous inequality ensures the weak-$\star$ convergence (up to an extraction of a subsequence) of (ϱ^n, w^n) to some (ϱ, w) in the space E_T.

Strong convergence properties are still needed, in order to pass to the limit in the weak formulation of the equations, and show that (ϱ, w) is indeed a solution of system (3.5)-(3.6). In order to glean strong compactness, it suffices to use the fact that the above uniform bound also provides a control on the first order time derivatives in sufficiently negative Sobolev spaces through the equation fulfilled by (ϱ^n, w^n). Then one can combine with Ascoli theorem and interpolation, to get strong convergence, which turns out to be enough to pass to the limit in the equation satisfied by w. In order to justify that ϱ satisfies the mass equation, one can repeat the arguments of [28] (see also [32]) which in particular imply that $\varrho^n \to \varrho$ (up to subsequence) in $\mathcal{C}([0, T]; L^q)$ for all $p \leq q < \infty$. The details are left to the reader.

Finally, once it is known that (ϱ, w) satisfy the desired equation, one can recover time continuity for w by taking advantage of Proposition 2.3.

To prove that the energy balance is fulfilled in the case where $\inf P' > 0$ on $[1 - 4\varepsilon, 1 + 4\varepsilon]$ and $\varrho_0, u_0 \in L^2$, we just have to observe that it is satisfied by (ρ^n, u^n) (with the regularized data) for all $n \in \mathbb{N}$ and that having $\varrho_0 \in L^2$ guarantees that $\varrho^n \to \varrho$ in $\mathcal{C}([0, T]; L^2)$. This implies that $v^n \to v$ in $\mathcal{C}([0, T]; H^1)$. Furthermore, the compactness properties of (w^n) that have been pointed out just above ensure that $w^n \to w$ in $L^2([0, T]; H^1_{loc}) \cap L^\infty([0, T]; L^2_{loc})$. Finally, since

$$\frac{1}{2}\left\|\sqrt{\rho(t)}\, u(t)\right\|_{L^2}^2 + \left\|\Pi\big(\rho(t)\big)\right\|_{L^1} + \mu\|\nabla u\|_{L_t^2(L^2)}^2 + \lambda\|\operatorname{div} u\|_{L_t^2(L^2)}^2$$
$$= \lim_{R \to \infty}\left(\frac{1}{2}\int_{B(0,R)} \left(\rho(t,x)|u(t,x)|^2 + \Pi\big(\rho(t,x)\big)\right)\, dx + \int_0^t \int_{B(0,R)} \left(\mu\,|\nabla u|^2 + \lambda\,(\operatorname{div} u)^2\right)\, dx\, d\tau\right),$$

and because the aforementioned properties of convergence enable us to pass to the limit in the right-hand side for any $R > 0$, we get the desired energy balance.

Proposition 3.1 is thus completely proven, and so is Theorem 1.1 (apart from uniqueness if $d = 1$ that will be discussed at the beginning of the next section).

4. TANGENTIAL REGULARITY AND UNIQUENESS

The main goal of this section is to prove uniqueness of solutions to (0.1) in the previous functional framework. According to the pioneering work by D. Hoff in [27] or to the recent paper [11] by the first author, the condition $\nabla u \in L_T^1(L^\infty)$ seems to be the minimal requirement in order to get uniqueness. Recall that (still assuming that $\nu = 1$ for notational simplicity)

$$(4.1) \qquad \nabla u = \nabla w + \nabla^2(\operatorname{Id} - \Delta)^{-1} P(\rho).$$

---

[4]Actually, [33] concentrates on the $\mathbb{R}^3$ case but very small changes allow to get a similar result in $\mathbb{R}^d$ provided $p > d$. One can alternately use [9] that proves local-in-time existence in Besov spaces with sub-critical regularity.

By Proposition 3.1 and Sobolev embeddings, we immediately get that ∇w is in $L_T^1(L^\infty)$. So is the last term of (4.1) if the space dimension is 1 since $P(\rho)$ is bounded and $\partial_{xx}^2(\mathrm{Id} - \partial_{xx}^2)^{-1}$ maps L^∞ to L^∞. If $d \geq 2$ however then the property that P is bounded ensures only (by Calderón-Zygmund theory) that $\nabla^2(\mathrm{Id} - \Delta)^{-1}P$ is in BMO. Having Proposition 2.4 in mind, this prompts us to make an additional tangential regularity type assumption so as to guarantee that, indeed, $\nabla^2(\mathrm{Id} - \Delta)^{-1}P$ belongs to L^∞.

In the rest of this section, we thus assume the following tangential regularity hypothesis: for $p \in\,]d, \infty[$, there exists a non-degenerate family $\mathcal{X}_0 = (X_{0,\lambda})_{1 \leq \lambda \leq m}$ of vector-fields in $\mathbb{L}^{\infty,p}$ such that the initial density ρ_0 belongs to the space $\mathbb{L}_{\mathcal{X}_0}^p$ (see Definition 1.1 above).

4.1. **Propagation of tangential regularity.** In this subsection we establish a priori estimates for striated regularity of the density of a smooth enough solution (ϱ, u) to system (3.5)-(3.6). From those bounds, we will infer a control on the Lipschitz norm of u. Throughout that section, we shall use the notation

$$U(t) := \int_0^t \|\nabla u(\tau)\|_{L^\infty}\, d\tau\,.$$

4.1.1. *Bounds for the tangential vector fields.* Let us generically denote by X_0 one of the vector-fields of the family $\mathcal{X}_0$. It is well known that the evolution of X_0 along the velocity flow is the solution to the transport equation

$$(4.2) \qquad \begin{cases} (\partial_t + u \cdot \nabla)\, X = \partial_X u \\ X_{|t=0} = X_0\,, \end{cases}$$

where the notation $\partial_X u$ has been introduced in (1.7).

For all $t \geq 0$, we then define the family $\mathcal{X}(t) := (X_\lambda(t))_{1 \leq \lambda \leq m}$, where X_λ stands for the solution to system (4.2), supplemented with initial datum $X_{0,\lambda}$.

Our goal, now, is to establish some bounds on each vector-field $X(t)$ of the family $\mathcal{X}(t)$. They are based on classical estimates for transport equations in the spirit of those of Paragraph 3.2.1.

First of all, the standard L^∞ estimate for equation (4.2) leads us to the inequality

$$(4.3) \qquad \|X(t)\|_{L^\infty} \leq \|X_0\|_{L^\infty}\, e^{U(t)}\,.$$

Next, arguing exactly as in Proposition 4.1 of [8] (one needs to pass through the flow associated to u) yields

$$(4.4) \qquad I\big(\mathcal{X}(t)\big) \geq I(\mathcal{X}_0)\, e^{-U(t)}\,,$$

which ensures that the family $\mathcal{X}(t)$ remains non-degenerate whenever $U(t)$ stays bounded.

Finally, differentiating equation (4.2) with respect to the space variable x_j, we get

$$\partial_t \partial_j X + u \cdot \nabla \partial_j X = -\partial_j u \cdot \nabla X + \partial_j \partial_X u\,,$$

which leads to

$$(4.5) \qquad \|\nabla X(t)\|_{L^p} \leq \|\nabla X(0)\|_{L^p} + \int_0^t \big(C\|\nabla u\|_{L^\infty}\|\nabla X\|_{L^p} + \|\nabla \partial_X u\|_{L^p}\big)\, d\tau.$$

Observe that, for all $1 \leq j \leq d$, we have the relation

$$\partial_j \partial_X u = \partial_j X \cdot \nabla u + \partial_X \partial_j u\,,$$

where the former term is easily bounded in L^p by the quantity $\|\nabla X\|_{L^p}\, \|\nabla u\|_{L^\infty}$. Then, taking advantage of Gronwall inequality, we get

$$(4.6) \qquad \|\nabla X(t)\|_{L^p} \leq e^{CU(t)} \left(\|\nabla X_0\|_{L^p} + \int_0^t e^{-CU(\tau)}\|\partial_X \nabla u\|_{L^p}\, d\tau \right).$$

4.1.2. *Propagation of striated regularity for the density.* We now show propagation of tangential regularity for the density function. To begin with, we recast the first equation of (0.1) in the form

$$
(4.7) \qquad \partial_t \rho \,+\, u \cdot \nabla \rho \,=\, -\, \rho \, \mathrm{div} \, u \,.
$$

Next, we multiply the previous relation by X: by virtue of (4.2), we find

$$
\partial_t \big(\rho \, X \big) \,+\, u \cdot \nabla \big(\rho \, X \big) \,+\, \rho \, X \, \mathrm{div} \, u \,-\, \rho \, \partial_X u \,=\, 0 \,.
$$

Taking the divergence of the obtained relation, straightforward computations lead to the equation

$$
\partial_t \, \mathrm{div} \big(\rho \, X \big) \,+\, u \cdot \nabla \, \mathrm{div} \big(\rho \, X \big) \,=\, -\, \mathrm{div} \, u \, \, \mathrm{div} \big(\rho \, X \big) \,.
$$

From it, repeating the computations of Paragraph 3.2.1, we deduce that

$$
(4.8) \qquad \big\| \mathrm{div} \big(\rho(t) \, X(t) \big) \big\|_{L^p} \,\leq\, e^{CU(t)} \, \big\| \mathrm{div} \big(\rho_0 \, X_0 \big) \big\|_{L^p} \,.
$$

Thanks to the previous estimate and to the relation

$$
\mathrm{div} \big(P(\rho) \, X \big) \,=\, \big(P(\rho) \,-\, \rho \, P'(\rho) \big) \, \mathrm{div} \, X \,+\, P'(\rho) \, \mathrm{div} \big(\rho \, X \big) \,,
$$

one easily gathers the propagation of tangential regularity also for the pressure term:

$$
\begin{aligned}
\big\| \mathrm{div} \big(P(\rho) \, X \big) \big\|_{L^p} &\,\leq\, \| P(\rho) \,-\, \rho P'(\rho) \|_{L^\infty} \, \| \, \mathrm{div} \, X \|_{L^p} \,+\, \| P'(\rho) \|_{L^\infty} \, \big\| \mathrm{div} \big(\rho \, X \big) \big\|_{L^p} \\
&\,\leq\, C \, \big(\| \nabla X \|_{L^p} \,+\, \big\| \mathrm{div} \big(\rho \, X \big) \big\|_{L^p} \big) \cdot
\end{aligned}
$$

where, in writing the last inequality, we have used (3.12) and (3.10). For future use, we also notice that, by the previous estimate and (1.7), we have

$$
(4.9) \qquad \| \partial_X P(\rho) \|_{L^p} \,\leq\, C \, \big(\| \nabla X \|_{L^p} \,+\, \big\| \mathrm{div} \big(\rho \, X \big) \big\|_{L^p} \big) \,.
$$

4.1.3. *Final estimates for the gradient of the velocity.* In this paragraph, we complete the proof of propagation of striated regularity, and exhibit a bound for ∇u in $L^1_{T_0}(L^\infty)$, for some time $T_0 > 0$ depending only on suitable norms of the data.

