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Jean-Pierre Conze, Stéphane Le Borgne. On the CLT for rotations and BV functions. *Comptes Rendus. Mathématique*, 2019, 357 (2), pp.212-215. 10.1016/j.crma.2019.01.008 . hal-01777584v2

HAL Id: hal-01777584

<https://hal.science/hal-01777584v2>

Submitted on 13 Jun 2018 (v2), last revised 10 Jan 2022 (v4)

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ON THE CLT FOR ROTATIONS AND BV FUNCTIONS

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ABSTRACT. Let $x \mapsto x + \alpha$ be a rotation on the circle and let φ be a step function. We denote by $\varphi_n(x)$ the corresponding ergodic sums $\sum_{j=0}^{n-1} \varphi(x + j\alpha)$. Under an assumption on α , for example when α has bounded partial quotients, and a Diophantine condition on the discontinuity points of φ , we show that $\varphi_n/\|\varphi_n\|_2$ is asymptotically Gaussian for n in a set of density 1. An important point is the control of the variance $\|\varphi_n\|_2^2$ for n belonging to a large set of integers. When α is a quadratic irrational, this information can be improved. The method is based on decorrelation inequalities for the ergodic sums taken at times q_k , where the q_k 's are the denominators of α .

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Date: June 13, 2018.

1. Introduction

Let us consider an irrational rotation $x \mapsto x + \alpha \bmod 1$ on $X = \mathbb{R}/\mathbb{Z}$. By the Denjoy-Koksma inequality, the ergodic sums $\varphi_L(x) = \sum_0^{L-1} \varphi(x + j\alpha)$ of a centered BV (bounded variation) function φ are uniformly bounded along the sequence (q_n) of denominators of α . But, besides, one has a stochastic behaviour at a certain scale along other sequences (L_n) . In a sense the process defined by the above sums $\varphi_L(x)$ presents more complexity than the ergodic sums under the action of hyperbolic maps for which a central limit theorem is usually satisfied. We propose a quantitative analysis of this phenomenon.

Several papers have been devoted to this topic. M. Denker and R. Burton (1987), M. Lacey (1993), M. Weber (2000) and other authors proved the existence of functions whose ergodic sums over rotations satisfy a CLT after self-normalization. D. Volný and P. Liardet in 1997 showed that, when α has unbounded partial quotients, for a dense G_δ set of functions f in the class of absolutely continuous, or Lipschitz continuous or differentiable functions, the distributions of the random variables $c_n^{-1} \sum_{j=0}^{n-1} f \circ T^j$, $c_n \uparrow \infty$ and $c_n/n \rightarrow 0$, are dense in the set of all probability measures on the real line.

Most often in these works the functions that are dealt with are not explicit. Here we consider ergodic sums of simple functions such as step functions. Let us mention the following related papers. For $\psi := 1_{[0, \frac{1}{2}[} - 1_{[\frac{1}{2}, 0[}$, F. Huveneers [Hu09] studied the existence of a sequence $(L_n)_{n \in \mathbb{N}}$ such that ψ_{L_n} after normalization is asymptotically normally distributed. In [CoIsLe17] it was shown that, when α has unbounded partial quotients, along some subsequences the ergodic sums of some step functions φ can be approximated by a Brownian motion.

Here we will use as in [Hu09] a method based on decorrelation inequalities which applies in particular to the bounded type case (bpq), i.e., when the sequence (a_n) of partial quotients of α is bounded. It relies on an abstract central limit theorem valid under some suitable decorrelation conditions. If φ is a step function, we give conditions which insure that for n in a set of density 1, the distribution of $\varphi_n / \|\varphi_n\|_2$ is close to a normal distribution. (Theorem 3.4). Beside the remarkable recent “temporal” limit theorems for rotations (see [Be10], [BrU17], [DoSa16]), this shows that a “spatial” asymptotic normal distribution can also be observed, for times restricted to a large set of integers.

An important point is the control of the variance $\|\varphi_n\|_2^2$ for n belonging to a set of density 1, at least in the case of α with bounded partial quotients. In the special case where α is a quadratic irrational, this information can be improved. A result for vectorial step functions can also be proven (Theorem 3.9). To apply the results to a step function, a Diophantine type condition is needed (cf. Condition (15)), which holds generically with respect to the discontinuities.

The results are presented in Section 2 and 3. There are based on the decorrelation of the ergodic sums taken at times q_k and on an abstract central limit theorem, whose proofs are given in Sections 4 and 5.

2. Variance of the ergodic sums

Notation The uniform measure on \mathbb{T}^1 identified with $X = [0, 1[$ is denoted by μ . The arguments of the functions are taken modulo 1. For a 1-periodic function φ , we denote by $V(\varphi)$ the *variation* of φ computed for its restriction to the interval $[0, 1[$ and use the shorthand BV for “bounded variation”. Let \mathcal{C} be the class of centered BV functions. By C we denote a numerical constant whose value may change from a line to the other or inside a line.

The number $\alpha = [0; a_1, a_2, \dots]$ is an irrational number in $[0, 1[$, with partial quotients (a_n) and denominators (q_n) : If φ belongs to \mathcal{C} , its Fourier coefficients $c_r(\varphi)$ satisfy:

$$(1) \quad c_r(\varphi) = \frac{\gamma_r(\varphi)}{r}, r \neq 0, \text{ with } K(\varphi) := \sup_{r \neq 0} |\gamma_r(\varphi)| < +\infty.$$

The class \mathcal{C} contains the step functions with a finite number of discontinuities.

The ergodic sums $\sum_{j=0}^{N-1} \varphi(x + j\alpha)$ are denoted by $\varphi_N(x)$. Their Fourier series are

$$(2) \quad \varphi_N(x) := \sum_{j=0}^{N-1} \varphi(x + j\alpha) = \sum_{r \neq 0} \frac{\gamma_r(\varphi)}{r} e^{\pi i(N-1)r\alpha} \frac{\sin \pi N r \alpha}{\sin \pi r \alpha} e^{2\pi i r x}.$$

If φ belongs to \mathcal{C} , then so do the sums φ_N and we have

$$|c_r(\varphi_N)| \leq \frac{K(\varphi)}{|r|} \frac{|\sin \pi N r \alpha|}{|\sin \pi r \alpha|} \leq \frac{N K(\varphi)}{|r|}.$$

If φ is a BV function, then so is φ_N and $V(\varphi_N) \leq NV(\varphi)$.

2.1. Reminders on continued fractions.

For $u \in \mathbb{R}$, $\|u\|$ denotes its distance to the integers: $\|u\| := \inf_{n \in \mathbb{Z}} |u - n| = \min(\{u\}, 1 - \{u\}) \in [0, \frac{1}{2}]$. Recall that

$$2\|x\| \leq |\sin \pi x| \leq \pi\|x\|, \quad \forall x \in \mathbb{R}.$$

Let $\alpha \in [0, 1[$ be an irrational number. Then, for each $n \geq 1$, we write $\alpha = \frac{p_n}{q_n} + \frac{\theta_n}{q_n}$, where p_n and q_n are the numerators and denominators of α . Recall that

$$(3) \quad \frac{1}{a_{n+1} + 2} \leq \frac{q_n}{q_{n+1} + q_n} \leq q_n \|q_n \alpha\| = q_n |\theta_n| \leq \frac{q_n}{q_{n+1}} = \frac{q_n}{a_{n+1} q_n + q_{n-1}} \leq \frac{1}{a_{n+1}},$$

$$(4) \quad \|k\alpha\| \geq \|q_{n-1} \alpha\| \geq \frac{1}{q_n + q_{n-1}} \geq \frac{1}{2q_n}, \text{ for } 1 \leq k < q_n,$$

$$(5) \quad \|q_n \alpha\| = (-1)^n (q_n \alpha - p_n), \quad \theta_n = (-1)^n \|q_n \alpha\|, \quad \alpha = \frac{p_n}{q_n} + (-1)^n \frac{\|q_n \alpha\|}{q_n}.$$

For $n \geq 1$ we put

$$(6) \quad m = m(n) := \ell, \quad \text{if } n \in [q_\ell, q_{\ell+1}[.$$

If α has bounded partial quotients ($K = \sup a_n < \infty$), then there is a constant $\lambda > 0$ such that $\frac{\ln n}{m(n)} \in [\lambda, \ln(K + 1)]$.

Thereafter we will need the following assumption which is satisfied by a.e. α :

Hypothesis 1. *There are two constants $A \geq 1$, $p \geq 0$ such that*

$$(7) \quad a_n \leq A n^p, \forall n \geq 1.$$

Recall that every integer $n \geq 1$ can be represented as follows (Ostrowski's expansion):

$$(8) \quad n = \sum_{k=0}^{m(n)} b_k q_k, \text{ with } 0 \leq b_0 \leq a_1 - 1, 0 \leq b_k \leq a_{k+1} \text{ for } 1 \leq k < m, 1 \leq b_m \leq a_{m+1}.$$

Indeed, if $n \in [q_m, q_{m+1}[$, we can write $n = b_m q_m + r$, with $1 \leq b_m \leq a_{m+1}$, $0 \leq r < q_m$. and by iteration, we get (8).

In this way, we can associate to every n its coding, which is a word $b_0 \dots b_m$, with $b_j \in \{0, 1, \dots, a_j\}$, $j = 0, \dots, m$. Let us call "admissible" a finite word $b_0 \dots b_m$, $b_j \in \{0, 1, \dots, a_j\}$, with $b_m \neq 0$, such that for two consecutive letters b_j, b_{j+1} , we have $b_j b_{j+1} \neq u a_{j+1}$, with $u \neq 0$.

Let us show that the Ostrowski's expansion of an integer n is admissible. The proof is by induction. Let n be in $[q_m, q_{m+1}[$. We start the construction of the Ostrowski's expansion of n as above. Now the following steps of the algorithm yield the Ostrowski's expansion of $n - b_m q_m$ (excepted some zero's which might be added at the end). Since $n - b_m q_m \in [0, q_m[$, the Ostrowski's expansion of $n - b_m q_m$ is admissible. It remains to check that, if $b_m = a_{m+1}$, then $b_{m-1} = 0$. If $b_{m-1} \neq 0$, we would have $n \geq a_{m+1} q_m + q_{m-1} = q_{m+1}$, a contradiction.

Therefore, if we associate to an admissible word the integer $n = b_0 + b_1 q_1 + \dots + b_m q_m$, there is a 1 to 1 correspondence between the Ostrowski's expansion of the integers n , when n runs in \mathbb{N} , and the set of (finite) admissible words starting with $b_0 \leq a_1 - 1$.

For n given by (8), putting $n_0 = b_0$, $n_k = \sum_{t=0}^k b_t q_t$, for $k \leq m(n)$ and

$$(9) \quad f_k(x) := \sum_{i=0}^{b_k-1} \varphi_{q_k}(x + (n_{k-1} + i q_k)\alpha),$$

the ergodic sum φ_n reads:

$$(10) \quad \begin{aligned} \varphi_n(x) &= \sum_{k=0}^{m(n)} \sum_{j=n_{k-1}}^{n_k-1} \varphi(x + j\alpha) = \sum_{k=0}^{m(n)} \sum_{j=0}^{b_k q_k-1} \varphi(x + n_{k-1}\alpha + j\alpha) \\ &= \sum_{k=0}^{m(n)} \sum_{i=0}^{b_k-1} \varphi_{q_k}(x + (n_{k-1} + i q_k)\alpha) = \sum_{k=0}^{m(n)} f_k(x). \end{aligned}$$

By convention, the expression $\sum_{i=0}^{b_k-1} \varphi_{q_k}(x + (n_{k-1} + i q_k)\alpha)$ is taken to be 0, if $b_k = 0$.

If q is a denominator of α and φ is a BV function, one has (Denjoy-Koksma inequality) :

$$(11) \quad \|\varphi_q\|_\infty = \sup_x \left| \sum_{i=0}^{q-1} \varphi(x + i\alpha) \right| \leq V(\varphi).$$

One can also show that if φ satisfies (1) then $\|\varphi_{q_n}\|_2 \leq 2\pi K(\varphi)$. By Denjoy-Koksma inequality, we have $\|f_k\| \leq b_k V(\varphi) \leq a_{k+1} V(\varphi)$.

2.2. Lower bound for the variance.

Let n be in $[q_{\ell-1}, q_\ell[$. Keeping only indices that belong to the sequence (q_n) , the variance at time n is bounded from below as follows, with the constant $c = \frac{8}{\pi^2}$,

$$\|\varphi_n\|_2^2 = 2 \sum_{k>1} |c_k(\varphi)|^2 \frac{(\sin \pi n k \alpha)^2}{(\sin \pi k \alpha)^2} \geq 2 \sum_{j=1}^{\ell} |c_{q_j}(\varphi)|^2 \frac{(\sin \pi n q_j \alpha)^2}{(\sin \pi q_j \alpha)^2} \geq c \sum_{j=1}^{\ell} |c_{q_j}(\varphi)|^2 \frac{\|n q_j \alpha\|^2}{\|q_j \alpha\|^2}.$$

Therefore we have, for $0 < \delta < \frac{1}{2}$:

$$(12) \quad \|\varphi_n\|_2^2 \geq c \sum_{j=1}^{\ell} |\gamma_{q_j}(\varphi)|^2 a_{j+1}^2 \|n q_j \alpha\|^2 \geq c \delta^2 \sum_{j=1}^{\ell} |\gamma_{q_j}(\varphi)|^2 a_{j+1}^2 \mathbf{1}_{\|n q_j \alpha\| \geq \delta}.$$

Bounds for the mean variance

An upper bound for the variance and a lower bound for the mean of the variance are shown in [CoIsLe17]):

Proposition 2.1. *There are constants $C, c > 0$ such that, if φ satisfies (1), then*

$$(13) \quad \|\varphi_n\|_2^2 \leq CK(\varphi)^2 \sum_{j=0}^{\ell} a_{j+1}^2, \quad \forall n \in [q_\ell, q_{\ell+1}[,$$

and the mean of the variances $\langle D\varphi \rangle_n := \frac{1}{n} \sum_{k=0}^{n-1} \|\varphi_k\|_2^2$ satisfies:

$$(14) \quad \langle D\varphi \rangle_n \geq c \sum_{j=0}^{\ell-1} |\gamma_{q_j}(\varphi)|^2 a_{j+1}^2, \quad \forall n \in [q_\ell, q_{\ell+1}[.$$

The proposition implies that, if $v_\ell \leq q_{\ell+1}$ is an integer such that $\|\varphi_{v_\ell}\|_2 = \max_{k < q_{\ell+1}} \|\varphi_k\|_2$, then $\|\varphi_{v_\ell}\|_2^2 \geq c \sum_{j=0}^{\ell-1} |\gamma_{q_j}(\varphi)|^2 a_{j+1}^2$. If the partial quotients of α are bounded, under a condition on the sequence (γ_k) , using the results below, it can be deduced that the behaviour of $\varphi_{v_\ell}/\|\varphi_{v_\ell}\|_2$ for the indices giving the record variances is approximately Gaussian. But our goal is to obtain such a behaviour for a large set of integers n . To do it, we need to obtain a lower bound of the variance for such a large set.

Bounds for the variance for a large set of integers

We will assume that φ satisfies the condition:

$$(15) \quad \exists N_0, \eta, \theta_0 > 0 \text{ such that } \frac{1}{N} \text{Card}\{j \leq N : |\gamma_{q_j}(\varphi)| \geq \eta\} \geq \theta_0 \quad \forall N \geq N_0.$$

This condition is clearly satisfied by $\varphi(x) = \{x\} - \frac{1}{2}$, since in this case $|\gamma_{q_j}(\varphi)| = \frac{1}{2\pi}, \forall j$. The validity of (15) for different step functions is discussed in subsection 3.2.

For $j < \ell$, we estimate how many times for $n \leq q_\ell$, we have $\{nq_j\alpha\} \in I_\delta := [0, \delta] \cup [1-\delta, 1]$. Since $\|nq_j\alpha\| = \{nq_j\alpha\}$ or $1 - \{nq_j\alpha\}$, depending whether $\{nq_j\alpha\} \in [0, \frac{1}{2}[$ or $\in]\frac{1}{2}, 1[$, we have $1_{\|nq_j\alpha\| < \delta} = I_\delta(\{nq_j\alpha\})$.

Remark: We are looking for values of n such that $\sum_{j=1}^{\ell} 1_{I_\delta}(\{nq_j\alpha\})$ is small. Of course there are special values of n , like $n = q_t$, such that this quantity is big, as shown by Denjoy-Koksma inequality. Let us check it directly.

Recall that for any irrational α , there are $\lambda \in]0, 1[$ and $C > 0$ such that

$$(16) \quad q_n/q_m \leq C\lambda^{m-n}, \quad \forall m \geq n.$$

We have $\|q_t q_j \alpha\| \leq q_j \|q_t \alpha\| < q_j/q_{t+1}$, for $j \leq t$, and $\|q_t q_j \alpha\| \leq q_t \|q_j \alpha\| < q_t/q_{j+1}$, for $j > t$. By (16), this implies $\|q_t q_j \alpha\| \leq C\lambda^{|t-j|}$, $\forall j, t$; hence, for $\delta > 0$, $\|q_t q_j \alpha\| \leq \delta$ if $|t-j| \geq M := \ln(\delta^{-1})/\ln(C^{-1}\delta^{-1})$.

For the complementary I_δ^c of I_δ on the circle, it follows: $\sum_{j \geq 1} 1_{I_\delta^c}(\{q_t q_j \alpha\}) \leq M$, $\forall t \geq 1$. This gives a bound for $n = q_t$ when j varies, as expected.

Nevertheless, we will show that $\sum_{j=1}^{\ell} 1_{I_\delta}(\{nq_j\alpha\})$ is small for a large set of values of n .

Lemma 2.2. *For every $\delta \in]0, \frac{1}{2}[$ and every interval of integers $I = [N_1, N_2[$ of length L , we have*

$$(17) \quad \sum_{n=N_1}^{N_2-1} 1_{I_\delta}(\{nq_j\alpha\}) \leq 20(\delta + q_{j+1}^{-1})L, \quad \forall j \text{ such that } q_{j+1} \leq 2L.$$

Proof. For a fixed j and $0 \leq N_1 < N_2$, let us describe the behaviour of the sequence $(\|nq_j\alpha\|, n = N_1, \dots, N_2 - 1)$.

