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ABSTRACT. Let $x \mapsto x + \alpha \mod 1$ be a rotation on the circle and let φ be a step function. We denote by $\varphi_n(x)$ the corresponding ergodic sums $\sum_{j=0}^{n-1} \varphi(x + j\alpha)$. For a class of irrational rotations (containing the class with bounded partial quotients) and under a Diophantine condition on the discontinuity points of φ, we show that $\varphi_n/\|\varphi_n\|_2$ is asymptotically Gaussian for n in a set of density 1. The proof is based on decorrelation inequalities for the ergodic sums taken at times q_k, where (q_k) is the sequence of denominators of α. Another important point is the control of the variance $\|\varphi_n\|_2^2$ for n belonging to a large set of integers. When α is a quadratic irrational, the size of this set can be precisely estimated.

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Introduction

For a dynamical system (X, μ, T) and an observable φ on X, a general question is the asymptotic behaviour in distribution of the ergodic sums $\sum_0^{L-1} \varphi \circ T^k$ after normalisation. For a large class of observables and chaotic systems, many results of convergence toward a Gaussian distribution are known.

When the dynamical system has zero entropy, in particular for a rotation, the situation is different. Nevertheless one can ask if, at least, there are observables satisfying a non degenerate Central Limit Theorem. In this direction there are positive answers: R. Burton and M. Denker [4] in 1987, then T. de la Rue, S. Ladouceur, G. Peskir and M. Weber [9], M. Lacey [18] proved for rotations the existence of functions whose ergodic sums satisfy a CLT after self-normalization. In general for a measure preserving aperiodic system, further results by D. Volný and P. Liardet [21], J.-P. Thouvenot and B. Weiss [23] showed that any distribution can appear as a limiting distribution of the ergodic sums of some functions after normalisation.

A different question is to ask if, for smooth systems, there is a CLT for explicit functions in a certain class of regularity. Here we consider step functions on $X = \mathbb{R}/\mathbb{Z}$ and their ergodic sums $\varphi_n(x) := \sum_0^{n-1} \varphi(x + j\alpha)$ over an irrational rotation $x \mapsto x + \alpha \bmod 1$.

By the Denjoy-Koksma inequality, if φ is a centered function with bounded variation, the sequence (φ_n) is uniformly bounded along the sub-sequence of denominators of α. But, besides, a stochastic behaviour at a certain scale can occur along other sub-sequences (n_k). We propose a quantitative analysis of this phenomenon.

Let us mention the following related papers. For $\psi := 1_{[0,\frac{1}{2}[} - 1_{[\frac{1}{2},0[}$, F. Huveneers [15] studied the existence of a sequence $(n_k)_{n\in\mathbb{N}}$ such that (ψ_{n_k}) after normalization is asymptotically normally distributed. In [6] it was shown that, when α has unbounded partial quotients, along some subsequences the ergodic sums of φ in a class of step functions can be approximated by a Brownian motion.

Here we will use as in [15] a method based on decorrelation inequalities which applies in particular when the sequence of partial quotients of α is bounded (α is said to be of bounded type or bpq) or under a slightly more general Diophantine assumption. It relies on an abstract central limit theorem valid under some suitable decorrelation conditions. If φ is a step function, we give conditions which ensure that for n in a set of integers of density 1, the distribution of $\varphi_n/\|\varphi_n\|_2$ is asymptotically Gaussian (Theorem 2.4). Beside the remarkable recent "temporal" limit theorems for rotations of bounded type (see [1], [2], [10], [3]), this shows that a "spatial" asymptotic normal distribution can also be observed for n in a large set of integers.

An important point is the control of the variance $\|\varphi_n\|_2^2$. In Section 1, we study the set of integers for which the variance $\|\varphi_n\|_2^2$ of the ergodic sums is big (expected to be of order $\ln n$ for n belonging to a set of density 1, in the case α bpq). The most precise information is obtained in the special case where α is a quadratic irrational in Subsection 1.4.

The central limit theorem is presented in Section 2. It is based on the decorrelation between the ergodic sums at times q_k (the denominators of α) and on an abstract central limit theorem. To apply the results to a step function, a Diophantine condition is needed on the discontinuities of φ which holds generically.

The proofs of the CLT and the decorrelation are given in Sections 3 and 4. In Appendix 1, we prove a proposition used for quadratic numbers in the study of the variance.

The results of this paper have been announced in [5]. The authors thank the referees for their helpful remarks.

1. Variance of the ergodic sums

Notation The uniform measure on $\mathbb{T}^1$ identified with $X = [0,1[$ is denoted by μ. A function φ on $\mathbb{T}^1$ is viewed as a 1-periodic function of a real variable. We denote by $V(\varphi)$ the *variation* of the restriction of φ to $[0,1]$ and write BV for "with bounded variation".

The class of real centered BV functions on $\mathbb{T}^1$ is denoted by $\mathcal{C}$. It contains the 1-periodic step functions with a finite number of discontinuities. The Fourier coefficients of a function φ are denoted by $\hat{\varphi}(r)$. For $\varphi \in \mathcal{C}$, we can write:

$$\hat{\varphi}(r) = \frac{\gamma_r(\varphi)}{r}, r \neq 0, \text{ with } K(\varphi) := \sup_{r\neq 0} |\gamma_r(\varphi)| \leq \frac{V(\varphi)}{2\pi} < +\infty. \tag{1}$$

Let $\alpha = [0; a_1, a_2, \ldots]$ be an irrational number in $]0,1[$, with partial quotients $a_n = a_n(\alpha)$, numerators p_n and denominators q_n, $n \geq 1$.

The ergodic sums $\sum_{j=0}^{n-1} \varphi(x + j\alpha)$ of a 1-periodic function φ for the rotation by α are denoted by $\varphi_n(x)$. Their Fourier expansion is $\varphi_n(x) = \displaystyle\sum_{r\neq 0} \frac{\gamma_r(\varphi)}{r}\, e^{\pi i(n-1)r\alpha}\, \frac{\sin \pi n r\alpha}{\sin \pi r\alpha}\, e^{2\pi i r x}$.

If $\varphi \in \mathcal{C}$, then $\mathrm{V}(\varphi_n) \leq nV(\varphi)$ and $|\widehat{\varphi_n}(r)| = \dfrac{|\gamma_r(\varphi)|}{|r|}\, \dfrac{|\sin \pi n r\alpha|}{|\sin \pi r\alpha|} \leq \dfrac{n\, K(\varphi)}{|r|}, r \neq 0.$

1.1. Reminders on continued fractions.

In this subsection, we recall some classical results on diophantine approximation. For this material we refer to [16] or [19], as well as J. Beck's book [2].

For $u \in \mathbb{R}$, $\{u\}$ denotes its fractional part and $\|u\| := \inf_{n\in\mathbb{Z}} |u - n| = \min(\{u\}, 1 - \{u\})$ its distance to $\mathbb{Z}$. Recall that $2\|x\| \leq |\sin \pi x| \leq \pi \|x\|,\ \forall x \in \mathbb{R}$.

For $n \geq 1$, writing $\alpha = \dfrac{p_n}{q_n} + \dfrac{\theta_n}{q_n}$, we have

$$\frac{1}{a_{n+1}+2} \leq \frac{q_n}{q_{n+1}+q_n} \leq q_n\|q_n\alpha\| = q_n|\theta_n| \leq \frac{q_n}{q_{n+1}} = \frac{q_n}{a_{n+1}q_n+q_{n-1}} \leq \frac{1}{a_{n+1}}, \tag{2}$$

$$\theta_n = (-1)^n\|q_n\alpha\|,\ \alpha = \frac{p_n}{q_n} + (-1)^n\frac{\|q_n\alpha\|}{q_n},\ \frac{1}{2}q_{n+1}^{-1} \leq \theta_n \leq q_{n+1}^{-1}, \tag{3}$$

$$q_{n+1}/q_{n+k} \leq C\,\rho^k,\ \forall n,k \geq 1,\ \text{with } C = \frac{5+\sqrt{5}}{2}, \rho = \frac{\sqrt{5}-1}{2} < 1. \tag{4}$$

Let us show the last inequality: for $n \geq 1$ fixed, putting $r_0 = q_n$, $r_1 = q_{n+1}$, $r_{k+1} = r_k + r_{k-1}$, for $k \geq 1$, we have $q_{n+k} \geq r_k$, $\forall k \geq 0$, by induction and (4) follows easily.

For $n \geq 1$, we denote by $m(n)$ the integer such that $n \in [q_{m(n)},\ q_{m(n)+1}[$.

If α has bounded partial quotients (i.e., $\sup a_n < \infty$), then $m(n)$ is of order $\ln n$.

Ostrowski's expansion ([22], [2])

Every integer $n \geq 1$ can be represented as follows (α-*Ostrowski's expansion*):

$$\text{if } n < q_{m+1},\ n = \sum_{k=0}^{m} b_k\,q_k, \text{with } 0 \leq b_0 \leq a_1 - 1,\ 0 \leq b_k \leq a_{k+1} \text{ for } 1 \leq k \leq m. \tag{5}$$

Indeed, if $n \in [q_0,\ q_1 = a_1[$, then (5) is satisfied, and if $n \in [q_m,\ q_{m+1}[$ with $m \geq 1$, we write $n = b_m q_m + r$, with $1 \leq b_m \leq a_{m+1}$, $0 \leq r < q_m$. By iteration, we get either $r = 0$ at some point and the algorithm stops, or $n \in [q_0, q_1[$. In either cases we obtain (5).

In this way, we can code every $n < q_{m+1}$ by a word $b_0...b_m$, with $b_0 \in \{0, 1, ..., a_1 - 1\}$ and $b_j \in \{0, 1, ..., a_{j+1}\}$, $j = 1, ..., m$.

In this representation, $b_{m(n)} \neq 0$ and $b_j = 0$ for $m(n) < j \leq m$ when $m > m(n)$. In the latter case, there are $m - m(n)$ zero's at the right end. For a given m and $n < q_{m+1}$, this Ostrowski's expansion is "proper" (without zeros at the end) if $m = m(n)$.

For $m \geq 0$, we call *admissible of length* $m+1$ a finite word $b_0...b_m$ such that $b_0 \in \{0, 1, ..., a_1 - 1\}$, $b_j \in \{0, 1, ..., a_{j+1}\}$, for $j = 1, ..., m$ and such that, for two consecutive letters b_{j-1}, b_j, if $b_j = a_{j+1}$ then $b_{j-1} = 0$.

Remark that if $b_0...b_m$ is admissible, $m \geq 1$, then $b_0...b_{m-1}$ is admissible. Let us show by induction that the Ostrowski's expansion of an integer n is admissible. Let n be in $[q_m,\ q_{m+1}[$. We start the construction of the expansion of n as above. Now the following steps of the algorithm yield the Ostrowski's expansion of $n - b_m q_m$. Since $n - b_m q_m \in [0,\ q_m[$, the Ostrowski's expansion of $n - b_m q_m$ is admissible. It remains to check that, if $b_m = a_{m+1}$, then $b_{m-1} = 0$. But if $b_{m-1} \neq 0$, we would have $n \geq a_{m+1}q_m + q_{m-1} = q_{m+1}$, a contradiction.

Conversely, if $b_0...b_m$ is admissible, one shows by induction that $b_0 + b_1q_1 + ... + b_mq_m < q_{m+1}$. This holds if $m = 0$, since $b_1 < q_1 = a_1$. Assume that this is true for the length m. Let $b_0...b_mb_{m+1}$ be admissible of length $m+1$.

If $b_{m+1} = a_{m+2}$, then $b_m = 0$ and $b_0 + b_1 q_1 + ... + b_m q_m = b_0 + b_1 q_1 + ... + b_{m-1} q_{m-1} < q_m$, so that $b_0 + b_1 q_1 + ... + b_{m+1} q_{m+1} < q_m + a_{m+2} q_{m+1} = q_{m+2}$.
If $b_{m+1} \leq a_{m+2} - 1$, then $b_0 + b_1 q_1 + ... + b_{m+1} q_{m+1} < q_{m+1} + (a_{m+2} - 1)\, q_{m+1} < q_{m+2}$.

Therefore, if we associate to an admissible word the integer $n = b_0 + b_1 q_1 + ... + b_m q_m$, there is a bijection between the Ostrowski's expansions of integers $n < q_{m+1}$ and the set of admissible words of length $m+1$. The number of admissible words of length m is $q_m - 1$.

For n given by (5), putting $n_0 = b_0$, $n_k = \sum_{t=0}^{k} b_t\, q_t$, for $k \leq m(n)$, we have

$$\begin{aligned} \varphi_n(x) &= \sum_{k=0}^{m(n)} \sum_{j=n_{k-1}}^{n_k - 1} \varphi(x + j\alpha) = \sum_{k=0}^{m(n)} \sum_{j=0}^{b_k\, q_k - 1} \varphi(x + n_{k-1}\alpha + j\alpha) \\ &= \sum_{k=0}^{m(n)} \sum_{i=0}^{b_k - 1} \varphi_{q_k}(x + (n_{k-1} + i q_k)\alpha) = \sum_{k=0}^{m(n)} f_k(x), \end{aligned} \tag{6}$$

$$\text{with } f_k(x) := \sum_{i=0}^{b_k - 1} \varphi_{q_k}(x + (n_{k-1} + i q_k)\alpha) = \varphi_{b_k q_k}(x + n_{k-1}\alpha), \tag{7}$$

By convention, we put $\sum_{i=0}^{b_k - 1} \varphi_{q_k}(x + (n_{k-1} + i q_k)\alpha) = 0$, if $b_k = 0$.
If φ is a BV centered function, then it holds (*Denjoy-Koksma inequality*):

$$\|\varphi_q\|_\infty = \sup_x |\sum_{i=0}^{q-1} \varphi(x + i\alpha)| \leq V(\varphi), \text{ if } q \text{ is a denominator of } \alpha. \tag{8}$$

One can also show that if φ satisfies (1) then $\|\varphi_{q_n}\|_2 \leq 2\pi\, K(\varphi)$. By (8), we have for f_k defined by (7): $\|f_k\|_\infty \leq b_k V(\varphi) \leq a_{k+1} V(\varphi)$.

1.2. **Bounds for the variance.**

Let $\varphi \in \mathcal{C}$ and $n \in [q_{\ell-1},\, q_\ell[$. The variance is bounded from below as follows:

$$\|\varphi_n\|_2^2 = 2 \sum_{k>1} |\widehat{\varphi}(k)|^2 \frac{(\sin \pi n k\alpha)^2}{(\sin \pi k\alpha)^2} \geq 2 \sum_{j=1}^{\ell} |\widehat{\varphi}(q_j)|^2 \frac{(\sin \pi n q_j\alpha)^2}{(\sin \pi q_j\alpha)^2} \geq c_0 \sum_{j=1}^{\ell} |\widehat{\varphi}(q_j)|^2 \frac{\|n q_j\alpha\|^2}{\|q_j\alpha\|^2},$$

with $c_0 = \frac{8}{\pi^2}$. Therefore, by (2) we have, for $0 < \delta < \frac{1}{2}$,

$$\|\varphi_n\|_2^2 \;\geq\; c_0 \sum_{j=1}^{\ell} |\gamma_{q_j}(\varphi)|^2\, a_{j+1}^2\, \|n q_j\alpha\|^2 \geq c_0\, \delta^2 \sum_{j=1}^{\ell} |\gamma_{q_j}(\varphi)|^2\, a_{j+1}^2\, 1_{\|n q_j\alpha\| \geq \delta}. \tag{9}$$

An upper bound for the variance and a lower bound for the mean of the variance are shown in [6]: there are constants $C, c > 0$ such that

$$\|\varphi_n\|_2^2 \leq C\,K(\varphi)^2 \sum_{j=0}^{m(n)} a_{j+1}^2, \tag{10}$$

$$\frac{1}{n}\sum_{k=0}^{n-1} \|\varphi_k\|_2^2 \geq c \sum_{j=0}^{m(n)-1} |\gamma_{q_j}(\varphi)|^2\, a_{j+1}^2. \tag{11}$$

Inequality (9) gives a semi explicit lower bound for the variance. Note that by (8), the variance is small if n is a denominator q_i of α. In this case, as expected, one finds that the lower bound given by (9) is small. Indeed, by (4), we have $\|q_i q_j \alpha\| \leq C_1 \rho^{|i-j|}$, with $\rho < 1$, for a constant C_1, so that, for a given $\delta > 0$, the number of j's less than ℓ such that $\|q_i q_j \alpha\| \geq \delta$ is bounded independently from ℓ.

Now our first goal will be to bound from below the variance $\|\varphi_n\|_2$ by a big value for n in a large set.

Bounds for the variance for n in a large set of integers

According to (9), a lower bound for $\|\varphi_n\|_2$ depends on two separate conditions:

First we need the following condition on the Fourier coefficients of φ:

$$\exists M, \eta, \theta > 0 \text{ such that } \mathrm{Card}\,\{j \leq N : a_{j+1}\,|\gamma_{q_j}(\varphi)| \geq \eta\} \geq \theta\, N,\ \forall N \geq M. \tag{12}$$

This condition clearly holds when φ is the function $\varphi^0(x) = \{x\} - \frac{1}{2}$, since in this case $|\gamma_{q_j}(\varphi^0)| = \frac{1}{2\pi}, \forall j$. Its validity, related to Diophantine conditions on the points of discontinuities, will be discussed for some step functions in Subsection 2.2.

For $j < \ell$, we will estimate how many times $\{nq_j\alpha\} \in I_\delta := [0, \delta] \cup [1-\delta, 1]$ for $n \leq q_\ell$ and deduce from this estimation that $\sum_{j=1}^{\ell} 1_{I_\delta}(\{n\,q_j\,\alpha\}) = \sum_{j=1}^{\ell} 1_{\|nq_j\alpha\| \leq \delta}$ is small for a large set of values of n.

Lemma 1.1. *For every $\delta \in]0, \frac{1}{2}[$ and every interval of integers $I = [N_1,\, N_1 + L[$, we have*

$$\sum_{n=N_1}^{N_1+L-1} 1_{I_\delta}(\{n\,q_j\,\alpha\}) \leq 20\,(\delta + q_{j+1}^{-1})\,L, \forall j \text{ such that } q_{j+1} \leq 2L. \tag{13}$$

Proof. For a fixed j and $0 \leq N_1 < N_1 + L$, let us describe the behaviour of the sequence $(\|n\,q_j\alpha\|, n = N_1, ..., N_1 + L - 1)$.

Recall that (modulo 1) we have $q_j\alpha = \theta_j$, with $\theta_j = (-1)^j \|q_j\alpha\|$ (see (3)). We treat the case j even (hence $\theta_j > 0$). The case j odd is analogous.

We are going to count how many times, for j even, we have $\{n\,\theta_j\} < \delta$ or $1 - \delta < \{n\,\theta_j\}$.

We start with $n_1 := N_1$. Putting $w(j,1) := \{n_1\,\theta_j\}$, we have $\{n\,\theta_j\} = w(j,1) + (n - n_1)\theta_j$, for $n = n_1, n_1 + 1, ..., n_2 - 1$, where n_2 is such that $w(j,1) + (n_2 - 1 - n_1)\,\theta_j < 1 < w(j,1) + (n_2 - n_1)\,\theta_j$.

Putting $w(j,2) := \{n_2\,\theta_j\}$, we have $w(j,2) = w(j,1) + (n_2 - n_1)\,\theta_j - 1 < \theta_j$. Starting now from n_2, we have $\{n\,\theta_j\} = w(j,2) + (n-n_2)\theta_j$ for $n = n_2, n_2+1, ..., n_3 - 1$, where n_3 is such that $w(j,2) + (n_3 - 1 - n_2)\,\theta_j < 1 < w(j,2) + (n_3 - n_2)\,\theta_j$.

We iterate up to $R(j)$, where $n_{R(j)-1} < N_1 + L \le n_{R(j)}$. This construction yields a sequence $n_1 < n_2 < ... < n_{R(j)}$ such that $\{n\theta_j\} = w(j,i) + (n-n_i)\theta_j$, $\forall n \in [n_i, n_{i+1}[$, and

$$w(j,i) + (n_{i+1} - 1 - n_i)\,\theta_j < 1 < w(j,i) + (n_{i+1} - n_i)\,\theta_j,$$

with $w(j,i)$ defined recursively by $w(j,i+1) = \{w(j,i) + (n_{i+1} - n_i)\,\theta_j\}$ and satisfying $w(j,i) < \theta_j$, for $i = 1, ..., R(j)$.