First, we want to control the L^p norm of ∇X which, in light of inequality (4.6), requires our bounding the quantity $\| \partial_X \nabla u \|_{L^p}$. Here lies the main difficulty, compared to the standard result of propagation of striated regularity for incompressible flows. On the one hand, the part of this term corresponding to $\mathbb{P} u$ may be bounded quite easily since $\mathbb{P} u = \mathbb{P} w$ (note that $u - w$ is a gradient) and estimates on the second derivatives are thus available through the maximal regularity results that have been proved before. On the other hand, $\mathbb{Q} u$ has a part involving w (which is fine, exactly as before) and another one depending on $P(\rho)$. Here Proposition 2.5 will come into play. More precisely, we use relation (4.1) to write

$$
(4.10) \qquad \partial_X \nabla u \,=\, \partial_X \nabla w \,+\, \partial_X \nabla^2 (\mathrm{Id} - \Delta)^{-1} P(\rho) \,.
$$

Bounding the first term is easy: thanks to Paragraph 3.2.2 and (4.3), we have

$$
(4.11) \qquad \| \partial_X \nabla w \|_{L^p} \,\leq\, \| X \|_{L^\infty} \, \| \nabla^2 w \|_{L^p} \,\leq\, e^U \, \| X_0 \|_{L^\infty} \, \| \nabla^2 w \|_{L^p} \,.
$$

Recall that, thanks to Theorem 1.1 and especially estimate (3.24), the quantity $\| \nabla^2 w \|_{L^p}$ is in L^r_T, and thus in L^1_T.

For the last term in (4.10), Proposition 2.5, guarantees that

$$
\begin{aligned}
\big\| \partial_X \nabla^2 (\mathrm{Id} - \Delta)^{-1} P(\rho) \big\|_{L^p} &\,\leq\, C \, \Bigg(\| \nabla \mathcal{X} \|_{L^p} \Big(1 + \frac{\| \mathcal{X} \|_{L^\infty}^{4d-5} \| \nabla \mathcal{X} \|_{L^p}}{\big(I(\mathcal{X}) \big)^{4d-4}} \Big) \| P(\rho) \|_{L^\infty} \\
&+ \Big(1 + \frac{\| \mathcal{X} \|_{L^\infty}^{4d-5} \| \nabla \mathcal{X} \|_{L^p}}{\big(I(\mathcal{X}) \big)^{4d-4}} \Big) \| \partial_{\mathcal{X}} P(\rho) \|_{L^p} \,+\, \frac{\| \mathcal{X} \|_{L^\infty}^{4d-4}}{\big(I(\mathcal{X}) \big)^{4d-4}} \| \nabla \mathcal{X} \|_{L^p} \| P(\rho) \|_{L^\infty} \Bigg) \,.
\end{aligned}
$$

In view of estimates (3.24) and (4.9), this implies that (taking C larger if need be)

$$\left\|\partial_X \nabla^2 (\mathrm{Id} - \Delta)^{-1} P(\rho)\right\|_{L^p} \leq C\left(\left(\|\nabla\mathcal{X}\|_{L^p} + \|\mathrm{div}(\rho\,\mathcal{X})\|_{L^p}\right)\left(1 + \frac{\|\mathcal{X}\|_{L^\infty}^{4d-5}\,\|\nabla\mathcal{X}\|_{L^p}}{\left(I(\mathcal{X})\right)^{4d-4}}\right)\right.$$
$$\left. + \frac{\|\mathcal{X}\|_{L^\infty}^{4d-4}}{\left(I(\mathcal{X})\right)^{4d-4}}\,\|\nabla\mathcal{X}\|_{L^p}\right).$$

At this point, we use the bounds (4.3), (4.4) and (4.8). Including a dependence on $\|\mathcal{X}_0\|_{L^\infty}$, $I(\mathcal{X}_0)$ and $\|\mathrm{div}(\rho_0\,\mathcal{X}_0)\|_{L^p}$ in C, we deduce for some constant c_d depending only on d,

$$\left\|\partial_X \nabla^2 (\mathrm{Id} - \Delta)^{-1} P(\rho)\right\|_{L^p} \leq C\left(\|\nabla\mathcal{X}\|_{L^p}\left(1 + e^{c_d U}\,\|\nabla\mathcal{X}\|_{L^p}\right)\right.$$
$$\left. + \left(1 + e^{c_d U}\,\|\nabla\mathcal{X}\|_{L^p}\right)\,e^U \, + \, e^{(c_d+1)U}\,\|\nabla\mathcal{X}\|_{L^p}\right).$$

Changing c_d to $c_d + 1$, the previous relation finally leads us to the bound

(4.12) $$\left\|\partial_X \nabla^2 (\mathrm{Id} - \Delta)^{-1} P(\rho)\right\|_{L^p} \leq C\,e^{c_d U}\left(1 + \|\nabla\mathcal{X}\|_{L^p}^2\right).$$

Putting estimates (4.11) and (4.12) together, we finally gather

(4.13) $$\|\partial_X \nabla u\|_{L^p} \leq C\,e^{c_d U}\left(1 + \|\nabla\mathcal{X}\|_{L^p}^2 + \|\nabla^2 w\|_{L^p}\right).$$

At this point, one define the time T_1 to satisfy

(4.14) $$T_1 := \sup\left\{t \in\,]0, T\right]\ \Big|\ \ U(t) \leq \log 2\right\},$$

where $T > 0$ is the time given by Proposition 3.1.

Then, changing C if need be, estimate (4.13) implies that, on $[0, T_1]$, one has

$$\|\partial_X \nabla u\|_{L^p} \leq C\left(1 + \|\nabla\mathcal{X}\|_{L^p}^2 + \|\nabla^2 w\|_{L^p}\right).$$

Inserting now this bound in (4.6) and using also (3.24), we find that for all $\lambda \in \{1, \cdots, m\}$,

$$\|\nabla X_\lambda(t)\|_{L^p} \leq 2\left(\|\nabla X_{0,\lambda}\|_{L^p} + \int_0^t \|\partial_{X_\lambda} \nabla u\|_{L^p}\,d\tau\right)$$
$$\leq 2\left(\|\nabla X_{0,\lambda}\|_{L^p} + C\int_0^t \left(1 + \|\nabla\mathcal{X}\|_{L^p}^2 + \|\nabla^2 w\|_{L^p}\right)d\tau\right).$$

Taking the supremum on λ, we find that for some C_0 depending only on the norms of the data, and for some 'absolute' constant C, we have for all $t \in [0, T_1]$,

$$\|\nabla\mathcal{X}(t)\|_{L^p} \leq C\left(C_0 + C\int_0^t \|\nabla\mathcal{X}(\tau)\|_{L^p}^2\,d\tau\right).$$

Then using a Gronwall type argument, we conclude that

(4.15) $$\|\nabla\mathcal{X}(t)\|_{L^p} \leq \frac{C_0}{1 - CC_0 t}\quad\text{for all}\ \ t \in [0, T_1]\ \ \text{satisfying}\ \ CC_0 t < 1.$$

This having been established, let us turn our attention to finding a bound for the quantity $U(t)$, as it is needed to close the estimates. Resorting to relation (4.1) again, we see that we have to control the $L_t^1(L^\infty)$ norm of each term appearing on its right-hand side. For the term in w, this is an easy task: thanks to Proposition 3.1 and Sobolev embeddings, a decomposition in low and high frequencies implies

(4.16) $$\|\nabla w\|_{L_t^1(L^\infty)} \leq C\left(t^{1-1/r_1}\,\|\nabla w\|_{L_T^{r_1}(L^p)} + t^{1-1/r}\,\|\nabla^2 w\|_{L_T^r(L^p)}\right)$$

for all $t \in [0, T]$, where we also used relation (1.4).

Bounding the latter term in (4.1) is based on Proposition 2.4 that gives

$$\left\|\nabla^2(\mathrm{Id}-\Delta)^{-1}P(\rho)\right\|_{L^\infty} \leq C\left(\left(1+\frac{\|\mathcal{X}\|_{L^\infty}^{4d-5}\|\nabla\mathcal{X}\|_{L^p}}{(I(\mathcal{X}))^{4d-4}}\right)\|P(\rho)\|_{L^\infty} + \frac{\|\mathcal{X}\|_{L^\infty}^{4d-5}}{(I(\mathcal{X}))^{4d-4}}\|\partial_\mathcal{X}P(\rho)\|_{L^p}\right).$$

In view of (3.24), (4.3), (4.4), (4.8) and (4.9), omitting once again the explicit dependence of the multiplicative constants on the norms of the initial data, the previous inequality allows us to get

$$\begin{aligned}(4.17)\qquad\left\|\nabla^2(\mathrm{Id}-\Delta)^{-1}P(\rho)\right\|_{L^\infty} &\leq C\left(1+e^{c_dU}\|\nabla\mathcal{X}\|_{L^p}+e^{c_dU}\left(\|\nabla\mathcal{X}\|_{L^p}+e^U\right)\right)\\ &\leq C\,e^{c_dU}\left(1+\|\nabla\mathcal{X}\|_{L^p}\right).\end{aligned}$$

Recalling definition (4.14) of T_1 and Inequality (4.15) and taking $0 < T_0 \leq T_1$ so that $2CC_0T_0 \leq 1$, we gather

$$(4.18)\qquad\left\|\nabla^2(\mathrm{Id}-\Delta)^{-1}P(\rho)\right\|_{L_{T_0}^\infty(L^\infty)} \leq C\left(1+C_0\right),$$

which implies, together with (4.16), the following control, for all fixed $t \in [0, T_0]$:

$$(4.19)\qquad\|\nabla u\|_{L_t^1(L^\infty)} \leq C\left(t^{1-1/r_1}\|\nabla w\|_{L_T^{r_1}(L^{p_1})}+t^{1-1/r}\|\nabla^2 w\|_{L_T^r(L^p)}+t(1+C_0)\right).$$

Up to taking a smaller T_0, we then see that the requirement $\|\nabla u\|_{L_{T_0}^1(L^\infty)} \leq \log 2$ can be fulfilled. Then, a classical bootstrap argument, which we do not detail here, finally allows us to deduce the boundedness of ∇u in $L_{T_0}^1(L^\infty)$.