Recall that (modulo 1) we have $q_j\alpha = \theta_j$, with $\theta_j = (-1)^j \|q_j\alpha\|$ (see (5)). We treat the case j even (hence $\theta_j > 0$). The case j odd is analogous.

Therefore the problem is to count how many times, for j even, we have $\{n\theta_j\} < \delta$ or $1 - \delta < \{n\theta_j\}$.

We start with $n = n_1 := N_1$. Putting $w(j, 1) := \{n_1\theta_j\}$, we have $\{n\theta_j\} = w(j, 1) + (n - n_1)\theta_j$, for $n = n_1, n_1 + 1, \dots, n_2 - 1$, where n_2 is such that $w(j, 1) + (n_2 - 1 - n_1)\theta_j < 1 < w(j, 1) + (n_2 - n_1)\theta_j$.

Putting $w(j, 2) := \{n_2\theta_j\}$, the previous inequality shows that $w(j, 2) = w(j, 1) + (n_2 - n_1)\theta_j - 1 < \theta_j$.

Starting now from $n = n_2$, we have $\{n\theta_j\} = w(j, 2) + (n - n_2)\theta_j$ for $n = n_2, n_2 + 1, \dots, n_3 - 1$, where n_3 is such that:

$$w(j, 2) + (n_3 - 1 - n_2)\theta_j < 1 < w(j, 2) + (n_3 - n_2)\theta_j.$$

Again we put $w(j, 3) := \{n_3\theta_j\} = \{w(j, 2) + (n_3 - n_2)\theta_j\} = w(j, 2) + (n_3 - n_2)\theta_j - 1 < \theta_j$.

We iterate this construction and obtain a sequence $n_1 < n_2 < \dots < n_{R(j)-1} < n_{R(j)}$ (with $n_{R(j)-1} < N_2 \leq n_{R(j)}$) such that

$$\begin{aligned} \{n\theta_j\} &= w(j, i) + (n - n_i)\theta_j, \quad \forall n \in [n_i, n_{i+1}[, \\ w(j, i) + (n_{i+1} - 1 - n_i)\theta_j &< 1 < w(j, i) + (n_{i+1} - n_i)\theta_j, \end{aligned}$$

where $(w(j, i), i = 1, \dots, R(j))$ is defined recursively by $w(j, i+1) = \{w(j, i) + (n_{i+1} - n_i)\theta_j\}$ and satisfies $w(j, i) < \theta_j$, for every i .

Since $(n_{i+1} - n_i + 1)\theta_j \geq w(j, i) + (n_{i+1} - n_i)\theta_j > 1$ for $i \neq 1$ and $i \neq R(j)$, we have $n_{i+1} - n_i \geq \theta_j^{-1} - 1$, for each $i \neq 1, R(j)$. This implies $R(j) \leq \frac{L}{\theta_j^{-1} - 1} + 2$.

For each i , the number of integers $n \in [n_i, n_{i+1} - 1[$ such that $\{n\theta_j\} \in [0, \delta[\cup]1 - \delta, 1[$ is bounded by $2(1 + \delta\theta_j^{-1})$. (This number is less than 2 if $\delta < \theta_j$.)

Altogether, the number of integers $n \in I$ such that $\{n\theta_j\} \in [0, \delta[\cup]1 - \delta, 1[$ is bounded by

$$\begin{aligned} 2R(j)(1 + \delta\theta_j^{-1}) &\leq 2\left(\frac{L}{\theta_j^{-1} - 1} + 2\right)(1 + \delta\theta_j^{-1}) \\ &\leq 4(L + \theta_j^{-1})(\delta + \theta_j) \leq 4(L + 2q_{j+1})(\delta + q_{j+1}^{-1}). \end{aligned}$$

If $q_{j+1} \leq L$, the previous term at right is less than $20L(\delta + q_{j+1}^{-1})$. This shows (17). \square

Lemma 2.3. *Suppose that φ satisfies Condition (15). Then there are positive constants c_1, c_2 (not depending on δ) such that, if I is an interval $[N_1, N]$ of length L and ℓ such that $q_\ell \leq 2L$, for every $\delta \in]0, \frac{1}{2}[$, the subset of I*

$$(18) \quad V(I, \delta) := \{n \in I : \|\varphi_n\|_2 \geq c_1 \delta \sqrt{\ell}\}$$

has a complementary in I satisfying:

$$(19) \quad \text{Card } V(I, \delta)^c \leq c_2 (\delta + \ell^{-1}) L.$$

Proof. Let $\zeta = \frac{1}{2}\eta_0$, where η_0 is the constant in (15), and let

$$A = A(I, \delta) := \left\{n \in I : \sum_{j=0}^{\ell-1} 1_{I_\delta}(\{n q_j \alpha\}) \leq \zeta \ell\right\}.$$

Let us bound the density $L^{-1} \text{Card } A(I, \delta)^c$ of the complementary of $A(I, \delta)$ in I by counting the number of values of n in I such that $\|nq_j\alpha\| < \delta$ in an array indexed by (j, n) .

By summation of (17) for $j = 0$ to $\ell - 1$ and the definition of $A(I, \delta)$, the following inequalities are satisfied:

$$\begin{aligned} 20(\delta\ell + \sum_{0 \leq j \leq \ell-1} q_{j+1}^{-1})L &\geq \sum_{0 \leq j \leq \ell-1, n \in I} 1_{I_\delta}(\{n q_j \alpha\}) \\ &\geq \sum_{n \in A^c} \sum_{0 \leq j \leq \ell-1} 1_{I_\delta}(\{n q_j \alpha\}) \geq \sum_{n \in A^c} \zeta \ell = \zeta \ell \text{Card } A^c. \end{aligned}$$

Let C be the finite constant $C := \sum_{j=0}^{\infty} q_j^{-1}$. Then the set $A = A(I, \delta)$ satisfies:

$$(20) \quad \text{Card } A^c \leq 20 \zeta^{-1} (\delta + C \ell^{-1}) L,$$

$$(21) \quad n \in A \Rightarrow \text{Card}\{j \leq \ell - 1 : \|nq_j \alpha\| \geq \delta\} \geq (1 - \zeta) \ell.$$

In view of (21) and Condition (15), we have, for $n \in A$:

$$\text{Card}(\{j \leq \ell - 1 : \|nq_j \alpha\| \geq \delta\} \cap \{j : |\gamma_{q_j}(\varphi)| \geq \eta\}) \geq (1 - (\zeta + 1 - \eta_0)) \ell = \frac{1}{2} \eta_0 \ell.$$

Putting $c_2 := 20 \zeta^{-1} \max(1, C)$ and $c_1 = \frac{1}{2} c \eta^2 \eta_0$ (c_1, c_2 do not depend on δ), this implies by (12):

$$(22) \quad \|\varphi_n\|_2^2 \geq \frac{1}{2} c \delta^2 \eta^2 \eta_0 \ell = c_1^2 \delta^2 \ell, \quad \forall n \in A, \quad \text{and } \text{Card } A \geq 1 - c_2 (\delta + \ell^{-1}) L;$$

hence $A = A(I, \delta) \subset V(I, \delta)$ and therefore $V(I, \delta)$ satisfies (25). \square

Proposition 2.4. *Suppose that φ satisfies Condition (15). The subset W of \mathbb{N} defined by*

$$(23) \quad W := \{n \in \mathbb{N} : \|\varphi_n\|_2 \geq c_1 \sqrt{m(n)}/\sqrt{\ln m(n)}\}$$

satisfies

$$(24) \quad \lim_{N \rightarrow \infty} \frac{\text{Card}(W \cap [0, N[)}{N} = 1.$$

Proof. By (16), there is a fixed integer u_0 such that $q_{m(N)-u} \leq \frac{1}{2} q_{m(N)}$, for $u_0 \leq u < m(N)$.

Let us take $u_N = \lfloor \frac{1}{2} m(N) \rfloor$ and let $N_1 := q_{m(N)-u_N}$. By this choice of N_1 , the condition of Lemma (2.3) is satisfied by $[N_1, N]$ and $\ell = m(N)$ for N big enough, since

$$q_{m(N)} \leq 2(q_{m(N)} - q_{m(N)-u_N}) \leq 2(N - N_1).$$

If $n \in [N_1, N]$, then $m(N) - u_N = m(N_1) \leq m(n) \leq m(N)$.

Let $B = \{n \in [N_1, N] : \|\varphi_n\|_2 \geq c_1 \sqrt{m(n)}/\sqrt{\ln m(n)}\}$.

If n is in the complementary B^c of B in $[N_1, N]$, then we have:

$$\|\varphi_n\|_2 \leq c_2 \sqrt{m(n)}/\sqrt{\ln m(n)} \leq c_2 \sqrt{m(N)}/\sqrt{\ln m(N_1)}.$$

Therefore, by Lemma (2.3) with $\delta = (\ln m(N_1))^{-\frac{1}{2}}$, we have

$$(25) \quad \text{Card } B^c \leq c_2 ((\ln m(N_1))^{-\frac{1}{2}} + m(N)^{-1}) (N - N_1)$$

Now, $W^c \cap [0, N[\subset [0, N_1[\cup B^c$, hence: given $\varepsilon > 0$, for N big enough,

$$\begin{aligned} \frac{\text{Card}(W^c \cap [0, N[)}{N} &\leq \frac{N_1}{N} + c_2 ((\ln m(N_1))^{-\frac{1}{2}} + m(N)^{-1}) \\ &= \frac{q_{m(N)-u_N}}{N} + c_2 ((\ln(m(N) - u_N))^{-\frac{1}{2}} + m(N)^{-1}) \leq \varepsilon. \quad \square \end{aligned}$$

Remark: if α is bpq, then $m(n)$ is of order $\ln n$ and the density of W satisfies, for some constant $C > 0$,

$$(26) \quad \frac{\text{Card}(W \cap [0, N])}{N} \geq 1 - C(\ln \ln N)^{-\frac{1}{2}}.$$

2.3. A special case: the quadratic numbers.

If α is a quadratic number, the variance $\|\varphi_n\|_2^2$ has the right order of magnitude for n in a big set of integers. More precisely, the following bounds hold for the ergodic sums of φ under the rotation by α :

Theorem 2.5. *If α is a quadratic number and φ satisfies Condition (15), there are positive constants ζ, η_1, η_2, R such that, for N big enough:*

$$(27) \quad \#\{n \leq N : \eta_1 \ln n \leq \|\varphi_n\|_2^2 \leq \eta_2 \ln n\} \geq N(1 - RN^{-\zeta}).$$

Proof. For every n , the upper bound follows from (13). By (12), the variance is bounded from below by $\sum_{j=1}^{\ell} \gamma_{q_j}^2 \|nq_j\alpha\|^2$ and we would like to bound from below these sums for a large set of integer n .

We will first make a few remarks on the Ostrowski expansion in base (q_k) , then study the quantity $\|nq_j\alpha\|^2$ before going back to the previous sum.

First step: Remarks on Ostrowski expansions.

The sequence (a_n) in this case is ultimately periodic: there exist integers n_0 and p such that

$$\forall n \geq n_0 \quad a_{n+p} = a_n.$$

The denominators of α are defined recursively: $(q_n) : q_0 = 1, q_1 = a_1$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$ for $n \geq 1$. Using matrices we can write:

$$\begin{pmatrix} q_n \\ q_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_{n+1} \end{pmatrix} \begin{pmatrix} q_{n-1} \\ q_n \end{pmatrix}$$

Let A_ℓ be the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & a_{n_0+\ell} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_{n_0+\ell-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{n_0+1} \end{pmatrix}.$$

For every $k \geq 1$ we have

$$\begin{pmatrix} q_{n_0+kp+\ell-1} \\ q_{n_0+kp+\ell} \end{pmatrix} = A_\ell A_p^k \begin{pmatrix} q_{n_0-1} \\ q_{n_0} \end{pmatrix}.$$

The matrix A_p is a 2×2 matrix with determinant $(-1)^p$ and non negative integer coefficients (positive if $p > 1$). Without loss of generality we may suppose that p is even (if not, just replace it by $2p$). The matrix A_p has two different eigenvalues that will be denoted by $\lambda > 1$ and (its inverse) λ^{-1} . The number λ also is a quadratic number and A_p is diagonal in a base of \mathbb{R}^2 with coordinates in $\mathbb{Q}[\lambda]$. We deduce that there exist integers $r, s_\ell, t_\ell, u_\ell, v_\ell$ such that, for every $k \geq 0, \ell \in \{0, \dots, p-1\}$,

$$(28) \quad q_{n_0+kp+\ell} = \frac{1}{r} [(s_\ell + t_\ell \lambda) \lambda^k + (u_\ell + v_\ell \lambda) \lambda^{-k}]$$

We will use a subshift of finite type defined by the admissible words corresponding to the periodic part of the sequence (a_n) . A finite word is admissible if $b_0 \dots b_m$, $b_j \in \{0, 1, \dots, a_j\}$, with $b_m \neq 0$, such that for two consecutive letters b_j, b_{j+1} , we have $b_j b_{j+1} \neq ua_j$, with $u \neq 0$.

Let us consider the sequences of infinite admissible sequences corresponding to the Ostrowski expansions for the periodic part of the sequence (a_n) (indices larger than n_0).

$$X := \{x = (x_i)_{i \in \mathbb{N}} \text{ such that } \forall i \ x_i x_{i+1} \neq ua_{n_0+i+1} \text{ with } u \neq 0\}.$$

The space X is not invariant under the action of the left shift σ but it is invariant under the action of σ^p (because (a_n) is periodic of period p for $n \geq n_0$). We define an irreducible, aperiodic subshift of finite type (Y, σ) as follows: the state space of Y is the set of words $(x_0 \dots x_{p-1})$ of X , a transition between two such words w_1 and w_2 can occur if the concatenation $w_1 w_2$ is the beginning of length $2p$ of a sequence in X .

The integers between 0 and q_n are coded by the Ostrowski expansions of length at most n . From (28) we see that the exponential growth rate of the number of Ostrowski expansions of length at most $n_0 + pk$ is λ (with respect to k). This rate depends only on the periodic part of the sequence (a_n) ; it is also the growth rate of the number of words of length pk of X . As these words corresponds to the words of length k in Y , the topological entropy of (Y, σ) is $\ln \lambda$ (where $\sigma = \sigma_Y$ is the shift to the left on Y). There is a unique invariant probability measure μ on (Y, σ) with entropy $\ln \lambda$. This measure can be constructed as follows. Let B be the matrix with entries 0 and 1 that gives the allowed transition between the elements of the alphabet of Y . This matrix has spectral radius λ . Let U and V two positive vectors such that

$$BU = \lambda U, \quad {}^t BV = \lambda V, \quad {}^t UV = 1.$$

The measure μ is the Markovian measure which is determined by its values on the cylinders by

$$\mu(C_{y_0 y_1 \dots y_n}) = V_{y_0} U_{y_n} \lambda^{-n},$$

when $(y_0 y_1 \dots y_n)$ is an admissible word (see [Ki98] (p. 21-23, 166) for more details on this classical construction). As there are only finitely many products $V_{y_0} U_{y_n}$, there exists a constant $C > 0$ such that, if a subset E of Y is a union of cylinders of length n , then

$$(29) \quad \frac{1}{C} \mu(E) \leq \#\{b \text{ cylinder of length } n : b \subset E\} \lambda^{-n} \leq C \mu(E).$$

We will use it, via an inequality of large deviations, to get an estimate of the number of values of $n < q_\ell$ giving a big variance.

Second step: Study of the numbers $\|nq_j \alpha\|^2$.

For an integer $n \in [q_\ell, q_{\ell+1}[$, we use its Ostrowski expansion $n = \sum_{j=0}^{\ell} b_j q_j$, with $0 \leq b_j \leq a_{j+1}$ for $1 \leq j < \ell$, $b_\ell > 0$, with an admissible “word” $b_0 \dots b_\ell$. As shown in Subsection 2.1, every admissible word $b_0 \dots b_\ell$ is the Ostrowski expansion of an integer $n < q_{\ell+1}$.

Let $0 \leq \kappa_0 \leq \ell$. From now on, j and ℓ will be supposed to satisfy the inequalities $n_0 + \kappa_0 p \leq j \leq n_0 + 2\kappa_0 p \leq \ell$.

We have $\|q_i q_j \alpha\| \leq q_j / q_i \leq C \lambda^{j-i}$, for $i \geq j$, and $\|q_i q_j \alpha\| \leq q_i / q_j \leq C \lambda^{i-j}$, for $i \leq j$. It follows:

$$(30) \quad \begin{aligned} & \left| \left\| \sum_{i=1}^{\ell} b_i q_i q_j \alpha \right\| - \left\| \sum_{i=j-\kappa_0}^{j+\kappa_0-1} b_i q_i q_j \alpha \right\| \right| \leq \left\| \sum_{i=j+\kappa_0}^{\ell} b_i q_i q_j \alpha \right\| + \left\| \sum_{i=1}^{j-\kappa_0-1} b_i q_i q_j \alpha \right\| \\ & \leq \sum_{i=j+\kappa_0}^{\ell} \frac{1}{\lambda^{i-j}} + \sum_{i=1}^{j-\kappa_0-1} \frac{1}{\lambda^{j-i}} \leq \frac{2}{\lambda^{\kappa_0} - \lambda^{\kappa_0-1}}. \end{aligned}$$

It is convenient for us to work with functions which can be defined on the subshift Y . To get this, we have to consider sums on intervals of indices of the type $\{n_0 + \ell p, \dots, n_0 + kp - 1\}$. Let us fix some notations: $\eta(j)$ denotes the integer part of $(j - n_0)/p$, \bar{j} denotes the unique integer between 0 and $p - 1$ equal to j modulo p .