Since $(n_{i+1} - n_i + 1)\,\theta_j \ge w(j,i) + (n_{i+1} - n_i)\,\theta_j > 1$ for $i \ne 1$ and $i \ne R(j)$, we have $n_{i+1} - n_i \ge \theta_j^{-1} - 1$, for each $i \ne 1, R(j)$. This implies $R(j) \le \dfrac{L}{\theta_j^{-1} - 1} + 2$.

For each i, the number of integers $n \in [n_i, n_{i+1} - 1[$ such that $\{n\theta_j\} \in [0, \delta[\cup]1-\delta, 1[$ is bounded by $2(1 + \delta\,\theta_j^{-1})$. (This number is less than 2 if $\delta < \theta_j$.)

Altogether, using (3) and the assumption $2L \ge q_{j+1}$, the number of integers $n \in I$ such that $\{n\theta_j\} \in [0, \delta[\cup]1-\delta, 1[$ is bounded by

$$2R(j)(1 + \delta\,\theta_j^{-1}) \le (\frac{2L}{\theta_j^{-1} - 1} + 4)\,(1 + \delta\,\theta_j^{-1}) \le 4\,(L + \theta_j^{-1})\,(\delta + \theta_j)$$
$$\le 4\,(L + 2q_{j+1})\,(\delta + q_{j+1}^{-1}) \le 20\,(\delta + q_{j+1}^{-1})\,L. \quad \square$$

Remark 1. For every $\delta \in]0, \frac{1}{2r}[$ and every interval $[N_1, N_1 + L]$, we have by a slight extension of Lemma 1.1:

$$\text{Card}\,\{n \in [N_1, N_1 + L] :\ d(nq_j\alpha, \mathbb{Z}/r) \le \delta\} \le 20r\,(\delta + q_{j+1}^{-1})L, \text{ if } q_{j+1} \le 2L. \tag{14}$$

Lemma 1.2. *Let $I = [N_1, N_1 + L]$ be an interval and ℓ such that $q_\ell \le 2L$.*

a) For all $\delta \in]0, \frac{1}{2}[$ and $\zeta \in]0, 1[$, the density of set

$$A := \{n \in I :\ \text{Card}\,(j < \ell : d(nq_j\alpha, \mathbb{Z}) \le \delta) \le \zeta\ell\} \tag{15}$$

satisfies

$$\text{Card}(A) \ge (1 - 20\,\zeta^{-1}\,(\delta + C\ell^{-1}))\,L. \tag{16}$$

b) Under Condition (12) on φ, there are positive constants η_0, c (not depending on δ) such that, for every $\delta \in]0, \frac{1}{2}[$, the subset $V(I,\delta,\ell) := \{n \in I : \|\varphi_n\|_2 \ge \eta_0\,\delta\,\sqrt{\ell}\}$ satisfies:

$$\text{Card}\,(V(I,\delta,\ell)) \ge (1 - c\,(\delta + \ell^{-1}))\,L. \tag{17}$$

Proof. a) Let $A^c = I \setminus A$ be the complementary of A. We will find an upper bound of the density $L^{-1}\,\text{Card}(A^c)$ by counting the number of values of n in I such that $\|nq_j\alpha\| < \delta$ in an array indexed by (j,n).

By summing (13) from $j=0$ to $j=\ell-1$ and using the definition of A, we get:

$$20\,(\delta\ell+\sum_{0\le j\le \ell-1} q_{j+1}^{-1})\,L \ge \sum_{0\le j\le \ell-1}\sum_{n\in I} 1_{I_\delta}(\{n\,q_j\,\alpha\})$$
$$\ge \sum_{n\in A^c}\sum_{0\le j\le \ell-1} 1_{I_\delta}(\{n\,q_j\,\alpha\}) \ge \sum_{n\in A^c} \zeta\,\ell = \zeta\,\ell\,\mathrm{Card}(A^c).$$

With $C:=\sum_{j=0}^{\infty} q_j^{-1}$, we have $\mathrm{Card}(A^c)\le 20\,\zeta^{-1}\,(\delta+C\ell^{-1})\,L$, so (16) is shown.

b) With $\zeta=\frac{1}{2}\theta$, where θ is the constant in (12), in view of the definition of A and (12), we have, for $n\in A$:

$$\mathrm{Card}\big(\{j\le \ell-1 : \|nq_j\alpha\|\ge\delta\}\bigcap\{j : |\gamma_{q_j}(\varphi)|\ge\eta\}\big)\ge (1-(\zeta+1-\theta))\,\ell=\frac{1}{2}\theta\,\ell.$$

Putting $c:=20\,\zeta^{-1}\max(1,C)$ and $\eta_0=(\frac{1}{2}c_0\,\eta^2\,\theta)^{\frac{1}{2}}$, this implies by (9) and (16):

$$\|\varphi_n\|_2^2\ge\frac{1}{2}c_0\,\delta^2\,\eta^2\,\theta\,\ell=\eta_0^2\,\delta^2\,\ell,\ \forall n\in A,\ \text{and}\ \mathrm{Card}\,(A)\ge 1-c\,(\delta+\ell^{-1})\,L; \tag{18}$$

hence $A\subset V(I,\delta,\ell)$ and therefore $V(I,\delta,\ell)$ satisfies (17). □

The constants c and η_0 below are those of Lemma 1.2.

Theorem 1.3. *Under Condition (12) on φ, the density of the subset*

$$W:=\{n\in\mathbb{N}:\ \|\varphi_n\|_2\ge\eta_0\,\big(\frac{m(n)}{\ln m(n)}\big)^{\frac{1}{2}}\}$$

satisfies for every $N\ge 1$:

$$\frac{\mathrm{Card}\,\big(W\cap[0,N[\big)}{N}\ge 1-2c\,(\ln m(N))^{-\frac{1}{2}}.$$

Proof. Since $t/\ln t$ is increasing for $t\ge e$, we have, after the first terms, for n in $W^c\cap[0,N[$:

$$\|\varphi_n\|_2<\eta_0\big(\frac{m(n)}{\ln m(n)}\big)^{\frac{1}{2}}\le\eta_0\big(\frac{m(N)}{\ln m(N)}\big)^{\frac{1}{2}}.$$

Therefore, by b) of Lemma 1.2 with $I=[0,N[$, $L=N$, $\delta=(\ln m(N))^{-\frac{1}{2}}$ and $\ell=m(N)$, it follows

$$\frac{\mathrm{Card}\,\big(W^c\cap[0,N[\big)}{N}\le c\,(\ln m(N))^{-\frac{1}{2}}+c\,m(N)^{-1}\le 2c\,(\ln m(N))^{-\frac{1}{2}}.\quad\square$$

1.3. **A counter-example.**

In the next sections we will show that, under a Diophantine condition on α, for a big set of n, the distribution of $\varphi_n/\|\varphi_n\|_2$ is approximately Gaussian. In particular, by (11), if $n_\ell\le q_{\ell+1}$ is an integer such that $\|\varphi_{n_\ell}\|_2=\max_{k<q_{\ell+1}}\|\varphi_k\|_2$, then we have the lower bound $\|\varphi_{n_\ell}\|_2^2\ge c\sum_{j=0}^{\ell-1}|\gamma_{q_j}|^2a_{j+1}^2$. Under Condition (12), it can be shown that, for these indices n_ℓ giving the record variances, when the partial quotients of α are bounded, the distribution of $\varphi_{n_\ell}/\|\varphi_{n_\ell}\|_2$ is asymptotically Gaussian.

Let us show by a counter-exemple that this is not true without a condition on α.

For a parameter $\gamma > 0$, let the sequence $(a_n)_{n\geq 1}$ be defined by

$$a_n = \lfloor n^\gamma \rfloor \text{ if } n \in \{2^k \ : \ k \geq 0\}, \ = 1 \text{ if } n \notin \{2^k \ : \ k \geq 0\}.$$

Let α be the number which has $(a_n)_{n\geq 1}$ for sequence of partial quotients. For simplicity, let us take for φ the sawtooth function φ^0 defined above for which $\gamma_k = \frac{-1}{2\pi i}$, $\forall k$.

For $\ell > 1$, let $n_\ell := \max\{n < q_{\ell+1} \ : \ \|\varphi_n\|_2 = \max_{k<q_{\ell+1}} \|\varphi_k\|_2\}$. We have

$$\|\varphi_n\|_2^2 \geq c \sum_{j=0}^{\ell-1} |\gamma_{q_j}|^2 a_{j+1}^2 \geq c \sum_{s=0}^{\lfloor \log_2(\ell) \rfloor} 2^{2\gamma s} \geq c\ell^{2\gamma}.$$

In the sum $\varphi_{n_\ell}(x) = \sum_{k=0}^{\ell} \sum_{j=0}^{b_k\, q_k - 1} \varphi(x + N_{k-1}\alpha + j\alpha)$ defined in (6), we can isolate the indices k for which $k+1$ is a power of 2 (for the other indices $a_{k+1} = 1$) and write $\varphi_{n_\ell}(x) = U_\ell + V_\ell$ with

$$U_\ell = \sum_{p=1}^{\lfloor \log_2(\ell) \rfloor} \sum_{j=0}^{b_{2^p}\, q_{2^p} - 1} \varphi(x + N_{2^p-1}\alpha + j\alpha),$$
$$V_\ell = \sum_{k\in[0,\ell]\cap\{a_{k+1}=1\}} \sum_{j=0}^{b_k\, q_k - 1} \varphi(x + N_{k-1}\alpha + j\alpha).$$

We will see in (49) that the variance of a sum where b_k equals 0 or 1 is bounded as follows

$$\| \sum_{k\in[0,\ell]\cap\{a_{k+1}=1\}} \sum_{j=0}^{b_k\, q_k - 1} \varphi(x + N_{k-1}\alpha + j\alpha)\|_2^2 \leq C\ell \log(\ell).$$

On the other side, we also have

$$| \sum_{p=1}^{\lfloor \log_2(\ell) \rfloor} \sum_{j=0}^{b_{2^{p-1}}\, q_{2^{p-1}} - 1} \varphi(x + N_{2^p-2}\alpha + j\alpha)| \leq \sum_{p=1}^{\lfloor \log_2(\ell) \rfloor} a_{2^p} V(\varphi) \leq C \sum_{p=1}^{\lfloor \log_2(\ell) \rfloor} 2^{\gamma p} \leq C\ell^\gamma.$$

The previous bounds imply $\frac{\varphi_{n_\ell}}{\|\varphi_{n_\ell}\|_2} = \frac{U_\ell}{\|\varphi_{n_\ell}\|_2} + \frac{V_\ell}{\|\varphi_{n_\ell}\|_2}$ with $\|\varphi_{n_\ell}\|_2 \geq c\ell^\gamma$, $\|U_\ell\|_\infty \leq C\ell^\gamma$, $\|V_\ell\|_2 \leq (C\ell\log(\ell))^{1/2}$.

Thus, if $\gamma > 1/2$, one has $\left\|\frac{U_\ell}{\|\varphi_{n_\ell}\|_2}\right\|_\infty \leq \frac{C}{c}$, $\left\|\frac{V_\ell}{\|\varphi_{n_\ell}\|_2}\right\|_2 \to 0$ and the limit points of the distributions of $\dfrac{\varphi_{n_\ell}}{\|\varphi_{n_\ell}\|_2}$ have all their supports included in $[-\frac{C}{c}, \frac{C}{c}]$, hence are not Gaussian.

1.4. **A special case: quadratic numbers.**

When α is a quadratic number, using the ultimate periodicity of the sequence $(a_n(\alpha))_{n\geq 1}$ and the good properties of the associated Ostrowski's expansion of the integers, it is possible to improve the result of Theorem 1.3 on the variance. In this subsection we show that the variance $\|\varphi_n\|_2^2$ of the ergodic sums of φ under the rotation by a quadratic number α is of order $\ln n$ for n in a big set of integers whose size is precisely estimated. For example, if we take $\varphi(x) = \{x\} - \frac{1}{2}$, Theorem 1.5 shows that there are positive constants

η_1, η_2, R and $\xi \in]0,1[$ such that,

$$\frac{1}{N}\,\mathrm{Card}\,\{n \le N :\ \eta_1\,\ln n \le \|\varphi_n\|_2^2 \le \eta_2\,\ln n\} \ge (1 - R\,N^{-\xi}). \tag{19}$$

The main step in the proof is the following proposition showing that, in case of a quadratic number, for most of the integers n (in a set whose size is precisely estimated), $\|nq_j\alpha\|$ is far from 0 for a big proportion of j's:

Proposition 1.4. *If α is a quadratic number, for every $\varepsilon_0 \in]0, \frac{1}{2}[$, there are $\delta \in]0, \frac{1}{2}[$ and positive constants C and ξ such that for every $\ell \ge 1$:*

$$\mathrm{Card}\,\{n < q_{\ell+1} :\ \mathrm{Card}\,(j < \ell : d(nq_j\alpha, \mathbb{Z}) \ge \delta) \ge (1 - \varepsilon_0)\,\ell\} \ge (1 - Cq_{\ell+1}^{-\xi})\,q_{\ell+1}. \tag{20}$$

The proof of Proposition 1.4 is given in Appendix.

Theorem 1.5. *If α is a quadratic number and if φ satisfies Condition (12), there are positive constants η_1, η_2, R and $\xi \in]0,1[$ such that, for N big enough, it holds :*

$$\mathrm{Card}\,\{n \le N :\ \eta_1\,\ln n \le \|\varphi_n\|_2^2 \le \eta_2\,\ln n\} \ge N\,(1 - R\,N^{-\xi}). \tag{21}$$

Proof. There is $\eta_2 > 0$ such that the upper bound in (21) holds for every $n \ge 1$: indeed, when α is quadratic, as (q_k) is equivalent to a geometric sequence, $m(n)$ is equivalent to $\ln(n)$ up to a multiplicative constant factor. Therefore, for $n \in [q_\ell, q_{\ell+1}[$ (i.e., $m(n) = \ell$), (10) implies $\|\varphi_n\|_2^2 \le CK(\varphi)^2 \sum_{j=0}^{\ell} a_{j+1}^2 \le \eta_2 \ln(n)$, for some positive constant η_2.

For the lower bound, by (9) we have $\|\varphi_n\|_2^2 \ge c_0\,\delta^2 \sum_{j=1}^{\ell} |\gamma_{q_j}(\varphi)|^2\,a_{j+1}^2\,1_{\|nq_j\alpha\| \ge \delta}$. Let φ in $\mathcal{C}$ be such that (12) is satisfied and, for $\varepsilon_0 = \frac{1}{2}\theta$, let $\delta = \delta(\varepsilon_0)$ be given by Proposition 1.4. According to (12) and (20), for ℓ big enough, the set of integers $n < q_{\ell+1}$ such that simultaneously $\|nq_j\alpha\| \ge \delta$ and $|\gamma_{q_j}(\varphi)| \ge \eta$, for at least $\frac{1}{2}\theta\ell$ different indices j, has a cardinal bigger than $q_{\ell+1}(1 - C\,q_{\ell+1}^{-\xi})$ for some constants $C > 0$, $\xi \in]0,1[$.

Therefore we have $\|\varphi_n\|_2^2 \ge \frac{c_0}{2}\eta^2\,\delta^2\theta\,\ell = \eta_1\ell$ for more than $q_{\ell+1}(1 - Cq_{\ell+1}^{-\xi})$ values of n between 1 and $q_{\ell+1}$.

This shows that, for $N \in [q_\ell, q_{\ell+1}[$, the cardinal of the set $\{n < N : \|\varphi_n\|_2^2 \le \eta_1\ell\}$ is less than $Cq_{\ell+1}^{1-\xi} \le C'N^{1-\xi}$ (because for a quadratic number $\sup_\ell q_{\ell+1}/q_\ell < +\infty$). Hence, the result. □

2. A central limit theorem and its application to rotations

2.1. Decorrelation and CLT.

An abstract CLT under a decorrelation property

Below Y_1 denotes a r.v. with a normal distribution $\mathcal{N}(0,1)$. Recall that, if X, Y are two real random variables, their mutual (Kolmogorov) distance in distribution is defined by: $d(X,Y) = \sup_{x\in\mathbb{R}} |\mathbb{P}(X \le x) - \mathbb{P}(Y \le x)|$.

The notation C denotes an absolute constant whose value may change from a line to the other.

Proposition 2.1. *Let N be a positive integer. Let $(q_k)_{1\le k\le N}$ be an increasing sequence of positive integers such that for a constant $\rho\in]0,1[$*

$$q_k/q_m \le C\,\rho^{m-k},\ 1\le k<m\le N. \tag{22}$$

Let $(f_k)_{1\le k\le N}$ be real centered BV functions such that for constants u_k

$$\|f_k\|_\infty \le u_k,\ \mathrm{V}(f_k)\le C\,u_k\,q_k,\ 1\le k\le N. \tag{23}$$

Moreover assume that, for some constant θ, the following decorrelation properties hold:

$$|\int_X \psi\, f_k\, d\mu| \le C\,\mathrm{V}(\psi)\,u_k\,\frac{k^\theta}{q_k},\ 1\le k\le N, \forall\psi\ BV, \tag{24}$$

$$|\int_X \psi\, f_k\, f_m\, d\mu| \le C\,V(\psi)\,u_k\,u_m\,\frac{m^\theta}{q_k},\ 1\le k\le m\le N, \forall\psi\ BV\ centered, \tag{25}$$

$$|\int_X \psi\, f_k\, f_m\, f_t\, d\mu| \le C\,V(\psi)\,u_k\,u_m\,u_t\,\frac{t^\theta}{q_k},\ 1\le k\le m\le t\le N, \forall\psi\ BV\ centered. \tag{26}$$

Then, putting $w_N:=\max_{j=1}^N u_j$, $S_N:=f_1+\cdots+f_N$, there is for every $\delta>0$ a constant $C(\delta)>0$ (depending only on δ) such that the condition

$$\frac{w_N}{\|S_N\|_2}\le N^{p-\frac12},\ \textit{with } p\in[0,\frac18[, \tag{27}$$

implies

$$d(\frac{S_N}{\|S_N\|_2},Y_1)\le C(\delta)N^{-\frac{1-8p}{12}+\delta}. \tag{28}$$

The proposition is proved in Section 3. We apply it to an irrational rotation by taking for q_k's the denominators of α (they satisfy (22)) and for f_k the ergodic sums $\varphi_{b_kq_k}$ of a function φ (composed by a translation), where the b_k's ($b_k\le a_{k+1}$) are given by the Ostrowski's expansion described above.

Decorrelation between partial ergodic sums

In order to apply the previous proposition we will prove decorrelation properties between the ergodic sums of $\varphi\in\mathcal{C}$ at time q_n under the following assumption on α:

Hypothesis 1. *There are two constants $A\ge1$, $p\ge0$ such that*

$$a_n\le A\,n^p, \forall n\ge1. \tag{29}$$

Remark 2. a) The case α of bounded type, i.e., with bounded partial quotients, corresponds to $p=0$. In this case, as we have seen, $m(n)$ is of order $\ln n$.

b) Observe that $m(n)$ can be smaller, but at least of order $\dfrac{\ln n}{\ln\ln n}$ up to a bounded factor, under the more general assumption 1.

Lemma 2.2. *a) For every $p>1$, for a.e. α, there is a finite constant $A(\alpha,p)$ such that*

$$a_n\le A(\alpha,p)\,n^p, \forall n\ge1. \tag{30}$$

b) If α satisfies (29), then there is $c > 0$ such that

$$\|k\alpha\| \geq \frac{c}{|k|\,(\log k)^p},\ \forall k > 1. \tag{31}$$

Proof. a) We have $a_{n+1}(\alpha) = \lfloor 1/\theta^n(\alpha) \rfloor$ where θ is the Gauss map. Let $\gamma > 1$. Since $\alpha \to (a_1(\alpha))^{\frac{1}{\gamma}}$ is integrable for the θ-invariant measure $\frac{dx}{1+x}$ on $]0,1]$, we have, for a constant $A(\gamma)$: $\mu\{\alpha : a_n(\alpha) > n^s\} \leq A(\gamma)\, n^{-\frac{s}{\gamma}}$.

By the Borel-Cantelli lemma, it follows that for a.e α there is $C(\alpha,\gamma)$ such that, if $s > \gamma$, $a_n(\alpha) \leq C(\alpha,\gamma)\, n^s, \forall n \geq 1$.

b) For every irrational α, there are $C > 0$ and $\lambda > 1$ such that the denominators of α satisfy $q_\ell \geq C\lambda^\ell$, for every $\ell \geq 1$. For $k \geq 2$, let n be such that $q_{n-1} \leq k < q_n$. Since $C\lambda^{n-1} \leq q_{n-1} \leq k$, it follows that $n \leq C' \log k$, for some constant C'. By (29), we have $a_n \leq An^p \leq A(C' \log k)^p$.