In order to prove rigorously the existence part of Theorem 1.2, one may proceed as in Subsection 3.3. There, we constructed a sequence (ϱ^n, u^n) of smooth solutions that is uniformly bounded in the space E_T. Therefore, it is only a matter of checking that one can get uniform bounds, too, for the striated regularity. To do this, we smooth out the reference family of vector fields $\mathcal{X}_0$ into $\mathcal{X}_0^n$ (paying attention to keep the nondegeneracy condition), then define the family $\mathcal{X}^n := (X_\lambda^n)_{1\leq\lambda\leq m}$ transported by the flow of u^n according to (4.2), taking u^n instead of u and starting from the initial vector field $X_{0,\lambda}^n$. Then, repeating the computations that have been carried out just above, we get uniform bounds for all the quantities involving the striated regularity, and thus also for ∇u^n in $L_{T_0}^1(L^\infty)$. That (ϱ^n, u^n) tends to some solution (ϱ, u) of (1.1) belonging to E_{T_0} has already been justified before. Furthermore, combining our new bounds with compactness arguments allows to pass to the limit in (4.2) as well, and to get the crucial information that ∇u is in $L_{T_0}^1(L^\infty)$.

4.2. The proof of uniqueness.
Property (4.19) having been established, one can now tackle the proof of uniqueness of solutions. The basic idea is to perform a Lagrangian change of coordinates in system (0.1), in order to by-pass the hyperbolic nature of the mass equation, which otherwise would cause the loss of one derivative in the stability estimates. In fact, we will perform stability estimates for the Lagrangian formulation of equations (0.1).

4.2.1. *Lagrangian formulation.* The goal of this paragraph is to recast system (0.1) in Lagrangian variables. Recall that in light of the estimates of Subsection 4.1, we know that for all $k_0 > 0$, there exists a time $T_0 > 0$ such that

$$(4.20)\qquad\int_0^{T_0}\|\nabla u(t)\|_{L^\infty}\,dt \leq k_0.$$

The value of k_0 will be determined in the course of the computations below.

First of all, we define the flow ψ_u associated to the velocity field u to be the solution of

$$(4.21)\qquad\psi_u(t,y) := y + \int_0^t u\big(\tau, \psi_u(\tau,y)\big)\,d\tau.$$

Thanks to that, any function $f = f(t,x)$ may be rewritten in Lagrangian coordinates (t,y) according to the relation

$$(4.22) \qquad\qquad \overline{f}(t,y) := f\big(t, \psi_u(t,y)\big).$$

A key observation is that, once passing in Lagrangian coordinates, one can forget about the reference *Eulerian* velocity u by rewriting Definition (4.21) in terms of the *Lagrangian* velocity $\overline{u}$, defining directly ψ_u by

$$\psi_u(t,y) = y + \int_0^t \overline{u}(\tau, y)\, d\tau\,.$$

In what follows, we set $J_u := \det\big(D\psi_u\big)$ and $A_u := \big(D\psi_u\big)^{-1}$. Observe that, by the standard chain rule, we obviously have

$$(4.23) \qquad\qquad \overline{D_x f} = D_y \overline{f} \cdot A_u\,.$$

Lemma A.2 of [11] provides us with the following alternative expressions[5].

Lemma 4.1. *For any $\mathcal{C}^1$ function K and any $\mathcal{C}^1$ vector-field H defined over $\mathbb{R}^d$, one has*

$$\overline{\nabla_x K} = J_u^{-1}\, \mathrm{div}_y\big(\mathrm{adj}\, D\psi_u\, \overline{K}\big)$$
$$\overline{\mathrm{div}_x H} = J_u^{-1}\, \mathrm{div}_y\big(\mathrm{adj}\, D\psi_u\, \overline{H}\big)\,.$$

Moreover, since our diffeomorphism ψ_u is the flow of the time-dependent vector-field u, we also get, for any function f,

$$(4.24) \qquad\qquad \overline{\partial_t f + \mathrm{div}(f\, u)} = J_u^{-1}\, \partial_t\big(J_u\, \overline{f}\big)\,.$$

The next statement is in the spirit of Lemma A.3 of [11]; also its proof follows the same steps, up to a straightforward adaptation to our functional framework.

Lemma 4.2. *Let u be a velocity field with $\nabla u \in L^1\big([0, T_0]; L^\infty(\mathbb{R}^d)\big)$, and let ψ_u be its flow, defined by (4.21). Suppose that condition (4.20) is fulfilled with $k_0 < 1$.*

Then there exists a constant $C > 0$, just depending on k_0, such that the following estimates hold true, for all time $t \in [0, T_0]$:

$$\|\mathrm{Id} - \mathrm{adj}\, D\psi_u(t)\|_{L^\infty} \leq C\, \|Du\|_{L^1_{T_0}(L^\infty)}$$

$$\|\mathrm{Id} - A_u(t)\|_{L^\infty} \leq C\, \|Du\|_{L^1_{T_0}(L^\infty)}$$

$$\big\|J_u^{\pm 1}(t) - 1\big\|_{L^\infty} \leq C\, \|Du\|_{L^1_{T_0}(L^\infty)}\,.$$

We also state the following lemma, the proof of which is straightforward.

Lemma 4.3. *For any function $f = f(x)$, define $\overline{f}$ according to (4.22). Then for any $p \in [1, \infty[$ one has*

$$\big\|\overline{f}\big\|_{L^p} \leq \|J_u\|_{L^\infty}^{1/p}\, \|f\|_{L^p} \qquad and \qquad \|f\|_{L^p} \leq \big\|J_u^{-1}\big\|_{L^\infty}^{1/p}\, \big\|\overline{f}\big\|_{L^p}\,.$$

After these preliminaries, we can recast our system in Lagrangian coordinates. First of all, from the mass equation in (0.1) and (4.24), we discover that

$$(4.25) \qquad\qquad \partial_t(J_u\, \overline{\rho}) = 0, \quad \text{whence} \quad J_u\, \overline{\rho} = \rho_0\,.$$

Second, we notice that, in Lagrangian coordinates, operator $\mathcal{L}$ reads

$$(4.26) \qquad \widetilde{\mathcal{L}} f := - J_u^{-1}\big(\mu\, \mathrm{div}\big(\mathrm{adj}\,(D\psi_u)\,{}^t A_u \nabla f\big) - \lambda\, \mathrm{div}\big(\mathrm{adj}\,(D\psi_u)\,({}^t A_u : \nabla f)\big)\big)$$

where we have used the notation $M : N := tr(MN) = \sum_{ij} M_{ij} N_{ji}$.

Hence, thanks to (4.24) and (4.25), the momentum equation in (0.1) recasts in

$$(4.27) \qquad\qquad \rho_0\, \partial_t \overline{u} + \widetilde{\mathcal{L}}\overline{u} = -\, \mathrm{div}\big(\mathrm{adj}\, D\psi_u\, P(J^{-1}\rho_0)\big).$$

---

[5]From now on we agree that $\mathrm{adj}\,(M)$ designates the adjugate matrix of M. Of course, if M is invertible, then $\mathrm{adj}\,(M) = (\det M)\, M^{-1}$.

4.2.2. *Stability estimates in Lagrangian coordinates.* In this section, we tackle the proof of uniqueness, by showing stability estimates for the Lagrangian formulation of our system.

More precisely, we consider initial data $\left(\rho_0^j, u_0^j\right)$, for $j = 1, 2$, verifying the hypotheses of Theorem 1.2. For the sake of simplicity and clarity, we focus on the case $\rho_0^1 = \rho_0^2 = \rho_0$, and suppose that ρ_0 satisfies the striated regularity assumption with respect to some fixed non-degenerate family of vector-fields $\mathcal{X}_0$. The initial velocities do not need to be equal.

Let $\left(\rho^1, u^1\right)$ and $\left(\rho^2, u^2\right)$ be two solutions to system (0.1) on the time interval $[0, T]$, fulfilling the properties given by Theorem 1.2 and corresponding to the data $\left(\rho_0, u_0^1\right)$ and $\left(\rho_0, u_0^2\right)$, respectively. Denoting $\varrho^j = \rho^j - 1$ for $j = 1, 2$, and defining w^j according to (3.3), the pairs $\left(\varrho^j, w^j\right)$ solve equations (3.5)-(3.6) and also enjoy the regularity properties stated in Theorem 1.1. Moreover, as shown in Subsection 4.1, for all j, tangential regularity is propagated with respect to the non-degenerate family $\mathcal{X}^j$, which correspond to the family $\mathcal{X}_0$ transported by the flow of u^j. Hence, for all $k_0 > 0$, there exists $T_0 > 0$ such that both ∇u^1 and ∇u^2 fulfill (4.20), which allows to pass in Lagrangian coordinates, as shown in Paragraph 4.2.1. Denoting, for $j = 1, 2$, the flow of u^j by ψ_j, setting $J_j := J_{u^j}$ and $A_j := A_{u^j}$, and taking advantage of the previous computations, we discover that $\left(\overline{\rho}^j, \overline{u}^j\right)_{j=1,2}$ satisfy the relations $J_j\, \overline{\rho}^j = \rho_0$ and

$$\rho_0\, \partial_t \overline{u}^j + \widetilde{\mathcal{L}}_j \overline{u}^j = -\operatorname{div}\left(\operatorname{adj} D\psi_j\, P(J_j^{-1}\, \rho_0)\right),$$

where $\widetilde{\mathcal{L}}_j$ is the operator corresponding to u^j that has been defined by formula (4.26).

Let $\delta\overline{u} := \overline{u}^1 - \overline{u}^2$ and use similar notations for the other quantities. Taking the difference of the equations respectively for $\overline{u}^1$ and $\overline{u}^2$, we find that $\delta\overline{u}$ satisfies

$$(4.28) \qquad \rho_0\, \partial_t \delta\overline{u} + \mathcal{L}\delta\overline{u} = \left(\mathcal{L} - \widetilde{\mathcal{L}}_1\right)\delta\overline{u} + \delta\mathcal{L}\,\overline{u}^2 - \operatorname{div}\left(\delta\operatorname{adj} P(J_1^{-1}\, \rho_0)\right)$$
$$- \operatorname{div}\left(\operatorname{adj} D\psi_2 \left(P(J_1^{-1}\, \rho_0) - P(J_2^{-1}\, \rho_0)\right)\right),$$

where we have set $\delta\operatorname{adj} := \operatorname{adj} D\psi_1 - \operatorname{adj} D\psi_2$. A slight adaptation of Lemma A.4 of [11] allows us to get the following bounds.