The following inequality follows from (30):

$$(31) \quad \left| \left\| \sum_{i=1}^{\ell} b_i q_i q_j \alpha \right\| - \left\| \sum_{i=n_0+(\eta(j)-\kappa_0)p}^{n_0+(\eta(j)+\kappa_0)p-1} b_i q_i q_j \alpha \right\| \right| \leq \frac{2}{\lambda^{(\kappa_0-1)p} - \lambda^{(\kappa_0-1)p-1}}.$$

Using the expression of the q_n given in (28) we get:

$$\begin{aligned} q_{n_0+kp+\ell} q_{n_0+k'p+\ell'} &= \frac{1}{r^2} \left[(s_\ell + t_\ell \lambda)(s_{\ell'} + t_{\ell'} \lambda) \lambda^{k+k'} + (u_\ell + v_\ell \lambda)(u_{\ell'} + v_{\ell'} \lambda) \lambda^{-(k+k')} \right] \\ &\quad + \frac{1}{r^2} \left[(s_\ell + t_\ell \lambda)(u_{\ell'} + v_{\ell'} \lambda) \lambda^{k-k'} + (u_\ell + v_\ell \lambda)(s_{\ell'} + t_{\ell'} \lambda) \lambda^{k'-k} \right]. \end{aligned}$$

Still using (28) we have

$$\begin{aligned} & \frac{1}{r^2} (s_\ell + t_\ell \lambda)(s_{\ell'} + t_{\ell'} \lambda) \lambda^{k+k'} = \frac{s_\ell (s_{\ell'} + t_{\ell'} \lambda)}{r} \lambda^{k+k'} + \frac{t_\ell (s_{\ell'} + t_{\ell'} \lambda)}{r} \lambda^{k+k'+1} \\ &= \frac{s_\ell}{r} (q_{n_0+(k'+k)p+\ell'} - \frac{1}{r} (u_{\ell'} + v_{\ell'} \lambda) \lambda^{-(k'+k)}) + \frac{t_\ell}{r} (q_{n_0+(k'+k+1)p+\ell'} - \frac{1}{r} (u_{\ell'} + v_{\ell'} \lambda) \lambda^{-(k'+k+1)}) \\ &= \frac{s_\ell}{r} q_{n_0+(k'+k)p+\ell'} + \frac{t_\ell}{r} q_{n_0+(k'+k+1)p+\ell'} - \frac{s_\ell}{r^2} (u_{\ell'} + v_{\ell'} \lambda) \lambda^{-(k'+k)} - \frac{t_\ell}{r^2} (u_{\ell'} + v_{\ell'} \lambda) \lambda^{-(k'+k+1)}. \end{aligned}$$

As we have seen $d(q_{n_0+(k'+k)p+\ell'} \alpha, \mathbb{Z}) \leq C \lambda^{-(k'+k)}$ and a similar inequality holds for $d(q_{n_0+(k'+k+1)p+\ell'} \alpha, \mathbb{Z})$. It follows

$$\begin{aligned} & d(q_{n_0+kp+\ell} q_{n_0+k'p+\ell'} \alpha - \frac{1}{r^2} \left[(s_\ell + t_\ell \lambda)(u_{\ell'} + v_{\ell'} \lambda) \lambda^{k-k'} + (u_\ell + v_\ell \lambda)(s_{\ell'} + t_{\ell'} \lambda) \lambda^{k'-k} \right] \alpha, \mathbb{Z}/r) \\ & \leq C \lambda^{-(k'+k)}. \end{aligned}$$

This means that for every i and j larger than n_0 we have

$$d(q_i q_j \alpha - [\zeta_{\bar{i}, \bar{j}} \lambda^{\eta(i)-\eta(j)} + \xi_{\bar{i}, \bar{j}} \lambda^{\eta(j)-\eta(i)}], \mathbb{Z}/r) \leq C \lambda^{-(\eta(j)+\eta(i))},$$

where \bar{i} and \bar{j} are the classes modulo p of $i - n_0$ and $j - n_0$, $\eta(i)$ and $\eta(j)$ are the integer parts of $(i - n_0)/p$ and $(j - n_0)/p$, and

$$\zeta_{\bar{i}, \bar{j}} = \frac{\alpha}{r^2} (s_{\bar{i}} + t_{\bar{i}} \lambda)(u_{\bar{j}} + v_{\bar{j}} \lambda) \quad \text{and} \quad \xi_{\bar{i}, \bar{j}} = \frac{\alpha}{r^2} (u_{\bar{i}} + v_{\bar{i}} \lambda)(s_{\bar{j}} + t_{\bar{j}} \lambda).$$

Thus we have

$$\begin{aligned}
& d\left(\sum_{i=1}^{\ell} b_i q_i q_j \alpha - \sum_{i=n_0+(\eta(j)-\kappa_0)p}^{n_0+(\eta(j)+\kappa_0)p-1} b_i \left[\zeta_{i,\bar{j}} \lambda^{\eta(i)-\eta(j)} + \xi_{i,\bar{j}} \lambda^{\eta(j)-\eta(i)} \right], \mathbb{Z}/r\right) \\
& \leq \frac{2}{\lambda^{(\kappa_0-1)p} - \lambda^{(\kappa_0-1)p-1}} + C \sum_{i=n_0+(\eta(j)-\kappa_0)p}^{n_0+(\eta(j)+\kappa_0)p-1} \lambda^{-\eta(j)-\eta(i)} \\
& \leq \frac{2}{\lambda^{(\kappa_0-1)p} - \lambda^{(\kappa_0-1)p-1}} + 2C\kappa_0 \lambda^{\kappa_0-2\eta(j)}.
\end{aligned}$$

Let $\delta \in]0, \frac{1}{2r}[$. We can chose κ_0 large enough for the following bound to hold for $j \geq n_0 + \kappa_0 p$:

$$(32) \quad d\left(\sum_{i=1}^{\ell} b_i q_i q_j \alpha - \sum_{i=n_0+(\eta(j)-\kappa_0)p}^{n_0+(\eta(j)+\kappa_0)p-1} b_i \left[\zeta_{i,\bar{j}} \lambda^{\eta(i)-\eta(j)} + \xi_{i,\bar{j}} \lambda^{\eta(j)-\eta(i)} \right], \mathbb{Z}/r\right) \leq \delta$$

The preceding bounding shows that if κ_0 is sufficiently large, we have the implications

$$\begin{aligned}
d\left(\sum_{i=1}^{\ell} b_i q_i q_j \alpha, \mathbb{Z}/r\right) \geq 3\delta & \Rightarrow d\left(\sum_{i=n_0+(\eta(j)-\kappa_0)p}^{n_0+(\eta(j)+\kappa_0)p-1} b_i \left[\zeta_{i,\bar{j}} \lambda^{\eta(i)-\eta(j)} + \xi_{i,\bar{j}} \lambda^{\eta(j)-\eta(i)} \right], \mathbb{Z}/r\right) \geq 2\delta \\
& \Rightarrow d\left(\sum_{i=1}^{\ell} b_i q_i q_j \alpha, \mathbb{Z}/r\right) \geq \delta.
\end{aligned}$$

For every $\delta \in]0, \frac{1}{2r}[$ and every interval $[N_1, N_2]$ of length L , we have by a slight extension of Lemma 2.2:

$$\#\{n \in [N_1, N_2] : d(nq_j \alpha, \mathbb{Z}/r) \leq \delta\} \leq 20r (\delta + q_j^{-1})L, \forall j \text{ such that } q_{j+1} \leq 2L.$$

This gives (if κ_0 is large enough):

$$\#\{n \in [1, q_{n_0+2\kappa_0 p}] : d(nq_{n_0+\kappa_0 p+m} \alpha, \mathbb{Z}/r) \leq 3\delta\} \leq 60r \delta q_{n_0+2\kappa_0 p}, \forall m \in \{0, \dots, p-1\}$$

It means that the proportion of words $b_0 \dots b_{n_0+2\kappa_0 p-1}$ for which

$$d\left(\sum_{i=1}^{n_0+2\kappa_0 p} b_i q_i q_{n_0+\kappa_0 p+m} \alpha, \mathbb{Z}/r\right) \geq 3\delta$$

is larger than $1 - 60r \delta$ and, because of (32), the proportion of words $(b_0 \dots b_{n_0+2\kappa_0 p})$ for which

$$d\left(\sum_{i=n_0+(\eta(j)-\kappa_0)p}^{n_0+(\eta(j)+\kappa_0)p-1} b_i \left[\zeta_{i,\bar{j}} \lambda^{\eta(i)-\eta(j)} + \xi_{i,\bar{j}} \lambda^{\eta(j)-\eta(i)} \right], \mathbb{Z}/r\right) \geq 2\delta$$

is larger than $1 - 60r \delta$.

But we remark that the last expression depends only on the ‘‘short word’’

$$(b_{n_0+(\eta(j)-\kappa_0)p} \dots b_{n_0+(\eta(j)+\kappa_0)p-1})$$

which is a sub-word of the “long” word $(b_0 \dots b_{n_0+2\kappa_0 p})$. The number of long words corresponding to a short one can vary. We have to distinguish different cases determined by the value of $b_{n_0+(\eta(j)-\kappa_0)p}$ (which can be $a_{n_0+(\eta(j)-\kappa_0)p+1}$ or not) and by the value of $b_{n_0+(\eta(j)+\kappa_0)p-1}$ (which can be 0 or not). So we see that the number of long words having the same coordinates from $n_0 + (\eta(j) - \kappa_0)p$ to $n_0 + (\eta(j) + \kappa_0)p - 1$ can take four different values. The ratio of the largest of these values by the smallest is at most $(1 + \max a_n)^2$. We obtain that the proportion of words $(b_{n_0+(\eta(j)-\kappa_0)p} \dots b_{n_0+(\eta(j)+\kappa_0)p-1})$ for which

$$(33) \quad d\left(\sum_{i=n_0+(\eta(j)-\kappa_0)p}^{n_0+(\eta(j)+\kappa_0)p-1} b_i [\zeta_{\bar{i},\bar{j}} \lambda^{\eta(i)-\eta(j)} + \xi_{\bar{i},\bar{j}} \lambda^{\eta(j)-\eta(i)}], \mathbb{Z}/r\right) \geq 2\delta$$

is larger than $1 - 60(1 + \max a_n)^2 r \delta$.

If we replace j by $j + p$ in the sum above, we get (by a change of index)

$$(34) \quad \begin{aligned} & \sum_{i=n_0+(\eta(j+p)-\kappa_0)p}^{n_0+(\eta(j+p)+\kappa_0)p-1} b_i [\zeta_{\bar{i},\bar{j+p}} \lambda^{\eta(i)-\eta(j+p)} + \xi_{\bar{i},\bar{j+p}} \lambda^{\eta(j+p)-\eta(i)}] \\ &= \sum_{i=n_0+(\eta(j)-\kappa_0)p}^{n_0+(\eta(j)+\kappa_0)p-1} b_{i+p} [\zeta_{\bar{i},\bar{j}} \lambda^{\eta(i)-\eta(j)} + \xi_{\bar{i},\bar{j}} \lambda^{\eta(j)-\eta(i)}], \end{aligned}$$

because \bar{j} do not change when we replace j by $j + p$ and $\eta(j + p) = \eta(j) + 1$.

If $y_k = (b_{n_0+kp} \dots b_{n_0+kp+p-1})$ we see that the sum inside (33) is a function of $(y_{\eta(j)-\kappa_0} \dots y_{\eta(j)+\kappa_0-1})$. For $m \in \{0, \dots, p-1\}$, let Γ_m be the function defined on Y by

$$\Gamma_m(y) = \sum_{i=n_0}^{n_0+2\kappa_0 p-1} b_i [\zeta_{\bar{i},m} \lambda^{\eta(i)-\kappa_0} + \xi_{\bar{i},m} \lambda^{\kappa_0-\eta(i)}].$$

The change of index performed above can be expressed by the equality

$$\sum_{i=n_0+(\eta(j)-\kappa_0)p}^{n_0+(\eta(j)+\kappa_0)p-1} b_i [\zeta_{\bar{i},\bar{j}} \lambda^{\eta(i)-\eta(j)} + \xi_{\bar{i},\bar{j}} \lambda^{\eta(j)-\eta(i)}] = \Gamma_{\bar{j}}(\sigma^{\eta(j)-\kappa_0} y).$$

Because of (33) and (29), for every $m \in \{0, \dots, p-1\}$, we have

$$\mu\{y \in Y : d(\Gamma_m(y), \mathbb{Z}/r) \geq 2\delta\} \geq 1 - 60C(1 + \max a_n)^2 r \delta,$$

and the set A_δ defined by

$$A_\delta := \{y : d(\Gamma_m(y), \mathbb{Z}/r) \geq 2\delta, \forall m \in \{0, \dots, p-1\}\}.$$

satisfies

$$(35) \quad \mu(A_\delta) \geq 1 - 60pC(1 + \max a_n)^2 r \delta.$$

Third step: We are interested in $\sum_{j=1}^\ell \gamma_{q_j}^2 \|nq_j \alpha\|^2$.

For μ -almost $y \in Y$, we have $\eta(\ell)^{-1} \sum_{j=0}^{\eta(\ell)} \mathbf{1}_{A_\delta}(\sigma^j(y)) \xrightarrow{\eta(\ell) \rightarrow +\infty} \mu(A_\delta)$. A more precise information is given by an inequality of large deviations for irreducible Markov chains with finite state space (see [Le98], Theorem 3.3).

We apply such an inequality to the Markov chain deduced from Y with state space the set of words of length $2\kappa_0$ in Y : for $\varepsilon \in]0, 1[$, there are two positive constants $R = R(\varepsilon)$, $r = r(\varepsilon)$ such that

$$\mu\left\{\frac{1}{\eta(\ell)} \sum_{j=0}^{\eta(\ell)} \mathbf{1}_{A_\delta}(\sigma^j(y)) \leq \mu(A_\delta)(1 - \varepsilon)\right\} \leq R e^{-r\eta(\ell)}.$$

Strictly speaking, this result is about infinite sequences y , but only a finite number of coordinates are involved. Now we translate it in terms of cardinals of sets of finite words. According to the definition of A_δ and because of (29), we get that the number of words $(b_{n_0} \dots b_\ell)$ such that

$$\#\{j \leq \ell : d\left(\sum_{i=n_0+(\eta(j)-\kappa_0)p}^{n_0+(\eta(j)+\kappa_0)p-1} b_i [\zeta_{\bar{i}, \bar{j}} \lambda^{\eta(i)-\eta(j)} + \xi_{\bar{i}, \bar{j}} \lambda^{\eta(j)-\eta(i)}], \mathbb{Z}/r\right) \geq 2\delta\} \leq \mu(A_\delta)(1 - \varepsilon) \ell$$

is smaller than $C R e^{-r\eta(\ell)} \lambda^{\eta(\ell)}$ and (by (32) satisfied for $j \geq n_0 + \kappa_0 p$) it implies that

$$\#\{(b_{n_0} \dots b_\ell) : \#\{j \leq \ell : d\left(\sum_{i=0}^{\ell} b_i q_i q_j \alpha, \mathbb{Z}/r\right) \geq \delta\} \leq \mu(A_\delta)(1 - \varepsilon) \ell - n_0 - \kappa_0 p\}$$

is smaller than $C R e^{-r\eta(\ell)} \lambda^{\eta(\ell)}$; so that, the number of words b of length ℓ such that

$$\#\{j \leq \ell : d\left(\sum_{i=0}^{\ell} b_i q_i q_j \alpha, \mathbb{Z}/r\right) \geq \delta\} \leq \mu(A_\delta)(1 - \varepsilon) \ell - n_0 - \kappa_0 p$$

is smaller than $M e^{-r\eta(\ell)} \lambda^{\eta(\ell)}$, for some new constant M .

If $\ell > (n_0 + \kappa_0 p) / (\mu(A_\delta)(1 - \varepsilon)\varepsilon)$ the number words b of length ℓ such that

$$(36) \quad \#\{j \leq \ell : d\left(\sum_{i=0}^{\ell} b_i q_i q_j \alpha, \mathbb{Z}/r\right) \geq \delta\} \leq \mu(A_\delta)(1 - \varepsilon)^2 \ell$$

is smaller than $M e^{-r\eta(\ell)} \lambda^{\eta(\ell)}$.

Let $\varphi \in \mathcal{C}$ satisfy Condition (15). For a lower bound of the variance we use (12) and (35). We can choose δ and ε such that $\eta_0 + \mu(A_\delta)(1 - \varepsilon)^2 > 1$. Then, because of (36) and (15), for ℓ large enough, the number of integers $n < q_{\ell+1}$ such that, for at least $(\eta_0 + \mu(A_\delta)(1 - \varepsilon)^2 - 1)\ell$ different j 's we have simultaneously

$$(37) \quad \left\| \sum_{i=0}^{\ell} b_i q_i q_j \alpha \right\| \geq \delta \text{ and } |\gamma_{q_j}(\varphi)| \geq \eta,$$

is larger than $q_{\ell+1}(1 - M' e^{-r\eta(\ell)}) \geq q_{\ell+1}(1 - M'' q_{\ell+1}^{-\zeta})$, for some $\zeta > 0$.

But when we have (37) for at least $(\eta_0 + \mu(A_\delta)(1 - \varepsilon)^2 - 1)\ell$ different j 's, the variance $\|\varphi_n\|_2^2$ is larger than $c\eta^2 \delta^2 (\eta_0 + \mu(A_\delta)(1 - \varepsilon)^2 - 1)\ell$. This proves Theorem 2.5. \square

3. A central limit theorem and its application to rotations

3.1. Decorrelation and CLT.

Decorrelation between partial ergodic sums

Proposition 3.1. *If ψ and φ are BV centered functions, under Hypothesis 1 there are constants $C, \theta_1, \theta_2, \theta_3$ such that the following decorrelation inequalities hold for every $1 \leq n \leq m \leq \ell$:*

$$(38) \quad \left| \int_X \psi \varphi_{b_n q_n} d\mu \right| \leq C V(\psi) V(\varphi) \frac{n^{\theta_1}}{q_n} b_n,$$

$$(39) \quad \left| \int_X \psi \varphi_{b_n q_n} \varphi_{b_m q_m} d\mu \right| \leq C V(\psi) V(\varphi)^2 \frac{m^{\theta_2}}{q_n} b_n b_m,$$

$$(40) \quad \left| \int_X \psi \varphi_{b_n q_n} \varphi_{b_m q_m} \varphi_{b_\ell q_\ell} d\mu \right| \leq C V(\psi) V(\varphi)^3 \frac{\ell^{\theta_3}}{q_n} b_n b_m b_\ell.$$

The proposition is proved in Section 4. From the proposition we will deduce a convergence toward a Gaussian distribution under a variance condition. The method is like in [Hu09], with here a quantitative form, bounding the distance to the normal distribution.