Since $\|k\alpha\| > \|q_{n-1}\alpha\| \geq \dfrac{1}{2q_n} \geq \dfrac{1}{4a_n q_{n-1}} \geq \dfrac{1}{4a_n k}$, this implies (31). □

As a corollary, using Theorem 1.3, it follows that for a.e. α, under the rotation by α, for a function $\varphi \in \mathcal{C}$ satisfying (12), the growth of the variance $\|\varphi_n\|_2^2$ is "roughly" of order $\ln n$. In Subsection 2.2 we will see that, if α satisfies (29) with $p < \frac{1}{4}$, (12) itself is a generic condition for some class of step functions φ.

Proposition 2.3. *Let ψ and φ be BV centered functions. Suppose that α satisfies Hypothesis 1. Then there are constants $C, \theta_1, \theta_2, \theta_3$ such that, for every $1 \leq k \leq m \leq \ell$:*

$$|\int_X \psi\, \varphi_{b_k q_k}\, d\mu| \leq C\, \mathrm{V}(\psi)\ \mathrm{V}(\varphi)\, \frac{k^{\theta_1}}{q_k}\, b_k, \tag{32}$$

$$|\int_X \psi\, \varphi_{b_k q_k} \varphi_{b_m q_m}\, d\mu| \leq C\, \mathrm{V}(\psi)\, \mathrm{V}(\varphi)^2\, \frac{m^{\theta_2}}{q_k}\, b_k b_m, \tag{33}$$

$$|\int_X \psi\, \varphi_{b_k q_k} \varphi_{b_m q_m} \varphi_{b_\ell q_\ell}\, d\mu| \leq C\, \mathrm{V}(\psi)\, \mathrm{V}(\varphi)^3\, \frac{\ell^{\theta_3}}{q_k}\, b_k b_m b_\ell. \tag{34}$$

The proposition is proved in Section 4. From the propositions 2.1 and 2.3 we will deduce a convergence toward a Gaussian distribution under a variance condition, by bounding the distance to the normal distribution.

Theorem 2.4. *Let φ be in $\mathcal{C}$ satisfying (12).*

1) The set defined (cf. Theorem 1.3) by

$$W := \{n \in \mathbb{N} :\ \|\varphi_n\|_2 \geq \eta_0\, (\log m(n))^{-\frac{1}{2}}\, m(n)^{\frac{1}{2}}\} \tag{35}$$

has density 1 in $\mathbb{N}$.

Suppose that α satisfies Hypothesis 1 (i.e., for constants $A \geq 1$, $p \geq 0$, $a_n \leq A\, n^p, \forall n \geq 1$) with $p < \frac{1}{8}$. Then, for $\delta \in]0, \frac{1-8p}{12}[$, there is a constant $C(\delta)$ such that, for n in W,

$$d(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1) \leq C(\delta)\, m(n)^{-\frac{1-8p}{12}+\delta} \underset{n\in W,\, n\to\infty}{\longrightarrow} 0. \tag{36}$$

In particular when α has bounded partial quotients, we have $p = 0$ and $m(n)$ can be replaced by $\log n$.

2) Suppose that α is a quadratic irrational. With the notation of Theorem 1.5, let

$$V := \{n \geq 1 : \eta_1 \sqrt{\log n} \leq \|\varphi_n\|_2 \leq \eta_2 \sqrt{\log n}\}.$$

Then, there are two constants $R, \xi > 0$ such that
- the density of V satisfies:

$$\text{Card}(V \bigcap [1, N]) \geq N\,(1 - R\,N^{-\xi}),\ \textit{for } N \geq N_0; \tag{37}$$

- for $\delta \in]0, \frac{1}{12}[$, there is a constant $C(\delta)$ such that, for $n \in V$:

$$d(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1) \leq C(\delta)\,(\log n)^{-\frac{1}{12}+\delta} \underset{n\in V,\, n\to\infty}{\longrightarrow} 0. \tag{38}$$

Proof. 1) The result on the density of the set W follows from Theorem 1.3.

For (36), we use Proposition 2.1 with $N = m(n)$ (where $m(n)$ is such that $n \in [q_{m(n)},\, q_{m(n)+1}[$), f_k defined by (7) and the decomposition of the ergodic sums given by (6), i.e.,

$$\varphi_n(x) = \sum_{k=0}^{m(n)} f_k(x),\ \text{where } f_k(x) := \sum_{i=0}^{b_k-1} \varphi_{q_k}(x + (n_{k-1} + iq_k)\alpha) = \varphi_{b_k q_k}(x + n_{k-1}\alpha).$$

The decorrelation inequalities in Proposition 2.3 are obtained for functions of the form $\varphi_{b_k q_k}$. But in the proof of the decorrelation inequalities, one sees that they remain valid for f_k, since translations on the variable do not change the modulus of the Fourier coefficients.

As $\|f_k\|_\infty \leq b_k V(\varphi) \leq a_{k+1} V(\varphi)$, up to a fixed factor the constant u_k in the statement of Proposition 2.3 can be taken to be $a_{k+1} \leq k^p$, for some constant $p > 0$, by Hypothesis 1.

With the notation of Proposition 2.1, we have $w_N := \max_{j=1}^N b_j$, $\varphi_n = S_N = f_1 + \cdots + f_N$. For $n \in W$ and under Hypothesis 1, we have

$$\frac{w_N}{\|S_N\|_2} \leq C N^{p-\frac{1}{2}} (\log N)^{\frac{1}{2}}.$$

The factor $(\log N)^{\frac{1}{2}}$ can be absorbed in the factor $N^{p-\frac{1}{2}}$ by taking p larger and we have (27). By (28) it follows:

$$d(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1) \leq C(\delta)\, m(n)^{-\frac{1-8p}{12}+\delta}.$$

2) In the quadratic case, $p = 0$ and the property of the set V is given by Theorem 1.5. □

Remark 3. The previous result is written with a self-normalisation. If α is quadratic, let us consider the ergodic sums normalised by $\sqrt{\ln n}$: $(\varphi_n/\sqrt{\ln n})_{n\geq 1}$. Then, for $n \in V$, the accumulation points of the sequence of distributions are Gaussian non degenerated with a variance belonging to a compact interval.

2.2. **Application to step functions, examples.**

If φ belongs to the class $\mathcal{C}$ of centered BV functions, with Fourier series $\sum_{r\neq 0}\frac{\gamma_r(\varphi)}{r}\,e^{2\pi ir.}$, to apply Theorem 2.4 we have to check Condition (12) on the coefficients $\gamma_{q_k}(\varphi)$, i.e.:

$$\exists M,\eta,\theta>0 \text{ such that } \frac{1}{N}\mathrm{Card}\,\{j\leq N : a_{j+1}\,|\gamma_{q_j}(\varphi)|\geq\eta\}\geq\theta,\ \forall N\geq M.$$

The functions $\{x\}-\frac{1}{2}=\frac{-1}{2\pi i}\sum_{r\neq 0}\frac{1}{r}\,e^{2\pi irx}$ and $1_{[0,\frac{1}{2}[}-1_{[\frac{1}{2},1[}=\sum_r\frac{2}{\pi i(2r+1)}\,e^{2\pi i(2r+1)\cdot}$ are immediate examples where this condition is satisfied. In the second case, one observes that $\gamma_{q_k}=0$ if q_k is even, $=\frac{2}{\pi i}$ if q_k is odd. Clearly, (12) is satisfied, because two consecutive q_k's are relatively prime and therefore cannot be both even.

In general, for a step function, Condition (12) (and therefore a lower bound for the variance $\|\varphi_n\|_2^2$ for a large set of integers n) is related to the Diophantine properties of its discontinuities with respect to α. We discuss now this point.

Let us consider a centered step function φ on $[0,1[$ taking a non null constant value $v_j\in\mathbb{R}$ on the interval $[u_j,u_{j+1}[$, $j=0,1,...,s$, with $u_0=0<u_1<...<u_s<u_{s+1}=1$:

$$\varphi=\sum_{j=0}^{s}v_j\,1_{[u_j,u_{j+1}[}-c. \tag{39}$$

The constant c above is such that φ is centered, but it plays no role below.

Lemma 2.5. *If φ is given by (39), there is a continuous periodic function $H_\varphi(u_1,...,u_s)\geq 0$ such that*

$$|\gamma_r(\varphi)|^2=\pi^{-2}\,H_\varphi(ru_1,...,ru_s). \tag{40}$$

Proof. Since $\widehat{\varphi}(r)=\sum_{j=0}^{s}\frac{v_j}{\pi r}\,e^{-\pi ir(u_j+u_{j+1})}\,\sin\pi r(u_{j+1}-u_j)$, $r\neq 0$, $H_\varphi(u_1,...,u_s)$ is

$$[\sum_{j=0}^{s}v_j\,\cos\pi(u_j+u_{j+1})\,\sin\pi(u_{j+1}-u_j)]^2+[\sum_{j=0}^{s}v_j\,\sin\pi(u_j+u_{j+1})\,\sin\pi(u_{j+1}-u_j)]^2.\quad\square$$

Examples: 1) $\varphi=\varphi(u,\cdot)=1_{[0,u[}-u$, $H_\varphi(u)=\sin^2(\pi u)$.

2) $\varphi=\varphi(w,u,\cdot)=1_{[0,\ u]}-1_{[w,\ u+w]}$, $H(\varphi)=4\,\sin^2(\pi u)\,\sin^2(\pi w)$.

We show now that (12) is satisfied generically by the family of step functions parametrised by $(u_1,...,u_s)$ defined by (39).

Corollary 2.6. *1) Suppose that φ is a step function given by (39) for $s\geq 1$, with parameter $(u_1,...,u_s)$. Then Condition (12) is satisfied if $(u_1,...,u_s)$ is such that the sequence $(q_ku_1,...,q_ku_s)_{k\geq 1}$ is uniformly distributed in $\mathbb{T}^s$.*

2) This latter condition holds for a.e. value of $(u_1,...,u_s)$ in $\mathbb{T}^s$.

Proof. 1) If the sequence $(q_k u_1, ..., q_k u_s)_{k\geq 1}$ is uniformly distributed in $\mathbb{T}^s$, we have with the notation of Lemma 2.5:

$$\lim_N \frac{1}{N}\sum_{k=1}^N |\gamma_{q_k}(\varphi)|^2 = \lim_n \frac{1}{n}\sum_{k=1}^n H_\varphi(q_k u_1, ..., q_k u_s)$$

$$= \int_{\mathbb{T}^s} H(x_1, ..., x_s)\, dx_1 ... dx_s > 0, \text{ for a.e. } (u_1, ..., u_s) \in \mathbb{T}^s. \tag{41}$$

Let N_0 and $\delta > 0$ be such that, for $N \geq N_0$, $\frac{1}{N}\sum_{k=1}^N |\gamma_{q_k}(\varphi)|^2 \geq \delta$. The sequence $(|\gamma_{q_k}(\varphi)|^2, k \geq 1)$ is bounded by $K := \pi^{-2}\|H_\varphi\|_\infty$. Therefore, we have, for $N \geq N_0$,

$$\delta \leq \frac{1}{N}\sum_{k=1}^N |\gamma_{q_k}(\varphi)|^2 \leq \frac{K}{N}\sum_{k=1}^N 1_{|\gamma_{q_j}(\varphi)|\geq\eta} + \frac{\eta^2}{N}\sum_{k=1}^N 1_{|\gamma_{q_j}(\varphi)|<\eta} \leq \frac{K}{N}\sum_{k=1}^N 1_{|\gamma_{q_j}(\varphi)|\geq\eta} + \eta^2.$$

This shows: $\frac{1}{N}\sum_{k=1}^N 1_{|\gamma_{q_j}(\varphi)|\geq\eta} \geq K^{-1}(\delta - \eta^2)$, for $N \geq N_0$.

It follows that (12) is satisfied with $M = N_0, \eta = (\frac{\delta}{2})^{\frac{1}{2}}, \theta = K^{-1}\frac{\delta}{2}$.

2) To prove the uniform distribution for a.e. value of $(u_1, ..., u_s)$ in $\mathbb{T}^s$, by Weyl equirepartition criterium it suffices to show, for all integers $r_1, ..., r_s$ not all 0,

$$\lim_k \frac{1}{N}\sum_{k=1}^N e^{2i\pi q_k(r_1u_1+...+r_su_s)} = 0, \text{ for a.e. } (u_1, ..., u_s) \in \mathbb{T}^s. \tag{42}$$

Since (q_k) is a strictly increasing sequence of integers, (42) follows from the law of large numbers for orthogonal bounded variables (Rajchman's theorem) which is recalled in Appendix 2 in a slightly more general formulation (Proposition 6.1). □

Besides a generic result, there are also specific values of the parameter $(u_1, ..., u_s)$ for which (12) holds. A simple example (for $s = 1$) is:

Example 3: $\varphi(\frac{r_1}{r_2}, \cdot) = 1_{[0,\frac{r_1}{r_2}[} - \frac{r_1}{r_2}$, for $r_1, r_2 \in \mathbb{N}, 0 < r_1 < r_2$.

We will give another example of special values related to the rectangular billiard model in example 4 below.

Remark 4. For the case of example 1, let us make some remarks about the degeneracy of the variance.

It is known that if α is bpq and if $\lim_k |\sin(\pi q_k u)| = 0$, where q_k are the denominators of α, then $u \in \mathbb{Z}\alpha + \mathbb{Z}$ (cf. for instance [8]). But it is easily seen that there is an uncountable set of u's such that $\lim_N \frac{1}{N}\sum_{k=1}^N \sin^2(\pi q_k u) = 0$ and thus for which Condition (12) does not hold.

Observe also that, if α is not bpq, there are many u's for which the sequence $(q_k u \bmod 1)$ does not satisfy the equidistribution property in a strong sense and (41) fails.

Indeed, let $u = \sum_{n\geq 0} b_n q_n \alpha \bmod 1$, $b_n \in \mathbb{Z}$, $0 \leq b_n \leq a_{n+1}$, be the so-called Ostrowski expansion of u associated to the denominators of α. It can be shown that, if $\lim_n \frac{|b_n|}{a_{n+1}} = 0$,

then $\lim_k \|q_k u\| = 0$ (Proposition 1 in [13]). There is an uncountable set of u's satisfying the condition $\lim_n \frac{|b_n|}{a_{n+1}} = 0$ if α is not bpq. For these values of u, we have $\lim_k \gamma_{q_k}(\varphi(u,.)) = 0$. Therefore Condition (12), which is used to get a lower estimate of the variance, fails, although, if u is not in the countable set $\mathbb{Z}\alpha + \mathbb{Z}$, $\varphi(u,.)$ is not a coboundary (and even generates an ergodic cocycle).

Remark 5. Another remark is about the "generic" validity of estimates of the variance.

As previous remarked, in Theorem 2.4 the CLT is written with self-normalisation (by $\|\varphi_n\|_2$). In Theorem 1.3 the lower bound given for the variance $\|\varphi_n\|_2^2$ for n in the set W can be smaller than the mean of the variance.

Inequalities (10) and (11) give a precise estimation of the variance in the mean when an information is available on $\gamma_{q_i}(\varphi)$.

For example in the case of the "saw-tooth" function, we get the estimate $\sum_{k=1}^{m(n)} a_k^2$ for the mean of the variance.

If we consider Example 1 or more generally $\varphi = \varphi(u,.)$ given by (39), the same estimate is valid "generically" with respect to u under a condition on α. This is a consequence of the equidistribution argument used previously and of Proposition 6.1. Namely, using this proposition and an approximation by trigonometric polynomials, we get:

If $1 \le a_n \le n^p$, with $p < \frac{1}{4}$, if $H_\varphi(u_1, ..., u_s)$ is a continuous periodic function on the torus $\mathbb{T}^s$, $s \ge 1$, then:

$$\lim_N \frac{\sum_{k=1}^N a_k^2 \, H(q_k u_1, ..., q_k u_s)}{\sum_{k=1}^N a_k^2} = \int_{\mathbb{T}^s} H(x_1, ..., x_s) dx_1 ... dx_s, \text{ for a.e. } u. \tag{43}$$

For instance, in Example 1, $\lim_N \dfrac{\sum_{k=1}^N a_k^2 \sin^2(\pi q_k u)}{\sum_{k=1}^N a_k^2} = \dfrac{1}{2}$, for a.e. u.

By (11), it follows that the mean of the variance, $\frac{1}{n}\sum_{k=0}^{n-1} \|\varphi_k(u,.)\|_2^2$, is of order $\sum_{k=1}^{m(n)} a_k^2$ generically with respect to u, if α satisfies Hypothesis 1, i.e., $a_n = O(n^p), \forall n \ge 1$, with $p < \frac{1}{4}$.

Vectorial case

For simplicity, we consider the case of two components. Let be given a vectorial function $\Phi = (\varphi^1, \varphi^2)$, where φ^1, φ^2 are two centered step functions with respectively s_1, s_2 discontinuities: $\varphi^i = \sum_{j=0}^{s_i} v_j^i \, 1_{[u_j^i, u_{j+1}^i[} - c_i$, for $i = 1, 2$.

Let the matrix Γ_n be defined by $\Gamma_n(a,b) := (\log n)^{-1} \|a\varphi_n^1 + b\varphi_n^2\|_2^2$ and denote by I_2 the 2-dimensional identity matrix.

Theorem 2.7. *If α has bounded partial quotients and if the condition (12) is satisfied uniformly with respect to (a,b) in the unit sphere, there are $0 < r_1, r_2 < +\infty$ two constants such that for a "large" set of integers n as in Theorem 2.4:*
- Γ_n satisfies inequalities of the form $r_1 I_2 \le \Gamma_n(a,b) \le r_1 I_2$;
- the distribution of $\Gamma_n^{-1}\Phi_n$ converges to the standard 2-dimensional normal law.

Proof. We only sketch the proof. The classical method of proof of a CLT for a vectorial function is to show a scalar CLT for all linear combinations of the components of the function. So the proof is like that of Theorem 2.1, but taking care of the bound from below of the variance for $a\varphi_n^1 + b\varphi_n^2$: (12) should be uniform for (a, b) on the unit sphere. This is done in the next proposition. □

Proposition 2.8. *Let Λ be a compact space and $(F_\lambda, \lambda \in \Lambda)$ be a family of nonnegative non identically null continuous functions on $\mathbb{T}^d$ depending continuously on λ. If a sequence (z_n) is equidistributed in $\mathbb{T}^d$, then*

$$\exists\, N_0, \eta > 0 \text{ such that } \mathrm{Card}\{n \le N : F_\lambda(z_n) \ge \eta\} \ge \theta\, N,\ \forall N \ge N_0, \forall \lambda \in \Lambda. \tag{44}$$

Proof. For $\lambda \in \Lambda$, let $u_\lambda \in \mathbb{T}^d$ be such that $F_\lambda(u_\lambda) = \sup_{u\in\mathbb{T}^d} F_\lambda(u)$. We have $F_\lambda(u_\lambda) > 0$ and there is $\eta_\lambda > 0$ and an open neighborhood U_λ of u_λ such that $F_\lambda(u) > 2\eta_\lambda$ for $u \in U_\lambda$. Using the continuity of F_λ with respect to the parameter λ, the inequality $F_\zeta(u) > \eta_\lambda$ holds for $u \in U_\lambda$ and ζ in an open neighborhood V_λ of λ. By compactness of Λ, there is a finite set $(\lambda_j, j \in J)$ such that $(V_{\lambda_j}, j \in J)$ is an open covering of Λ. Let $\theta := \frac{1}{2}\inf_{j\in J} Leb(U_{\lambda_j})$.

By equidistribution of (z_n), there is N_0 such that $\frac{1}{N}\sum_{n=1}^N 1_{U_{\lambda_j}}(z_n) \ge \theta, \forall N \ge N_0,\ \forall j \in J$.