Lemma 4.4. *If* (4.20) *is fulfilled by* u^1 *and* u^2 *for some* $k_0 \in]0, 1]$, *then there exists a constant* $C > 0$ *just depending on* k_0, *such that the following estimates hold true, for all time* $t \in [0, T_0]$ *and all* $p \in [1, \infty]$:

$$\|\operatorname{adj} D\psi_1(t) - \operatorname{adj} D\psi_2(t)\|_{L^p} \leq C \int_0^t \|\nabla\delta u(\tau)\|_{L^p}\, d\tau$$

$$\|A_1(t) - A_2(t)\|_{L^p} \leq C \int_0^t \|\nabla\delta u(\tau)\|_{L^p}\, d\tau$$

$$\left\|J_1^{\pm 1}(t) - J_2^{\pm 1}(t)\right\|_{L^p} \leq C \int_0^t \|\nabla\delta u(\tau)\|_{L^p}\, d\tau.$$

We now perform energy estimates for equation (4.28): take the L^2 scalar product of both sides with $\delta\overline{u}$ and integrate by parts. In view of Lemmas 4.2 and 4.4, we deduce the following controls for the terms coming from the right-hand side:

$$\left|\int \left(\mathcal{L} - \widetilde{\mathcal{L}}_1\right)\delta\overline{u} \cdot \delta\overline{u}\, dx\right| \leq C\, k_0\, \|\nabla\delta\overline{u}\|_{L^2}^2$$

$$\left|\int \delta\mathcal{L}\,\overline{u}^2 \cdot \delta\overline{u}\, dx\right| \leq C\left(\int_0^t \|\nabla\delta\overline{u}\|_{L^2}\, d\tau\right)\left\|\nabla\overline{u}^2\right\|_{L^\infty} \|\nabla\delta\overline{u}\|_{L^2}$$

$$\left|\int \operatorname{div}\left(\delta\operatorname{adj} P(J_1^{-1}\, \rho_0)\right) \cdot \delta\overline{u}\, dx\right| \leq C\left(\int_0^t \|\nabla\delta\overline{u}\|_{L^2}\, d\tau\right)\|\rho_0\|_{L^\infty} \|\nabla\delta\overline{u}\|_{L^2}$$

$$\left| \int \operatorname{div}\Big(\operatorname{adj} D\psi_2 \left(P(J_1^{-1}\rho_0) - P(J_2^{-1}\rho_0))\right)\Big) \cdot \delta\overline{u}\, dx \right| \leq C\Big(\int_0^t \|\nabla\delta\overline{u}\|_{L^2}\, d\tau\Big)\|\rho_0\|_{L^\infty}\, \|\nabla\delta\overline{u}\|_{L^2}\, .$$

Now, if k_0 in (4.20) has been taken small enough, then a repeated use of Young and Cauchy-Schwarz inequalities leads us to the estimate

$$(4.29) \qquad \frac{d}{dt}\int \rho_0\, |\delta\overline{u}|^2\, dx \, + \, \int |\nabla\delta\overline{u}|^2\, dx \, \leq \, C\, t \, \Big(1 + \big\|\nabla\overline{u}^2\big\|_{L^\infty}^2\Big)\int_0^t \|\nabla\delta\overline{u}\|_{L^2}^2\, d\tau$$

for a new constant $C > 0$ that depends only on k_0, $\|\rho_0\|_{L^\infty}$, P, λ and μ.

In order to conclude to uniqueness on $[0, T_0]$ by applying Gronwall lemma, we need that

$$(4.30) \qquad \int_0^{T_0} t \, \Big(1 + \big\|\nabla\overline{u}^2\big\|_{L^\infty}^2\Big)\, dt < \infty.$$

In view of (4.23) and Lemma 4.3, it suffices to show that $t^{1/2}\nabla u^2$ is in $L^2_{T_0}(L^\infty)$. Now, recall that

$$\nabla u^2 \, = \, \nabla w^2 \, - \, \nabla^2(\operatorname{Id} - \Delta)^{-1}P(\rho^2)\,,$$

and that, thanks to the estimates of Subsection 4.1, one has that $\nabla^2(\operatorname{Id} - \Delta)^{-1}P(\rho^2)$ belongs to $L^\infty\big([0, T_0] \times \mathbb{R}^d\big)$. Therefore, our next (and final) goal is to show that there exists $T_0 > 0$ such that

$$(4.31) \qquad \int_0^{T_0} t \, \big\|\nabla w^2\big\|_{L^\infty}^2\, dt < \infty.$$

This will be achieved thanks to the following proposition, the proof of which is postponed to the next paragraph.

Proposition 4.1. *Under the hypotheses of Theorem* 1.2, *let us fix some*

$$R_2 > \max\{r_0/2\,,\, r_1/2\,,\, 2\} \qquad \textit{such that} \qquad \frac{1}{2\,R_2} < \frac{3}{2} - \frac{1}{r_2} + \frac{d}{2\,p} \qquad \textit{with} \quad r_2 := r,$$

and set $R_0 = R_1 = 2\,R_2$. For $j \in \{0, 1, 2\}$, define moreover the indices

$$(4.32) \qquad \alpha_j := \frac{1}{r_j} - \frac{1}{R_j} \qquad \textit{and} \qquad \gamma_j := \frac{1}{r_j} - \frac{1}{2\,R_j}.$$

Then, there exists a positive time T_* such that one has the properties

$$t^{\alpha_2} \, \nabla^2 w \, \in \, L_{T_*}^{R_2}(L^p)\,, \qquad t^{\gamma_1} \, \nabla w \, \in \, L_{T_*}^{R_1}(L^p)\,, \qquad t^{\gamma_0} \, w \, \in \, L_{T_*}^{R_0}(L^\infty)\,.$$

From the previous proposition, we immediately deduce the following corollary.

Corollary 4.1. *Under the assumptions of Theorem* 1.2, *one has*

$$\int_0^{T_*} t \, \|\nabla w(t)\|_{L^\infty}^2\, dt \, < \, \infty\,.$$

Proof. By Sobolev embedding, the stated inequality is a consequence of the following computation:

$$\int_0^{T_*} t \, \|\nabla^2 w(t)\|_{L^p}^2\, dt \, = \, \int_0^{T_*} t^{1-\eta}\, t^\eta \, \|\nabla^2 w(t)\|_{L^p}^2\, dt$$

$$\leq \, \Big(\int_0^{T_*} t^{\eta q}\, \|\nabla^2 w(t)\|_{L^p}^{2q}\, dt\Big)^{1/q} \Big(\int_0^{T_*} t^{(1-\eta)\, q'}\, dt\Big)^{1/q'}\,,$$

where q' is the conjugate exponent of q. Take $q = R_2/2$, so that $1/q' = 1 - 2/R_2$ and impose the relation $q\,\eta = R_2\,\alpha_2$, getting in this way $\eta = 2\,\alpha_2$. With these choices and because $r_2 = r \in (1, 2)$, the condition $(1 - \eta)\, q' > -1$ is verified, which completes the proof of the corollary. $\qquad\square$

At this point, one can finish the proof of Theorem 1.2, by establishing the uniqueness of solutions. Let us define the function

$$E(t) := \|\sqrt{\rho_0}\,\delta\overline{u}(t)\|_{L^2}^2 + \int_0^t \|\nabla\delta\overline{u}(\tau)\|_{L^2}^2\,d\tau\,.$$

Up to choosing a smaller T_0, we can suppose that $T_0 = T_*$. Then, applying Gronwall inequality to (4.29), we get, for all $t \in [0, T_*]$, the bound

$$E(t) \leq E(0)\exp\left(C\int_0^t f(\tau)\,d\tau\right), \qquad \text{where} \qquad f(t) := t\left(1 + \left\|\nabla\overline{u}^2(t)\right\|_{L^\infty}^2\right).$$

Since $E(0) \equiv 0$ and, by Corollary 4.1, $f \in L^1([0, T_*])$, we get uniqueness on $[0, T_*]$. Combining with a standard continuation argument, we then conclude to uniqueness on the whole interval $[0, T]$.

4.2.3. *Maximal regularity with time weights.* For completeness, we still have to prove Proposition 4.1. As a first, we need the following lemma that concerns the maximal regularity issue with time weights for the heat semi-group, and is strongly inspired by Lemma 3.2 of [28].

Lemma 4.5. *Let the exponents* $(R_j,\,\alpha_j,\,\gamma_j)_{j\in\{0,1,2\}}$ *be chosen as in Proposition 4.1. Let the operators* $\mathcal{A}_1$ *and* $\mathcal{A}_0$ *be defined as in Lemmas 2.3 and 2.4. Fix some* $T > 0$, *and assume that* $t^{\alpha_2} f$ *belongs to* $L_T^{R_2}(L^p)$.
Then one has $t^{1/r_1}\mathcal{A}_1 f \in L_T^\infty(L^p)$ *and* $t^{1/r_0}\mathcal{A}_0 f \in L_T^\infty(L^\infty)$, *together with the estimates*

$$\left\|t^{1/r_1}\mathcal{A}_1 f\right\|_{L_T^\infty(L^p)} + \left\|t^{1/r_0}\mathcal{A}_0 f\right\|_{L_T^\infty(L^\infty)} \leq C\left\|t^{\alpha_2} f\right\|_{L_T^{R_2}(L^p)}.$$

Moreover, we have $t^{\alpha_1}\mathcal{A}_1 f(t) \in L_T^{R_1/(1+\delta)}(L^p)$ *and* $t^{\alpha_0}\mathcal{A}_0 f(t) \in L_T^{R_0/(1+\delta)}(L^\infty)$ *for all* $\delta > 0$, *with the bounds*

$$\|t^{\alpha_1}\mathcal{A}_1 f\|_{L_T^{R_1/(1+\delta)}(L^p)} + \|t^{\alpha_0}\mathcal{A}_0 f\|_{L_T^{R_0/(1+\delta)}(L^\infty)} \leq C\left(T^{\delta/R_1} + T^{\delta/R_0}\right)\|t^{\alpha_2} f\|_{L_T^{R_2}(L^p)}.$$

In particular, defining γ_0 *and* γ_1 *according to (4.32), we have* $t^{\gamma_1}\mathcal{A}_1 f \in L_T^{R_1}(L^p)$ *and* $t^{\gamma_0}\mathcal{A}_0 f \in L_T^{R_0}(L^\infty)$, *and the following estimate is verified:*

$$\|t^{\gamma_1}\mathcal{A}_1 f\|_{L_T^{R_1}(L^p)} + \|t^{\gamma_0}\mathcal{A}_0 f\|_{L_T^{R_0}(L^\infty)} \leq C\left(T^{1/(2R_1)} + T^{1/(2R_0)}\right)\|t^{\alpha_2} f\|_{L_T^{R_2}(L^p)}.$$