Recall that, if X, Y are two r.r.v.'s defined on the same probability space, their mutual (Kolmogorov) distance in distribution is defined by: $d(X, Y) = \sup_{x \in \mathbb{R}} |\mathbb{P}(X \leq x) - \mathbb{P}(Y \leq x)|$. Below Y_1 denotes a r.v. with a normal distribution $\mathcal{N}(0, 1)$.

Proposition 3.2. *Let $(q_k)_{1 \leq k \leq n}$ be an increasing sequence of positive integers satisfying the lacunary condition: there exists $\rho < 1$ such that*

$$(41) \quad q_k/q_m \leq C \rho^{m-k}, \quad 1 \leq k < m \leq n.$$

Let $(f_k)_{1 \leq k \leq n}$ be real centered BV functions and u_k constants such that $\|f_k\|_\infty \leq u_k$ and

$$(42) \quad V(f_k) \leq C u_k q_k, \quad 1 \leq k \leq n.$$

Suppose that there are finite constants $C > 0$ and $\theta \in \mathbb{R}$ such that the following conditions hold for every centered BV function:

$$(43) \quad \left| \int_X \psi f_k d\mu \right| \leq C V(\psi) u_k \frac{k^\theta}{q_k}, \quad 1 \leq k \leq n,$$

$$(44) \quad \left| \int_X \psi f_k f_m d\mu \right| \leq C V(\psi) u_k u_m \frac{m^\theta}{q_k}, \quad 1 \leq k \leq m \leq n,$$

$$(45) \quad \left| \int_X \psi f_k f_m f_t d\mu \right| \leq C V(\psi) u_k u_m u_t \frac{t^\theta}{q_k}, \quad 1 \leq k \leq m \leq t \leq n.$$

Then, for every $\delta > 0$, there is a constant $C(\delta) > 0$ such that

$$(46) \quad d\left(\frac{f_1 + \cdots + f_n}{\|f_1 + \cdots + f_n\|_2}, Y_1\right) \leq C(\delta) \left(\frac{\max_{j=1}^n u_j}{\|f_1 + \cdots + f_n\|_2}\right)^{\frac{2}{3}} n^{\frac{1}{4} + \delta}.$$

The proposition is proved in Section 5.

We apply this abstract proposition to an irrational rotation: the q_k 's will be the denominators of α and the functions f_k (defined by (9)) will be translated ergodic sums of a BV function φ .

More precisely, let $\varphi \in \mathcal{C}$ and φ_n the ergodic sum $\sum_{j=0}^{n-1} \varphi(x + j\alpha)$. We use the decomposition $\varphi_n = \sum_{k \leq m(n)} f_k$ given by (10), with the notation $m(n) = \ell$, where ℓ is such that $n \in [q_\ell, q_{\ell+1}[$.

The notation n in Proposition 3.2 has been replaced by ℓ and in the application we have $\ell = m(n)$. Observe also that the decorrelation inequalities in Proposition 3.1 are obtained for functions of the form $\varphi_{b_n q_n}$. But looking at the proof of the decorrelation inequalities, one sees that they remain valid for the functions f_k in the previous proposition, since translations on the variable do not change the modulus of the Fourier coefficients.

As already observed, we have $\|f_k\| \leq b_k V(\varphi) \leq a_{k+1} V(\varphi)$. Up to a fixed factor, the constant u_k in the statement of the proposition can be taken to be $a_{k+1} \leq k^p$, for some constant $p > 0$, by Hypothesis 1.

Therefore, in (46), we have $\max_{j=1}^{m(n)} u_j / \|f_1 + \dots + f_{m(n)}\|_2 \leq m(n)^p / \|\varphi_n\|_2$. It follows:

$$d\left(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1\right) \leq C(\delta) m(n)^{\frac{1}{4} + \frac{2}{3}p + \delta} \|\varphi_n\|_2^{-\frac{2}{3}}.$$

If we restrict to a set of values of n such that the b_k 's in the Ostrowski expansion of n are uniformly bounded, or (without restriction on n) if we take α bpq, we get the bound:

$$d\left(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1\right) \leq C(\delta) m(n)^{\frac{1}{4} + \delta} \|\varphi_n\|_2^{-\frac{2}{3}}.$$

Therefore, in order to have an effective rate in the previous formula, we need to bound from below the variance $\|\varphi_n\|_2^2$.

For example, when α is bpq, we have the implication: if $\|\varphi_n\|_2^2 \geq cm(n)$ for a constant $c > 0$, then $d\left(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1\right) \leq c^{\frac{1}{3}} C(\delta) m(n)^{-\frac{1}{12} + \delta}$.

As we have seen, there are two methods to obtain a lower bound for the variance. This leads to the following two formulations of a CLT along a subset.

1) First method

From (13) and (14) one can get an information on the density of a set of integers n for which $\|\varphi_n\|_2^2$ is large.

For a given BV function φ , define $c_0 \geq 0$ by

$$(47) \quad c_0 = c_0(\varphi, \alpha) := \liminf_{\ell \geq 1} \frac{c \sum_{j=0}^{\ell-1} |\gamma_{q_j}(\varphi)|^2 a_{j+1}^2}{C K(\varphi)^2 \sum_{j=0}^{\ell} a_{j+1}^2},$$

where c and C are the constants in (13) and (14). The a_k 's depend on α and so does c_0 .

Theorem 3.3. *Let α be an irrational number satisfying Hypothesis 1. Let φ be a BV centered function such that $c_0(\varphi, \alpha) > 0$. Then there is a set $E_0(\ell)$ in $[1, q_\ell]$ with $\frac{1}{q_\ell} \#E_0(\ell) \geq$*

$\frac{1}{2}c_0(\varphi, \alpha)$ such that, for $n \in E_0(\ell)$, the ergodic sums φ_n for the rotation by α satisfy:

$$\|\varphi_n\|_2^2 \geq \frac{1}{2}c_0 C K(\varphi)^2 \sum_{j=1}^{\ell} a_j^2.$$

Moreover, for every $\delta > 0$, there is a constant $C_0(\delta) > 0$ such that

$$d\left(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1\right) \leq C_0(\delta) \left(\frac{\max_{j=1}^{\ell} a_j}{\left(\sum_{j=1}^{\ell} a_j^2\right)^{\frac{1}{2}}}\right)^{\frac{2}{3}} \ell^{\frac{1}{4}+\delta}, \quad \forall n \in E_0(\ell).$$

In particular, if $\max_{j=1}^{\ell} a_j / \min_{j=1}^{\ell} a_j \leq C \ell^{\gamma}$, with $\gamma < \frac{1}{8}$, then

$$d\left(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1\right) \leq C_0(\delta) \ell^{\frac{8\gamma-1}{12}+\delta} \ll 1, \quad \forall n \in E_0(\ell).$$

If α is bpq, the above bound holds with $\gamma = 0$.

Proof. Let $E_0(\ell)$ be the set $\{k \leq q_{\ell} : \|\varphi_k\|_2^2 \geq \frac{1}{2}c_0(\varphi, \alpha) C K(\varphi)^2 \sum_{j=0}^{\ell} a_{j+1}^2\}$. For ℓ big enough, its density satisfies $\frac{1}{q_{\ell}} \#E_0(\ell) \geq \frac{1}{2}c_0(\varphi, \alpha)$.

Indeed, let c_1 be such that $c_0 - \frac{1}{4}c_0^2 < c_1 < c_0$. For ℓ big enough, by (13), (14) and the definition of $c_0(\varphi, \alpha)$, we have for $E_0(\ell)$ denoted simply by E_0 :

$$\begin{aligned} c_1 C K(\varphi)^2 \sum_{j=0}^{\ell} a_{j+1}^2 &\leq c \sum_{j=0}^{\ell-1} |\gamma_{q_j}(\varphi)|^2 a_{j+1}^2 \leq \frac{1}{q_{\ell}} \sum_{k=1}^{q_{\ell}} \|\varphi_k\|_2^2, \\ \frac{1}{q_{\ell}} \sum_{k=1}^{q_{\ell}} \|\varphi_k\|_2^2 &\leq \frac{\#E_0}{q_{\ell}} C K(\varphi)^2 \sum_{j=0}^{\ell} a_{j+1}^2 + \frac{\#E_0^c}{q_{\ell}} \frac{c_0}{2} C K(\varphi)^2 \sum_{j=0}^{\ell} a_{j+1}^2; \end{aligned}$$

hence: $c_1 \leq \frac{1}{q_{\ell}} \#E_0 + \frac{1}{q_{\ell}} \frac{c_0}{2} \#E_0^c = \frac{1}{q_{\ell}} \#E_0 + \frac{1}{q_{\ell}} \frac{c_0}{2} (q_{\ell} - \#E_0)$, which implies: $\frac{\#E_0}{q_{\ell}} \geq (c_1 - \frac{1}{2}c_0) / (1 - \frac{1}{2}c_0) > \frac{1}{2}c_0$.

Now the conclusion of the theorem follows from the discussion after Proposition 3.2. \square

The drawback of this first method is that it yields only a subset of integers of positive density.

2) Second method

Another method to show a CLT is to use Proposition 2.4, or for a quadratic α Theorem 2.5.

We state the result in an increasing order of generality for α and decreasing strength for the conclusion.

Theorem 3.4. (CLT with rate holds along large subsets of integers) Let φ be a function in \mathcal{C} satisfying Condition (15).

1) Let α be a quadratic irrational. For two positive constants $0 < b < B$, let

$$V_{b,B} := \{n \geq 1 : b \sqrt{\log n} \leq \|\varphi_n\|_2 \leq B \sqrt{\log n}\}.$$

Then, there are b, B and N_0 and two constants $R, \zeta > 0$ such that
- the density of V_b satisfies:

$$(48) \quad \text{Card}(V_{b,B} \cap [1, N]) \geq N(1 - RN^{-\zeta}), \text{ for } N \geq N_0;$$

- for $\delta_0 \in]0, \frac{1}{2}[$, there is a constant $K(\delta_0)$ such that for $n \in V_{b,B}$,

$$(49) \quad d\left(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1\right) \leq K(\delta_0) (\log n)^{-\frac{1}{12} + \delta_0}.$$

2) Suppose that α is bpq. For two positive constants $0 < b < B$, let

$$W_{b,B} := \{n \geq 1 : b(\log \log n)^{-\frac{1}{2}} \sqrt{\log n} \leq \|\varphi_n\|_2 \leq B \sqrt{\log n}\}.$$

Then, there are b, B such that

- $W_{b,B}$ has density 1 in the integers;

- for $\delta_0 \in]0, \frac{1}{2}[$, there is a constant $K(\delta_0)$ such that (49) holds for $n \in W_{b,B}$.

3) Suppose that α is such that $a_n \leq Cn^p, \forall n \geq 1$, for a constant C . For two positive constants $0 < b < B$, with notation (6) let

$$(50) \quad Z_{b,B} := \{n \in \mathbb{N} : b \sqrt{m(n)}/\sqrt{\ln m(n)} \leq \|\varphi_n\|_2 \leq B \sqrt{m(n)}\}.$$

There are b, B such that

- $Z_{b,B}$ has density 1 in the integers;

- for $\delta_0 \in]0, \frac{1}{2}[$, there is a constant $K(\delta_0)$ such that, if $p < \frac{1}{8}$,

$$(51) \quad d\left(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1\right) \leq K(\delta_0) m(n)^{-\frac{1}{12} + \frac{2}{3}p + \delta_0}, \forall n \in Z_{b,B}.$$

Proof. We use Proposition 3.2 with f_n defined by (9) (see comments after the statement of Proposition 3.2).

For a lower bound of the variance we use Theorem 2.5 in case 1) and Proposition 2.4 in case 2) or 3).

For 3), we have, if $n \in Z_{b,B}$,

$$\begin{aligned} d\left(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1\right) &\leq C(\delta) \left(\frac{m(n)^p}{\|\varphi_n\|_2}\right)^{\frac{2}{3}} m(n)^{\frac{1}{4} + \delta} \leq C(\delta) (m(n)^{p - \frac{1}{2}} (\ln m(n))^{\frac{1}{2}})^{\frac{2}{3}} m(n)^{\frac{1}{4} + \delta} \\ &\leq C(\delta) m(n)^{-\frac{1}{12} + \frac{2}{3}p + \delta} (\ln m(n))^{\frac{1}{3}} \ll 1. \end{aligned}$$

The factor $(\ln m(n))^{\frac{1}{3}}$ can be absorbed in the factor $m(n)^\delta$ by taking δ smaller. \square

Remark 1. The previous result is written with a self-normalisation. In the first case (α quadratic), let us consider the sequence of variables (φ_n/\sqrt{n}) , with a fixed normalisation. Then, for $n \in V_b$, the accumulation points of the sequence of its distributions are Gaussian non degenerated with a variance belonging to a compact interval.

3.2. Application to step functions, examples.

If φ belongs to the class \mathcal{C} of centered BV functions, with Fourier series $\sum_{r \neq 0} \frac{\gamma_r(\varphi)}{r} e^{2\pi i r x}$, to be able to apply Theorems 3.3 and 3.4 we have to check a condition on the coefficients $\gamma_{q_k}(\varphi)$. In Theorem 3.4, the condition is (15). Recall that it reads:

$$\exists N_0, \eta, \theta_0 > 0 \text{ such that } \frac{1}{N} \text{Card}\{j \leq N : |\gamma_{q_j}(\varphi)| \geq \eta\} \geq \theta_0 \forall N \geq N_0.$$

The functions $\{x\} - \frac{1}{2} = \frac{-1}{2\pi i} \sum_{r \neq 0} \frac{1}{r} e^{2\pi i r x}$ and $1_{[0, \frac{1}{2}[} - 1_{[\frac{1}{2}, 1[} = \sum_r \frac{2}{\pi i(2r+1)} e^{2\pi i(2r+1)x}$ are immediate examples where (15) are satisfied. In the second case, one observes that $\gamma_{q_k} = 0$ if q_k is even, $= \frac{2}{\pi i}$ if q_k is odd. Clearly, (15) is satisfied because two consecutive q_k 's are relatively prime and therefore cannot be both even.

In general, for a step function, (15) and therefore a lower bound for the variance $\|\varphi_n\|_2^2$ for many n 's are related to the Diophantine properties of its discontinuities with respect to α . We discuss now the validity of (15).

Let us consider a centered step function φ on $[0, 1[$ taking a constant value $v_j \in \mathbb{R}$ on the interval $[u_j, u_{j+1}[$, with $u_0 = 0 < u_1 < \dots < u_s < u_{s+1} = 1$:

$$(52) \quad \varphi = \sum_{j=0}^s v_j 1_{[u_j, u_{j+1}[} - c.$$

The constant c above is such that φ is centered, but it plays no role below.

Lemma 3.5. *If φ is given by (52), there is a function $H_\varphi(u_1, \dots, u_s) \geq 0$ such that*

$$(53) \quad |\gamma_r(\varphi)|^2 = \pi^{-2} H_\varphi(ru_1, \dots, ru_s).$$

Proof. For $r \neq 0$, the Fourier coefficients of φ are: $c_r(\varphi) = \sum_{j=0}^s \frac{v_j}{\pi r} e^{-\pi i r(u_j + u_{j+1})} \sin \pi r(u_{j+1} - u_j)$. Therefore, we obtain (53) with

$$H_\varphi(u_1, \dots, u_s) := \left[\sum_{j=0}^s v_j \cos \pi(u_j + u_{j+1}) \sin \pi(u_{j+1} - u_j) \right]^2 + \left[\sum_{j=0}^s v_j \sin \pi(u_j + u_{j+1}) \sin \pi(u_{j+1} - u_j) \right]^2. \quad \square$$

Example 1: $\varphi = \varphi(u, \cdot) = 1_{[0, u[} - u$, $H_\varphi(u) = \sin^2(\pi u)$.

Example 2: $\varphi = \varphi(w, u, \cdot) = 1_{[0, u[} - 1_{[w, u+w[}$, $H(\varphi) = 4 \sin^2(\pi u) \sin^2(\pi w)$.

Corollary 3.6. *Condition (15) is satisfied if the parameter (u_1, \dots, u_s) is such that the sequence $(q_k u_1, \dots, q_k u_s)_{k \geq 1}$ is uniformly distributed in \mathbb{T}^s .*

Since (q_k) is a strictly increasing sequence of integers, for almost every (u_1, \dots, u_s) in \mathbb{T}^s , the sequence $(q_k u_1, \dots, q_k u_s)_{k \geq 1}$ is uniformly distributed in \mathbb{T}^s . Hence:

$$\lim_n \frac{1}{N} \sum_{k=1}^n H_\varphi(q_k u_1, \dots, q_k u_s) = \int_{\mathbb{T}^s} H(u_1, \dots, u_s) du_1 \dots du_s > 0, \text{ for a.e. } (u_1, \dots, u_s) \in \mathbb{T}^s.$$

Therefore, we have the following generic property:

Corollary 3.7. *Condition (15) is satisfied for a.e. value of (u_1, \dots, u_s) in \mathbb{T}^s .*

Besides a generic result, there are also specific values of the parameter (u_1, \dots, u_s) for which (15) holds. A simple example is:

Example 3: $\varphi(\frac{r}{s}, \cdot) = 1_{[0, \frac{r}{s}[} - \frac{r}{s}$, for $r, s \in \mathbb{N}, 0 < r < s$.