Let $\eta := \inf_{j\in J} \eta_{\lambda_j}$. For every $\lambda \in \Lambda$, there is $j \in J$ such that $\lambda \in V_{\lambda_j}$ and therefore $F_\lambda(z_n) \ge \eta_{\lambda_j} \ge \eta$, if $z_n \in U_{\lambda_j}$. This implies:

$$\mathrm{Card}\{n \le N : F_\lambda(z_n) \ge \eta\} \ge \mathrm{Card}\{n \le N : z_n \in U_{\lambda_j}\} \ge \theta N,\ \forall N \ge N_0. \quad \square$$

A generic result

By Proposition 2.8 applied for (a, b) in the unit sphere, for a.e. values of the parameter $(u_1^1, ..., u_{s_1}^1, u_1^2, ..., u_{s_2}^2)$, the functions $a\varphi^1 + b\varphi^2$ satisfy Condition (12) uniformly in (a, b) in the unit sphere. Hence Theorem 2.7 applies generically with respect to the discontinuities.

Special values: an application to the rectangular billiard in the plane

Example 4 Now, for an application to the periodic billiard, we consider the vectorial function $\psi = (\varphi^1, \varphi^2)$ with

$$\begin{aligned}
\varphi^1 &= 1_{[0,\frac{\alpha}{2}]} - 1_{[\frac{1}{2},\frac{1}{2}+\frac{\alpha}{2}]} = \frac{2}{\pi}\sum_{r\in\mathbb{Z}} e^{-\pi i(2r+1)\frac{\alpha}{2}} \frac{\sin(\pi(2r+1)\frac{\alpha}{2})}{2r+1} e^{2\pi i(2r+1)\cdot},\\
\varphi^2 &= 1_{[0,\frac{1}{2}-\frac{\alpha}{2}]} - 1_{[\frac{1}{2},1-\frac{\alpha}{2}]} = \frac{-2i}{\pi}\sum_{r\in\mathbb{Z}} e^{\pi i(2r+1)\frac{\alpha}{2}} \frac{\cos(\pi(2r+1)\frac{\alpha}{2})}{2r+1} e^{2\pi i(2r+1)\cdot}.
\end{aligned}$$

The Fourier coefficients of φ^1 and φ^2 of order r are null for r even.

Let us consider a linear combination $\varphi_{a,b} = a\varphi^1 + b\varphi^2$. For $r = 2t+1$ odd, we have:

$$c_{2t+1}(a\varphi^1 + b\varphi^2) = \frac{2}{\pi}\frac{1}{2t+1}\, e^{-\pi i(2t+1)\frac{\alpha}{2}}\, [a\sin(\pi(2t+1)\,\frac{\alpha}{2}) - ib\cos(\pi(2t+1)\,\frac{\alpha}{2})].$$

If q_j is even, $\gamma_{q_j}(\varphi_{a,b})$ is null. If q_j is odd, we have

$$|\gamma_{q_j}(\varphi_{a,b})|^2 = |a\sin(\pi q_j\,\frac{\alpha}{2}) - ib\sin(\pi(\frac{1}{2} + q_j\,\frac{\alpha}{2}))|^2,$$

For q_j odd, we have by (2),

$$\begin{aligned} \|q_j\frac{\alpha}{2}\| &= \|\frac{p_j}{2}+\frac{\theta_j}{2}\|, \text{ hence } \left|\|q_j\frac{\alpha}{2}\|-\|\frac{p_j}{2}\|\right| \leq \left|\frac{\theta_j}{2}\right| \leq \frac{1}{2q_{n+1}}, \\ \|\frac{1}{2}+q_j\frac{\alpha}{2}\| &= \|\frac{q_j}{2}-\frac{p_j}{2}-\frac{\theta_j}{2}\|, \text{ hence } \left|\|q_j\beta_2\|-\left(\|\frac{1}{2}+\frac{p_j}{2}\|\right|\right) \leq \left|\frac{\theta_j}{2}\right| \leq \frac{1}{2q_{n+1}}, \end{aligned}$$

This implies, for q_j odd: $\gamma_{q_j}(\varphi_{a,b}) = a(1+O(\frac{1}{q_{j+1}}))$, if p_j is odd, $= b(1+O(\frac{1}{q_{j+1}}))$, if p_j is even.

The computation shows that, if α is such that, in average, there is a positive proportion of pairs (p_j, q_j) which are (even, odd) and a positive proportion of pairs (p_j, q_j) which are (odd, odd), then the condition of Theorem 2.7 is fulfilled by the vectorial step function $\psi = (\varphi^1, \varphi^2)$.

For an application to the model of rectangular periodic billiard in the plane described in [7], we refer to [6].

3. **Proof of Proposition 2.1 (CLT)**

The difference $H_{X,Y}(\lambda) := |\mathbb{E}(e^{i\lambda X}) - \mathbb{E}(e^{i\lambda Y})|$ can be used to get an upper bound of the distance $d(X,Y)$ thanks to the following inequality ([12], Chapter XVI, Inequality (3.13)): if X has a vanishing expectation, then, for every $U > 0$,

$$d(X,Y) \leq \frac{1}{\pi}\int_{-U}^{U} H_{X,Y}(\lambda)\frac{d\lambda}{\lambda} + \frac{24}{\pi}\frac{1}{\sigma\sqrt{2\pi}}\frac{1}{U}. \tag{45}$$

Using (45), we get an upper bound of the distance between the distribution of X and the normal law by bounding $|\mathbb{E}(e^{i\lambda X}) - e^{-\frac{1}{2}\sigma^2\lambda^2}|$.

We will use the following remarks:

$$\mathrm{V}(fg) \leq \|f\|_\infty \mathrm{V}(g) + \|g\|_\infty \mathrm{V}(f),\ \forall f, g \text{ BV}, \tag{46}$$

$$\text{if } g \in \mathcal{C}^1(\mathbb{R},\mathbb{R}) \text{ and } u \text{ is BV, then } \mathrm{V}(g\circ u) \leq \|g'\|_\infty \mathrm{V}(u). \tag{47}$$

Let $w_k := \max_{j=1}^k u_j$, where u_j is larger than $\|f_j\|_\infty$ (see Proposition 2.1).

Since $\mathrm{V}(f_k) \leq C u_k\, q_k$, (24) implies

$$|\int_X f_k f_m\, d\mu| \leq C\,\frac{q_k}{q_m} m^\theta\, w_m^2, \text{ for } k \leq m. \tag{48}$$

Bounding the moments

Lemma 3.1. *Under the assumption of Proposition 2.1, there is C_1 such that*

$$\int_X |\sum_{k=m}^{m+\ell} f_k|^2 \le C_1 \ln(m+\ell) \sum_{j\in[m,m+\ell]} u_j^2 \ \le C_1\, \ell \ln(m+\ell)\, w_{m+\ell}^2. \tag{49}$$

$$\int_X |\sum_{k=m}^{m+\ell} f_k|^3 \le C_1\, \ell \ln^2(m+\ell)\, w_{m+\ell}^3,\ \forall m,\ell \ge 1, \tag{50}$$

$$\int_X |\sum_{k=m}^{m+\ell} f_k|^4 \le C_1\, \ell^2 \ln^2(m+\ell)\, w_{m+\ell}^3,\ \forall m,\ell \ge 1. \tag{51}$$

Proof. We show (50) and (51). The proof of (49) is the same.

1) For (50), it suffices to bound the sums $\sum_{m\le s\le t\le u\le m+\ell} |\int_X f_s f_t f_u\, d\mu|$.

Replacing f_k by $w_{m+\ell}^{-1} f_k$, we will deduce the bound (50) from the inequalities (23), (24), (25) when $w_k \le 1$, for $1 \le k \le m+\ell$. By (23) and (46), we have

$$|\int_X f_s f_t f_u\, d\mu| \le C \ \text{ and } \ \mathrm{V}(f_s f_t) \le C(q_s + q_t) \le 3Cq_t.$$

From (24) and (22), then from (25) and (22), we obtain

$$|\int_X (f_s f_t).f_u\, d\mu| \le C\frac{V(f_s f_t)}{q_u} u^\theta \le C\frac{q_t}{q_u} u^\theta \le C\rho^{(u-t)} u^\theta,$$
$$|\int_X f_s.(f_t f_u)\, d\mu| \le C\frac{V(f_s)}{q_t} u^\theta \le C\frac{q_s}{q_t} u^\theta \le C\rho^{(t-s)} u^\theta.$$

Set $\kappa = \frac{\theta+3}{\ln(1/\rho)} \ln(m+\ell)$. If $t-s$ or $u-t \ge \kappa$, the previous inequalities imply:

$$|\int_X f_s f_t f_u\, d\mu| \le C\rho^\kappa u^\theta \le C(m+\ell)^{-\theta-3} u^\theta \le C(m+\ell)^{-3}.$$

It implies:

$$\sum_{m\le s\le t\le u\le m+\ell\,:\,\max(t-s,u-t)>\kappa} |\int_X f_s f_t f_u\, d\mu| \le C\ell^3(\ell+m)^{-3} \le C.$$

Now the result follows from:

$$\sum_{m\le s\le t\le u\le m+\ell\,:\,\max(t-s,u-t)\le\kappa} |\int_X f_s f_t f_u\, d\mu| \le C\ell\kappa^2.$$

2) For (51), we bound the sums $\sum_{m\le s\le t\le u\le v\le m+\ell} |\int_X f_s f_t f_u f_v\, d\mu|$ using (22) and successively (24), (25), (26).

We obtain (because f_v is centered for the first inequality):

$$|\int_X (f_s f_t f_u).f_v \, d\mu| \leq C\frac{V(f_s f_t f_u)}{q_v}v^\theta \leq C\frac{q_u}{q_v}v^\theta \leq C\rho^{(v-u)}v^\theta,$$
$$|\int_X [f_s f_t - \mathbb{E}(f_s f_t)]\, f_u f_v \, d\mu| \leq C\frac{V(f_s f_t)}{q_u}v^\theta \leq C\frac{q_t}{q_u}v^\theta \leq C\rho^{(u-t)}v^\theta,$$
$$|\int_X f_s f_t f_u f_v \, d\mu| \leq C\frac{V(f_s)}{q_t}v^\theta \leq C\frac{q_s}{q_t}v^\theta \leq C\rho^{(t-s)}v^\theta.$$

Putting $\kappa = \frac{\theta+4}{\ln(1/\rho)}\, \log(m+\ell)$, we get by the previous inequalities, for constants C, C_2, C_3:

$$\sum_{m\leq s\leq t\leq u\leq v\leq m+\ell\,:\,\max(t-s,u-t,v-u)>\kappa} |\int_X f_s f_t f_u f_v \, d\mu|$$
$$\leq C\ell^4(\ell+m)^{-4} + \sum_{m\leq s\leq t\leq u\leq v\leq m+\ell} |\int_X f_s f_t \, d\mu|\, |\int_X f_u f_v \, d\mu|$$
$$\leq C + (\sum_{m\leq s\leq t\leq m+\ell} |\int_X f_s f_t \, d\mu|)^2 \leq C + C_2\, \ell^2\, (\ln(m+\ell))^2.$$

The remaining terms give a bound which can be absorbed in the previous one, namely:

$$\sum_{m\leq s\leq t\leq u\leq m+\ell\,:\,\max(t-s,u-t,v-u)\leq\kappa} |\int_X f_s f_t f_u f_v \, d\mu| \leq C\ell\kappa^3 \leq C_3\, \ell\, \log(m+\ell)^3. \quad \square$$

Proof of Proposition 2.1

The proof is given in several steps.

Defining blocks

We split the sum $S_n := f_1 + \cdots + f_n$ into small and large blocks. The small ones will be removed, providing gaps and allowing to take advantage of the decorrelation properties assumed in the statement of the proposition.

Let τ, δ be parameters (δ *close to 0*) such that $0 < \delta < \frac{1}{2}$ and $\delta < \tau$. We set for $n \geq 1$:

$$n_1 = n_1(n) := \lfloor n^\tau \rfloor, n_2 = n_2(n) := \lfloor n^\delta \rfloor,$$
$$\nu = \nu(n) := n_1 + n_2,\ p(n) := \lfloor n/\nu(n) \rfloor + 1 = n^{1-\tau} + h_n \sim n^{1-\tau},\ \text{with } |h_n| \text{ bounded.}$$

For $0 \leq k < p(n)$, we put (with $f_j = 0$, if $n < j \leq n+\nu$)

$$F_{n,k} = f_{k\,\nu(n)+1} + \cdots + f_{k\,\nu(n)+n_1(n)},\quad G_{n,k} = f_{k\,\nu(n)+n_1(n)+1} + \cdots + f_{(k+1)\,\nu(n)}.$$

The sums $F_{n,k}, G_{n,k}$ have respectively $n_1 \sim n^\tau, n_2 \sim n^\delta$ terms and S_n reads

$$S_n = \sum_{k=0}^{p(n)-1} (F_{n,k} + G_{n,k}).$$

We put $S'_n := \sum_{k=0}^{p(n)-1} F_{n,k},\ v_k = v_{n,k} := (\int_X F_{n,k}^2 \, d\mu)^{\frac{1}{2}}$.

The following inequalities are implied by (49):

$$v_k^2 = v_{n,k}^2 = \|F_{n,k}\|_2^2 \le C n^\tau \ln n\, w_n^2,\ \|G_{n,k}\|_2^2 \le C\, n^\delta \ln n\, w_n^2,\ 0 \le k < p(n). \tag{52}$$

Since $q_1 + q_2 + ... + q_n \le C q_{n+1}, \forall n \ge 1$, by (22), it follows by (47) and hypothesis (23):

$$\mathrm{V}(e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}) \le C|\zeta|\, w_n\, q_{(k-1)\,\nu+n_1}. \tag{53}$$

Lemma 3.2.

$$|\|S_n\|_2^2 - \sum_{k=0}^{p(n)-1} v_k^2| = |\|S_n\|_2^2 - \sum_{k=0}^{p(n)-1} \|F_{n,k}\|_2^2| \le C\, n^{1-\frac{\tau-\delta}{2}} \ln n\, w_n^2, \tag{54}$$

$$\|S_n - S_n'\|_2^2 = \|\sum_{k=0}^{p(n)-1} G_k\|^2 \le C\, n^{1-\tau+\delta} \ln n\, w_n^2. \tag{55}$$

Proof. It follows from (48) and (22), with $C_0 = \frac{C\rho}{(1-\rho)^2}$,

$$|\int_X (\sum_{u=a}^{b} f_u)\,(\sum_{t=c}^{d} f_t)\, d\mu| \le C_0\, \rho^{c-b}\, d^\theta\, w_d^2,\ \forall a \le b < c \le d.$$

Therefore, we have, with $C_1 = C_0 \sum_{i\ge 0} \rho^{i\nu}$, writing simply F_k, G_k instead of $F_{n,k}, G_{n,k}$,

$$\sum_{0\le j<k<p(n)} |\int F_j\, F_k\, d\mu| \le C_0\, n^\theta\, w_n^2 \sum_{0\le j<k<p(n)} \rho^{k\nu+1-(j\nu+n_1)}$$
$$\le C_0\, n^\theta\, w_n^2\, \rho^{n_2} \sum_{0\le j<k<p(n)} \rho^{(k-1)\nu-j\nu} \le C_0\, \rho^{n_2}\, n^\theta\, w_n^2\, p(n) \sum_{i\ge 0} \rho^{i\nu} \le C_1\, n^{\frac{1}{2}-\delta+\theta}\, w_n^2\, \rho^{n^\delta}.$$

The LHS of (54) is less than the sum for $k = 0$ to $p(n) - 1$ of

$$\int G_k^2\, d\mu + |\int G_k\, F_k\, d\mu| + |\int G_k\, F_{k+1}\, d\mu|$$
$$+2 \sum_{0\le j<k} [|\int F_j\, (F_k + G_k)\, d\mu| + |\int G_j\, G_k\, d\mu|] + 2 \sum_{0\le j<k-1} |\int G_j F_k\, d\mu|.$$

The first term is bounded by $C\, n^\delta \ln n\, w_n^2$, the second one and the third one bounded by $C\, n^{\frac{\delta+\tau}{2}} \ln n\, w_n^2$ are the biggest. The other terms are negligible as shown by the preliminary computation: $n^\theta\, \rho^{n^\delta}$ is small compared to a power of n, for n big.

Therefore the LHS of (54) is less than: $C_1\, n^{1-\tau}\, n^{\frac{\delta+\tau}{2}} \ln n\, w_n^2 = C_1\, n^{1-\frac{\tau-\delta}{2}} \ln n\, w_n^2$.

An analogous computation shows that the LHS of (55) behaves like $\sum_{k=0}^{p(n)-1} \int G_k^2\, d\mu$ which gives the bound $C\, n^{1-\tau+\delta} \ln n\, w_n^2$ of (55). □

Approximation of the characteristic function of the sum S_n' by a product

For $\zeta \in \mathbb{R}$, let $\mathrm{I}_{n,-1}(\zeta) := 1,\ \mathrm{I}_{n,k}(\zeta) := \int_X e^{i\zeta\,(F_{n,0}+\cdots+F_{n,k})}\, d\mu,\ 0 \le k < p(n)$.

Lemma 3.3. *For $0 \leq k < p(n)$, we have*

$$(56)\ |\mathrm{I}_{n,k}(\zeta) - \big(1 - \frac{\zeta^2}{2}\, v_{n,k}^2\big)\ \mathrm{I}_{n,k-1}(\zeta)| \leq C\,\big(|\zeta|^3\, w_n^3\, n^\tau\ \ln^2(n) + \zeta^4\, w_n^4\, n^{2\tau}\ \ln^2(n)\big)\,.$$

Proof. We use $e^{iu} = 1 + iu - \frac{1}{2}u^2 - \frac{i}{6}u^3 + u^4 r(u),$ with $|r(u)| \leq \frac{1}{24}$, for $u \in \mathbb{R}$. Let $k \geq 1$. We have $\mathrm{I}_{n,k}(\zeta) = \int_X e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}\,[1 + i\zeta F_{n,k} - \frac{\zeta^2}{2}\,F_{n,k}^2 - \frac{i}{6}\zeta^3 F_{n,k}^3 + \zeta^4 F_{n,k}^4\, r(\zeta F_{n,k})]\,d\mu$. For the first term, using (53), we have:

$$|\int_X e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}\,F_{n,k}\,d\mu| \leq \sum_{j=1}^{n_1} |\int_X e^{i\zeta\,(F_{n,0}+\cdots+F_{n,k-1})}\,f_{k\,\nu+j}\,d\mu|$$

$$\leq C|\zeta|\,w_n \sum_{j=1}^{n_1} \frac{q_{(k-1)\,\nu+n_1}}{q_{k\,\nu+j}}\,(k\,\nu+j)^\theta\, w_{(k-1)\,\nu+n_1} \leq C|\zeta|\,w_n^2\,n^\theta \sum_{j=1}^{n_1} \rho^{\nu+j-n_1}$$

$$(57)\qquad \leq \frac{C\rho}{1-\rho}\,|\zeta|\,w_n^2\,n^\theta\,\rho^{n_2}.$$

Similarly, for the second term we apply (25) and (53) and we get:

$$|\int_X e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}\,F_{n,k}^2\,d\mu - \mathrm{I}_{n,k-1}\int_X F_{n,k}^2\,d\mu|$$

$$(58)\qquad \leq C\mathrm{V}(e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})})\,w_n^2 \sum_{j'=1}^{n_1}\sum_{j=1}^{j'} \frac{((k-1)\nu+j')^\theta}{q_{(k-1)\nu+j}} \leq C|\zeta|\,w_n^3\,n^{\theta+\tau}\,\rho^{n_2}.$$

Likewise (26) and Lemma 3.1 imply: $|\int_X e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}\,F_{n,k}^3\,d\mu|$

$$\leq |\int_X (e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})} - \mathbb{E}(e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})})\,F_{n,k}^3\,d\mu| + |\int_X F_{n,k}^3\,d\mu|$$

$$(59)\qquad \leq C\,|\zeta|\,n^{1+2\tau+\theta}\,w_n^3\,\rho^{n_2} + Cn^\tau\,w_n^3\ \ln^2(n).$$

At last, by (51) we have

$$(60)\qquad |\int_X e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}\,F_{n,k}^4\,r(\zeta F_{n,k})\,d\mu| \leq \int_X F_{n,k}^4\,d\mu \leq w_n^4 n^{2\tau}\ \ln(n)^2.$$

From (57), (58), (59) and (60), we deduce that $|\mathrm{I}_{n,k}(\zeta) - \big(1 - \frac{\zeta^2}{2}\,v_{n,k}^2\big)\ \mathrm{I}_{n,k-1}(\zeta)|$ is bounded up to a constant factor C by

$$|\zeta|^2\,w_n^2\,n^\theta\,\rho^{n_2} + |\zeta|^3\,w_n^3\,n^{\theta+\tau}\,\rho^{n_2} + |\zeta|^4\,n^{1+2\tau+\theta}\,w_n^4\,\rho^{n_2} + |\zeta|^3\,w_n^3\,n^\tau\ \ln^2(n) + |\zeta|^4\,w_n^4\,n^{2\tau}\ \ln^2(n).$$