Proof. Regarding operator $\mathcal{A}_1$, going along the lines of the proof to Lemma 2.3, one gets

$$\left\|\nabla e^{(t-s)\Delta} f(s,\cdot)\right\|_{L^p} \leq C\,(t-s)^{-1/2}\,\|f(s)\|_{L^p} \quad \text{for all} \ \ 0 \leq s \leq t \leq T,$$

which implies, after setting $1/R_2' = 1 - 1/R_2$, the inequality

$$\|\mathcal{A}_1 f(t)\|_{L^p} \leq C\int_0^t (t-s)^{-1/2}\,s^{-\alpha_2}\,\|s^{\alpha_2} f(s)\|_{L^p}\,ds$$

$$\leq C\left(\int_0^t (t-s)^{-R_2'/2}\,s^{-\alpha_2 R_2'}\,ds\right)^{1/R_2'}\|s^{\alpha_2} f\|_{L_T^{R_2}(L^p)}.$$

Since $R_2 > 2$, we have $R_2'/2 < 1$, while, by our definition of α_2 in (4.32), we have $\alpha_2 R_2' < 1$. Therefore, performing the change of variable $s = t\tau$ inside the integral yields

$$\|\mathcal{A}_1 f(t)\|_{L^p} \leq C\,t^{1/2-\alpha_2-1/R_2}\,\|s^{\alpha_2} f\|_{L_T^{R_2}(L^p)}.$$

On the one hand, since $1/2 - \alpha_2 - 1/R_2 = 1/2 - 1/r_2 = -1/r_1$, we have

(4.33) $$\left\|t^{1/r_1}\,\|\mathcal{A}_1 f(t)\|_{L^p}\right\|_{L_T^\infty} \leq C\,\|s^{\alpha_2} f\|_{L_T^{R_2}(L^p)}.$$

On the other hand, since $1/2 - \alpha_2 - 1/R_2 = -\alpha_1 - 1/R_1$, we also get that $t^{\alpha_1}\,\|\mathcal{A}_1 f(t)\|_{L^p}$ belongs to $L_T^{R_1/(1+\delta)}$ for all $\delta > 0$, and verifies

$$(4.34) \qquad \left\| t^{\alpha_1}\,\|\mathcal{A}_1 f(t)\|_{L^p} \right\|_{L_T^{R_1/(1+\delta)}} \leq C_\delta\,\|s^{\alpha_2} f\|_{L_T^{R_2}(L^p)}\,T^{\delta/R_1}\,.$$

Taking $\delta = 1$ and interpolating between estimates (4.33) and (4.34), we get that $t^{\gamma_1}\,\|\mathcal{A}_1 f(t)\|_{L^p} \in L_T^{R_1}$, with the estimate

$$(4.35) \qquad \left\| t^{\gamma_1}\,\|\mathcal{A}_1 f(t)\|_{L^p} \right\|_{L_T^{R_1}} \leq C\,\|s^{\alpha_2} f\|_{L_T^{R_2}(L^p)}\,T^{1/(2R_1)}\,.$$

Proving the claimed bound for the term $\mathcal{A}_0$ follows from the same lines. First of all, setting p' to be the conjugate exponent of p, we can write

$$\left\| e^{(t-s)\Delta} f(s, \cdot) \right\|_{L^\infty} \leq C\,(t-s)^{-d/2} \left\| K_0\left(\frac{\cdot}{\sqrt{4\pi(t-s)}} \right) \right\|_{L^{p'}} \|f(s)\|_{L^p}$$

$$\leq C\,(t-s)^{-d/(2p)}\,s^{-\alpha_2}\,\|s^{\alpha_2} f(s)\|_{L^p}\,.$$

Integrating this expression in time and applying Hölder inequality once give us, similarly as above,

$$\|\mathcal{A}_0 f(t)\|_{L^\infty} \leq C\left(\int_0^t (t-s)^{-d/(2p)\,R_2'}\,s^{-\alpha_2\,R_2'}\,ds \right)^{1/R_2'} \|s^{\alpha_2} f\|_{L_T^{R_2}(L^p)}\,.$$

Once again, thanks to our choice of R_2 we have that $d/(2p)\,R_2' < 1$ (recall that $p > d$); hence, repeating the change of variable $s = t\tau$ we find

$$\|\mathcal{A}_0 f(t)\|_{L^\infty} \leq C\,t^{1-d/(2p)-\alpha_2-1/R_2}\,\|t^{\alpha_2} f\|_{L_T^{R_2}(L^p)}\,.$$

Now, first we remark that $1 - d/(2p) - \alpha_2 - 1/R_2 = -1/r_0$, and hence $t^{1/r_0}\,\|\mathcal{A}_0 f(t)\|_{L^\infty} \in L_T^\infty$, with

$$(4.36) \qquad \left\| t^{1/r_0}\,\|\mathcal{A}_0 f(t)\|_{L^\infty} \right\|_{L_T^\infty} \leq C\,\|s^{\alpha_2} f\|_{L_T^{R_2}(L^p)}\,.$$

Then, we also notice that $1 - d/(2p) - \alpha_2 - 1/R_2 = -\alpha_0 - 1/R_0$, so that $t^{\alpha_0}\,\|\mathcal{A}_0 f(t)\|_{L^\infty}$ belongs to $L_T^{R_0/(1+\delta)}$ for all $\delta > 0$, and verifies the estimate

$$(4.37) \qquad \left\| t^{\alpha_0}\,\|\mathcal{A}_0 f(t)\|_{L^\infty} \right\|_{L_T^{R_0/(1+\delta)}} \leq C_\delta\,\|s^{\alpha_2} f\|_{L_T^{R_2}(L^p)}\,T^{\delta/R_0}\,.$$

As above, taking $\delta = 1$ and interpolating between estimates (4.36) and (4.37), we finally deduce the property $t^{\gamma_0}\,\|\mathcal{A}_0 f(t)\|_{L^\infty} \in L_T^{R_0}$, together with the estimate

$$(4.38) \qquad \left\| t^{\gamma_0}\,\|\mathcal{A}_0 f(t)\|_{L^\infty} \right\|_{L_T^{R_0}} \leq C\,\|s^{\alpha_2} f\|_{L_T^{R_2}(L^p)}\,T^{1/(2R_0)}\,.$$

The lemma is now proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Finally, we need the following lemma that has been established in [28]:

Lemma 4.6. *Let $1 < R, p < \infty$, and let $\alpha \geq 0$ be such that $\alpha + 1/R < 1$. Suppose that $t^\alpha f$ belongs to $L_T^R(L^p)$, for some $T \in\,]0, \infty]$.*
Then also $t^\alpha\,\mathcal{A}_2 f$ belongs to $L_T^R(L^p)$, and one has the estimate

$$\|t^\alpha\,\mathcal{A}_2 f\|_{L_T^R(L^p)} \leq C\,\|t^\alpha f\|_{L_T^R(L^p)}\,.$$

Now, we are in the position of proving Proposition 4.1.

Proof of Proposition 4.1. Recall that w satisfies (3.6), and thus

$$(4.39) \qquad w(t) = e^{-t\mathcal{L}} w_0 - \int_0^t e^{(s-t)\mathcal{L}} (\rho F)(s)\, ds \quad \text{with} \quad F \text{ given by (3.7)}.$$

Let us first study the term containing the initial data. By hypothesis, $\nabla^2 w_0 \in \dot{B}_{p,r}^{-2/r} \hookrightarrow \dot{B}_{p,R_2}^{-2/r}$, since $R_2 > 2 > r$ by our definitions. Thanks to Proposition 2.3, this implies that $t^{\alpha_2} \nabla^2 e^{-t\mathcal{L}} w_0$ belongs to $L^{R_2}(\mathbb{R}_+; L^p(\mathbb{R}^d))$.

In the same way, we have that $\nabla w_0 \in \dot{B}_{p,r_1}^{-2/r_1} \hookrightarrow \dot{B}_{p,R_1}^{-2/r_1}$ and $w_0 \in \dot{B}_{\infty,r_0}^{-2/r_0} \hookrightarrow \dot{B}_{\infty,R_0}^{-2/r_0}$, because we have taken $R_1 = R_0 = 2R_2 > \max\{r_0, r_1\}$. From these properties we deduce that $t^{\alpha_1} \nabla e^{-t\mathcal{L}} w_0 \in L^{R_1}(\mathbb{R}_+; L^p(\mathbb{R}^d))$ and that $t^{\alpha_0} e^{-t\mathcal{L}} w_0 \in L^{R_0}(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$. Since now both γ_1 and γ_0 are greater than α_1 and α_0 respectively, we get that, for all $T > 0$ fixed, $t^{\gamma_1} \nabla e^{-\mathcal{L}t} w_0 \in L_T^{R_1}(L^p)$ and $t^{\gamma_0} e^{-t\mathcal{L}} w_0 \in L_T^{R_0}(L^\infty)$.

As for the forcing term of (4.39), we apply Lemma 4.6 with $R = R_2$ and $\alpha = \alpha_2$ (note that $\alpha_2 + 1/R_2 = 1/r < 1$). We also apply Lemma 4.5. Therefore, if we set

$$\widetilde{\mathcal{N}}(T) := \|\varrho\|_{L_T^\infty(L^p \cap L^\infty)} + \|t^{\gamma_0} w\|_{L_T^{R_0}(L^\infty)} + \|t^{\gamma_1} \nabla w\|_{L_T^{R_1}(L^p)} + \|t^{\alpha_2} \nabla^2 w\|_{L_T^{R_2}(L^p)},$$

arguing exactly as in Paragraph 3.2.2, we get for some constant C_T bounded by a positive power of T,

$$(4.40) \qquad \widetilde{\mathcal{N}}(T) \leq C_T \left(\|\varrho_0\|_{L^p \cap L^\infty} + \|w_0\|_{\dot{B}_{p,r}^{2-2/r}} + \|t^{\alpha_2} \rho F\|_{L_T^{R_2}(L^p)} \right),$$

where F is defined in (3.7). At this point, we bound the term $\|t^{\alpha_2} \rho(t) F(t)\|_{L_T^{R_2}(L^p)}$ by following the computations of Paragraph 3.2.2: first of all, (3.19) is now replaced by the control

$$\|t^{\alpha_2} (\mathrm{Id} - \Delta)^{-1} \nabla P\|_{L_T^{R_2}(L^p)} \leq C\, T^{\alpha_2 + 1/R_2} \|\varrho\|_{L_T^\infty(L^p)} \leq C_T\, \widetilde{\mathcal{N}}(T).$$