Remark 1: In example 1, it is known that if α is bpq and if $\lim_k \sin^2(\pi q_k u) = 0$, then $u \in \mathbb{Z}\alpha + \mathbb{Z}$. But there is an uncountable set of u 's such that $\lim_N \frac{1}{N} \sum_{k=1}^N \sin^2(\pi q_k u) = 0$.

Observe also that, if α is not bpq, there are many β 's which do not satisfy the previous equidistribution property. Let $\beta = \sum_{n \geq 0} b_n q_n \alpha \bmod 1$, $b_n \in \mathbb{Z}$, be the so-called Ostrowski expansion of β , where q_n are the denominators of α . It can be shown that, if $\sum_{n \geq 0} \frac{|b_n|}{a_{n+1}} < \infty$, then $\lim_k \|q_k \beta\| = 0$. There is an uncountable set of β 's satisfying this condition if α is not bpq. Therefore we have degeneracy of the variance for these β 's, although ergodicity of the cocycle holds if β is not in the countable set $\mathbb{Z}\alpha + \mathbb{Z}$.

An extension of the equidistribution property

Consider for instance example 1. The equidistribution property reads then as follows: Since (q_k) is a strictly increasing sequence of integers, for almost every β in \mathbb{T} , the sequence $(q_k \beta)$ is uniformly distributed modulo 1 in \mathbb{T}^1 :

$$\lim_N \frac{1}{N} \sum_{k=1}^N \sin^2(\pi q_k \beta) = \frac{1}{2}, \text{ for a.e. } \beta.$$

Recall that a way to proof it is to use Weyl equirepartition criterium and the law of large numbers for orthogonal bounded random variables. For the case 3) of Theorem 3.4, we need a slight improvement of an equidistribution argument, namely:

If $1 \leq a_n \leq n^\gamma$, with $\gamma < \frac{1}{6}$, then:

$$\lim_N \frac{\sum_{k=1}^N a_k^2 \sin^2(\pi q_k \beta)}{\sum_{k=1}^N a_k^2} = \frac{1}{2}, \text{ for almost every } \beta.$$

This is an easy consequence of the following lemma:

Lemma 3.8. *Let (u_n) be a sequence of real numbers such that $1 \leq u_n \leq n^\gamma$, with $0 \leq \gamma < \frac{1}{3}$. If (X_n) is a sequence of bounded orthogonal random variables on a probability space (Ω, μ) , then*

$$(54) \quad \lim_N \frac{\sum_{k=1}^N u_k X_k}{\sum_{k=1}^N u_k} = 0, \text{ for a.e. } \omega.$$

Proof. By orthogonality, we have $\frac{\int_{\Omega} |\sum_{k=1}^N u_k X_k|^2 d\mu}{(\sum_{k=1}^N u_k)^2} \leq N^{2\gamma-1}$.

Setting $R_N(\omega) := \frac{\sum_{k=1}^N u_k X_k(\omega)}{\sum_{k=1}^N u_k}$, it implies $\sum_{n=1}^{\infty} \|R_{n^p}\|_2^2 < +\infty$, if $p(1 - 2\gamma) > 1$, hence $\lim_n R_{n^p}(\omega) = 0$, for a.e. ω , if $\gamma < \frac{1-\frac{1}{p}}{2}$. Therefore, it suffices to show that:

$$\lim_n \sup_{0 \leq \ell < n^{p+1} - n^p} |R_{n^p+\ell}(\omega) - R_{n^p}(\omega)| = 0.$$

We have $|R_{n^p+\ell}(\omega) - R_{n^p}(\omega)|$

$$\begin{aligned} &= \left| \frac{(\sum_{k=1}^{n^p} u_k)(\sum_{k=n^p+1}^{n^p+\ell} u_k X_k) - (\sum_{k=n^p+1}^{n^p+\ell} u_k)(\sum_{k=1}^{n^p} u_k X_k)}{(\sum_{k=1}^{n^p+\ell} u_k)(\sum_{k=1}^{n^p} u_k)} \right| \\ &= \left| \frac{\sum_{k=n^p+1}^{n^p+\ell} u_k X_k}{\sum_{k=1}^{n^p+\ell} u_k} - \frac{(\sum_{k=n^p+1}^{n^p+\ell} u_k)(\sum_{k=1}^{n^p} u_k X_k)}{(\sum_{k=1}^{n^p+\ell} u_k)(\sum_{k=1}^{n^p} u_k)} \right| \\ &= \left| \frac{\sum_{k=n^p+1}^{n^p+\ell} u_k X_k}{\sum_{k=1}^{n^p+\ell} u_k} \right| + \left| \frac{(\sum_{k=n^p+1}^{n^p+\ell} u_k)(\sum_{k=1}^{n^p} u_k X_k)}{(\sum_{k=1}^{n^p+\ell} u_k)(\sum_{k=1}^{n^p} u_k)} \right| \\ &\leq C \frac{\sum_{k=n^p+1}^{n^p+\ell} u_k}{\sum_{k=1}^{n^p+\ell} u_k} + C \frac{\sum_{k=n^p+1}^{n^p+\ell} u_k}{\sum_{k=1}^{n^p} u_k} \leq 2C \frac{\sum_{k=n^p+1}^{(n+1)^p} u_k}{\sum_{k=1}^{n^p} u_k}. \end{aligned}$$

Therefore, $\sup_{0 \leq \ell < n^{p+1} - n^p} |R_{n^p+\ell}(\omega) - R_{n^p}(\omega)| \leq Cpn^{p\gamma-1} \rightarrow 0$, if $p\gamma - 1 < 0$.

Taking $p = 3$, It follows that (54) holds if $\gamma < \min(\frac{1}{p}, \frac{1-\frac{1}{p}}{2}) = \frac{1}{3}$. \square

Vectorial case

For simplicity, we consider the case of two components. Let be given a vectorial function $\Phi = (\varphi^1, \varphi^2)$, where φ^1, φ^2 are two centered step functions with respectively s_1, s_2 discontinuities: $\varphi^i = \sum_{j=0}^{s_i} v_j^i 1_{[u_j^i, u_{j+1}^i]} - c_i$, for $i = 1, 2$.

Let the covariance matrix Γ_n be defined by $\Gamma_n(a, b) := (\log n)^{-1} \|a\varphi_n^1 + b\varphi_n^2\|_2^2$ and I_2 the 2-dimensional identity matrix.

Theorem 3.9. *If α is bpq and if the condition (15) is satisfied uniformly with respect to (a, b) in the unit sphere, there are $0 < r_1, r_2 < +\infty$ two constants such that for a “large” set of integers n as in Theorem 3.4:*

- Γ_n satisfies inequalities of the form $r_1 I_2 \leq \Gamma_n(a, b) \leq r_2 I_2$;
- the distribution of $\Gamma_n^{-1} \Phi_n$ is close to the standard 2-dimensional normal law.

Proof. We only give an idea of the proof.

The classical method of proof of a CLT for a vectorial function is to show a scalar CLT for all linear combinations of the components. Likewise, we start again the proof of Theorem (3.2), but we take care of the bound from below of the variance for $a\varphi_n^1 + b\varphi_n^2$ (which should be uniform for (a, b) in the unit sphere). \square

In order to apply the theorem, we need (15) uniformly on the unit sphere. To get it, we can use the following proposition.

Proposition 3.10. *Let Λ be a compact space and $(F_\lambda, \lambda \in \Lambda)$ be a family of nonnegative non identically null continuous functions on \mathbb{T}^d depending continuously on λ . If a sequence (z_n) is equidistributed in \mathbb{T}^d , then*

$$(55) \quad \exists N_0, \eta, \eta_0 > 0 \text{ such that } \text{Card}\{n \leq N : F_\lambda(z_n) \geq \eta\} \geq \theta N, \forall N \geq N_0, \forall \lambda \in \Lambda.$$

Proof. For $\lambda \in \Lambda$, let $u_\lambda \in \mathbb{T}^d$ be such that $F_\lambda(u_\lambda) = \sup_{u \in \mathbb{T}^d} F_\lambda(u)$. We have $F_\lambda(u_\lambda) > 0$ and there is $\eta_\lambda > 0$ and an open neighborhood U_λ of u_λ such that $F_\lambda(u) > 2\eta_\lambda$ for $u \in U_\lambda$. Using the continuity of F_λ with respect to the parameter λ , the inequality $F_\zeta(u) > \eta_\lambda$ holds for $u \in U_\lambda$ and ζ in an open neighborhood V_λ of λ . By compactness of Λ , there is a finite set $(\lambda_j, j \in J)$ such that $(V_{\lambda_j}, j \in J)$ is an open covering of Λ . Let $\theta := \frac{1}{2} \inf_{j \in J} \text{Leb}(U_{\lambda_j})$.

By equidistribution of (z_n) , there is N_0 such that

$$\frac{1}{N} \sum_{n=1}^N 1_{U_{\lambda_j}}(z_n) \geq \theta, \forall N \geq N_0, \forall j \in J.$$

Let $\eta := \inf_{j \in J} \eta_{\lambda_j}$. For every $\lambda \in \Lambda$, there is $j \in J$ such that $\lambda \in V_{\lambda_j}$ and therefore $F_\lambda(z_n) \geq \eta_{\lambda_j} \geq \eta$, if $z_n \in U_{\lambda_j}$. This implies:

$$\text{Card}\{n \leq N : F_\lambda(z_n) \geq \eta\} \geq \text{Card}\{n \leq N : z_n \in U_{\lambda_j}\} \geq \theta N, \forall N \geq N_0. \quad \square$$

A generic result

By Proposition 3.10 applied for (a, b) in the unit sphere, for a.e. values of the parameter $(u_1^1, \dots, u_{s_1}^1, u_1^2, \dots, u_{s_2}^2)$, the functions $a\varphi^1 + b\varphi^2$ satisfy Condition (15) uniformly in (a, b) in the unit sphere. Hence Theorem 3.9 applies generically with respect to the discontinuities.

Special values: an application to the rectangular billiard in the plane

Example 4 Now, in order to apply it to the periodic billiard, we consider the vectorial function $\psi = (\varphi^1, \varphi^2)$ with

$$\begin{aligned} \varphi^1 &= 1_{[0, \frac{\alpha}{2}]} - 1_{[\frac{1}{2}, \frac{1}{2} + \frac{\alpha}{2}]} = \frac{2}{\pi} \sum_{r \in \mathbb{Z}} e^{-\pi i(2r+1)\frac{\alpha}{2}} \frac{\sin(\pi(2r+1)\frac{\alpha}{2})}{2r+1} e^{2\pi i(2r+1)\cdot}, \\ \varphi^2 &= 1_{[0, \frac{1}{2} - \frac{\alpha}{2}]} - 1_{[\frac{1}{2}, 1 - \frac{\alpha}{2}]} = \frac{-2i}{\pi} \sum_{r \in \mathbb{Z}} e^{\pi i(2r+1)\frac{\alpha}{2}} \frac{\cos(\pi(2r+1)\frac{\alpha}{2})}{2r+1} e^{2\pi i(2r+1)\cdot}. \end{aligned}$$

The Fourier coefficient of φ^1 and φ^2 of order r are null for r even.

Let us consider a linear combination $\varphi_{a,b} = a\varphi^1 + b\varphi^2$. For $r = 2t + 1$ odd, we have:

$$c_{2t+1}(a\varphi^1 + b\varphi^2) = \frac{2}{\pi} \frac{1}{2t+1} e^{-\pi i(2t+1)\frac{\alpha}{2}} \left[a \sin(\pi(2t+1)\frac{\alpha}{2}) - ib \cos(\pi(2t+1)\frac{\alpha}{2}) \right].$$

If q_j is even, $\gamma_{q_j}(\varphi_{a,b})$ is null. If q_j is odd, we have

$$|\gamma_{q_j}(\varphi_{a,b})|^2 = \left| a \sin(\pi q_j \frac{\alpha}{2}) - ib \sin(\pi(\frac{1}{2} + q_j \frac{\alpha}{2})) \right|^2,$$

For q_j odd, we have by (3),

$$\begin{aligned} \|q_j \frac{\alpha}{2}\| &= \left\| \frac{p_j}{2} + \frac{\theta_j}{2} \right\|, \text{ hence } \left| \|q_j \frac{\alpha}{2}\| - \left\| \frac{p_j}{2} \right\| \right| \leq \left| \frac{\theta_j}{2} \right| \leq \frac{1}{2q_{n+1}}, \\ \left\| \frac{1}{2} + q_j \frac{\alpha}{2} \right\| &= \left\| \frac{q_j}{2} - \frac{p_j}{2} - \frac{\theta_j}{2} \right\|, \text{ hence } \left| \|q_j \beta_2\| - \left(\left\| \frac{1}{2} + \frac{p_j}{2} \right\| \right) \right| \leq \left| \frac{\theta_j}{2} \right| \leq \frac{1}{2q_{n+1}}, \end{aligned}$$

This implies, for q_j odd: $\gamma_{q_j}(\varphi_{a,b}) = a(1 + O(\frac{1}{q_{j+1}}))$, if p_j is odd, $= b(1 + O(\frac{1}{q_{j+1}}))$, if p_j is even.

The computation shows that, if α is such that, in average, there is a positive proportion of pairs (p_j, q_j) (even, odd) and a positive proportion of pairs (p_j, q_j) (odd, odd), then the condition of Theorem 3.9 is fulfilled by the vectorial step function $\psi = (\varphi^1, \varphi^2)$.

For an application to the model of rectangular periodic billiard in the plane described in [CoGu12], we refer to [CoIsLe17].

3.3. A counter-example.

For a parameter $\gamma > 0$, let the sequence $(a_n)_{n \geq 1}$ be defined by

$$a_n = \lfloor n^\gamma \rfloor \text{ if } n \in \{2^k : k \geq 0\}, = 1 \text{ if } n \notin \{2^k : k \geq 0\}.$$

Let α be the number which has $(a_n)_n$ for sequence of partial quotients. In order not to have to discuss the relations between the q_n and the γ_{q_n} , let us take for φ the function φ^0 defined above for which the numbers $\gamma_k = \frac{1}{2\pi i}$, $\forall k$. For a given ℓ , let n_ℓ be the largest of the integers $n < q_{\ell+1}$ such that

$$\text{Var}(\varphi_n) = \max_{k < q_{\ell+1}} \text{Var}(\varphi_k).$$

We have

$$\text{Var}(\varphi_{n_\ell}) \geq c \sum_{k=0}^{\ell-1} |\gamma_{q_k}|^2 a_{k+1}^2 \geq c \sum_{j=0}^{\lfloor \log_2(\ell) \rfloor} 2^{2\gamma j} \geq c\ell^{2\gamma}.$$

In the sum $\varphi_{n_\ell}(x) = \sum_{k=0}^{\ell} \sum_{j=0}^{b_k q_k - 1} \varphi(x + N_{k-1}\alpha + j\alpha)$ defined in (10), we can isolate the indices k for which $k+1$ is a power of 2 (for the other indices $a_{k+1} = 1$) and write $\varphi_{n_\ell}(x) = U_\ell + V_\ell$ with

$$\begin{aligned} U_\ell &= \sum_{p=1}^{\lfloor \log_2(\ell) \rfloor} \sum_{j=0}^{b_{2^p} q_{2^p} - 1} \varphi(x + N_{2^p-1}\alpha + j\alpha), \\ V_\ell &= \sum_{k \in [0, \ell] \cap \{a_{k+1}=1\}} \sum_{j=0}^{b_k q_k - 1} \varphi(x + N_{k-1}\alpha + j\alpha). \end{aligned}$$

We will see in (78) that the variance of a sum where b_k equals 0 or 1 is bounded as follows

$$\text{Var}\left(\sum_{k \in [0, \ell] \cap \{a_{k+1}=1\}} \sum_{j=0}^{b_k q_k - 1} \varphi(x + N_{k-1}\alpha + j\alpha) \right) \leq C\ell \log(\ell).$$

On the other side, we also have

$$\left| \sum_{p=1}^{\lfloor \log_2(\ell) \rfloor} \sum_{j=0}^{b_{2^p-1} q_{2^p-1}-1} \varphi(x + N_{2^p-2} \alpha + j\alpha) \right| \leq \sum_{p=1}^{\lfloor \log_2(\ell) \rfloor} a_{2^p} V(\varphi) \leq C \sum_{p=1}^{\lfloor \log_2(\ell) \rfloor} 2^{\gamma p} \leq C \ell^\gamma.$$

The previous bounds imply that

$$(56) \quad \frac{\varphi_{n_\ell}}{\|\varphi_{n_\ell}\|_2} = \frac{U_\ell}{\|\varphi_{n_\ell}\|_2} + \frac{V_\ell}{\|\varphi_{n_\ell}\|_2}$$

with $\|\varphi_{n_\ell}\|_2 \geq c \ell^\gamma$, $\|U_\ell\|_\infty \leq C \ell^\gamma$, $\|V_\ell\|_2 \leq (C \ell \log(\ell))^{1/2}$. Thus, if $\gamma > 1/2$, one has

$$\left\| \frac{U_\ell}{\|\varphi_{n_\ell}\|_2} \right\|_\infty \leq \frac{C}{c} \text{ and } \left\| \frac{V_\ell}{\|\varphi_{n_\ell}\|_2} \right\|_2 \rightarrow 0,$$

and the limit points of the distributions of the $\frac{\varphi_{n_\ell}}{\|\varphi_{n_\ell}\|_2}$ have all their supports included in $[-\frac{C}{c}, \frac{C}{c}]$. It implies that none is gaussian.

4. Proof of the decorrelation (Proposition 3.1)

To prove the decorrelation properties 3.1, we first truncate the Fourier series of the ergodic sums φ_q . For functions in \mathcal{C} , we easily control the remainders and it suffices to treat the case of trigonometric polynomials.

4.1. Some preliminary inequalities.

Lemma 4.1. *For each irrational α and $n \geq 1$, we have*

$$(57) \quad q_{n+1}/q_{n+k} \leq C \rho^k, \forall k \geq 1, \text{ with } \rho = (\sqrt{5} - 1)/2, C = (3\sqrt{5} + 5)/2.$$

Proof. For $n \geq 1$ fixed, set $r_0 = q_n$, $r_1 = q_{n+1}$, $r_{k+1} = r_k + r_{k-1}$, for $k \geq 1$. It follows immediately by induction that $q_{n+k} \geq r_k$, $\forall k \geq 0$.