In the sum above, for n big, we keep only the last two terms, since for n big enough the first terms are smaller than the last ones. □

If X and Y are two real square integrable random variables, then $|\mathbb{E}(e^{iX}) - \mathbb{E}(e^{iY})| \leq \|X - Y\|_2$. Therefore, using (55), we have for $\mathrm{J}_n(\zeta) := \int_X e^{i\zeta\,S_n}\,d\mu$:

$$(61)\qquad |\mathrm{J}_n(\zeta) - \mathrm{I}_{n,p(n)}(\zeta)| \leq\ |\zeta|\,\|S_n - S'_n\|_2 \leq C\,|\zeta|\,w_n\,n^{\frac{1-\tau+\delta}{2}}\,(\ln n)^{\frac{1}{2}},$$

then, by (61) and (56) of Lemma 3.3, we get

$$\begin{aligned}|\mathrm{J}_n(\zeta) - \textstyle\prod_{k=1}^{p(n)}(1-\frac{1}{2}\zeta^2 v_k^2)| \le\ & |\mathrm{J}_n(\zeta) - \mathrm{I}_{n,p(n)}(\zeta)| + \textstyle\sum_{k=0}^{p(n)-1} |\mathrm{I}_{n,k}(\zeta) - (1-\frac{\zeta^2}{2}\,v_k^2)\ \mathrm{I}_{n,k-1}(\zeta)| \\ &\le C\,[|\zeta|\,w_n\,n^{\frac{1-\tau+\delta}{2}}\,(\ln n)^{1/2} + n^{1-\tau}|\zeta|^3\,w_n^3\,n^{\tau}\ \ln^2(n) + n^{1-\tau}\zeta^4 w_n^4 n^{2\tau}\ \ln(n)^2 \\ &\le C\,[|\zeta|\,w_n\,n^{\frac{1-\tau+\delta}{2}}\,(\ln n)^{\frac{1}{2}} + |\zeta|^3\,w_n^3\,n\,(\ln n)^2 + \zeta^4\,w_n^4\,n^{1+\tau}\,(\ln n)^2]. \qquad (62)\end{aligned}$$

Approximation of the exponential by a product

Below, ζ will be such that $|\zeta|\,v_{n,k} \le 1$. This is satisfied if

$$|\zeta|\,n^{\frac{\tau}{2}} w_n (\log n)^{\frac{1}{2}} \le 1. \qquad (63)$$

Lemma 3.4. *If $(\rho_k)_{k\in J}$ is a finite family of real numbers in $[0,1[$, then*

$$0 \le e^{-\sum_{k\in J}\rho_k} - \prod_{k\in J}(1-\rho_k) \le \sum_{k\in J}\rho_k^2, \ \text{if}\ \ 0 \le \rho_k \le \frac{1}{2}, \forall k. \qquad (64)$$

Proof. We have $\ln(1-u) = -u - u^2\,v(u)$, with $\frac{1}{2} \le v(u) \le 1$, for $0 \le u \le \frac{1}{2}$ and $1 - e^{-\sum\varepsilon_k} \le \sum\varepsilon_k$, if $\sum_k \varepsilon_k \ge 0$.

Writing $1-\rho_k = e^{-\rho_k - \varepsilon_k}$, with $\varepsilon_k = -(\ln(1-\rho_k) + \rho_k)$, the previous inequality implies $0 \le \varepsilon_k \le \rho_k^2$, if $0 \le \rho_k \le \frac{1}{2}$. Therefore, under this condition, we have:

$$0 \le e^{-\sum_J \rho_k} - \textstyle\prod_J(1-\rho_k) = e^{-\sum_J\rho_k}\big(1 - e^{-\sum_J\varepsilon_k}\big) \le e^{-\sum\rho_k}\ \sum\varepsilon_k \le \sum\varepsilon_k \le \sum\rho_k^2. \quad \square$$

We apply (64) with $\rho_k = \frac{1}{2}\zeta^2 v_{n,k}^2$, under Condition (63). In view of (52) it follows:

$$|e^{\frac{1}{2}\zeta^2\sum v_k^2} - \prod_{k=0}^{p(n)-1}\ (1-\frac{1}{2}\zeta^2 v_k^2)| \le \frac{1}{4}\zeta^4 \sum_{k=0}^{p(n)-1}\ v_k^4 \le C\zeta^4\,w_n^4\,n^{1+\tau}\ \ln^2(n). \qquad (65)$$

The bound is like the last term in (62).

Conclusion

From (62) and (65), it follows:

$$|J_n(\zeta) - e^{-\frac{1}{2}\zeta^2\sum_{k=0}^{p(n)-1} v_k^2}| \le C\,[|\zeta|\,w_n\,n^{\frac{1-\tau+\delta}{2}}\,(\ln n)^{\frac{1}{2}} + |\zeta|^3\,w_n^3\,n\,(\ln n)^2 + \zeta^4\,w_n^4\,n^{1+\tau}\,(\ln n)^2].$$

We replace ζ by $\frac{\lambda}{\|S_n\|_2}$; hence Condition (63) becomes

$$\frac{\lambda}{\|S_n\|_2}\,n^{\frac{\tau}{2}}\,w_n(\log n)^{\frac{1}{2}} \le 1. \qquad (66)$$

We get: $|\int_X e^{i\lambda\frac{S_n(x)}{\|S_n\|_2}}\,d\mu(x) - e^{-\frac{1}{2}\frac{\lambda^2}{\|S_n\|_2^2}\sum_k v_k^2}|$

$$\le C\,[|\lambda|\frac{w_n}{\|S_n\|_2}\,n^{1-\tau+\delta} + |\lambda|^3\frac{w_n^3}{\|S_n\|_2^3}\,n\ln^2(n)\ + \lambda^4\frac{w_n^4}{\|S_n\|_2^4}\,n^{1+\tau}\ln^2(n)].$$

Since $|e^{-a}-e^{-b}|\leq |a-b|$, for any $a,b\geq 0$, we have, by (54):

$$|e^{-\frac{1}{2}\lambda^2}-e^{-\frac{1}{2}\frac{\lambda^2}{\|S_n\|_2^2}\sum_{k=0}^{p(n)-1}v_k^2}|\leq \frac{1}{2}\frac{\lambda^2}{\|S_n\|_2^2}\,|\|S_n\|_2^2-\sum_{k=0}^{p(n)-1}v_k^2|\leq C\lambda^2\frac{w_n^2}{\|S_n\|_2^2}n^{1-\frac{\tau-\delta}{2}}\,\ln n.$$

Let us call respectively E_1 the error in neglecting the sums on the small blocks, E_2 the error in the replacement of $e^{-\frac{1}{2}\lambda^2}$ by $\exp(-\frac{1}{2}\lambda^2\frac{\sum v_k^2}{\|S_n\|_2^2})$, E_3 the error of order 3 in the expansion, E_4 the approximation error of the exponential by the product.

Finally we get the bound $|\int_X e^{i\lambda\frac{S_n(x)}{\|S_n\|_2}}\,d\mu(x)-e^{-\frac{1}{2}\lambda^2}|\leq E_1+E_2+E_3+E_4\leq$

$$C\,[|\lambda|\frac{w_n}{\|S_n\|_2}\,n^{\frac{1-\tau+\delta}{2}}\,\ln^{\frac{1}{2}}(n)+\lambda^2\frac{w_n^2}{\|S_n\|_2^2}n^{1-\frac{\tau-\delta}{2}}\,\ln n+|\lambda|^3\frac{w_n^3}{\|S_n\|_2^3}\,n\ln^2(n)\,+\lambda^4\frac{w_n^4}{\|S_n\|_2^4}\,n^{1+\tau+\delta}\ln^2(n)].$$

Denote by Y_1 a r.v. with $\mathcal{N}(0,1)$-distribution. Putting $R_n:=\frac{w_n}{\|S_n\|_2}$, the bound reads:

$$C\,[|\lambda|\,R_n\,n^{\frac{1-\tau+\delta}{2}}+\lambda^2\,R_n^2n^{1-\frac{\tau-\delta}{2}}\,\ln n+|\lambda|^3\,R_n^3\,n\ln^2(n)\,+\lambda^4\,R_n^4\,n^{1+\tau+\delta}\ln^2(n)]. \tag{67}$$

Notice that δ can be taken arbitrary small. A change of its value modifies the generic constant C in the previous inequalities. Therefore we take $\delta=0$ in the optimisation below, keeping in mind that the constant factor in the inequalities depends on δ. Likewise the $\ln n$ factors can be neglected.

We have an inequality of the form $H_{\frac{S_n}{\|S_n\|_2},Y_1}(\lambda)\leq C\sum_{i=1}^4|\lambda|^{\alpha_i}\,R_n^{\alpha_i}\,n^{\gamma_i}$, where the exponents are given by the previous inequality. In view of (45), it follows that, up to a constant factor, $d(\frac{S_n}{\|S_n\|_2},Y_1)\leq$

$$\frac{1}{U_n}+\sum_{i=1}^4\alpha_i^{-1}\,U_n^{\alpha_i}\,R_n^{\alpha_i}\,n^{\gamma_i}\leq\frac{1}{U_n}+U_n\,R_n\,n^{\frac{1-\tau}{2}}+\frac{1}{2}U_n^2\,R_n^2\,n^{1-\frac{\tau}{2}}+\frac{1}{3}U_n^3\,R_n^3\,n+\frac{1}{4}U_n^4\,R_n^4\,n^{1+\tau}.$$

Now, we optimize the choice of $U=U_n$. As R_n is less than $n^{-\beta}$ for some $\beta>0$, if we take $U_n=n^\gamma$ with $\gamma>0$, then the previous inequality gives inside the bracket the bound:

$$n^{\frac{1-\tau}{2}-\beta+\gamma}+\frac{1}{2}\,n^{1-\frac{\tau}{2}-2\beta+2\gamma}+\frac{1}{3}\,n^{1-3\beta+3\gamma}+\frac{1}{4}\,n^{1+\tau-4\beta+4\gamma}.$$

We choose U_n such that $1/U_n$ is of the same order as the second term, i.e., we take $n^{-\gamma}=n^{1-\frac{\tau}{2}-2\beta+2\gamma}$, i.e., $\gamma=\frac{\frac{\tau}{2}+2\beta-1}{3}$. If $\tau=\frac{1}{2}$ and if $\beta=\frac{1}{2}-p$ with $p>0$, then it gives:

$$\gamma=\frac{1-8p}{12}\;>0\text{ if }p<\frac{1}{8}.$$

The four terms in the bound are respectively:

$$(A)=-\frac{1}{6}+\frac{p}{3},\ (B)=-\gamma=-\frac{1}{12}+\frac{2p}{3},\ (C)=-\frac{1}{4}+p,\ (D)=-\frac{1}{6}+\frac{4p}{3}.$$

We check that (B) is the biggest term: $(B)-(A)=\frac{1}{12}+\frac{p}{3}>0$, $(B)-(C)=\frac{1}{6}-\frac{p}{3}>0$, if $p<\frac{1}{2}$, $(B)-(D)=\frac{1}{12}-\frac{2p}{3}>0$, if $p<\frac{1}{8}$.

This gives the bound stated in Proposition 2.1 for the distance to the normal law:

For every $\delta > 0$, for N big enough, there is a constant $C(\delta) > 0$ (depending only on δ) such that, if $\frac{w_N}{\|S_N\|_2} \leq N^{p-\frac{1}{2}}$ with $p \in [0, \frac{1}{8}[$, then

$$d(\frac{S_N}{\|S_N\|_2}, Y_1) \leq C(\delta)\, N^{-\frac{1-8p}{12}+\delta} << 1.$$

To conclude, observe that, if $w_n \leq n^p$ with $p < \frac{1}{8}$ and $\|S_n\|_2 \geq Cn^{\frac{1}{2}}/(\log n)^{\frac{1}{2}}$, (66) is satisfied for $|\lambda| \leq U_n = n^\gamma$, since

$$n^\gamma\, n^{\frac{\tau}{2}}\, \frac{w_n(\log n)^{\frac{1}{2}}}{\|S_n\|_2} \leq Cn^{\frac{1-8p}{12}} n^{p-\frac{1}{4}}\, (\log n)^{\frac{1}{2}} = Cn^{-\frac{1}{6}+\frac{p}{3}}\, (\log n)^{\frac{1}{2}} \leq 1, \text{ for } n \text{ big.} \quad \square$$

4. **Proof of Proposition 2.3 (decorrelation)**

For the proof of Proposition 2.3, by homogeneity, we may assume that ψ and φ are BV centered functions with variation ≤ 1. Moreover, we may also assume $b_i = 1, \forall i$. Indeed, the decorrelation inequalities will follow from bounds on sums of products of quantities like $|\widehat{\varphi_{b_n q_n}}(j)| \leq b_n\, |\widehat{\varphi_{q_n}}(j)|$ or $\|\varphi_{b_n q_n}\|_2 \leq b_n\, \|\varphi_{q_n}\|_2$.

First we truncate the Fourier series of the ergodic sums φ_q. For functions in $\mathcal{C}$, the remainders are easily controlled and it suffices to treat the case of trigonometric polynomials.

For $\varphi \in \mathcal{C}$, the Fourier coefficients of order $j \neq 0$ of the ergodic sum φ_n satisfy:

$$|\widehat{\varphi_n}(j)| = \frac{|\gamma_j(\varphi)|}{|j|}\, \frac{|\sin \pi n j\alpha|}{|\sin \pi j\alpha|} \leq \frac{\pi}{2}\, \frac{K(\varphi)|}{|j|}\, \frac{\|nj\alpha\|}{\|j\alpha\|}. \tag{68}$$

Recall also (cf. (8)) that, if q is a denominator of α, then

$$\|\varphi_q\|_\infty = \sup_x |\sum_{\ell=0}^{q-1} \varphi(x+\ell\alpha)| \leq V(\varphi) \text{ and } \|\varphi_q\|_2 \leq 2\pi\, K(\varphi).$$

We will use the notations: $S_L f$ for the partial sum of order $L \geq 1$ of the Fourier series of $f \in L^2(\mathbb{T})$, $R_L f := f - S_L f$ for the remainder and, q_n denoting the denominators of α,

$$a'_n := \frac{q_{n+1}}{q_n} \leq a_{n+1} + 1,\ c_n := \frac{q_{n+1}}{q_n}\, \ln q_{n+1}. \tag{69}$$

Preliminary inequalities and truncation

We begin by some inequalities which are valid for any irrational number α.

Lemma 4.1. *There is a constant C such that, if q is a denominator of α,*

$$\sum_{|j|\geq q} \frac{1}{j^2} \frac{\|Lj\alpha\|^2}{\|j\alpha\|^2} \leq C\, \frac{L}{q},\ \forall L \in [1,\, q]. \tag{70}$$

Proof. If f is a non negative BV function with integral $\mu(f)$, by Denjoy-Koksma inequality applied to $f-\mu(f)$, we have

$$\sum_{j=q}^{\infty}\frac{f(j\alpha)}{j^2}\leq\sum_{i=1}^{\infty}\frac{1}{(iq)^2}\sum_{r=0}^{q-1}f((iq+r)\alpha)\leq\frac{1}{q^2}(\sum_{i=1}^{\infty}\frac{1}{i^2})\,(q\,\mu(f)+V(f))=\frac{\pi^2}{6}(\frac{\mu(f)}{q}+\frac{V(f)}{q^2}).$$

Taking for $f(x)$ respectively $1_{[0,\frac{1}{L}]}(|x|)$ and $\frac{1}{x^2}1_{[\frac{1}{L},\frac{1}{2}[}(|x|)$, we obtain:

$$\sum_{j:\,\|j\alpha\|\leq\frac{1}{L},\,j\geq q}\frac{1}{j^2}\leq C\,(\frac{1}{q^2}+\frac{1}{Lq}),\qquad\sum_{j:\,\|j\alpha\|\geq\frac{1}{L},\,j\geq q}\frac{1}{j^2}\frac{1}{\|j\alpha\|^2}\leq C\,(\frac{L^2}{q^2}+\frac{L}{q}).$$

This implies (70), since for $L\leq q$:

$$\frac{1}{2}\sum_{|j|\geq q}\frac{1}{j^2}\frac{\|Lj\alpha\|^2}{\|j\alpha\|^2}\leq L^2\sum_{\|j\alpha\|\leq\frac{1}{L},\,j\geq q}\frac{1}{j^2}+\sum_{\|j\alpha\|>\frac{1}{L},\,j\geq q}\frac{1}{j^2}\frac{1}{\|j\alpha\|^2}\leq 2C\,(\frac{L}{q}+\frac{L^2}{q^2})\leq 4C\,\frac{L}{q}.\quad\square$$

We will use the good equirepartition of the numbers $\|k\alpha\|$ when k varies between 1 and q_n through two inequalities given in the following lemma, which will be used several times.

Lemma 4.2. *We have*

$$\sum_{j=q_t}^{q_{t+1}-1}\frac{1}{\|j\alpha\|}\leq\sum_{j=1}^{q_{t+1}-1}\frac{1}{\|j\alpha\|}\leq Cq_{t+1}\,\ln q_{t+1},\ \forall t\geq 0, \tag{71}$$

$$\sum_{1\leq j<q_{r+1}}\frac{1}{j\,\|j\alpha\|}\leq C\sum_{t=0}^{r}\frac{q_{t+1}}{q_t}\,\ln q_{t+1}=C\sum_{t=0}^{r}c_t,\ \forall r\geq 0. \tag{72}$$

Proof. There is exactly one element of the set $\{j\alpha\mod 1, j=1,...,q_{t+1}-1\}$ in each interval $[\frac{\ell}{q_{t+1}},\frac{\ell+1}{q_{t+1}}[$, $\ell=1,...,q_{t+1}-1$. Moreover, for $1\leq j<q_{t+1}$, one has $\|j\alpha\|\geq\frac{1}{2q_{t+1}}$.

This implies: $\sum_{j=1}^{q_{t+1}-1}\frac{1}{\|j\alpha\|}\leq 2q_{t+1}+\sum_{\ell=1}^{q_{t+1}-1}\frac{1}{\ell/q_{t+1}}\leq Cq_{t+1}\,\ln q_{t+1}$.

From (71) applied for $t=1,...,r$, we deduce (72):

$$\sum_{1\leq j<q_{r+1}}\frac{1}{j\,\|j\alpha\|}=\sum_{t=0}^{r}\sum_{q_t\leq j<q_{t+1}}\frac{1}{j\,\|j\alpha\|}\leq\sum_{t=0}^{r}\frac{1}{q_t}\sum_{q_t\leq j<q_{t+1}}\frac{1}{\|j\alpha\|}\leq C\sum_{t=0}^{r}\frac{q_{t+1}}{q_t}\,\ln q_{t+1}.\quad\square$$

Lemma 4.3. *For $\varphi\in\mathcal{C}$, it holds*

$$\|S_{q_r}\varphi_{q_n}\|_\infty\leq C\,V(\varphi)\,\ln(q_r). \tag{73}$$

Proof. Using the Fejér kernel, we get

$$\|S_{q_r}\varphi_{q_n}\|_\infty\leq\|\varphi_{q_n}\|_\infty+\frac{1}{q_r}\sum_{|j|<q_r}|j\widehat{\varphi_{q_n}}(j)|\leq\|\varphi_{q_n}\|_\infty+CK(\varphi)\frac{1}{q_r}\sum_{j=1}^{q_r-1}\frac{1}{||j\alpha||}.$$

(73) follows by (8) and (71). $\square$

Truncation

Now we bound the truncation error for the Fourier series of the ergodic sums $\varphi_{b_n\, q_n}$.