Next, we have, noting that our conditions on the exponents imply that $\alpha_2 > \gamma_0 + \gamma_1$,

$$\|t^{\alpha_2} (w \cdot \nabla w + w \cdot \nabla v + v \cdot \nabla w + v \cdot \nabla v)\|_{L_T^{R_2}(L^p)}$$

$$\leq C\, T^{\alpha_2 - (\gamma_0 + \gamma_1)} \left(\|t^{\gamma_0} v\|_{L_T^{R_0}(L^\infty)} + \|t^{\gamma_0} w\|_{L_T^{R_0}(L^\infty)} \right) \left(\|t^{\gamma_1} \nabla v\|_{L_T^{T_1}(L^p)} + \|t^{\gamma_1} \nabla w\|_{L_T^{R_1}(L^p)} \right)$$

$$\leq C\, T^{\alpha_2 - (\gamma_0 + \gamma_1)} (1 + T^{\gamma_0})(1 + T^{\gamma_1})\, \widetilde{\mathcal{N}}^2(T),$$

and estimate (3.22) becomes

$$\|t^{\alpha_2} \nabla (\mathrm{Id} - \Delta)^{-1} \mathrm{div}(P\, u)\|_{L_T^{R_2}(L^p)} \leq C\, T^{\alpha_2 - \gamma_0} \|P\|_{L_T^{R_1}(L^p)} \left(\|t^{\gamma_0} v\|_{L_T^{R_0}(L^\infty)} + \|t^{\gamma_0} w\|_{L_T^{R_0}(L^\infty)} \right)$$

$$\leq C\, T^{\alpha_2 - \gamma_0 + 1/R_1} (1 + T^{\gamma_0})\, \widetilde{\mathcal{N}}^2(T).$$

Finally, we have

$$\|t^{\alpha_2} \nabla (\mathrm{Id} - \Delta^{-1}) \mathrm{div}\, u\|_{L_T^{R_2}(L^p)} \leq C\, T^{\alpha_2 - \gamma_1} \left(\|t^{\gamma_1} \nabla w\|_{L_T^{R_2}(L^p)} + \|t^{\gamma_1} P\|_{L_T^{R_2}(L^p)} \right)$$

$$\leq C\, T^{\alpha_2 - \gamma_1 + 1/(2R_2)} \left(1 + T^{1/(2R_2) + \gamma_1} \right) \widetilde{\mathcal{N}}(T),$$

and, arguing in a pretty similar way, we also get

$$\left\| t^{\alpha_2} \nabla (\mathrm{Id} - \Delta)^{-1} \left((g(\rho) - g(1))\, \mathrm{div}\, u \right) \right\|_{L_T^{R_2}(L^p)}$$

$$\leq C\, T^{\alpha_2 - \gamma_1} \|\varrho\|_{L_T^\infty(L^\infty)} \left(\|t^{\gamma_1} \Delta (\mathrm{Id} - \Delta)^{-1} P(\rho)\|_{L_T^{R_2}(L^p)} + \|t^{\gamma_1} \mathrm{div}\, w\|_{L_T^{R_2}(L^p)} \right)$$

$$\leq T^{\alpha_2 - \gamma_1 + 1/(2R_2)} \left(1 + T^{1/(2R_2) + \gamma_1} \right) \widetilde{\mathcal{N}}^2(T).$$

Putting all these bounds together, we end up with

$$\|t^{\alpha_2}\rho F\|_{L^{R_2}_T(L^p)} \le C_T\left(\widetilde{\mathcal{N}}(T) + \widetilde{\mathcal{N}}^2(T)\right).$$

Therefore, we can insert the previous inequality into (4.40): the application of a standard bootstrap argument allows us to find a time $T_* > 0$ such that, for all $t \in [0, T_*]$, one has

$$\widetilde{\mathcal{N}}(t) \le C\left(\|\varrho_0\|_{L^p\cap L^\infty} + \|w_0\|_{\dot{B}^{2-2/r}_{p,r}}\right),$$

for a suitable positive constant C, which completes the proof of Proposition 4.1. $\square$

APPENDIX A. HARMONIC ANALYSIS ESTIMATES

This appendix is devoted to the proofs of Lemma 2.5 and Proposition 2.5.

A.1. **Proof of Lemma 2.5.** It is based on the following Bony's paraproduct decomposition (first introduced in [3]) for the (formal) product of two tempered distributions u and v:

(A.1) $$u\,v = T_u v + T_v u + R(u,v),$$

where we have defined

$$T_u v := \sum_j S_{j-1}u\,\Delta_j v \quad \text{and} \quad R(u,v) := \sum_j \widetilde{\Delta}_j u\,\Delta_j v \quad \text{with} \quad \widetilde{\Delta}_j := \sum_{|j'-j|\le 1}\Delta_{j'}.$$

The above operator T is called *paraproduct* whereas R is called *remainder*. We refer to Chapter 2 of [1] for a presentation of continuity properties of the previous operators in the class of Besov spaces. For the time being, we limit ourselves to pointing out that the generic term $S_{j-1}u\,\Delta_j v$ of $T_u v$ is spectrally supported on dyadic annuli with radius of size about 2^j, while the generic term $\widetilde{\Delta}_j u\,\Delta_j v$ of $R(u,v)$ is supported on dyadic balls of size about 2^j.

One can now start the proof to Lemma 2.5. By using Bony's decomposition (A.1) and a commutator's process, we get, denoting $\widetilde{X} := (\mathrm{Id} - S_0)X$,

(A.2) $$\partial_X \sigma(D)g = \sigma(D)\mathrm{div}(Xg) + \left[T_{X^k}; \sigma(D)\partial_k\right]g - \sigma(D)\partial_k T_g X^k - \sigma(D)\partial_k R(\widetilde{X}^k, g)$$
$$+ T_{\sigma(D)\partial_k g}X^k + R(\widetilde{X}^k, \sigma(D)\partial_k g) + \left(R(S_0 X^k, \sigma(D)\partial_k g) - \sigma(D)\partial_k R(S_0 X^k, g)\right).$$

Bounding the first term relies on the fact that multiplier operators in S^{-1} map $B^0_{p,\infty}$ to $B^1_{p,\infty}$ (see [1, Prop. 2.78]) and that L^p is embedded in $B^0_{p,\infty}$. We thus have

$$\|\sigma(D)\mathrm{div}(Xg)\|_{B^1_{p,\infty}} \le C\,\|\mathrm{div}(Xg)\|_{B^0_{p,\infty}}$$
$$\le C\,\|\mathrm{div}(Xg)\|_{L^p} \le C\left(\|\partial_X g\|_{L^p} + \|\nabla X\|_{L^p}\|g\|_{L^\infty}\right).$$

Next, to handle the third term of (A.2), we use the fact that, being in S^0, the operator $\sigma(D)\partial_k$ maps $B^1_{p,\infty}$ to itself (again, see [1, Prop. 2.78]), that the paraproduct operator T maps $L^\infty \times B^1_{p,\infty}$ to $B^1_{p,\infty}$ and [1, Rem. 2.83], and that L^p is embedded in $B^0_{p,\infty}$. We eventually get

$$\|\sigma(D)\partial_k T_g X^k\|_{B^1_{p,\infty}} \le C\|T_g X\|_{B^1_{p,\infty}}$$
$$\le C\|g\|_{L^\infty}\|\nabla X\|_{B^0_{p,\infty}} \le C\|g\|_{L^\infty}\|\nabla X\|_{L^p}.$$

Similarly, since the remainder operator R maps $L^\infty \times B^1_{p,\infty}$ to $B^1_{p,\infty}$ and because, owing to the low frequency cut-off, we have

(A.3) $$\|\widetilde{X}\|_{B^1_{p,\infty}} \le C\|\nabla X\|_{B^0_{p,\infty}} \le C\|\nabla X\|_{L^p},$$

we readily get

$$\left\|\sigma(D)\partial_k R(\widetilde{X}^k, g)\right\|_{B^1_{p,\infty}} \le C\,\|g\|_{L^\infty}\,\|\nabla X\|_{L^p}.$$

Regarding the term $R(\widetilde{X}^k, \sigma(D)\partial_k g)$, we just have to use (A.3) and that R maps also $B^0_{\infty,\infty} \times B^1_{p,\infty}$ to $B^1_{p,\infty}$, to get

$$\|R(\widetilde{X}^k, \sigma(D)\partial_k g)\|_{B^1_{p,\infty}} \leq C \|\sigma(D)\partial_k g\|_{B^0_{\infty,\infty}} \|\nabla X\|_{L^p}.$$

Since $\sigma(D)\partial_k$ maps $B^0_{\infty,\infty}$ to itself, and because $L^\infty \hookrightarrow B^0_{\infty,\infty}$, that term also satisfies the required inequality.

The term $T_{\sigma(D)\partial_k g} X^k$ turns out to be the only one that cannot be bounded in $B^1_{p,\infty}$ under our assumptions. In fact, for that term, we use that the paraproduct maps $B^{s-1}_{\infty,\infty} \times B^1_{p,\infty}$ to $B^s_{p,\infty}$ (as $s - 1 < 0$) to write (still using [1, Rem. 2.83]),

$$\|T_{\sigma(D)\partial_k g} X^k\|_{B^s_{p,\infty}} \leq C\|\sigma(D)\partial_k g\|_{B^{s-1}_{\infty,\infty}} \|\nabla X^k\|_{L^p}.$$

Because $\sigma(D)\partial_k$ maps $B^{s-1}_{\infty,\infty}$ to itself, and $L^\infty \hookrightarrow B^{s-1}_{\infty,\infty}$, we get

$$\|T_{\sigma(D)\partial_k g} X^k\|_{B^s_{p,\infty}} \leq C\|g\|_{L^\infty} \|\nabla X\|_{L^p}.$$

To conclude the proof, it is only a matter of bounding suitably the two commutators terms in (A.2). First of all, notice that since the general term of the paraproduct is spectrally supported in dyadic annuli, one may find a smooth function ψ supported in some annulus centered at the origin, and such that

$$(A.4) \qquad [T_{X^k}; \sigma(D)\partial_k] g = \sum_{j \in \mathbb{Z}} \left[S_{j-1} X^k, \psi(2^{-j} D)\sigma(D)\partial_k \right] \Delta_j g.$$

For each fixed $j \in \mathbb{Z}$ and $k \in \{1, \cdots, d\}$, let us define $h^k_j := i\mathcal{F}^{-1}(\xi_k \psi(2^{-j}\cdot)\sigma)$. Then we have, thanks to the definition of $h^k_j(D)$ and the mean value formula,

$$
\begin{aligned}
\left[S_{j-1} X^k, \psi(2^{-j}D)\sigma(D)\partial_k \right] \Delta_j g(x) &= \int_{\mathbb{R}^d} h^k_j(y) \big(S_{j-1}X^k(x) - S_{j-1}X^k(x-y)\big)\Delta_j g(x-y)\, dy \\
&= -\int_0^1 \int_{\mathbb{R}^d} h^k_j(y)\, y \cdot \nabla S_{j-1}X^k(x - \tau y)\, \Delta_j g(x-y)\, dy\, d\tau \\
&= -\int_0^1 \int_{\mathbb{R}^d} h^k_j\Big(\frac{z}{\tau}\Big)\Big(\frac{z}{\tau}\Big) \cdot \nabla S_{j-1}X^k(x-z)\, \Delta_j g\Big(x - \frac{z}{\tau}\Big) \frac{dz}{\tau^d}\, d\tau.
\end{aligned}
$$