Denote $c = \frac{1}{\sqrt{5}}$ and let $\lambda_1 = \frac{\sqrt{5}}{2} + \frac{1}{2}$, $\lambda_2 = -\frac{\sqrt{5}}{2} + \frac{1}{2}$ be the two roots of the polynomial $\lambda^2 - \lambda - 1$. Since $\lambda_j^{\ell+1} = \lambda_j^\ell + \lambda_j^{\ell-1}$ for each $j = 1, 2$ and $\ell \geq 1$, we obtain by induction that:

$$(58) \quad q_{n+k} \geq r_k = c \lambda_1^k (q_{n+1} - \lambda_2 q_n) - c \lambda_2^k (q_{n+1} - \lambda_1 q_n), \quad k \geq 0, n \geq 1.$$

Take $k \geq 1$. Since $\lambda_2 < 0$ and $\left| 1 - \frac{\lambda_1 q_n}{q_{n+1}} \right| < \lambda_1$ (as $\lambda_1 > 1$), from (58), we obtain

$$q_{n+k} \geq c \lambda_1^k \left(1 - \lambda_1 \left(\frac{|\lambda_2|}{\lambda_1} \right)^k \right) q_{n+1} = c \lambda_1^k q_{n+1} \left(1 - |\lambda_2| \left(\frac{|\lambda_2|}{\lambda_1} \right)^{k-1} \right) \geq c \lambda_1^k q_{n+1} (1 - |\lambda_2|).$$

It follows that, for $k \geq 1$, we have $q_{n+k} \geq c_1 \lambda_1^k q_{n+1}$, with $c_1 := \frac{3\sqrt{5}-5}{10}$, which gives (57). \square

For the frequently used quantities $\frac{q_{n+1}}{q_n}$ and $\frac{q_{n+1}}{q_n} \ln q_{n+1}$, we introduce the notation:

$$(59) \quad a'_n := \frac{q_{n+1}}{q_n}, \quad c_n := \frac{q_{n+1}}{q_n} \ln q_{n+1}.$$

Observe that $a'_n \leq a_{n+1} + 1$.

We will prove the decorrelation inequalities of Lemma 4.3 under Hypothesis 1 on α , that is: there are two constants $A \geq 1$, $p \geq 0$ such that $a_n \leq A n^p, \forall n \geq 1$.

It follows from it: for constants B, C , $b_n \leq B n^p$ and, since $q_n \leq B^n (n!)^p$,

$$(60) \quad \ln q_n \leq C n \ln n, \quad c_n \leq C n^{p+1} \ln n.$$

The case when α is bpq corresponds to $p = 0$ in the previous inequalities and we have then $\ln q_n \leq C n$.

Lemma 4.2. *If q_n, q_ℓ are denominators of α , we have with an absolute constant C :*

$$(61) \quad \sum_{|j| \geq q_\ell} \frac{1}{j^2} \frac{\|q_n j \alpha\|^2}{\|j \alpha\|^2} \leq C \frac{q_n}{q_\ell}, \quad \text{for } n \leq \ell.$$

Proof. Observe first that, if f is a nonnegative BV function with integral $\mu(f)$ and if q is a denominator of α , then:

$$\sum_{j=q}^{\infty} \frac{f(j\alpha)}{k^2} \leq \frac{2\mu(f)}{q} + \frac{2V(f)}{q^2}.$$

Indeed, by Denjoy-Koksma inequality applied to $f - \mu(f)$, $\sum_{j=q}^{\infty} \frac{f(j\alpha)}{j^2}$ is less than

$$\sum_{i=1}^{\infty} \frac{1}{(iq)^2} \sum_{r=0}^{q-1} f((iq+r)\alpha) \leq \frac{1}{q^2} \left(\sum_{i=1}^{\infty} \frac{1}{i^2} \right) (q\mu(f) + V(f)) = \frac{\pi^2}{6} \left(\frac{\mu(f)}{q} + \frac{V(f)}{q^2} \right).$$

Taking for $f(x)$ respectively $1_{[0, \frac{1}{L}]}(|x|)$ and $\frac{1}{x^2} 1_{[\frac{1}{L}, \frac{1}{2}]}(|x|)$, we obtain with $L = 1/q_\ell$:

$$(62) \quad \sum_{j: \|j\alpha\| \leq 1/q_n, j \geq q_\ell} \frac{1}{j^2} \leq C \frac{q_n + q_\ell}{q_n q_\ell^2}, \quad \sum_{j: \|j\alpha\| \geq 1/q_n, j \geq q_\ell} \frac{1}{j^2} \frac{1}{\|j\alpha\|^2} \leq C q_n \frac{q_n + q_\ell}{q_\ell^2}.$$

Now, (61) follows, since for $n \leq \ell$:

$$\begin{aligned} \frac{1}{2} \sum_{|j| \geq q_\ell} \frac{1}{j^2} \frac{\|q_n j \alpha\|^2}{\|j \alpha\|^2} &\leq q_n^2 \sum_{\|j\alpha\| \leq 1/q_n, j \geq q_\ell} \frac{1}{j^2} + \sum_{\|j\alpha\| > 1/q_n, j \geq q_\ell} \frac{1}{j^2} \frac{1}{\|j\alpha\|^2} \\ &\leq \frac{q_n^2}{q_n q_\ell} + \frac{q_n^2}{q_\ell^2} + \frac{q_n}{q_\ell} + \frac{q_n^2}{q_\ell^2} \leq 4 \frac{q_n}{q_\ell}. \quad \square \end{aligned}$$

4.2. Truncation.

The Fourier coefficients of the ergodic sum φ_q satisfy:

$$|\widehat{\varphi_{q_n}}(j)| = \frac{|\gamma_j(\varphi)|}{|j|} \frac{|\sin \pi q_n j \alpha|}{|\sin \pi j \alpha|} \leq \frac{\pi}{2} \frac{K(\varphi)}{|j|} \frac{\|q_n j \alpha\|}{\|j \alpha\|}.$$

Recall also (cf. (11)) that

$$\|\varphi_{q_n}\|_\infty = \sup_x \left| \sum_{\ell=0}^{q_n-1} \varphi(x + \ell\alpha) \right| \leq V(\varphi) \quad \text{et} \quad \|\varphi_{q_n}\|_2 \leq 2\pi K(\varphi).$$

Let $S_L f$ denote the partial sum of order $L \geq 1$ of the Fourier series of a function f and let $R_L f = f - S_L f$ be the remainder. We will take $L = q_\ell$ and bound the truncation error for the ergodic sums $\varphi_{b_n q_n}$.

We use the bound (61) which gives, for $q_n \leq L = q_\ell$,

$$(63) \quad \begin{aligned} \|R_L \varphi_{b_n q_n}\|_2 &= \left[\sum_{|j| \geq L} |\widehat{\varphi_{b_n q_n}}(j)|^2 \right]^{\frac{1}{2}} = \left[\sum_{|j| \geq L} \frac{|\gamma_j(\varphi)|^2}{j^2} \frac{\|b_n q_n j \alpha\|^2}{\|j \alpha\|^2} \right]^{\frac{1}{2}} \\ &\leq b_n \left[\sum_{|j| \geq L} \frac{|\gamma_j(\varphi)|^2}{j^2} \frac{\|q_n j \alpha\|^2}{\|j \alpha\|^2} \right]^{\frac{1}{2}} \leq C K(\varphi) b_n \left(\frac{q_n}{L}\right)^{\frac{1}{2}}. \end{aligned}$$

For every bounded ψ , we have, for $q_n \leq q_m \leq q_\ell$ and $L = q_\ell$:

$$\begin{aligned} & \left| \int \psi [\varphi_{b_n q_n} \varphi_{b_m q_m} - S_L \varphi_{b_n q_n} S_L \varphi_{b_m q_m}] d\mu \right| \\ & \leq \|\psi\|_\infty [\|\varphi_{b_n q_n}\|_2 \|R_L \varphi_{b_m q_m}\|_2 + \|R_L \varphi_{b_n q_n}\|_2 \|S_L \varphi_{b_m q_m}\|_2] \end{aligned}$$

and, as $\|\varphi_{b_n q_n}\|_2 \leq C b_n K(\varphi)$, we have by (63):

$$\begin{aligned} & \left| \int \psi [\varphi_{b_n q_n} \varphi_{b_m q_m} - S_L \varphi_{b_n q_n} S_L \varphi_{b_m q_m}] d\mu \right| \\ & \leq \|\psi\|_\infty [\|\varphi_{b_n q_n}\|_2 \|R_L \varphi_{b_m q_m}\|_2 + \|R_L \varphi_{b_n q_n}\|_2 \|\varphi_{b_m q_m}\|_2] \\ & \leq C K(\varphi)^2 \|\psi\|_\infty b_n b_m \left[\left(\frac{q_n}{L}\right)^{\frac{1}{2}} + \left(\frac{q_m}{L}\right)^{\frac{1}{2}} \right]. \end{aligned}$$

This gives the following truncation error, where the constant $C_1 = C_1(\psi, \varphi)$ is equal to $K(\varphi)^2 \|\psi\|_\infty$, up to a universal factor,

$$(64) \quad \left| \int \psi [\varphi_{b_n q_n} \varphi_{b_m q_m} - S_{q_\ell} \varphi_{b_n q_n} S_{q_\ell} \varphi_{b_m q_m}] d\mu \right| \leq C_1 b_n b_m \left(\frac{q_m}{q_\ell}\right)^{\frac{1}{2}} \quad \text{for } q_n \leq q_m \leq q_\ell.$$

Using the Fejer kernel, we get

$$\|S_{q_r} \varphi_{b_n q_n}\|_\infty \leq \|\varphi_{b_n q_n}\|_\infty + \frac{1}{q_r} \sum_{|j| < q_r} |j \widehat{\varphi_{b_n q_n}}(j)| \leq C K(\varphi) \frac{1}{q_r} \sum_{j=1}^{q_r-1} \frac{1}{\|j \alpha\|},$$

and, by (11) and (70),

$$\|S_{q_r} \varphi_{b_n q_n}\|_\infty \leq b_n V(\varphi) + C K(\varphi) \ln(q_r).$$

With this inequality, by computations similar to the ones used to get (64), we have:

$$(65) \quad \left| \int \psi [\varphi_{b_n q_n} \varphi_{b_m q_m} \varphi_{b_\ell q_\ell} - S_{q_r} \varphi_{b_n q_n} S_{q_r} \varphi_{b_m q_m} S_{q_r} \varphi_{b_\ell q_\ell}] d\mu \right| \leq C b_n b_m b_\ell \left(\frac{q_\ell}{q_r}\right)^{\frac{1}{2}} \ln(q_r),$$

when $q_n \leq q_m \leq q_\ell \leq q_r$.

Lemma 4.3. *If $a_{k+1} \leq Ak^p, \forall k \geq 1$ and $n \leq m \leq \ell$, we have for every $\Lambda \geq 1$:*

$$(66) \quad \sum_{j=1}^{\infty} \frac{\|q_n j \alpha\|}{j^2 \|j \alpha\|} \leq C \frac{n^{p+2} \ln n}{q_{n+1}},$$

$$(67) \quad \sum_{1 \leq j, k < q_\Lambda, j \neq k} \frac{\|q_n j \alpha\| \|q_m k \alpha\|}{|k-j| k j \|j \alpha\| \|k \alpha\|} \leq \frac{C}{q_{n+1}} \Lambda^{2p+4} (\ln \Lambda)^2,$$

$$(68) \quad \sum_{-q_\Lambda < i, j, k < q_\Lambda, i+j+k \neq 0} \frac{\|q_n i \alpha\| \|q_m j \alpha\| \|q_\ell k \alpha\|}{|i+j+k| i j k \|i \alpha\| \|j \alpha\| \|k \alpha\|} \leq \frac{C}{q_{n+1}} \Lambda^{3p+8}.$$

Now we apply this lemma which is proved in the next subsection. Let us mention that Hardy and Littlewood in [HaLi30] considered similar quantities for α bpq. One of their motivations was to study the asymptotic of the number of integral points contained in homothetic of triangles.

Proof of Proposition 3.1

By (66) we have, with $K = CV(\psi)V(\varphi)$,

$$\left| \int \psi \varphi_{b_n q_n} d\mu \right| \leq \sum_{j \neq 0} |\widehat{\varphi_{b_n q_n}}(j)| |\hat{\psi}(-j)| \leq K \sum_{j \geq 1} \frac{\|b_n q_n j \alpha\|}{j^2 \|j \alpha\|} \leq \frac{K b_n}{q_{n+1}} n^{p+2} \ln n.$$

This implies (38). We prove now (39).

With $L = q_\Lambda$, we have:

$$\int \psi S_L \varphi_{b_n q_n} S_L \varphi_{b_m q_m} d\mu = \sum_{|j|, |k| \leq L, j \neq k} \widehat{\varphi_{b_n q_n}}(j) \widehat{\varphi_{b_m q_m}}(k) \hat{\psi}(j-k).$$

In what follows, the constant C is equal to $V(\psi)V(\varphi)^2$ up to a factor not depending on ψ and φ , which may change.

Recall that, by (57), there is a constant B such that $m \leq B \ln q_m, \forall m \geq 1$.

By (67), it holds

$$\begin{aligned} \left| \int \psi S_L \varphi_{b_n q_n} S_L \varphi_{b_m q_m} d\mu \right| &\leq \sum_{|j|, |k| \leq L, j \neq k} |\widehat{\varphi_{b_n q_n}}(j)| |\widehat{\varphi_{b_m q_m}}(k)| |\hat{\psi}(j-k)| \\ &\leq C \sum_{1 \leq j, k \leq L} \frac{\|b_n q_n j \alpha\| \|b_m q_m k \alpha\|}{|k-j| k j \|j \alpha\| \|k \alpha\|} \leq \frac{C b_n b_m}{q_{n+1}} \Lambda^{2p+4} (\ln \Lambda)^2. \end{aligned}$$

Putting it together with (64) and replacing q_{n+1} by q_n , we get, for $n \leq m \leq \Lambda$,

$$(69) \quad \begin{aligned} \left| \int_X \psi \varphi_{b_n q_n} \varphi_{b_m q_m} d\mu \right| &\leq C b_n b_m \left[\frac{\Lambda^{2p+4} (\ln \Lambda)^2}{q_n} + \left(\frac{q_m}{q_\Lambda} \right)^{\frac{1}{2}} \right] \\ &\leq C b_n b_m \left[\frac{\Lambda^{2p+4} (\ln \Lambda)^2}{q_n} + \rho^{\frac{\Lambda-m}{2}} \right]. \end{aligned}$$

Let us take $\Lambda - m$ of order $2(\ln \frac{1}{\rho})^{-1} \ln q_n$, i.e., such that the second term in the RHS in (69) is of order $1/q_n$. The first term is then less than

$$\frac{C_1}{q_n} \max((\ln q_n)^{2p+5}, m^{2p+5}) \leq \frac{C_2}{q_n} \max((\ln q_n)^{2p+5}, (\ln q_m)^{2p+5}) \leq \frac{C_2}{q_n} (\ln q_m)^{2p+5}.$$

As $\ln q_n \leq Cn \ln n$ by Hypothesis 1, we have $\max((\ln q_n)^\theta, m^\theta) \leq C m^{\theta+1}$.

This shows (39) with $\theta = 2p + 5$. \square

In the same way, (40) follows from (65) and (68).

4.3. Proof of Lemma 4.3.

As already mentioned, Lemma 4.3 is a consequence of the good repartition of the numbers $\|k\alpha\|$ when k varies between 1 and q_n . We will use this information through two inequalities given in the following lemma, which will be used several times.