Lemma 4.4. *If ψ is bounded and $\varphi \in \mathcal{C}$, with $C_1 = V(\varphi)^2\|\psi\|_\infty$, $C_2 = V(\varphi)^3\|\psi\|_\infty$, up to a numerical factor, we have, for $q_n \le q_m \le q_r \le q_\ell$:*

$$|\int \psi\,[\varphi_{q_n}\varphi_{q_m} - S_{q_\ell}\varphi_{q_n}\, S_{q_\ell}\varphi_{q_m}]\,d\mu| \le C_1\,(\frac{q_m}{q_\ell})^{\frac12}, \tag{74}$$

$$|\int \psi\,[\varphi_{q_n}\varphi_{q_m}\varphi_{q_r} - S_{q_\ell}\varphi_{q_n}\, S_{q_\ell}\varphi_{q_m}\, S_{q_\ell}\varphi_{q_r}]\,d\mu| \le C_2\,(\frac{q_r}{q_\ell})^{\frac12}\ln^2(q_\ell). \tag{75}$$

Proof. We use the bound (70) which gives, for $q_n \le q_\ell$,

$$\|R_L\varphi_{q_n}\|_2^2 = \sum_{|j|\ge q_\ell} |\widehat{\varphi_{q_n}}(j)|^2 = \sum_{|j|\ge q_\ell} \frac{|\gamma_j(\varphi)|^2}{j^2}\,\frac{\|q_n j\alpha\|^2}{\|j\alpha\|^2} \le C^2\,K(\varphi)^2\,\frac{q_n}{q_\ell}.$$

For ψ bounded, as $\|\varphi_{q_n}\|_2 \le CK(\varphi)$, this implies: $|\int \psi\,[\varphi_{q_n}\varphi_{q_m} - S_{q_\ell}\varphi_{q_n}\, S_{q_\ell}\varphi_{q_m}]\,d\mu| \le$

$$\|\psi\|_\infty[\|\varphi_{q_n}\|_2\,\|R_{q_\ell}\varphi_{q_m}\|_2 + \|R_{q_\ell}\varphi_{q_n}\|_2\,\|\varphi_{q_m}\|_2] \le C\,V(\varphi)^2\|\psi\|_\infty\,[(\frac{q_n}{q_\ell})^{\frac12} + (\frac{q_m}{q_\ell})^{\frac12}].$$

This proves (74). For (75), in each term of the expansion of $(S_{q_\ell}\varphi_{q_n} + R_{q_\ell}\varphi_{q_n})\,(S_{q_\ell}\varphi_{q_m} + R_{q_\ell}\varphi_{q_m})\,(S_{q_\ell} + R_{q_\ell}) - S_{q_\ell}\varphi_{q_n}\, S_{q_\ell}\varphi_{q_m}\, S_{q_\ell}\varphi_{q_r}$, we bound one factor in L^2-norm and the others in uniform norm using (73). □

Inequalities under Hypothesis 1

Recall that the decorrelation inequalities of Lemma 4.5 are based on Hypothesis 1 on α.

From (29) in Hypothesis 1, one deduces: for constants B, C, the coefficients in Ostrowski's expansion satisfy $b_n \le B\,n^p$ and, since $q_n \le B^n\,(n!)^p$,

$$\ln q_n \le C\,n\,\ln n,\ \ c_n \le C\,n^{p+1}\,\ln n. \tag{76}$$

The case when α has bounded partial quotients corresponds to $p = 0$ and we have then $\ln q_n \le C\,n$.

Let us mention that Hardy and Littlewood in [14] considered quantities similar to that in the lemma below. One of their motivations was to study asymptotically the number of integral points contained in homothetic triangles.

Lemma 4.5. *If $a_{k+1} \le Ak^p, \forall k \ge 1$ and $n \le m \le \ell$, we have for every $\Lambda \ge 1$:*

$$\sum_{j=1}^{\infty} \frac{\|q_n j\alpha\|}{j^2\,\|j\alpha\|} \le C\,\frac{n^{p+2}\,\ln n}{q_{n+1}}, \tag{77}$$

$$\sum_{1\le j,k<q_\Lambda,\, j\ne k} \frac{\|q_n j\alpha\|\|q_m k\alpha\|}{|k-j|\,k\,j\,\|j\alpha\|\,\|k\alpha\|} \le \frac{C}{q_{n+1}}\,\Lambda^{2p+4}(\ln\Lambda)^2, \tag{78}$$

$$\sum_{-q_\Lambda<i,j,k<q_\Lambda,\, i+j+k\ne 0} \frac{\|q_n i\alpha\|\,\|q_m j\alpha\|\,\|q_\ell k\alpha\|}{|i+j+k|\,i\,j\,k\,\|i\alpha\|\,\|j\alpha\|\,\|k\alpha\|} \le \frac{C}{q_{n+1}}\,\Lambda^{3p+8}. \tag{79}$$

Proof. 1) Proof of (77)

We use the inequalities: $\frac{\|jq_k\alpha\|}{j} \leq \|q_k\alpha\| \leq \frac{1}{q_{k+1}}$ for $j < q_{k+1}$, $\|jq_k\alpha\| \leq 1$ for $j \geq q_{k+1}$. For $\ell > n$, we write $\displaystyle\sum_{j=1}^{q_\ell-1} \frac{\|q_n j\alpha\|}{j^2\,\|j\alpha\|} = (A) + (B)$, with

$$(A) := \sum_{j=1}^{q_{n+1}-1} \frac{1}{j}\,\frac{\|q_n j\alpha\|}{j\,\|j\alpha\|} \leq \frac{1}{q_{n+1}} \sum_{j=1}^{q_{n+1}-1} \frac{1}{j}\,\frac{1}{\|j\alpha\|}$$
$$\leq \frac{1}{q_{n+1}} \sum_{k=0}^{n} \frac{1}{q_k} \sum_{j=q_k}^{q_{k+1}-1} \frac{1}{\|j\alpha\|} \leq C\frac{1}{q_{n+1}} \sum_{k=0}^{n} \frac{q_{k+1}}{q_k}\,\ln q_{k+1},\ \text{by (72)};$$

$$(B) := \sum_{j=q_{n+1}}^{q_\ell-1} \frac{\|q_n j\alpha\|}{j^2\,\|j\alpha\|} \leq \sum_{k=n+1}^{\ell-1} \sum_{j=q_k}^{q_{k+1}-1} \frac{1}{j^2\,\|j\alpha\|}$$
$$\leq \sum_{k=n+1}^{\ell-1} \frac{1}{q_k^2} \sum_{j=q_k}^{q_{k+1}-1} \frac{1}{\|j\alpha\|} \leq C \sum_{k=n+1}^{\ell-1} \frac{1}{q_k}\,\frac{q_{k+1}}{q_k}\,\ln q_{k+1},\ \text{by (71)}.$$

By (4), we know that $\frac{q_{n+1}}{q_k} \leq C\rho^{k-n}$, with $\rho < 1$, for $k \geq n+1$. By hypothesis, $a_{k+1} \leq Ak^p$. It follows with the notation (69): $(A) \leq \dfrac{C}{q_{n+1}} \displaystyle\sum_{k=0}^{n} c_k \leq \frac{C\,n^{p+2}\ln n}{q_{n+1}}$ and for (B), with a bound which doesn't depend on $\ell \geq n$:

$$\frac{1}{q_{n+1}} \sum_{k=n+1}^{\ell-1} \frac{q_{n+1}}{q_k}\,\frac{q_{k+1}}{q_k}\,\ln q_{k+1} \leq C\frac{1}{q_{n+1}} \sum_{j=0}^{\infty} \rho^j\,(j+n+1)^{p+1}\,\ln(j+n+1) \leq \frac{C\,n^{p+1}\ln n}{q_{n+1}}. \quad \square$$

2) Proof of (78)

To bound the sum in (78), we cover the square $[1, q_\Lambda[\times[1, q_\Lambda[$ in $\mathbb{N}\times\mathbb{N}$ by rectangles $R_{r,s} = [q_r, q_{r+1}[\times[q_s, q_{s+1}[$ for r and s varying between 0 and $\Lambda - 1$ and then we bound the sum on each of these rectangles (minus the diagonal if $r = s$).

Distinguishing different cases according to the positions of r and s with respect to $n+1$ and $m+1$, we have, for $j \in [q_r, q_{r+1}[$, $k \in [q_s, q_{s+1}[$, $j \neq k$.

$$\frac{\|q_n j\alpha\|\,\|q_m k\alpha\|}{|k-j|\,j\,k\,\|j\alpha\|\,\|k\alpha\|} \leq \frac{1}{q_{\max(r,n+1)}q_{\max(s,m+1)}}\frac{1}{|k-j|\|j\alpha\|\|k\alpha\|}.$$

By (71) and (72), using $\|(k-j)\alpha\| \leq \|j\alpha\| + \|k\alpha\|$), we have

$$\sum_{(j,k)\in R_{r,s}} \frac{1}{|k-j|\|j\alpha\|\|k\alpha\|} \leq \sum_{(j,k)\in R_{r,s}} \left(\frac{1}{|k-j|\,\|(k-j)\alpha\|\,\|j\alpha\|} + \frac{1}{|k-j|\,\|(k-j)\alpha\|\,\|k\alpha\|}\right)$$
$$\leq q_{\max(r,s)+1}\ln(q_{\max(r,s)+1}) \sum_{t=0}^{\max(r,s)} c_t.$$

It follows

$$\sum_{(j,k)\in R_{r,s},\, j\neq k} \frac{\|q_n j\alpha\|\|q_m k\alpha\|}{|k-j|\, k\, j\, \|j\alpha\|\, \|k\alpha\|} \leq \frac{q_{\max(r,s)+1}}{q_{\max(r,n+1)} q_{\max(s,m+1)}} \ln(q_{\max(r,s)+1}) \sum_{t=0}^{\max(r,s)} c_t$$
$$\leq \frac{1}{q_{n+1}} \ln(q_{\Lambda+1}) \sum_{t=0}^{\Lambda} c_t \max_{k=1,\dots,\Lambda} a'_k.$$

The square $[1, q_\Lambda[\times[1, q_\Lambda[$ is covered by Λ^2 rectangles $R_{r,s}$ and the sums on these rectangles are bounded by the same quantity. It follows, with Hypothesis 1,

$$\sum_{1\leq j,k<q_\Lambda,\, j\neq k} \frac{\|q_n j\alpha\|\|q_m k\alpha\|}{|k-j|\, k\, j\, \|j\alpha\|\, \|k\alpha\|} \leq \Lambda^2 \frac{C}{q_{n+1}} \ln(q_{\Lambda+1}) \sum_{t=0}^{\Lambda} c_t \max_{k=1,\dots,\Lambda} a'_k \leq \frac{C}{q_{n+1}} \Lambda^{2p+5} \ln(\Lambda)^2. \quad \square$$

3) Proof of (79)

Here we consider sums with three indices i, j, k. Though we do not write it explicitly, these sums are to be understood to be taken on non zero indices i, j, k such that $i + j + k \neq 0$. We cover the set of indices by sets of the form

$$R_{\pm r,\pm s,\pm t} = \{(i,j,k) :\ \pm i \in [q_r, q_{r+1}[, \pm j \in [q_s, q_{s+1}[, \pm k \in [q_t, q_{t+1}[\}$$

Distinguishing different cases according to the positions of r, s and t with respect to $n+1$, we get: if $(i,j,k) \in R_{\pm r,\pm s,\pm t}$ and $n \leq m \leq \ell$,

$$\frac{\|q_n i\alpha\|\|q_m j\alpha\|\|q_\ell k\alpha\|}{|i|\, |j|\, |k|} \leq \frac{1}{q_{\max(r,n+1)} q_{\max(s,n+1)} q_{\max(t,n+1)}}. \tag{80}$$

We have $\dfrac{1}{\|i\alpha\|\, \|j\alpha\|\, \|k\alpha\|} \leq \dfrac{1}{\|(i+j+k)\alpha\|} \left[\dfrac{1}{\|j\alpha\|\, \|k\alpha\|} + \dfrac{1}{\|i\alpha\|\, \|k\alpha\|} + \dfrac{1}{\|i\alpha\|\, \|j\alpha\|}\right]$.

We then use (71) and (72) three times, sum over $R_{\pm r,\pm s,\pm t}$ and get:

$$\sum_{(i,j,k)\in R_{\pm r,\pm s,\pm t}} \frac{1}{|i+j+k|\, \|i\alpha\|\, \|j\alpha\|\, \|k\alpha\|}$$
$$\leq \Big(\sum_{v=0}^{3\max(r,s,t)} c_v\Big) \ln^2(q_{\max(r,s,t)+1}) \left(q_{s+1}q_{t+1} + q_{r+1}q_{t+1} + q_{r+1}q_{s+1}\right).$$

By (80) we then have:

$$\sum_{R_{\pm r,\pm s,\pm t}} \frac{\|q_n i\alpha\|\|q_m j\alpha\|\|q_\ell k\alpha\|}{|i+j+k|\, |i|\, |j|\, |k|\, \|i\alpha\|\, \|j\alpha\|\, \|k\alpha\|}$$
$$\leq C \frac{\left(\sum_{v=0}^{3\max(r,s,t)} c_v\right) \ln^2(q_{\max(r,s,t)})}{q_{\max(r,n+1)} q_{\max(s,n+1)} q_{\max(t,n+1)}} \left(q_{s+1}q_{t+1} + q_{r+1}q_{t+1} + q_{r+1}q_{s+1}\right)$$
$$\leq \frac{C}{q_{n+1}} \Big(\sum_{v=0}^{3\Lambda} c_v\Big) \ln^2(q_{\Lambda+1}) \Big(\max_{k=1,\dots,\Lambda} a'_k\Big)^2.$$

One needs $8\Lambda^3$ boxes $R_{\pm r,\pm s,\pm t}$ to cover the set $\{-q_\Lambda < i,j,k < q_\Lambda,\, i+j+k \neq 0\}$. This implies for a constant C:

$$\sum_{-q_\Lambda<i,j,k<q_\Lambda,\, i+j+k\neq 0} \frac{\|q_n i\alpha\|\|q_m j\alpha\|\|q_\ell k\alpha\|}{|i+j+k|\,|i|\,|j|\,|k|\,\|i\alpha\|\,\|j\alpha\|\,\|k\alpha\|} \leq \frac{C}{q_{n+1}}\Lambda^{3p+8}. \quad \square$$

Proof of Proposition 2.3

By (68) we have $|\int \psi\,\varphi_{q_n}\,d\mu| \leq \sum_{j\neq 0} |\widehat{\varphi_{q_n}}(j)|\,|\widehat{\psi}(-j)| \leq K\sum_{j\geq 1}\frac{\|q_n j\alpha\|}{j^2\,\|j\alpha\|}$ and (32) follows from (77): $\sum_{j\geq 1}\frac{\|q_n j\alpha\|}{j^2\,\|j\alpha\|} \leq C\,\frac{n^{p+2}\,\ln n}{q_{n+1}}$.

We prove now (33). With $L = q_\Lambda$, we have:

$$\int \psi\, S_L\varphi_{q_n}\, S_L\varphi_{q_m}\,d\mu = \sum_{|j|,|k|\leq L, j\neq k} \widehat{\varphi_{q_n}}(j)\,\widehat{\varphi_{q_m}}(k)\,\widehat{\psi}(j-k).$$

In what follows, the constant C is equal to $V(\psi)V(\varphi)^2$ (up to a factor not depending on ψ and φ) which may change.

Recall that, by (4), there is a constant B such that $m \leq B\ln q_m, \forall m \geq 1$.

The functions ψ, φ are real valued. By (78), it holds $|\int \psi\, S_L\varphi_{q_n}\, S_L\varphi_{q_m}\,d\mu| \leq$

$$\sum_{|j|,|k|\leq L, j\neq k} |\widehat{\varphi_{q_n}}(j)|\,|\widehat{\varphi_{q_m}}(k)|\,|\hat{\psi}(j-k)| \leq C\sum_{1\leq j,k\leq L}\frac{\|q_n j\alpha\|\,\|q_m\,k\alpha\|}{|k-j|\,j\,k\,\|j\alpha\|\,\|k\alpha\|} \leq \frac{C}{q_{n+1}}\,\Lambda^{2p+4}(\ln\Lambda)^2.$$

Putting it together with the truncation error term (74) and replacing q_{n+1} by q_n, we get

$$(81) \qquad |\int_X \psi\,\varphi_{q_n}\varphi_{q_m}\,d\mu| \leq C\,[\frac{\Lambda^{2p+4}(\ln\Lambda)^2}{q_n} + (\frac{q_m}{q_\Lambda})^{\frac{1}{2}}],\ \text{for } n \leq m \leq \Lambda.$$

Recall that $(\frac{q_m}{q_\Lambda})^{\frac{1}{2}} \leq \rho^{\frac{\Lambda-m}{2}}$. Let us take $\Lambda - m$ of order $2(\ln\frac{1}{\rho})^{-1}\ln q_n$, i.e., such that the second term in the bracket of the RHS of (81) is of order $1/q_n$. We have then $\Lambda \leq \max(m, C_1\log q_n)$ and with Hypothesis 1 the first term in the bracket is less than

$$\frac{C_1}{q_n}\,\max(\,(\ln q_n)^{2p+5}, m^{2p+5}) \leq \frac{C_2}{q_n}\,\max(\,(\ln q_n)^{2p+5}, (\ln q_m)^{2p+5}) \leq \frac{C_2}{q_n}\,(\ln q_m)^{2p+5} \leq C_3\frac{m^{2p+5}}{q_n}.$$

This shows (33) with $\theta_2 = 2p+5$.

In the same way, (34) follows from (75) and (79). $\square$

5. Appendix 1: proof of Proposition 1.4

Proof of Proposition 1.4

The proof consists in several steps. To bound from below $d(nq_j\alpha, \mathbb{Z})$, successively we code n as an admissible word (Ostrowski's coding), reduce long words to short words, then interpret cardinals in terms of cylinders and invariant measure for a subshift. Finally we use a result of large deviations recalled in Lemma 5.1.

For the reader's convenience, at each step we will consider first the simpler special case of the golden mean $\alpha = \frac{\sqrt{5}+1}{2}$ (the corresponding rotation number is $\frac{\sqrt{5}-1}{2} \in]0,1[$). Then the general case is treated between the signs "$\diamondsuit$" and "$\triangle$" and may be skipped if α is the golden mean.

When α is the golden mean, its partial quotients are equal to 1 and (q_n) (the Fibonacci sequence with $q_{-1} = 0, q_0 = 1, q_1 = 1, ...$) is almost a geometric sequence of ratio α. We have

$$q_n = \frac{1}{5}[(2+\alpha)\,\alpha^n + (-1)^n(3-\alpha)\,\alpha^{-n}],\ n \geq 0, \tag{82}$$

$$\alpha^n + (-\alpha)^{-n} \in \mathbb{Z},\ d(\alpha^n, \mathbb{Z}) = \alpha^{-n},\ n \geq 1. \tag{83}$$

$\diamondsuit$ For a general quadratic number α, the sequence (a_n) is ultimately periodic: there are integers n_0, p such that $a_{n+p} = a_n$, $\forall n \geq n_0$.

Let $A_1 := \begin{pmatrix} 0 & 1 \\ 1 & a_{n_0+1} \end{pmatrix}$, $A_i := \begin{pmatrix} 0 & 1 \\ 1 & a_{n_0+i} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_{n_0+i-1} \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ 1 & a_{n_0+1} \end{pmatrix}$, for $i > 1$.

From the recursive relation $\begin{pmatrix} q_n \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_{n+1} \end{pmatrix} \begin{pmatrix} q_{n-1} \\ q_n \end{pmatrix}$, between the denominators (q_n) of α, it follows, $\forall k \geq 1$, $\begin{pmatrix} q_{n_0+kp+m-1} \\ q_{n_0+kp+m} \end{pmatrix} = A_m A_p^k \begin{pmatrix} q_{n_0-1} \\ q_{n_0} \end{pmatrix}$, $m = 1, ..., p$.

The matrix A_p is a 2×2 matrix with determinant $(-1)^p$ and non negative integer coefficients (positive if $p > 1$). It has two distinct eigenvalues $\lambda > 1$ and $-\lambda^{-1}$ (where λ is a quadratic number) and it is diagonal in a basis of $\mathbb{R}^2$ with coordinates in $\mathbb{Q}[\lambda]$. We have $\lambda^p + (-\lambda^{-1})^p \in \mathbb{Z}$.

Without loss of generality we may suppose that p is even (otherwise, we replace it by $2p$). Therefore there are integers $r, s_\ell, t_\ell, u_\ell, v_\ell$ for $\ell \in \{0, \dots, p-1\}$ such that

$$q_{n_0+kp+\ell} = \frac{1}{r}\left[(s_\ell + t_\ell\lambda)\,\lambda^k + (u_\ell + v_\ell\lambda)\,\lambda^{-k}\right],\ \forall k \geq 0. \tag{84}$$

For every ℓ, $(q_{n_0+kp+\ell})_{k\geq 1}$ behaves like a geometric progression with ratio λ.