From the last line and convolution inequalities, we get

$$\left\| \left[S_{j-1}X^k, \psi(2^{-j}D)\sigma(D)\partial_k \right] \Delta_j g \right\|_{L^p} \leq \| |\cdot| h^k_j \|_{L^1} \|\Delta_j g\|_{L^\infty} \|\nabla S_{j-1}X^k\|_{L^p},$$

which, admitting for a while that

$$(A.5) \qquad \| |\cdot| h^k_j \|_{L^1} \leq C2^{-j}$$

and using the definition of the norm in $B^1_{p,\infty}$ implies that

$$\left\| [T_{X^k}; \sigma(D)\partial_k] g \right\|_{B^1_{p,\infty}} \leq C\|g\|_{L^\infty} \|\nabla X\|_{L^p}.$$

In order to prove (A.5), we use the fact that performing integration by parts,

$$(1 + |z|^2)^d (z h^k_j(z)) = (2\pi)^{-d} \int e^{iz\cdot\xi} (\mathrm{Id} - \Delta)^d \nabla\big(\xi_k \psi(2^{-j}\cdot)\sigma\big)(\xi)\, d\xi.$$

As integration may be restricted to those $\xi \in \mathbb{R}^d$ such that $|\xi| \sim 2^j$ and since σ is in S^{-1}, routine computations lead to

$$(1 + |z|^2)^d |z h^k_j(z)| \leq C2^{-j} \quad \text{for all } z \in \mathbb{R}^d,$$

whence Inequality (A.5).

In order to bound the last term of (A.2), we use the fact that, owing to the properties of the localization of the Littlewood-Paley decomposition, we have for some suitable smooth function ψ with compact support and value 1 on some neighborhood of the origin,

$$R(S_0 X^k, \sigma(D)\partial_k g) - \sigma(D)\partial_k R(S_0 X^k, g) = \sum_{j=-1}^{0} [\Delta_j S_0 X^k, \sigma(D)\psi(D)\partial_k]\widetilde{\Delta}_j g.$$

Then, arguing as above and setting $h^k := \mathcal{F}^{-1}(i\xi^k \psi \sigma)$, we find that

$$[\Delta_j S_0 X^k, \sigma(D)\psi(D)\partial_k]\widetilde{\Delta}_j g(x) = \int_0^1 \int_{\mathbb{R}^d} h^k(y)\, y \cdot \nabla\Delta_{j-1} S_0 X^k(x - \tau y)\, \widetilde{\Delta}_j g(x - y)\, dy\, d\tau.$$

Hence convolution inequalities and the fact that the only nonzero terms above correspond to $j = 0, 1$, lead us to

$$\|R(S_0 X^k, \sigma(D)\partial_k g) - \sigma(D)\partial_k R(S_0 X^k, g)\|_{L^p} \leq C2^{-j}\|\nabla\Delta_{j-1} S_0 X^k\|_{L^p}\|\widetilde{\Delta}_j g\|_{L^\infty}$$
$$\leq C2^{-j}\|\nabla X\|_{L^p}\|g\|_{L^\infty}.$$

This completes the proof to Lemma 2.5. $\square$

A.2. **Proof of Proposition 2.5.** For all $1 \leq j \leq d$ and $\eta > 0$, let us introduce the following *modified Riesz transform*:

(A.6) $$\mathcal{R}_j^{(\eta)} := \partial_j(\eta\,\mathrm{Id} - \Delta)^{-1/2},$$

so that $\mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)} = \partial_i\partial_j(\eta\,\mathrm{Id} - \Delta)^{-1}$.

Proposition 2.5 follows from Proposition 2.4 and the following lemma involving the tangential regularity with respect to *only one* vector field.

Lemma A.1. *Let $p \in\,]1, \infty[$ and take a vector-field $X \in \mathbb{L}^{\infty,p}$. Let $g \in L^\infty$ be such that $g \in \mathbb{L}_X^p$ and $\mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)} g \in L^\infty$ for some $\eta > 0$. There exists a constant $C > 0$ such that*

$$\left\|\partial_X \mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)} g\right\|_{L^p} \leq C\left((\|\mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)} g\|_{L^\infty} + \|g\|_{L^\infty})\|\nabla X\|_{L^p} + \|\partial_X g\|_{L^p}\right).$$

For proving that lemma, a few reminders concerning the Hardy-Littlewood *maximal function* $M[f]$ of a function f in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ are in order. Recall that it is defined by

$$M[f](x) := \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\, dy,$$

where $B(x, r)$ denotes the ball in $\mathbb{R}^d$ of center x and radius r, and $|B(x, r)|$ its Lebesgue measure.

The following statement is classical (for the proof, see e.g. Chapter 1 of [37]).

Lemma A.2. *The following properties hold true.*
(a) For any $1 < p \leq \infty$, there exist constants $0 < c < C$ such that for any function g in $L^p(\mathbb{R}^d)$,

$$c\,\|g\|_{L^p} \leq \|M[g]\|_{L^p} \leq C\,\|g\|_{L^p}.$$

(b) Let $p, q \in\,]1, \infty[$ or $p = q = \infty$. Let $\{f_j\}_{j\in\mathbb{Z}}$ be a sequence of functions in $L^p(\mathbb{R}^d)$ such that $(f_j)_{\ell^q(\mathbb{Z})} \in L^p(\mathbb{R}^d)$. Then there holds

$$\left\|(M[f_j])_{\ell^q}\right\|_{L^p} \leq C\left\|(f_j)_{\ell^q}\right\|_{L^p}.$$

(c) For any fixed $\Phi \in L^1(\mathbb{R}^d)$ such that $\Psi(x) = \sup_{|y|\geq|x|} |\Phi(y)| \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \Psi(x)dx = A$, and any function g, we have for all $x \in \mathbb{R}^d$,

$$\sup_{t>0}\left|g * \Phi_t(x)\right| \leq C\,M[g](x) \quad \text{with} \quad \Phi_t(x) := t^{-d}\,\Phi(x/t).$$

We shall also need the following definition.

Definition A.1. *Let $s \in \mathbb{R}$ and $p \in]1, \infty[$. The homogeneous Sobolev space $\dot{W}^{s,p}$ is defined as the set of $u \in \mathcal{S}'_h$ such that*

$$\|u\|_{\dot{W}^{s,p}} := \left\| (-\Delta)^{s/2} u \right\|_{L^p} < \infty.$$

The spaces L^p and $\dot{W}^{s,p}$ may be characterized in terms of Littlewood-Paley decomposition as they come up as special Triebel-Lizorkin spaces (see e.g. [36], Chapter 2),.

Proposition A.1. *Let $s \in \mathbb{R}$ and $p \in]1, \infty[$. Then one has the following equivalence of norms:*

$$\|u\|_{\dot{W}^{s,p}} \sim \left\| \left\| \left(2^{js} \dot{\Delta}_j u \right)_j \right\|_{\ell^2(\mathbb{Z})} \right\|_{L^p} \quad and \quad \|u\|_{L^p} \sim \left\| \left\| \left(\Delta_j u \right)_j \right\|_{\ell^2(j \geq -1)} \right\|_{L^p}.$$

Proof of Lemma A.1. We start the proof by remarking that

$$\partial_X \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g = \operatorname{div} \left(X \, \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g \right) - \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g \operatorname{div} X.$$

Since

$$\left\| \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g \operatorname{div} X \right\|_{L^p} \leq C \, \|\nabla X\|_{L^p} \, \|\mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g\|_{L^\infty},$$

we just need to bound the term $\operatorname{div}(X \, \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g)$ in L^p.

To this end, we resort again to Bony's decomposition (A.1) and get

$$
\begin{aligned}
(A.7) \quad \operatorname{div}(X \, \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g) &= \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} \operatorname{div}(X \, g) + \partial_k \left[T_{X^k}; \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} \right] g \\
&\quad + \partial_k T_{\mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g} X^k - \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} \partial_k T_g X^k + \partial_k R(X^k, \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g) - \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} \partial_k R(X^k, g).
\end{aligned}
$$

The first term in the right-hand side of the previous relation may be bounded by means of Corollary 2.1 and identity (1.7):

$$\left\| \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} \operatorname{div}(X \, g) \right\|_{L^p} \leq C \, \|\operatorname{div}(X \, g)\|_{L^p} \leq C \left(\|\partial_X g\|_{L^p} + \|\nabla X\|_{L^p} \, \|g\|_{L^\infty} \right).$$

Next, since we have

$$\left\| S_{m-1} \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g \right\|_{L^\infty} \leq C \, \|\mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g\|_{L^\infty},$$

we get for all $\ell \geq -1$, thanks to Lemma A.2(c), and using the fact that $S_{m-1} = 0$ for $m \leq 0$,

$$
\begin{aligned}
\left| (\Delta_\ell \partial_k T_{\mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g} X^k)(x) \right| &\leq C \, 2^\ell \sum_{|m-\ell| \leq 4} M \left[S_{m-1} \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g \, \Delta_m X^k \right](x) \\
&\leq C \sum_{|m-\ell| \leq 4} \left\| S_{m-1} \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g \right\|_{L^\infty} M \left[2^m \Delta_m X^k \right](x) \\
&\leq C \left\| \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g \right\|_{L^\infty} \sum_{|m-\ell| \leq 4, \, m \geq 1} M \left[2^m \Delta_m X^k \right](x).
\end{aligned}
$$

As a result, thanks to Proposition A.1 (where we take $s = 0$) and point (b) of Lemma A.2, we get

$$
\begin{aligned}
\left\| \partial_k T_{\mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g} X^k \right\|_{L^p} &\leq C \left\| \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g \right\|_{L^\infty} \left\| \left(\sum_{\ell \geq 1} \left(M \left[2^\ell \dot{\Delta}_\ell X^k \right] \right)^2 \right)^{1/2} \right\|_{L^p} \\
&\leq C \left\| \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} g \right\|_{L^\infty} \|\nabla X\|_{L^p}.
\end{aligned}
$$

Using the same strategy for handling $\partial_k T_g X^k$, we also obtain

$$\left\| \mathcal{R}_i^{(\eta)} \mathcal{R}_j^{(\eta)} \partial_k T_g X^k \right\|_{L^p} \leq C \left\| \partial_k T_g X^k \right\|_{L^p} \leq C \, \|g\|_{L^\infty} \, \|\nabla X\|_{L^p}.$$

For the third term of the second line in (A.7), we remove the low frequencies of X and consider the *modified* remainder defined by

$$\widetilde{R}(X^k, \mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)}g) = \sum_{m\geq 0} \Delta_m X^k \, \widetilde{\Delta}_m \mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)}g.$$

Then we write that for all $\ell \geq -1$, taking advantage of Lemma A.2,

$$\left|\Delta_\ell \partial_k \widetilde{R}(X^k, \mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)}g)(x)\right| \leq C\, 2^\ell \sum_{m\geq \max(0,\ell-5)} M\left[\Delta_m X^k \, \widetilde{\Delta}_m \mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)}g\right](x)$$

$$\leq C\, \|\mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)}g\|_{L^\infty} \sum_{m\geq \max(0,\ell-5)} 2^{\ell-m} M\left[2^m \Delta_m X^k\right](x),$$

Hence, from Proposition A.1 and Young inequality for convolutions, we infer that

$$\left\|\partial_k \widetilde{R}(X^k, \mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)}g)\right\|_{L^p} \leq C\, \|g\|_{L^\infty}\left(\sum_{\ell\geq -5} 2^{-\ell}\right)\left\|\left\|\left(M\left[2^j \Delta_j X\right](x)\right)_{j\in\mathbb{N}}\right\|_{\ell^2}\right\|_{L^p}$$

$$\leq C\, \|g\|_{L^\infty} \|\nabla X\|_{L^p}.$$

Obviously, the same estimate holds true for the term $\mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)}\partial_k \widetilde{R}(X^k, g)$.