Lemma 4.4. *We have*

$$(70) \quad \sum_{j=q_t}^{q_{t+1}-1} \frac{1}{\|j\alpha\|} \leq \sum_{j=1}^{q_{t+1}-1} \frac{1}{\|j\alpha\|} \leq C q_{t+1} \ln q_{t+1}, \quad \forall t \geq 0,$$

$$(71) \quad \sum_{1 \leq j < q_{r+1}} \frac{1}{j \|j\alpha\|} \leq C \sum_{t=0}^r \frac{q_{t+1}}{q_t} \ln q_{t+1} = C \sum_{t=0}^r c_t, \quad \forall r \geq 0.$$

Proof. For $1 \leq j < q_{t+1}$, one has $\|j\alpha\| \geq \frac{1}{q_{t+1}+q_t} \geq \frac{1}{2q_{t+1}}$. There is exactly one value of $j\alpha \bmod 1$, for $1 \leq j < q_{t+1}$, in each interval $[\frac{\ell}{q_{t+1}}, \frac{\ell+1}{q_{t+1}}[$, $\ell = 1, \dots, q_{t+1} - 1$. This implies:

$$\sum_{j=1}^{q_{t+1}-1} \frac{1}{\|j\alpha\|} \leq 2q_{t+1} + \sum_{\ell=1}^{q_{t+1}-1} \frac{1}{\ell/q_{t+1}} \leq C q_{t+1} \ln q_{t+1}.$$

From (70), applied for $t = 1, \dots, r$, we deduce (71):

$$\sum_{1 \leq j < q_{r+1}} \frac{1}{j \|j\alpha\|} = \sum_{t=0}^r \sum_{q_t \leq j < q_{t+1}} \frac{1}{j \|j\alpha\|} \leq \sum_{t=0}^r \frac{1}{q_t} \sum_{q_t \leq j < q_{t+1}} \frac{1}{\|j\alpha\|} \leq C \sum_{t=0}^r \frac{q_{t+1}}{q_t} \ln q_{t+1}. \quad \square$$

4.3.1. Bound (66) in Lemma 4.3.

We want to bound $\sum_{j=1}^{\infty} \frac{\|q_n j \alpha\|}{j^2 \|j\alpha\|}$. For $n < \ell$, we write $\sum_{j=1}^{q_\ell-1} \frac{\|q_n j \alpha\|}{j^2 \|j\alpha\|} = (A) + (B)$, with

$$\begin{aligned} (A) &:= \sum_{j=1}^{q_{n+1}-1} \frac{1}{j} \frac{\|q_n j \alpha\|}{j \|j\alpha\|} \leq \frac{1}{q_{n+1}} \sum_{j=1}^{q_{n+1}-1} \frac{1}{j} \frac{1}{\|j\alpha\|} \\ &\leq \frac{1}{q_{n+1}} \sum_{k=0}^n \frac{1}{q_k} \sum_{j=q_k}^{q_{k+1}-1} \frac{1}{\|j\alpha\|} \leq C \frac{1}{q_{n+1}} \sum_{k=0}^n \frac{q_{k+1}}{q_k} \ln q_{k+1}, \quad \text{by (71);} \end{aligned}$$

$$\begin{aligned}
(B) &:= \sum_{j=q_{n+1}}^{q_\ell-1} \frac{\|q_n j \alpha\|}{j^2 \|j \alpha\|} \leq \sum_{k=n+1}^{\ell-1} \sum_{j=q_k}^{q_{k+1}-1} \frac{1}{j^2 \|j \alpha\|} \\
&\leq \sum_{k=n+1}^{\ell-1} \frac{1}{q_k^2} \sum_{j=q_k}^{q_{k+1}-1} \frac{1}{\|j \alpha\|} \leq C \sum_{k=n+1}^{\ell-1} \frac{1}{q_k} \frac{q_{k+1}}{q_k} \ln q_{k+1}, \text{ by (70)}.
\end{aligned}$$

By (57), we know that $\frac{q_{n+1}}{q_k} \leq C\rho^{k-n}$, with $\rho < 1$, for $k \geq n+1$. By hypothesis, $a_{k+1} \leq Ak^p$. It follows with the notation (59):

$$(A) \leq \frac{C}{q_{n+1}} \sum_{k=0}^n c_k \leq \frac{C}{q_{n+1}} n^{p+2} \ln n$$

and, for (B), with a bound which doesn't depend on $\ell \geq n$:

$$\sum_{k=n+1}^{\ell-1} \frac{q_{n+1}}{q_k} \frac{q_{k+1}}{q_k} \ln q_{k+1} \leq C \sum_{j=0}^{\infty} \rho^j (j+n+1)^{p+1} \ln(j+n+1) \leq C' n^{p+1} \ln n. \quad \square$$

4.3.2. *Bounding (67) of the lemma 4.3.* We want to bound the sum:

$$\sum_{1 \leq j, k < q_\Lambda, j \neq k} \frac{\|q_n j \alpha\| \|q_m k \alpha\|}{|k-j| k j \|j \alpha\| \|k \alpha\|}$$

We cover the square of integers $[1, q_\Lambda[\times [1, q_\Lambda[$ by rectangles $R_{r,s} = [q_r, q_{r+1}[\times [q_s, q_{s+1}[$ for r and s varying between 0 and $\Lambda-1$ and then we bound the sum on each of these rectangles.

We use the inequalities

- for $j < q_{k+1}$, $\|j q_k \alpha\|/j \leq \|q_k \alpha\| \leq 1/q_{k+1}$,
- for $j \geq q_{k+1}$ the (trivial) inequality $\|j q_k \alpha\| \leq 1$
- and the inequality (which is direct consequence of $\|(k-j)\alpha\| \leq \|j\alpha\| + \|k\alpha\|$)

$$\frac{1}{|k-j| \|j \alpha\| \|k \alpha\|} \leq \frac{1}{|k-j| \|(k-j)\alpha\| \|j \alpha\|} + \frac{1}{|k-j| \|(k-j)\alpha\| \|k \alpha\|}.$$

On one hand, we have, distinguishing different cases according to the positions of r and s with respect to $n+1$ and $m+1$:

$$\frac{\|q_n j \alpha\| \|q_m k \alpha\|}{|k-j| k j \|j \alpha\| \|k \alpha\|} \leq \frac{1}{q_{\max(r, n+1)} q_{\max(s, m+1)}} \frac{1}{|k-j| \|j \alpha\| \|k \alpha\|}.$$

On the other hand, by (70) and (71), we have

$$\begin{aligned}
&\sum_{(j,k) \in R_{r,s}} \frac{1}{|k-j| \|j \alpha\| \|k \alpha\|} \\
&\leq \sum_{R_{r,s}} \left(\frac{1}{|k-j| \|(k-j)\alpha\| \|j \alpha\|} + \frac{1}{|k-j| \|(k-j)\alpha\| \|k \alpha\|} \right) \\
&\leq q_{\max(r,s)+1} \ln(q_{\max(r,s)+1}) \sum_{t=0}^{\max(r,s)} c_t.
\end{aligned}$$

It follows

$$\begin{aligned} \sum_{(j,k) \in R_{r,s}} \frac{\|q_n j \alpha\| \|q_m k \alpha\|}{|k-j| k j \|j \alpha\| \|k \alpha\|} &\leq \frac{q_{\max(r,s)+1}}{q_{\max(r,n+1)} q_{\max(s,m+1)}} \ln(q_{\max(r,s)+1}) \sum_{t=0}^{\max(r,s)} c_t \\ &\leq \frac{1}{q_{n+1}} \ln(q_{\Lambda+1}) \sum_{t=0}^{\Lambda} c_t \max_{k=1, \dots, \Lambda} a'_k. \end{aligned}$$

The square $[1, q_\Lambda[\times [1, q_\Lambda[$ is covered by Λ^2 rectangles $R_{r,s}$ and the sums on these rectangles are bounded by the same quantity. It follows

$$\sum_{1 \leq j, k < q_\Lambda, j \neq k} \frac{\|q_n j \alpha\| \|q_m k \alpha\|}{|k-j| k j \|j \alpha\| \|k \alpha\|} \leq \Lambda^2 \frac{C}{q_{n+1}} \ln(q_{\Lambda+1}) \sum_{t=0}^{\Lambda} c_t \max_{k=1, \dots, \Lambda} a'_k,$$

and, with Hypothesis 1,

$$\sum_{1 \leq j, k < q_\Lambda, j \neq k} \frac{\|q_n j \alpha\| \|q_m k \alpha\|}{|k-j| k j \|j \alpha\| \|k \alpha\|} \leq \frac{C}{q_{n+1}} \Lambda^{2p+4} \ln(\Lambda)^2. \quad \square$$

4.3.3. Bounding (68) in Lemma 4.3.

We want to bound the sum

$$(72) \quad \sum_{-q_\Lambda < i, j, k < q_\Lambda, i+j+k \neq 0} \frac{\|q_n i \alpha\| \|q_m j \alpha\| \|q_\ell k \alpha\|}{|i+j+k| i j k \|i \alpha\| \|j \alpha\| \|k \alpha\|}.$$

Here we consider sums with three indices i, j, k . Though we do not write it explicitly these sums are to be understood on non zero indices i, j, k such that $i+j+k \neq 0$. We cover the set of indices by sets of the form

$$R_{\pm r, \pm s, \pm t} = \{(i, j, k) / \pm i \in [q_r, q_{r+1}[, \pm j \in [q_s, q_{s+1}[, \pm k \in [q_t, q_{t+1}[]\}$$

Distinguishing different cases according to the positions of r, s and t with respect to $n+1$, we get: if $(i, j, k) \in R_{\pm r, \pm s, \pm t}$,

$$(73) \quad \begin{aligned} &\frac{\|q_n i \alpha\| \|q_m j \alpha\| \|q_\ell k \alpha\|}{|i+j+k| i j k \|i \alpha\| \|j \alpha\| \|k \alpha\|} \\ &\leq \frac{1}{q_{\max(r,n+1)} q_{\max(s,n+1)} q_{\max(t,n+1)}} \frac{1}{|i+j+k| \|i \alpha\| \|j \alpha\| \|k \alpha\|}. \end{aligned}$$

We have

$$\begin{aligned} \frac{1}{\|i \alpha\| \|j \alpha\| \|k \alpha\|} &\leq \frac{1}{\|(i+j+k) \alpha\| \|j \alpha\| \|k \alpha\|} + \frac{1}{\|(i+j+k) \alpha\| \|i \alpha\| \|k \alpha\|} \\ &\quad + \frac{1}{\|(i+j+k) \alpha\| \|i \alpha\| \|j \alpha\|}. \end{aligned}$$

We then use (70) and (71) three times, sum over $R_{\pm r, \pm s, \pm t}$ and get:

$$\begin{aligned} & \sum_{(i,j,k) \in R_{\pm r, \pm s, \pm t}} \frac{1}{|i+j+k| \|i\alpha\| \|j\alpha\| \|k\alpha\|} \\ & \leq \left(\sum_{v=0}^{3 \max(r,s,t)} c_v \right) \ln^2(q_{\max(r,s,t)+1}) (q_{s+1}q_{t+1} + q_{r+1}q_{t+1} + q_{r+1}q_{s+1}). \end{aligned}$$

By (73) we then have:

$$\begin{aligned} & \sum_{R_{\pm r, \pm s, \pm t}} \frac{\|q_n i \alpha\| \|q_m j \alpha\| \|q_\ell k \alpha\|}{|i+j+k| i j k \|i\alpha\| \|j\alpha\| \|k\alpha\|} \\ & \leq C \frac{(\sum_{v=0}^{3 \max(r,s,t)} c_v) \ln^2(q_{\max(r,s,t)})}{q_{\max(r,n+1)} q_{\max(s,n+1)} q_{\max(t,n+1)}} (q_{s+1}q_{t+1} + q_{r+1}q_{t+1} + q_{r+1}q_{s+1}) \\ & \leq \frac{C}{q_{n+1}} \left(\sum_{v=0}^{3\Lambda} c_v \right) \ln^2(q_{\Lambda+1}) \left(\max_{k=1, \dots, \Lambda} a'_k \right)^2. \end{aligned}$$

One needs $8\Lambda^3$ boxes $R_{\pm r, \pm s, \pm t}$ to cover the set $\{-q_\Lambda < i, j, k < q_\Lambda, i+j+k \neq 0\}$. This implies that there exists C such that

$$\sum_{-q_\Lambda < i, j, k < q_\Lambda, i+j+k \neq 0} \frac{\|q_n i \alpha\| \|q_m j \alpha\| \|q_\ell k \alpha\|}{|i+j+k| i j k \|i\alpha\| \|j\alpha\| \|k\alpha\|} \leq \frac{C}{q_{n+1}} \Lambda^{3p+8}. \quad \square$$

5. Proof of the CLT (Proposition 3.2)

The difference $H_{X,Y}(\lambda) := |\mathbb{E}(e^{i\lambda X}) - \mathbb{E}(e^{i\lambda Y})|$ can be used to get an upper bound of the distance $d(X, Y)$ thanks to the following inequality ([Fe], Chapter XVI, Inequality (3.13)): if X has a vanishing expectation, then, for every $U > 0$,

$$(74) \quad d(X, Y) \leq \frac{1}{\pi} \int_{-U}^U H_{X,Y}(\lambda) \frac{d\lambda}{\lambda} + \frac{24}{\pi} \frac{1}{\sigma \sqrt{2\pi}} \frac{1}{U}.$$

Using (74), we get an upper bound of the distance between the distribution of X and the normal law by bounding $|\mathbb{E}(e^{i\lambda X}) - e^{-\frac{1}{2}\sigma^2 \lambda^2}|$.

We will use the following remarks:

$$(75) \quad V(fg) \leq \|f\|_\infty V(g) + \|g\|_\infty V(f), \quad \forall f, g \text{ BV},$$

$$(76) \quad \text{if } g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}) \text{ and } u \text{ is BV, then } V(g \circ u) \leq \|g'\|_\infty V(u).$$

Let $w_k := \max_{j=1}^k u_j$, where u_j is larger than $\|f_j\|_\infty$ (see the proposition 3.2).

Since $V(f_k) \leq C \|f_k\|_\infty q_k$, (43) implies

$$(77) \quad \left| \int_X f_k f_m d\mu \right| \leq C \frac{q_k}{q_m} m^\theta w_m^2, \quad \text{for } k \leq m.$$

5.1. Bounding the moments.

Lemma 5.1. *There is C_1 such that, for all $m, \ell \geq 1$,*

$$(78) \quad \left\| \sum_{k=m}^{m+\ell} f_k \right\|_2^2 \leq C_1 \ln(m+\ell) \sum_{j \in [m, m+\ell]} u_j^2 \leq C_1 w_{m+\ell}^2 \ell \ln(m+\ell).$$

Proof. The computation is as in [Hu09]. Let $r_0 := \lfloor \frac{\theta}{\ln(1/\rho)} \ln(m+\ell) \rfloor + 1$, so that $(m+\ell)^\theta \rho^{r_0} \leq 1$. We have $\left\| \sum_{k=m}^{m+\ell} f_k \right\|_2^2 \leq 2(A) + 2(B)$, with

$$(A) = \sum_{0 \leq r < r_0} \sum_{j \in [m, m+\ell] \cap [m-r, m+\ell-r]} \left| \int f_j f_{j+r} d\mu \right|,$$

$$(B) = \sum_{r_0 \leq r < m+\ell} \sum_{j \in [m, m+\ell] \cap [m-r, m+\ell-r]} \left| \int f_j f_{j+r} d\mu \right|.$$

From (77) and (41), then by the choice of r_0 , it follows:

$$(B) \leq C \sum_{r_0 \leq r < m+\ell} \sum_{j \in [m, m+\ell] \cap [m-r, m+\ell-r]} \rho^r (m+\ell)^\theta u_j u_{j+r}$$

$$\leq \frac{1}{2} C \sum_{r_0 \leq r < m+\ell} \sum_{j \in [m, m+\ell] \cap [m-r, m+\ell-r]} \rho^{r-r_0} [u_j^2 + u_{j+r}^2]$$

$$\leq C \sum_{r_0 \leq r < m+\ell} \sum_{j \in [m, m+\ell]} \rho^{r-r_0} u_j^2 \leq \frac{C}{1-\rho} \sum_{j \in [m, m+\ell]} u_j^2.$$

For (A) we have:

$$(A) = \sum_{0 \leq r < r_0} \sum_{j \in [m, m+\ell] \cap [m-r, m+\ell-r]} \left| \int f_j f_{j+r} d\mu \right|,$$

$$\leq \frac{1}{2} \sum_{0 \leq r < r_0} \sum_{j \in [m, m+\ell] \cap [m-r, m+\ell-r]} [u_j^2 + u_{j+r}^2]$$

$$\leq \sum_{0 \leq r < r_0} \sum_{j \in [m, m+\ell]} u_j^2 \leq \frac{\theta}{\ln(1/\rho)} \ln(m+\ell) \sum_{j \in [m, m+\ell]} u_j^2.$$

Therefore we get,

$$\left\| \sum_{k=m}^{m+\ell} f_k \right\|_2^2 \leq \left(\frac{2\theta}{\ln(1/\rho)} \ln(m+\ell) + \frac{2C}{1-\rho} \right) \sum_{j \in [m, m+\ell]} u_j^2. \quad \square$$

We will also need a bound for the terms of degree 3.

Lemma 5.2. *Under the assumption of Proposition 3.2, there exists $C > 0$ such that*

$$(79) \quad \int_X \left(\sum_{k=m}^{m+\ell} f_k \right)^3 \leq C \ell w_{m+\ell}^3 \ln^2(m+\ell), \quad \forall m, n \geq 1.$$

Proof. It suffices to show the result for the sum $\sum_{m \leq s \leq t \leq u \leq m+\ell} \left| \int_X f_s f_t f_u d\mu \right|$.

Replacing f_k by $w_{m+\ell}^{-1} f_k$, we will deduce the bound (79) from the inequalities (42), (43), (44) when $w_k \leq 1$, for $1 \leq k \leq m + \ell$. By (42) and (75), we have

$$\left| \int_X f_s f_t f_u d\mu \right| \leq C \quad \text{and} \quad V(f_s f_t) \leq C(q_s + q_t) \leq 3Cq_t.$$

From (43) and (41), we obtain $\left| \int_X (f_s f_t) \cdot f_u d\mu \right| \leq C \frac{q_t}{q_u} u^\theta \leq C \rho^{(u-t)} u^\theta$ and $\left| \int_X f_s \cdot (f_t f_u) d\mu \right| \leq C \rho^{(t-s)} u^\theta$.

Set $\kappa = \frac{\theta+3}{\ln(1/\rho)} \ln(\ell + m)$. If one of the two differences $t - s$ or $u - t$ is larger than κ then

$$\left| \int_X f_s f_t f_u d\mu \right| \leq C \rho^\kappa u^\theta \leq (\ell + m)^{-\theta-3} u^\theta \leq C(\ell + m)^{-3}.$$

This gives

$$\sum_{m \leq s \leq t \leq u \leq m+\ell: \max(t-s, u-t) > \kappa} \left| \int_X f_s f_t f_u d\mu \right| \leq C \ell^3 (\ell + m)^{-3} \leq C.$$

We have also

$$\sum_{m \leq s \leq t \leq u \leq m+\ell: \max(t-s, u-t) \leq \kappa} \left| \int_X f_s f_t f_u d\mu \right| \leq C \ell \kappa^2,$$

hence the result. \square

5.2. Proof of Proposition 3.2.

We divide the proof in several steps.

5.2.1. *Defining blocks.* We split the sum $S_n := f_1 + \dots + f_n$ into several blocks, small ones and large ones. The small ones will be removed, providing gaps and allowing to take advantage of the decorrelation properties assumed in the statement of the proposition.