For the golden mean, (84) corresponds to (82) for n even and $r = 5$. $\triangle$

1) Ostrowski's coding, invariant measure for a subshift of finite type and counting

As recalled in Subsection 1.1, every $n < q_{m+1}$ is coded by an "admissible" word $b_0...b_m$, with $b_0 \in \{0, 1, ..., a_1 - 1\}$, $b_j \in \{0, 1, ..., a_{j+1}\}$, $j = 1, ..., m$, where for two consecutive letters b_{j-1}, b_j, if $b_j = a_{j+1}$, then $b_{j-1} = 0$.

For α the golden mean, a finite word $b_0...b_m$ is admissible if it is composed of 0's and 1's and two consecutive letters b_i, b_{i+1} cannot be both 1. We denote by X the space of one-sided infinite admissible sequences, that is sequences of 0, 1 without two consecutive 1's. For simplicity the letter b will denote an admissible word, either finite or infinite. The context will make clear if b is finite or not.

If $b = b_0...b_\ell$ is an admissible word, we put $n_{b,\ell} := \sum_{i=0}^{\ell} b_i q_i$.

When α is the golden mean, we use the sub-shift (X, σ), where $\sigma = \sigma_X$ is the shift on X. Let μ be the σ-invariant probability measure on X of maximal entropy. Let $C_{x_0...x_n}$ denote the cylinder composed of sequences starting with $x_0 \dots x_n$. For $n \geq 1$, depending whether x_0 and x_n are both equal to 1, or only one of them, or none, we have

$$\mu(C_{x_0...x_n}) = \frac{1}{\alpha+2}\alpha^{-n}, \ \frac{\alpha}{\alpha+2}\alpha^{-n} \ \text{ or } \ \frac{\alpha^2}{\alpha+2}\alpha^{-n}.$$

If $E \subset X$ is a union of cylinders of length n, its measure can be compared to the number of cylinders which compose it:

$$\frac{\alpha+2}{\alpha+1}\,\mu(E) \leq \alpha^{-n}\,\text{Card}\,\{\text{cylinder } \mathcal{W} \text{ of length } \ n: \ \mathcal{W} \subset E\} \leq (\alpha+2)\,\mu(E). \tag{85}$$

$\diamondsuit$ In the general case, let us consider the set of infinite admissible sequences corresponding to the Ostrowski expansions for the periodic part of the sequence (a_n):

$$X := \{x = (x_i)_{i\in\mathbb{N}} \text{ such that } \forall i \ x_{i-1}x_i \neq u a_{n_0+i+1} \text{ with } u \neq 0\}.$$

The space X is invariant under the action of σ_X^p (because (a_n) is p-periodic for $n \geq n_0$). We define an irreducible aperiodic sub-shift of finite type as follows: the state space of Y is the set of words $x_0 \dots x_{p-1}$ of X, a transition between two such words w_1 and w_2 is allowed if the concatenation w_1w_2 is the beginning of length $2p$ of a sequence in X.

From (84) we see that the exponential growth rate of the number of Ostrowski expansions of length at most $n_0 + pk$ is $\ln\lambda$ (with respect to k). It is also the growth rate of the number of words of length pk of X. As these words correspond to the words of length k in Y, the topological entropy of (Y, σ) is $\ln\lambda$ (where $\sigma = \sigma_Y$ is the shift to the left on Y). There is a unique invariant probability measure μ on (Y, σ) with entropy $\ln\lambda$. This measure can be constructed as follows. Let B be the matrix with entries 0 and 1 that gives the allowed transitions between elements of the alphabet of Y. As the topological entropy of Y is the logarithm of the spectral radius of B, this spectral radius is λ. Let U and V be two positive vectors such that $BU = \lambda U$, ${}^tBV = \lambda V$, ${}^tUV = 1$. The measure μ is the Markovian measure determined by its values on cylinders given by

$$\mu(C_{y_0y_1...y_n}) = V_{y_0}U_{y_n}\lambda^{-n},$$

when $y_0y_1 \dots y_n$ is an admissible word (see [17] pp.21-23 and p.166 for more details on this classical construction). As there are only finitely many products $V_{y_0}U_{y_n}$, there exists

a constant $c' > 0$ such that, if a subset E of Y is a union of cylinders of length n, then

$$\frac{1}{c'}\,\mu(E) \le \mathrm{Card}\,\{\mathcal{W} \text{ cylinders of length } n:\ \mathcal{W} \subset E\}\,\lambda^{-n} \le c'\,\mu(E). \quad \triangle \tag{86}$$

Lemma of large deviations

We will use the following inequality of large deviations for irreducible Markov chains with finite state space (see [20], Theorem 3.3):

Lemma 5.1. *Let A be finite union of cylinders. For every $\varepsilon \in]0,1[$, there are two positive constants $R(\varepsilon)$, $\xi(\varepsilon)$ depending on A such that*

$$\mu\{x \in X:\ \frac{1}{L}\sum_{k=0}^{L-1} \mathbf{1}_A(\sigma^k x) \le \mu(A)(1-\varepsilon)\} \le R(\varepsilon)\, e^{-\xi(\varepsilon)L},\ \forall L \ge 1. \tag{87}$$

2) Reduction of the Ostrowski expansion to a "window"

By (2) and (4) we have, for a constant $\rho < 1$, $\|q_i q_j \alpha\| \le C\rho^{|j-i|}$. Hence, for $0 \le j \le \ell$, if κ is such that $0 \le j-\kappa \le j+\kappa \le \ell$:

$$|\|\sum_{i=0}^{\ell} b_i q_i q_j \alpha\| - \|\sum_{i=j-\kappa}^{j+\kappa-1} b_i q_i q_j \alpha\|| \le \|\sum_{i=j+\kappa}^{\ell} b_i q_i q_j \alpha\| + \|\sum_{i=0}^{j-\kappa-1} b_i q_i q_j \alpha\| \le C\,\rho^{-\kappa}. \tag{88}$$

It means that $\|\sum_{i=0}^{\ell} b_i q_i q_j \alpha\| = \|n_{b,\ell} q_j \alpha\|$ is well approximated by $\|\sum_{i=j-\kappa}^{j+\kappa-1} b_i q_i q_j \alpha\|$ which depends on a word with indices belonging to a window around j, with a precision depending on the size of the window. This is valid for any irrational α.

The quantity introduced in the next definition can be viewed as a function of an infinite word b or of a finite word $b_{j-\kappa_0}, ..., b_{j+\kappa_0}$. We put

$$\Gamma(b,j) := \frac{1}{5}\sum_{i=j-\kappa_0}^{j+\kappa_0} (-1)^i b_i \left(\alpha^{j-i} + (-\alpha)^{i-j}\right)\alpha. \tag{89}$$

A simple computation shows that $\Gamma(b, j+1) = -\Gamma(\sigma b, j)$. Therefore we have:

$$\Gamma(\sigma^k b, \kappa_0) = (-1)^k\,\Gamma(b, k+\kappa_0). \tag{90}$$

Lemma 5.2. *Let α be the golden mean For every $\delta > 0$, there is $\kappa_0 = \kappa_0(\delta)$ such that*

$$d(n_{b,\ell} q_j \alpha - \Gamma(b,j), \mathbb{Z}/5) \le \delta,\ \textit{if } j \ge \kappa_0. \tag{91}$$

Proof. We can restrict the sum $n_{b,\ell} q_j \alpha = \sum_{i=0}^{\ell} b_i q_i q_j \alpha$ to the sum $\sum_{i=j-\kappa_0}^{j+\kappa_0} b_i q_i q_j \alpha$, since their distance modulo 1 is small for κ_0 big enough by (88).

By (82), we have $q_i q_j = \dfrac{1+\alpha}{5}\alpha^{i+j} + \dfrac{2-\alpha}{5}(-\alpha)^{-(i+j)} + \dfrac{(-1)^i}{5}\left(\alpha^{j-i} + (-\alpha)^{i-j}\right)$;

$$\text{hence: } \sum_{i=j-\kappa_0}^{j+\kappa_0} b_i q_i q_j \alpha = \frac{1}{5}\sum_{i=j-\kappa_0}^{j+\kappa_0} [b_i\,(1+\alpha)\alpha^{i+j+1} + b_i\,(-1)^{i+j}\,(2-\alpha)\alpha^{1-(i+j)}] + \Gamma(b,j).$$

The distance to $\mathbb{Z}$ of the first sum above at right is small by (83). $\square$

The lemma shows that for the golden mean the distance to $\mathbb{Z}/5$ of $\sum_{j=0}^{\ell} b_i\, q_i\, q_j \alpha$ is almost the distance to $\mathbb{Z}/5$ of $\Gamma(b,j)$, which depends on the "short" word $b_{j-\kappa_0}...b_{j+\kappa_0}$ (reduction to a window of width $2\kappa_0$ of the "long" word $b_0...b_\ell$) in such a way that its values, when j varies, are the values of a fixed function computed for shifted words.

$\diamondsuit$ The lemma extends to a general quadratic number. We need some notation.

For an integer i, we write $i = \underline{i} + p\eta_i + n_0$, where $\underline{i}$ is the class of $i - n_0$ modulo p and η_i the integer part of $(i-n_0)/p$. The classes mod p are identified with the integers $0, ..., p-1$. With the notation introduced in (84), we put

$$T(\underline{i},\underline{j}) := \frac{\alpha}{r^2}(s_{\underline{i}} + t_{\underline{i}}\lambda)(u_{\underline{j}} + v_{\underline{j}}\lambda),\ U(\underline{i},\underline{j}) := \frac{\alpha}{r^2}(u_{\underline{i}} + v_{\underline{i}}\lambda)(s_{\underline{j}} + t_{\underline{j}}\lambda).$$

Lemma 5.3. *Let $\delta \in]0, \frac{1}{2r}[$. There is $\kappa_0 = \kappa_0(\delta)$ such that, if $j \geq n_0 + \kappa_0 p$,*

$$d(n_{b,\ell} q_j \alpha - \sum_{i=n_0+(\eta_j-\kappa_0)p}^{n_0+(\eta_j+\kappa_0)p-1} b_i \left[T(\underline{i},\underline{j})\lambda^{\eta_i-\eta_j} + U(\underline{i},\underline{j})\lambda^{\eta_j-\eta_i}\right], \mathbb{Z}/r) \leq \delta. \tag{92}$$

Proof. Recall that (a_n) is p-periodic for $n \geq n_0$. We consider indices $j \geq n_0$ and take sums on windows union of blocks of length p, hence of the form $n_0 + mp, \ldots, n_0 + qp - 1$. Using (84), the product $q_{n_0+kp+m}\, q_{n_0+k'p+m'}$ is equal to

$$\begin{aligned}&\frac{1}{r^2}\left[(s_m + t_m\lambda)(s_{m'} + t_{m'}\lambda)\lambda^{k+k'} + (u_m + v_m\lambda)(u_{m'} + v_{m'}\lambda)\lambda^{-(k+k')}\right]\\ &+\frac{1}{r^2}\left[(s_m + t_m\lambda)(u_{m'} + v_{m'}\lambda)\lambda^{k-k'} + (u_m + v_m\lambda)(s_{m'} + t_{m'}\lambda)\lambda^{k'-k}\right].\end{aligned}$$

Still using (84), we have

$$\frac{s_m}{r}\frac{(s_{m'} + t_{m'}\lambda)}{r}\lambda^{k+k'} = \frac{s_m}{r}(q_{n_0+(k'+k)p+m'} - \frac{1}{r}(u_{m'} + v_{m'}\lambda)\lambda^{-(k'+k)}).$$

From this (and a similar equality) we obtain

$$\begin{aligned}&q_{n_0+kp+m}\, q_{n_0+k'p+m'}\alpha\\ &\quad - \frac{1}{r^2}\left[(s_m + t_m\lambda)(u_{m'} + v_{m'}\lambda)\lambda^{k-k'} + (u_m + v_m\lambda)(s_{m'} + t_{m'}\lambda)\lambda^{k'-k}\right]\alpha\\ &= \frac{s_m}{r}q_{n_0+(k'+k)p+m'}\alpha + \frac{t_m}{r}q_{n_0+(k'+k+1)p+m'}\alpha\\ &\quad - [\frac{s_m}{r^2}(u_{m'} + \frac{v_{m'}}{r^2}\lambda)\lambda^{-(k'+k)}\alpha + \frac{t_m}{r^2}(u_{m'} + v_{m'}\lambda)\lambda^{-(k'+k+1)}\alpha].\end{aligned}$$

Since $d(q_{n_0+(k'+k)p+m'}\alpha, \mathbb{Z}/r) \leq d(q_{n_0+(k'+k)p+m'}\alpha, \mathbb{Z}) \leq C\lambda^{-(k'+k)}$ by (2), the distance of the left side term above to $\mathbb{Z}/r$ is bounded by $C\lambda^{-(k'+k)}$. It follows:

$$d(q_i q_j \alpha - \left[T(\underline{i},\underline{j})\lambda^{\eta_i-\eta_j} + U(\underline{i},\underline{j})\lambda^{\eta_j-\eta_i}\right], \mathbb{Z}/r) \leq C\lambda^{-p(\eta_j+\eta_i)}, \forall i, j \geq n_0.$$

Thus, using (88), for κ_0 large enough and if $j \geq n_0 + \kappa_0 p$, we have:

$$d(n_{b,\ell} q_j \alpha - \sum_{i=n_0+(\eta_j-\kappa_0)p}^{n_0+(\eta_j+\kappa_0)p-1} b_i \left[T(\underline{i},\underline{j})\lambda^{\eta_i-\eta_j} + U(\underline{i},\underline{j})\lambda^{\eta_j-\eta_i}\right], \mathbb{Z}/r)$$
$$\leq C\lambda^{-\kappa_0 p} + C \sum_{i=n_0+(\eta_j-\kappa_0)p}^{n_0+(\eta_j+\kappa_0)p-1} \lambda^{-\eta_j-\eta_i} \leq C\lambda^{-\kappa_0 p} + 2C\kappa_0 \lambda^{\kappa_0 - 2\eta_j} \leq \delta. \quad \square$$

$\triangle$

3) From long words to short words

Lemma 5.4. *Let $1 \leq \ell_0 \leq \ell_1 \leq \ell$ be three integers and let $\Lambda : b_0 \dots b_\ell \mapsto b_{\ell_0} \dots b_{\ell_1}$ be the "restriction" map from the set $\mathcal{L}$ of admissible words to shortened words. There is a constant $c > 0$ such that, if $\mathcal{S}$ is the image of Λ, for any subset $\mathcal{P}$ of $\mathcal{S}$, we have*

$$\frac{\mathrm{Card}\,(\mathcal{P})}{\mathrm{Card}\,(\mathcal{S})} \leq c\, \frac{\mathrm{Card}\,\{w \in \mathcal{L} :\, \Lambda(w) \in \mathcal{P}\}}{\mathrm{Card}\,(\mathcal{L})}.$$

We can take $c = 4$ for the golden mean, $c = u_0^{-2}$, with $u_0 = \inf_{k>1} \frac{q_{k-1}-1}{q_k - 1}$, in the general case.

Proof. The proof is given for the golden mean. The general case is analogous.

The ways of completing a short word into a long one depend only on the first letter b_{ℓ_0} and the last letter b_{ℓ_1}: if $b_{\ell_0} \neq 1$, any admissible beginning fits; if $b_{\ell_0} = 1$, then only the admissible beginnings finishing by 0 fit; if $b_{\ell_1} = 0$ then any admissible ending fits; if $b_{\ell_1} = 1$, only endings with 0 as first letter fit.

The number of admissible words of length r is q_{r+1}, the number of admissible words of length r beginning (or ending) by 0 is q_r.

Let denote $\mathcal{S}_i$, $i = 1, ..., 4$, the set of short words $b_{\ell_0} \dots b_{\ell_1}$ such that $b_{\ell_0} = b_{\ell_1} = 0$, $b_{\ell_0} = 0$ and $b_{\ell_1} = 1$, $b_{\ell_0} = 1$ and $b_{\ell_1} = 0$, $b_{\ell_0} = b_{\ell_1} = 1$, respectively.

Depending on the set $\mathcal{S}_i$ to which $\Lambda(w)$ belongs, the cardinal of $\mathrm{Card}\,\Lambda^{-1}(\Lambda(w))$ is $D_1 = q_{\ell_0} q_{\ell-\ell_1+1}$, $D_2 = q_{\ell_0} q_{\ell-\ell_1}$, $D_3 = q_{\ell_0-1} q_{\ell-\ell_1+1}$, or $D_4 = q_{\ell_0-1} q_{\ell-\ell_1}$ respectively.

Since, $\frac{1}{2} \leq q_r/q_{r+1} \leq 1$, for all r, we have $D_1 = \max_i D_i$, $D_4 = \min_i D_i$, $D_4 \leq D_1 \leq 4D_4$ and finally

$$\begin{aligned}
\mathrm{Card}\,(\mathcal{P}) &= \sum_{i=1}^4 \mathrm{Card}\,(\mathcal{P} \cap \mathcal{S}_i) = \sum_{i=1}^4 \frac{1}{D_i} \mathrm{Card}\,\{w \in \mathcal{L} :\, \Lambda(w) \in \mathcal{P} \cap \mathcal{S}_i\} \\
&\leq \frac{1}{D_4} \sum_{i=1}^4 \mathrm{Card}\,\{w \in \mathcal{L} :\, \Lambda(w) \in \mathcal{P} \cap \mathcal{S}_i\} = \frac{1}{D_4} \mathrm{Card}\,\{w \in \mathcal{L} :\, \Lambda(w) \in \mathcal{P}\}, \\
\mathrm{Card}\,(\mathcal{S}) &= \sum_{i=1}^4 \mathrm{Card}\,(\mathcal{S}_i) = \sum_{i=1}^4 \frac{1}{D_i} \mathrm{Card}\,\{w \in \mathcal{L} :\, \Lambda(w) \in \mathcal{S}_i\} \geq \frac{1}{D_1} \mathrm{Card}\,(\mathcal{L}). \quad \square
\end{aligned}$$

4a) End of the proof of Proposition 1.4 when α is the golden mean

Let δ be a small positive number. Its value will be chosen later. It follows from (14) for ℓ big enough that, if $q_{j+1}^{-1} < \delta$:

$$\text{Card}\,\{n \in [1, q_{\ell+1}[:\ d(nq_j\alpha, \mathbb{Z}/5) \leq 3\delta\} \leq C_1\,\delta q_{\ell+1}, \forall j \leq \ell. \tag{93}$$

If κ_0 is big enough, from (91) in Lemma 5.2, we have with $\Gamma(b,j)$ defined in (89):

$$d(n_{b,\ell}q_j\alpha, \mathbb{Z}/5) \geq 3\delta \Rightarrow d(\Gamma(b,j), \mathbb{Z}/5) \geq 2\delta \Rightarrow d(n_{b,\ell}q_j\alpha, \mathbb{Z}/5) \geq \delta. \tag{94}$$

By taking κ_0 large enough, we can suppose $q_{\kappa_0+1}^{-1} < \delta$. By (93) (translated in terms of words) for each $j \in [\kappa_0, \ell]$, the proportion of words $b = b_0 \dots b_\ell$ of length $\ell+1$ for which $d(n_{b,\ell}q_j\alpha, \mathbb{Z}/5) \geq 3\delta$, is smaller than $C_1\delta$. Therefore, if $\ell \geq \kappa_0$, we get

$$\text{Card}\,\{b_0 \dots b_\ell :\ d(\Gamma(b,j), \mathbb{Z}/5) \leq 2\delta\} \leq C_1\,\delta q_\ell, \forall j \in [\kappa_0, \ell]. \tag{95}$$

But $\Gamma(b,j)$ depends only on the short word $b_{j-\kappa_0} \dots b_{j+\kappa_0}$, part of the long word $b = b_0 \dots b_\ell$. It follows, using Lemma 5.4 that

$$\text{Card}\,\{b_{j-\kappa_0} \dots b_{j+\kappa_0} :\ d(\Gamma(b,j), \mathbb{Z}/5) \leq 2\delta\} \leq C_2\,\delta q_{2\kappa_0+2}, \forall j \in [\kappa_0, \ell]. \tag{96}$$

Putting $A_\delta := \{b :\ d(\Gamma(b,\kappa_0), \mathbb{Z}/5) \geq 2\delta\}$, it follows from (96) and (85):

$$\mu(A_\delta^c) \leq \alpha^{-2\kappa_0-2}\text{Card}\,\{b_{j-\kappa_0} \dots b_{j+\kappa_0} :\ d(\Gamma(b,j), \mathbb{Z}/5) \leq 2\delta\} \leq C_2\,\delta q_{2\kappa_0+2}\alpha^{-2\kappa_0-2} \leq C_3\,\delta.$$

Let C_4 be a constant $> C_3$ and $\varepsilon = C_4\delta$. Observe that we can chose ℓ large enough so that $\mu(A_\delta)\,(\ell - \kappa_0) \geq (1-\varepsilon)\,\ell$: indeed, we have $\mu(A_\delta) - (1-\varepsilon) > 0$ and by taking $\ell > \mu(A_\delta)\kappa_0/(\mu(A_\delta) - (1-\varepsilon))$ we obtain the required inequality.