The low frequencies terms that have been discarded have to be treated together, that is, we have to bound $\partial_k[\Delta_{-1}X^k, \mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)}]\widetilde{\Delta}_{-1}g$ in L^p. and this may be done in the same way as at the end of the proof of Lemma 2.5. We end up with

$$\|\partial_k[\Delta_{-1}X^k, \mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)}]\widetilde{\Delta}_{-1}g\|_{L^p} \leq C\|\Delta_{-1}\nabla X\|_{L^p}\|\widetilde{\Delta}_{-1}g\|_{L^\infty} \leq C\|\nabla X\|_{L^p}\|g\|_{L^\infty}.$$

Finally, it remains us to handle the commutator term on the right-hand side of (A.7). We start by decomposing $\mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)}$ into

(A.8) $$\mathcal{R}_i^{(\eta)}\mathcal{R}_j^{(\eta)} = \mathcal{R}_i\mathcal{R}_j - \eta(\eta\,\mathrm{Id} - \Delta)^{-1}\mathcal{R}_i\mathcal{R}_j,$$

where $\mathcal{R}_j$ stands for the classical Riesz transform (which corresponds to take $\eta = 0$ in (A.6)). Let us first consider the part of the commutator corresponding to $\mathcal{R}_i\mathcal{R}_j$. We have

$$\left([T_{X^k}; \mathcal{R}_i\mathcal{R}_j]\, g\right)(x) = \sum_{m\geq 1}[S_{m-1}X^k, \mathcal{R}_i\mathcal{R}_j]\Delta_m g.$$

Let $\theta(\xi) = -\xi_i\xi_j|\xi|^{-2}\varphi(\xi)$, with φ being the function used in the Littlewood-Paley decomposition. Since the generic term $[S_{m-1}X^k, \mathcal{R}_i\mathcal{R}_j]\Delta_m g$ is supported on dyadic annuli of size 2^m, one can write

$$[S_{m-1}X^k, \mathcal{R}_i\mathcal{R}_j]\Delta_m g =$$

$$= \sum_{|m-\ell|\leq 4}\left(S_{m-1}X^k\,\Delta_m\mathcal{R}_i\mathcal{R}_j\Delta_\ell g \,-\, \mathcal{R}_i\mathcal{R}_j\Delta_\ell\left(S_{m-1}X^k\,\Delta_m g\right)\right)$$

$$= \sum_{|m-\ell|\leq 4}\left(S_{m-1}X^k\left(S_{m+1}\theta(2^{-\ell}D)g \,-\, S_m\theta(2^{-\ell}D)g\right) \,-\, \theta(2^{-\ell}D)\left(S_{m-1}X^k\left(S_{m+1}g - S_m g\right)\right)\right)$$

Therefore, by applying Abel rearrangement techniques an using that $\Delta_j = \dot{\Delta}_j$ for $j \in \mathbb{N}$, we get

$$[T_{X^k}; \mathcal{R}_i\mathcal{R}_j]\, g = -\sum_{m\geq 2}\sum_{|m-\ell|\leq 4}\left[\dot{\Delta}_{m-2}X, \theta(2^{-\ell}D)\right]S_m g.$$

The general term of the above series is spectrally supported in dyadic annuli of size about 2^m. Therefore, there exists some universal integer N_0 so that for all $q \in \mathbb{Z}$,

(A.9) $$\dot{\Delta}_q\partial_k\left[T_{X^k}; \mathcal{R}_i\mathcal{R}_j\right]g = \sum_{|m-q|\leq N_0}\sum_{|\ell-m|\leq 4}\dot{\Delta}_q\partial_k\left[\theta(2^{-\ell}D), \dot{\Delta}_{m-2}X\right]S_m g.$$

Now, the point (c) of Lemma A.2 ensures that for all $x \in \mathbb{R}^d$,

$$(A.10) \qquad \left| \dot{\Delta}_q \partial_k \Big[\theta(2^{-\ell}D), \dot{\Delta}_{m-2}X \Big] S_m g(x) \right| \le C 2^q M \Big[\big[\theta(2^{-\ell}D), \dot{\Delta}_{m-2}X \big] S_m g \Big](x)$$

and the mean value formula gives us, denoting $\check{\theta} := \mathcal{F}^{-1}\theta$,

$$\Big(\big[\theta(2^{-\ell}D), \dot{\Delta}_{m-2}X^k \big] S_m g \Big)(x) = -2^{d\ell} \int_0^1 \int_{\mathbb{R}^d} \check{\theta}(2^\ell z)\, z \cdot \nabla \dot{\Delta}_{m-2}X^k(x - \tau z) S_m g(x - z)\, dz\, d\tau\,,$$

whence, performing a change of variables and setting $\Psi(z) := z\check{\theta}(z)$,

$$\Big(\big[\theta(2^{-\ell}D), \dot{\Delta}_{m-2}X^k \big] S_m g \Big)(x) = -2^{-\ell} \int_0^1 \int_{\mathbb{R}^d} \Big(\frac{2^\ell}{\tau} \Big)^d \Psi\Big(\frac{2^\ell z}{\tau} \Big) \cdot \nabla \dot{\Delta}_{m-2}X^k(x - z) S_m g\Big(x - \frac{z}{\tau} \Big) dz\, d\tau\,.$$

From that latter relation, we deduce that

$$M \Big[\big[\theta(2^{-\ell}D), \dot{\Delta}_{m-2}X \big] S_m g \Big](x) \le C 2^{-\ell} \|g\|_{L^\infty} \int_0^1 \int_{\mathbb{R}^d} \Big(\frac{2^\ell}{\tau} \Big)^d \Big| \Psi\Big(\frac{2^\ell z}{\tau} \Big) \Big| M\big[\nabla \dot{\Delta}_{m-2}X^k(x - z) \big] dz\, d\tau$$

$$\le C 2^{-\ell} \|g\|_{L^\infty} M\big[\dot{\Delta}_{m-2}\nabla X \big](x)\,.$$

We now plug that inequality in (A.10) and (A.9), then take the norm in $\ell^2(\mathbb{Z})$ with respect to q and eventually compute the norm in $L^p(\mathbb{R}^d)$. We end up with

$$\Big\| \partial_k \dot{\Delta}_q \big[T_{X^k}; \mathcal{R}_i \mathcal{R}_j \big] g \Big\|_{L^p(\ell^2(\mathbb{Z}))} \le C \| M(\dot{\Delta}_q \nabla X) \|_{L^p(\ell^2(\mathbb{Z}))}$$

Therefore, by applying Proposition A.1 with $s = 0$ and the point (b) of Lemma A.2, we finally get

$$(A.11) \qquad \big\| \partial_k \big[T_{X^k}; \mathcal{R}_i \mathcal{R}_j \big] g \big\|_{L^p} \le C \, \|g\|_{L^\infty} \, \|\nabla X\|_{L^p}\,.$$

To complete the proof, we have to bound the commutator term corresponding to the last part of (A.8). To do this, we use the fact that

$$\eta \big[T_{X^k}; (\eta \,\mathrm{Id} - \Delta)^{-1}\mathcal{R}_i \mathcal{R}_j \big] g = \eta \big[T_{X^k}; (\eta \,\mathrm{Id} - \Delta)^{-1} \big] \mathcal{R}_i \mathcal{R}_j g + \eta (\eta \,\mathrm{Id} - \Delta)^{-1} \big[T_{X^k}; \mathcal{R}_i \mathcal{R}_j \big] g.$$

To handle the last term, we just have to use (A.11) and the fact that $\eta(\eta \,\mathrm{Id} - \Delta)^{-1}$ maps L^p to itself (uniformly with respect to η). For the other term, we use that, by embedding,

$$\| \eta \partial_k \big[T_{X^k}; (\eta \,\mathrm{Id} - \Delta)^{-1} \big] \mathcal{R}_i \mathcal{R}_j g \|_{L^p} \le C \eta \| \big[T_{X^k}; (\eta \,\mathrm{Id} - \Delta)^{-1} \big] \mathcal{R}_i \mathcal{R}_j g \|_{B_{p,1}^1}.$$

Then, using the fact that the multiplier $\eta(\eta \,\mathrm{Id} - \Delta)^{-1}$ is in S^{-2} (uniformly with respect to $\eta \le 1$) and arguing as for bounding (A.4), one ends up with

$$\| \eta \partial_k \big[T_{X^k}; (\eta \,\mathrm{Id} - \Delta)^{-1} \big] \mathcal{R}_i \mathcal{R}_j g \|_{L^p} \le C \|\nabla X\|_{L^p} \|\mathcal{R}_i \mathcal{R}_j g\|_{B_{\infty,1}^{-1}} \le C \|\nabla X\|_{L^p} \|\mathcal{R}_i \mathcal{R}_j g\|_{B_{\infty,\infty}^0}.$$

Clearly, in the above computations, the low frequencies of g are not involved. Hence, we actually have, using that $\mathcal{R}_i$ maps $\dot{B}_{\infty,\infty}^0$ to itself and that $\dot{B}_{\infty,\infty}^0 \hookrightarrow L^\infty$,

$$\| \eta \partial_k \big[T_{X^k}; (\eta \,\mathrm{Id} - \Delta)^{-1} \big] \mathcal{R}_i \mathcal{R}_j g \|_{L^p} \le C \|\nabla X\|_{L^p} \|g\|_{L^\infty}.$$

Summing up all the above estimate concludes the proof of the lemma. $\qquad \square$

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