Now we use $\sum_{k=0}^{L-1} \mathbf{1}_{A_\delta}(\sigma^k b) = \text{Card}\,\{k < L :\ d(\Gamma(\sigma^k b, \kappa_0), \mathbb{Z}/5) \geq 2\delta\}$ and (90). According (94) with $j = k + \kappa_0$ and Lemma 5.1 with $A = A_\delta$ and $\varepsilon = C_4\delta$ (we assume $\delta < C_4^{-1}$), there are two positive constants $R = R(\varepsilon)$, $\xi = \xi(\varepsilon)$ such that

$$\mu\{b \in X :\ \text{Card}\,\{j \in [\kappa_0, L+\kappa_0[:\ d(\Gamma(b,j), \mathbb{Z}/5) \geq 2\delta\} \leq \mu(A_\delta)(1-\varepsilon)L\} \leq R(\varepsilon)\,e^{-\xi L}.$$

Using "$\Rightarrow$" in (94), we have therefore, taking $L = \ell - \kappa_0$, for $\ell - \kappa_0 \geq j \geq \kappa_0$,

$$\mu\{b \in X :\ \text{Card}\,\{j \in [\kappa_0, \ell[:\ d(n_{b,\ell}q_j\alpha, \mathbb{Z}/5) \geq \delta\} \leq \mu(A_\delta)(1-\varepsilon)(\ell - \kappa_0)\} \leq R\,e^{-\xi\,(\ell-\kappa_0)}.$$

By (85), the previous inequality translated in terms of cardinal yields for a constant C_5:

$$\text{Card}\,\{b_0 \dots b_\ell :\ \text{Card}\,\{j < \ell :\ d(n_{b,\ell}q_j\alpha, \mathbb{Z}/5) \geq \delta\} \leq (1 - C_4\delta)^2\ell\} \leq C_5 e^{-\xi\ell} q_{\ell+1}.$$

If δ is taken small enough to get $(1-\varepsilon_0) \leq (1-C_4\delta)^2$ and using that $e^{\xi\ell}$ is equivalent to a power of $q_{\ell+1}$ (because $(q_\ell)_\ell$ is equivalent to a geometric progression), the previous inequality shows (20) of Proposition 1.4. □

♢ 4b) End of the proof of Proposition 1.4 for a general quadratic number

As for the golden number, we take a positive number δ whose value will be fixed later. By (14), if κ_0 is large enough, we have for some $C_1 > 0$

$$\mathrm{Card}\,\{n \in [1, q_{\ell+1}[:\ d(nq_j\alpha, \mathbb{Z}/r) \le 3\delta\} \le C_1\,\delta q_{\ell+1}, \forall j \in [n_0+\kappa_0, \ell];$$

hence, in terms of admissible words $b_0 \dots b_\ell$, if $j \in [n_0+\kappa_0, \ell]$,

$$\mathrm{Card}\{b_0 \dots b_\ell :\ d(n_{b,\ell}q_j\alpha, \mathbb{Z}/r) \le 3\delta\} \le C_1\,\delta q_{\ell+1}. \tag{97}$$

Let $\Gamma_j, \Gamma^0_{\underline{j}}$ be the functions on Y

$$\begin{aligned}
\Gamma_j(b) &:= \sum_{i=n_0+(\eta_j-\kappa_0)p}^{n_0+(\eta_j+\kappa_0)p-1} b_i \left[T(\underline{i}, \underline{j})\lambda^{\eta_i-\eta_j} + U(\underline{i}, \underline{j})\lambda^{\eta_i-\eta_j}\right], \\
\Gamma^0_{\underline{j}}(b) &:= \sum_{i=n_0}^{n_0+2\kappa_0 p-1} b_i \left[T(\underline{i}, \underline{j})\lambda^{\eta_i-\kappa_0} + U(\underline{i}, \underline{j})\lambda^{\kappa_0-\eta_i}\right].
\end{aligned}$$

Remark that the sums on the right can be viewed as functions of y through the b_i's. Letting $y_k := b_{n_0+kp} \dots b_{n_0+kp+p-1}$, we see that the sum inside the definition of Γ_j is a function of $y_{\eta_j-\kappa_0} \dots y_{\eta_j+\kappa_0-1}$.

Let A_δ be the subset of Y defined by

$$A_\delta := \{y :\ d(\Gamma^0_{\underline{j}}(y), \mathbb{Z}/r) \ge 2\delta,\ \text{for } \underline{j} = 0, \dots, p-1\}.$$

By (92) in Lemma 5.3, if κ_0 is sufficiently large, we have the implication

$$d(n_{b,\ell}q_j\alpha, \mathbb{Z}/r) \ge 3\delta \Rightarrow d(\Gamma_j(b), \mathbb{Z}/r) \ge 2\delta. \tag{98}$$

As $\underline{j} = \underline{j-p}$, $\eta_{j+p} = \eta_j + 1$ and $\eta_{i+p} = \eta_i + 1$, we obtain by $\eta_j - \kappa_0$ iterations:

$$\begin{aligned}
&\sum_{i=n_0+(\eta_j-\kappa_0)p}^{n_0+(\eta_j+\kappa_0)p-1} b_i T(\underline{i}, \underline{j})\lambda^{\eta_i-\eta_j} = \sum_{i=n_0+(\eta_j-1-\kappa_0)p}^{n_0+(\eta_j-1+\kappa_0)p-1} b_{i+p} T(\underline{i+p}, \underline{j})\lambda^{\eta_{i+p}-\eta_j} \\
&= \sum_{i=n_0+(\eta_{j-p}-\kappa_0)p}^{n_0+(\eta_{j-p}+\kappa_0)p-1} b_{i+p} T(\underline{i}, \underline{j})\lambda^{\eta_i+1-\eta_j} = \sum_{i=n_0+(\eta_{j-p}-\kappa_0)p}^{n_0+(\eta_{j-p}+\kappa_0)p-1} b_{i+p} T(\underline{i}, \underline{j})\lambda^{\eta_i-\eta_{j-p}} \\
&= \dots = \sum_{i=n_0+(\eta_{j-(\eta_j-\kappa_0)p}-\kappa_0)p}^{n_0+\eta_{j-(\eta_j-\kappa_0)p}+\kappa_0 p-1} b_{i+(\eta_j-\kappa_0)p} T(\underline{i}, \underline{j})\lambda^{\eta_i-\eta_{j-(\eta_j-\kappa_0)p}}.
\end{aligned}$$

Since $\eta_{j-(\eta_j-\kappa_0)p} = \kappa_0$, the last quantity reduces to $\sum_{i=n_0}^{n_0+2\kappa_0 p-1} b_{i+(\eta_j-\kappa_0)p} T(\underline{i}, \underline{j})\lambda^{\eta_i-\kappa_0}$.

The same computation can be done for $\sum_{i=n_0+(\eta_j-\kappa_0)p}^{n_0+(\eta_j+\kappa_0)p-1} b_i U(\underline{i},\underline{j})\lambda^{\eta_i-\eta_j}$. Taking the sum for the T's and U's, we get

$$\Gamma_j(y) = \Gamma^0_{\underline{j}}(\sigma^{\eta_j-\kappa_0}y). \tag{99}$$

From (98), (97) and (99), it follows, if $\ell \geq n_0 + 2\kappa_0 p$ and $j \geq n_0 + \kappa_0$,

$$\mathrm{Card}\{b_0 \dots b_\ell :\ d(\Gamma_{\underline{j}}(\sigma^{\eta_j-\kappa_0}y), \mathbb{Z}/r) \leq 2\delta\} \leq C_1\, \delta q_{\ell+1}.$$

But $\Gamma^0_{\underline{j}}(\sigma^{\eta_j-\kappa_0}y)$ depends only on the short word $b_{n_0+(\eta_j-\kappa_0)p} \dots b_{n_0+(\eta_j+\kappa_0)p-1}$, which is a sub-word of the "long" word $b_0 \dots b_\ell$. By Lemma 5.4 we obtain for constants $C_2, C_3 > 0$:

$$\mathrm{Card}\{b_{n_0+(\eta_j-\kappa_0)p} \dots b_{n_0+(\eta_j+\kappa_0)p-1} :\ d(\Gamma^0_{\underline{j}}(\sigma^{\eta_j-\kappa_0}y), \mathbb{Z}/r) \leq 2\delta\} \leq C_2\delta\lambda^{2\kappa_0}. \tag{100}$$

Then, (100) and (86) imply that

$$\mu(A_\delta^c) = \mu\{y \in Y :\ d(\Gamma^0_m(y), \mathbb{Z}/r) < 2\delta,\ m = 0, ..., p-1\} \leq C_3\delta. \tag{101}$$

Now, we have

$$\sum_{k=0}^{n-1} \mathbf{1}_{A_\delta}(\sigma^k y) = \mathrm{Card}\{k < n :\ d(\Gamma^0_m(\sigma^k y), \mathbb{Z}/r) \geq 2\delta,\ m = 0, ..., p-1\},$$

$$\Gamma^0_m(\sigma^k y) = \sum_{i=n_0+kp}^{n_0+(k+2\kappa_0)p-1} b_i \left[T(\underline{i}, m)\lambda^{\eta_i-k-\kappa_0} + U(\underline{i}, m)\lambda^{k+\kappa_0-\eta_i}\right],$$

and, if $j = (k+\kappa_0)p + m \in [n_0 + \kappa_0, \ell]$ (i.e., $\eta_j = k + \kappa_0$, $\underline{j} = m$),

$$d(\Gamma^0_m(\sigma^k y), \mathbb{Z}/r) \geq 2\delta \Rightarrow d(n_{b,\ell} q_j \alpha, \mathbb{Z}/r) \geq \delta.$$

In particular:

$$\begin{aligned} p\,\mathrm{Card}\{k < \eta_\ell - \kappa_0 :\ d(\Gamma^0_m(\sigma^k y), \mathbb{Z}/r) \geq 2\delta,\ m = 0, ..., p-1\} \\ \leq \mathrm{Card}\,\{j < (\eta_\ell - \kappa_0)p :\ d(n_{b,\ell} q_j \alpha, \mathbb{Z}/r) \geq \delta\}. \end{aligned}$$

By Lemma 5.1, for the Markov chain deduced from Y with state space the set of words of length $2\kappa_0$ in Y, we get from (87):

$$\mu\{y :\ \mathrm{Card}\,\{j < (\eta_\ell - \kappa_0)p :\ d(n_{b,\ell} q_j \alpha, \mathbb{Z}/r) \geq \delta\} \leq \mu(A_\delta)(1-\varepsilon)p(\eta_\ell - \kappa_0)\} \leq Re^{-\xi(\eta_\ell-\kappa_0)}.$$

This can be translated in terms of cardinal using (86):

$$\mathrm{Card}\,\{y_0 \dots y_{\eta_\ell-\kappa_0} :\ \mathrm{Card}\,\{j < (\eta_\ell - \kappa_0)p :\ d(n_{b,\ell} q_j \alpha, \mathbb{Z}/r) \geq \delta\} \leq \mu(A_\delta)(1-\varepsilon)p(\eta_\ell - \kappa_0)\}$$

is smaller than $C_4 e^{-\xi(\eta_\ell-\kappa_0)}\lambda^{\eta_\ell-\kappa_0}$. It implies

$$\mathrm{Card}\,\{b_0 \dots b_\ell :\ \mathrm{Card}\,\{j < \ell :\ d(n_{b,\ell} q_j \alpha, \mathbb{Z}/r) \geq \delta\} \leq \mu(A_\delta)(1-\varepsilon)p(\eta_\ell - \kappa_0)\} \leq C_5 e^{-\xi(\eta_\ell-\kappa_0)}\lambda^{\eta_\ell-\kappa_0}.$$

If $\eta_\ell > (\kappa_0 + 1)/\varepsilon$ (that is $\ell \geq p(\kappa_0 + 2)/\varepsilon$), then $p(\eta_\ell - \kappa_0) \geq (1-\varepsilon)\ell$ and, for some $C_6 > 0$, there are less than $C_6 e^{-\xi\eta_\ell}\lambda^{\eta_\ell}$ words b of length ℓ such that

$$\mathrm{Card}\,\{j \leq \ell :\ d(\sum_{i=0}^{\ell} b_i q_i q_j \alpha, \mathbb{Z}/r) \geq \delta\} \leq \mu(A_\delta)(1-\varepsilon)^2\ell. \tag{102}$$

By (101), for $\varepsilon_0 > 0$, we can choose ε and δ such that $\mu(A_\delta)(1-\varepsilon)^2 = 1-\varepsilon_0$. On the other hand, since $c^{-1}q_{\ell+1} \le \lambda^{\eta_\ell} \le cq_{\ell+1}$ for some $c > 0$, $C_6 e^{-\xi\eta_\ell}\lambda^{\eta_\ell} \le C_7 q_{\ell+1}^{1-\zeta}$, for some positive constants ζ, C_7. Finally, we have obtained (20) (in terms of number of admissible words):

$$\text{Card}\,\{b_0 \dots b_\ell : \text{Card}\,\{j < \ell : d(n_{b,\ell}q_j\alpha, \mathbb{Z}/r) \ge \delta\} \le (1-\varepsilon_0)\ell\} \le C_7 q_{\ell+1}^{1-\zeta}. \quad \square$$

$\triangle$

6. **Appendix 2: weighted orthogonal functions**

Let (g_n) be a sequence of orthogonal real functions in L^2 of a probability space (X, μ) and (u_n) be a sequence of positive constants. By the Lebesgue dominated convergence theorem, if the functions g_n are uniformly bounded, the following condition

$$\lim_n \frac{\sum_{k=1}^N u_k^2}{(\sum_{k=1}^N u_k)^2} \to 0 \tag{103}$$

is necessary for

$$\lim_N \frac{\sum_{k=1}^N u_k\, g_k(x)}{\sum_{k=1}^N u_k} = 0, \text{ for a.e. } x. \tag{104}$$

(103) is satisfied if the following condition holds:

$$1 \le u_n \le n^\gamma, \forall n \ge 1, \text{ with } 0 \le \gamma < 1. \tag{105}$$

Indeed we have $\dfrac{\sum_{k=1}^N u_k^2}{(\sum_{k=1}^N u_k)^2} \le (\max_{k=1}^N u_k)\dfrac{\sum_{k=1}^N u_k}{(\sum_{k=1}^N u_k)^2} \le \dfrac{\max_{k=1}^N u_k}{(\sum_{k=1}^N u_k)} \le N^{\gamma-1} \to 0.$

But (103) and the result of Proposition 6.1 can fail if the parameter γ in (105) is taken ≥ 1. Indeed, suppose that $\|g_k\|_2 = 1$, and let us take $u_k = k$ if k is a power of 2, else $u_k = 1$.

Then, we have $1 \le u_k \le k$, $\sum_{k=1}^{2^n} u_k^2 \ge \frac{4}{3}2^{2n}$ and $\sum_{k=1}^{2^n} u_k = 2^{n+1} - (n+1)$, so that

$$\frac{\sum_{k=1}^{2^n} u_k^2}{(\sum_{k=1}^{2^n} u_k)^2} \ge \frac{1}{3}(1 - 2^{-(n+1)}(n+1))^2 \to \frac{1}{3}.$$

Proposition 6.1. *Let $(g_k)_{k\ge1}$ be a sequence of orthogonal functions in $L^2(X,\mu)$, bounded in L^2 norm. Under the condition*

$$1 \le u_n \le n^\gamma, \text{ with } 0 \le \gamma < \frac{1}{2}, \tag{106}$$

it holds

$$\lim_N \frac{\sum_{k=1}^N u_k\, g_k(x)}{\sum_{k=1}^N u_k} = 0, \text{ for a.e. } x. \tag{107}$$

Proof. 1) Setting $R_N(x) := \dfrac{\sum_{k=1}^N u_k\, g_k(x)}{\sum_{k=1}^N u_k}$, by orthogonality and the conditions on u_k, there is a constant C such that $\displaystyle\int_X |R_N(x)|^2\, d\mu \leq C\, N^{2\gamma-1}$, which implies $\sum_{n=1}^\infty \|R_{n^p}\|_2^2 < +\infty$, if $p(1-2\gamma) > 1$.

As $1-2\gamma > 0$, we can choose p such that $p(1-2\gamma) > 1$. We have then: $\lim_n R_{n^p}(x) = 0$, for a.e. x. Therefore, it suffices to show that:

$$\lim_n \sup_{\ell \in J_n} |R_{n^p+\ell}(x) - R_{n^p}(x)| = 0, \text{ where } J_n = \{0, 1, ..., (n+1)^p - n^p - 1\}.$$

2) For reals A, C, B_ℓ, D_ℓ, $\ell \in J_n$, with $C, D_\ell > 0$, it holds:

$$\max_{\ell \in J_n} |\frac{A+B_\ell}{C+D_\ell} - \frac{A}{C}| \leq \frac{\max_{\ell\in J_n} |B_\ell| + |A|}{C}.$$

This implies, with $A = \sum_{k=1}^{n^p} u_k\, g_k$, $B_\ell = \sum_{k=n^p+1}^{n^p+\ell} u_k\, g_k$, $C = \sum_{k=1}^{n^p} u_k$, $D_\ell = \sum_{k=n^p+1}^{n^p+\ell} u_k$,

$$\max_{\ell \in J_n} |R_{n^p+\ell}(x) - R_{n^p}(x)| \leq \frac{\max_{\ell\in J} |\sum_{k=n^p+1}^{n^p+\ell} u_k\, g_k|}{\sum_{k=1}^{n^p} u_k} + \frac{|\sum_{k=1}^{n^p} u_k\, g_k|}{\sum_{k=1}^{n^p} u_k}. \tag{108}$$

By a lemma of Rademacher-Mensov ([11], p. 156), if $Y_1, ..., Y_L$ are mutually orthogonal functions in a probability space (X, μ) with finite variances $\sigma_1^2, ..., \sigma_L^2$, then

$$\mathbb{E}[(\max_{\ell=1}^L (\sum_{j=1}^\ell Y_j))^2] \leq C(\log(4L))^2 \sum_{\ell=1}^L \sigma_\ell^2. \tag{109}$$

If we put $M_{n,p} := \max_{\ell\in J_n} |\sum_{k=n^p}^{n^p+\ell} u_k g_k|$, then by (109) we have

$$\begin{aligned} \mathbb{E}(M_{n,p}^2) &\leq C(\log(4\,p\,n^{p-1}))^2 \sum_{j=n^p}^{(n+1)^p-1} u_j^2 \leq C(\log(4\,p\,n^{p-1}))^2 \sum_{j=n^p}^{(n+1)^p-1} j^{2\gamma} \\ &\leq C'(\log n)^2\, n^{p-1}\, n^{2p\gamma} = C'(\log n)^2\, n^{p(2\gamma+1)-1}. \end{aligned}$$

It follows:

$$\mathbb{E}[\big(\frac{\max_{\ell\in J} |\sum_{k=n^p}^{n^p+\ell} u_k g_k|}{\sum_{j=1}^{n^p} u_j}\big)^2] \leq C' \frac{(\log n)^2\, n^{p(2\gamma+1)-1}}{n^{2p}} = C'(\log n)^2\, n^{p(2\gamma-1)-1}.$$

Therefore, since $2\gamma - 1 < 0$, we have

$$\sum_n \mathbb{E}[\big(\frac{\max_{\ell\in J} |\sum_{k=n^p}^{n^p+\ell} u_k g_k|}{\sum_{k=1}^{n^p} u_k}\big)^2] < +\infty,$$

so that $\lim_n \dfrac{\max_{\ell\in J} |\sum_{k=n^p}^{n^p+\ell} u_k g_k|}{\sum_{k=1}^{n^p} u_k} = 0$, a.e.

Both terms in the right side of (108) converge a.e. to 0, which implies a.e.:

$$\lim_n \max_{\ell\in J_n} |R_{n^p+\ell}(x) - R_{n^p}(x)| \to 0. \quad \square$$

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