Let δ be a parameter (*which will be chosen close to 0*) such that $0 < \delta < \frac{1}{2}$. We set

$$(80) \quad n_1 = n_1(n) := \lfloor n^{\frac{1}{2}+\delta} \rfloor, n_2 = n_2(n) := \lfloor n^\delta \rfloor, n \geq 1,$$

$$(81) \quad \nu = \nu(n) := n_1 + n_2, p(n) := \lfloor n/\nu(n) \rfloor + 1, n \geq 1.$$

We have:

$$(82) \quad p(n) = n^{\frac{1}{2}-\delta} + h_n \sim n^{\frac{1}{2}-\delta}, \quad \text{with } |h_n| \leq 1.$$

For $0 \leq k < p(n)$, we put (with $f_j = 0$, if $n < j \leq n + \nu$)

$$(83) \quad F_{n,k} = f_{k\nu+1} + \dots + f_{k\nu+n_1}, \quad G_{n,k} = f_{k\nu+n_1+1} + \dots + f_{(k+1)\nu},$$

$$(84) \quad v_k = v_{n,k} := \left(\int_X F_{n,k}^2 d\mu \right)^{\frac{1}{2}}, \quad S'_n = \sum_{k=0}^{p(n)-1} F_{n,k}.$$

The sums $F_{n,k}, G_{n,k}$ have respectively $n_1 \sim n^{\frac{1}{2}+\delta}, n_2 \sim n^\delta$ terms and S_n reads

$$(85) \quad S_n = \sum_{k=0}^{p(n)-1} (F_{n,k} + G_{n,k}).$$

The following inequalities are implied by (78):

$$(86) \quad v_{n,k} = \|F_{n,k}\|_2 \leq C n^{\frac{1}{4}+\frac{1}{2}\delta} (\ln n)^{1/2} w_n, \quad \|G_{n,k}\|_2 \leq C n^{\frac{\delta}{2}} (\ln n)^{1/2} w_n, \quad 1 \leq k \leq p(n).$$

Lemma 5.3.

$$(87) \quad \left| \|S_n\|_2^2 - \sum_{k=0}^{p(n)-1} v_k^2 \right| = \left| \|S_n\|_2^2 - \sum_{k=0}^{p(n)-1} \|F_{n,k}\|_2^2 \right| \leq C n^{\frac{3}{4}} \ln n w_n^2,$$

$$(88) \quad \|S_n - S'_n\|_2^2 = \left\| \sum_{k=0}^{p(n)-1} G_k \right\|^2 \leq C n^{\frac{1}{2}-\delta} \ln n w_n^2.$$

Proof. We write simply $F_k = F_{n,k}, G_k = G_{n,k}$. The following inequality follows easily from (77) and (41):

$$\left| \int_X \left(\sum_{u=a}^b f_u \right) \left(\sum_{t=c}^d f_t \right) d\mu \right| \leq \frac{C\rho}{(1-\rho)^2} \rho^{c-b} d^\theta w_d^2, \quad \forall a \leq b < c \leq d.$$

Therefore, we have, with $C_1 = \frac{C\rho}{(1-\rho)^2} \sum_{i \geq 0} \rho^{i\nu}$,

$$\begin{aligned} \sum_{0 \leq j < k < p(n)} \left| \int F_j F_k d\mu \right| &\leq \frac{C\rho}{(1-\rho)^2} n^\theta w_n^2 \sum_{0 \leq j < k < p(n)} \rho^{k\nu+1-(j\nu+n_1)} \\ &\leq \frac{C\rho}{(1-\rho)^2} n^\theta w_n^2 \rho^{n_2} \sum_{0 \leq j < k < p(n)} \rho^{(k-1)\nu-j\nu} \\ &\leq \frac{C\rho}{(1-\rho)^2} \rho^{n_2} n^\theta w_n^2 p(n) \sum_{i \geq 0} \rho^{i\nu} \leq C_1 n^{\frac{1}{2}-\delta+\theta} w_n^2 \rho^{n^\delta}. \end{aligned}$$

The same bound holds for

$$\sum_{0 \leq j < k-1 < p(n)} \left| \int G_j F_k d\mu \right|, \quad \sum_{0 \leq j < k < p(n)} \left| \int F_j G_k d\mu \right|, \quad \sum_{0 \leq j < k < p(n)} \left| \int G_j G_k d\mu \right|.$$

The LHS of (87) is less than

$$\begin{aligned} & \sum_{k=0}^{p(n)-1} \left[\int G_k^2 d\mu + \left| \int G_k F_k d\mu \right| + \left| \int G_k F_{k+1} d\mu \right| \right] \\ & + 2 \sum_{k=0}^{p(n)-1} \sum_{0 \leq j < k < p(n)} \left[\left| \int F_j (F_k + G_k) d\mu \right| + \left| \int G_j G_k d\mu \right| \right] \\ & + 2 \sum_{k=0}^{p(n)-1} \sum_{0 \leq j < k-1 < p(n)} \left| \int G_j F_k d\mu \right|. \end{aligned}$$

The first term is bounded by $C n^{\frac{1}{2}-\delta} n^\delta \ln n w_n^2 = C n^{\frac{1}{2}} \ln n w_n^2$, the second one and the third one by $C n^{\frac{1}{2}-\delta} n^{\frac{1}{4}+\delta} \ln n w_n^2 = C n^{\frac{3}{4}} \ln n w_n^2$. For the other terms, a gap of length n^δ is available and we use the previous bounds which are of the form $C_1 n^{\frac{1}{2}-\delta+\theta} w_n^2 \rho^{n^\delta}$.

Therefore the LHS of (87) is less than: $C n^{\frac{1}{2}-\delta} \ln n w_n^2 + 2C n^{\frac{3}{4}} \ln n w_n^2 + 3C_1 n^{\frac{1}{2}-\delta+\theta} w_n^2 \rho^{n^\delta}$. The biggest term in this sum is $2C n^{\frac{3}{4}} \ln n w_n^2$, which gives (87).

An analogous computation shows that the LHS of (88) behaves like $\sum_{k=0}^{p(n)-1} \int G_k^2 d\mu$ which gives the bound $C n^{\frac{1}{2}-\delta} \ln n w_n^2$; hence (88). \square

5.2.2. *Approximation of the characteristic function of the sum (after removing the small blocks) by a product.*

Let $I_{n,-1}(\lambda) := 1$ and, for $0 \leq k \leq p(n) - 1$ and $\zeta \in \mathbb{R}$,

$$(89) \quad J_n(\zeta) := \int_X e^{i\zeta S_n} d\mu, \quad I_{n,k}(\zeta) := \int_X e^{i\zeta (F_{n,0} + \dots + F_{n,k})} d\mu.$$

Lemma 5.4. *For $0 \leq k < p(n)$, we have*

$$(90) \quad |I_{n,k}(\zeta) - (1 - \frac{\zeta^2}{2} v_{n,k}^2) I_{n,k-1}(\zeta)| \leq C (|\zeta|^3 w_n^3 n^{1/2+\delta} \ln^2(n) + \zeta^4 w_n^4 n^{1+2\delta} \ln(n)^2).$$

Proof. Let $k \geq 1$. For $u \in \mathbb{R}$ we use the inequality:

$$(91) \quad e^{iu} = 1 + iu - \frac{1}{2}u^2 - \frac{i}{6}u^3 + u^4 r(u), \quad \text{with } |r(u)| \leq \frac{1}{24}.$$

By (91), one has:

$$I_{n,k}(\zeta) = \int_X e^{i\zeta (F_{n,0} + \dots + F_{n,k-1})} \left[1 + i\zeta F_{n,k} - \frac{\zeta^2}{2} F_{n,k}^2 - \frac{i}{6} \zeta^3 F_{n,k}^3 + \zeta^4 F_{n,k}^4 r(\zeta F_{n,k}) \right] d\mu.$$

Recall the inequality (41) : $q_1 + q_2 + \dots + q_n \leq C q_{n+1}, \forall n \geq 1$. Using (76) and hypothesis (42), one has $V(e^{i\zeta (F_{n,0} + \dots + F_{n,k-1})}) \leq C |\zeta| w_n q_{(k-2)\nu+n_1}$. Using (76) and hypothesis (42),

one obtains

$$\begin{aligned}
& \left| \int_X e^{i\zeta(F_{n,0}+\dots+F_{n,k-1})} F_{n,k} d\mu \right| \leq \sum_{j=1}^{n_1} \left| \int_X e^{i\zeta(F_{n,0}+\dots+F_{n,k-1})} f_{(k-1)\nu+j} d\mu \right| \\
& \leq C|\zeta| w_n \sum_{j=1}^{n_1} \frac{q^{(k-2)\nu+n_1}}{q^{(k-1)\nu+j}} ((k-1)\nu+j)^\theta w_{((k-2)\nu+n_1)} \\
& \leq C|\zeta| w_n^2 n^\theta \sum_{j=1}^{n_1} \frac{q^{(k-2)\nu+n_1}}{q^{(k-1)\nu+j}} \\
(92) \quad & \leq C|\zeta| w_n^2 n^\theta \sum_{j=1}^{n_1} \rho^{\nu+j-n_1} \leq \frac{C\rho}{1-\rho} |\zeta| w_n^2 n^\theta \rho^{n_2}.
\end{aligned}$$

Similarly, we apply (44) to bound $\int (e^{i\zeta(F_{n,0}+\dots+F_{n,k-1})} - I_{n,k-1}) f_k f_{k'} d\mu$ and get:

$$(93) \quad \left| \int_X e^{i\zeta(F_{n,0}+\dots+F_{n,k-1})} F_{n,k}^2 d\mu - I_{n,k-1} \left(\int_X F_{n,k}^2 d\mu \right) \right| \leq C|\zeta| n^{\theta+\frac{1}{2}+\delta} e^{\rho n_2}.$$

Likewise inequality (45) implies

$$\begin{aligned}
& \left| \int_X (e^{i\zeta(F_{n,0}+\dots+F_{n,k-1})} - \mathbb{E}(e^{i\zeta(F_{n,0}+\dots+F_{n,k-1})})) F_{n,k}^3 d\mu \right| \\
& \leq CV(e^{i\zeta(F_{n,0}+\dots+F_{n,k-1})}) n^{3\delta} w_n^3 \rho^{n^\delta}.
\end{aligned}$$

One deduces

$$\left| \int_X e^{i\zeta(F_{n,0}+\dots+F_{n,k-1})} F_{n,k}^3 d\mu \right| \leq CV(\varphi) n^{1+3\delta} w_n^3 \rho^{n^\delta} + \left| \int_X F_{n,k}^3 d\mu \right|,$$

then by the lemma 5.2,

$$(94) \quad \left| \int_X e^{i\zeta(F_{n,0}+\dots+F_{n,k-1})} F_{n,k}^3 d\mu \right| \leq C(V(\varphi) n^{1+3\delta} w_n^3 \rho^{n^\delta} + n^{1/2+\delta} w_n^3 \ln^2(n)).$$

At last we have

$$(95) \quad \left| \int_X e^{i\zeta(F_{n,0}+\dots+F_{n,k-1})} \zeta^4 F_{n,k}^4 r(\zeta F_{n,k}) d\mu \right| \leq \zeta^4 \int_X F_{n,k}^4 d\mu \leq \zeta^4 w_n^4 n^{1+2\delta} \ln(n)^2$$

From (92), (93), (94) and (95) one deduces

$$\begin{aligned}
& \left| I_{n,k}(\zeta) - \left(1 - \frac{\zeta^2}{2} v_{n,k}^2\right) I_{n,k-1}(\zeta) \right| \\
& \leq C w_n^2 \rho^{-n^\delta} (|\zeta|^2 n^\theta + |\zeta|^3 n^{\theta+\frac{1}{2}+\delta}) + |\zeta|^3 w_n^3 n^{1/2+\delta} \ln^2(n) + \zeta^4 w_n^4 n^{1+2\delta} \ln(n)^2.
\end{aligned}$$

In the sum above, we keep only the last two terms, since we can take n big enough such that the two other terms are smaller than the last ones. \square

If X and Y are two real square integrable random variables, then $|\mathbb{E}(e^{iX}) - \mathbb{E}(e^{iY})| \leq \|X - Y\|_2$. Therefore, using (88) in claim 1, we have

$$(96) \quad |J_n(\zeta) - I_{n,p(n)}(\zeta)| \leq |\zeta| \|S_n - S'_n\|_2 \leq C |\zeta| w_n n^{\frac{1}{4}-\frac{1}{2}\delta} (\ln n)^{1/2},$$

then, by (96) and (90) of Lemma 5.4, we get

$$\begin{aligned}
|J_n(\zeta) - \prod_{k=1}^{p(n)} (1 - \frac{1}{2} \zeta^2 v_k^2)| &\leq |J_n(\zeta) - I_{n,p(n)}(\zeta)| + \sum_{k=0}^{p(n)-1} |I_{n,k}(\zeta) - (1 - \frac{\zeta^2}{2} v_k^2) I_{n,k-1}(\zeta)| \\
&\leq C [|\zeta| w_n n^{\frac{1}{4}-\frac{1}{2}\delta} (\ln n)^{1/2} + n^{\frac{1}{2}-\delta} |\zeta|^3 w_n^3 n^{1/2+\delta} \ln^2(n) + n^{\frac{1}{2}-\delta} \zeta^4 w_n^4 n^{1+2\delta} \ln(n)^2] \\
(97) \quad &\leq C [|\zeta| w_n n^{\frac{1}{4}} + |\zeta|^3 w_n^3 n \ln^2(n) + \zeta^4 w_n^4 n^{\frac{3}{2}+\delta} \ln(n)^2].
\end{aligned}$$

5.2.3. Approximation of the exponential by a product.

Below, ζ will be such that $|\zeta| v_k \leq 1$. This condition implies: $0 \leq 1 - \frac{1}{2} \zeta^2 v_k^2 \leq 1$.

Lemma 5.5. *If $(\rho_k)_{k \in J}$ is a finite family of real numbers in $[0, 1[$, then*

$$(98) \quad 0 \leq e^{-\sum_{k \in J} \rho_k} - \prod_{k \in J} (1 - \rho_k) \leq \sum_{k \in J} \rho_k^2, \text{ if } 0 \leq \rho_k \leq \frac{1}{2}, \forall k.$$

Proof. We have $\ln(1 - u) = -u - u^2 v(u)$, with $\frac{1}{2} \leq v(u) \leq 1$, for $0 \leq u \leq \frac{1}{2}$.

Writing $1 - \rho_k = e^{-\rho_k - \varepsilon_k}$, with $\varepsilon_k = -(\ln(1 - \rho_k) + \rho_k)$, the previous inequality implies $0 \leq \varepsilon_k \leq \rho_k^2$, if $0 \leq \rho_k \leq \frac{1}{2}$. We have: $e^{-\sum_{k \in J} \rho_k} - \prod_{k \in J} (1 - \rho_k) = e^{-\sum_{k \in J} \rho_k} (1 - e^{-\sum_{k \in J} \varepsilon_k})$.

Since $0 \leq \sum \varepsilon_k \leq \sum \rho_k^2$, if $0 \leq \rho_k \leq \frac{1}{2}$, and $1 - e^{-\sum \varepsilon_k} \leq \sum \varepsilon_k$, if $\sum_k \varepsilon_k \geq 0$, the following inequalities hold and imply (98):

$$0 \leq e^{-\sum \rho_k} - \prod (1 - \rho_k) \leq e^{-\sum \rho_k} \sum \varepsilon_k \leq \sum \varepsilon_k \leq \sum \rho_k^2, \text{ if } \rho_k \leq \frac{1}{2}, \forall k. \quad \square$$

We apply (98) with $\rho_k = \frac{1}{2} \zeta^2 v_k^2$. The condition $\zeta^2 v_k^2 \leq 1$ is satisfied by the assumption on ζ . In view of (86) it follows:

$$(99) \quad |e^{\frac{1}{2} \zeta^2 \sum v_k^2} - \prod_{k=0}^{p(n)-1} (1 - \frac{1}{2} \zeta^2 v_k^2)| \leq \frac{1}{4} \zeta^4 \sum_{k=0}^{p(n)-1} v_k^4 \leq C \zeta^4 w_n^4 n^{\frac{3}{2}+\delta}.$$

5.2.4. *Conclusion.* From (97) and (99), it follows:

$$(100) \quad |J_n(\zeta) - e^{-\frac{1}{2} \zeta^2 \sum_{k=0}^{p(n)-1} v_k^2}| \leq C [|\zeta| w_n n^{\frac{1}{4}} + |\zeta|^3 w_n^3 n \ln^2(n) + \zeta^4 w_n^4 n^{\frac{3}{2}+\delta} \ln^2(n)].$$

Replacing ζ by $\frac{\lambda}{\|S_n\|_2}$, we get:

$$\begin{aligned}
&| \int_X e^{i\lambda \frac{S_n(x)}{\|S_n\|_2}} d\mu(x) - e^{-\frac{1}{2} \frac{\lambda^2}{\|S_n\|_2^2} \sum_k v_k^2} | \\
&\leq C [|\lambda| \frac{w_n}{\|S_n\|_2} n^{\frac{1}{4}} + |\lambda|^3 \frac{w_n^3}{\|S_n\|_2^3} n \ln^2(n) + \lambda^4 \frac{w_n^4}{\|S_n\|_2^4} n^{\frac{3}{2}+\delta} \ln^2(n)].
\end{aligned}$$

Since $|e^{-a} - e^{-b}| \leq |a - b|$, for any $a, b \geq 0$, we have, by (87):

$$|e^{-\frac{1}{2} \lambda^2} - e^{-\frac{1}{2} \frac{\lambda^2}{\|S_n\|_2^2} \sum_{k=0}^{p(n)-1} v_k^2}| \leq \frac{1}{2} \frac{\lambda^2}{\|S_n\|_2^2} | \|S_n\|_2^2 - \sum_{k=0}^{p(n)-1} v_k^2 | \leq C \lambda^2 \frac{w_n^2}{\|S_n\|_2^2} n^{\frac{3}{4}} \ln n.$$

Finally we get the bound

$$(101) \quad \left| \int_X e^{i\lambda \frac{S_n(x)}{\|S_n\|_2}} d\mu(x) - e^{-\frac{1}{2}\lambda^2} \right| \leq E_1 + E_2 + E_3 + E_4$$

$$= C \left[|\lambda| \frac{w_n}{\|S_n\|_2} n^{\frac{1}{4}} + |\lambda|^3 \frac{w_n^3}{\|S_n\|_2^3} n \ln^2(n) + \lambda^4 \frac{w_n^4}{\|S_n\|_2^4} n^{\frac{3}{2}+\delta} \ln^2(n) + \lambda^2 \frac{w_n^2}{\|S_n\|_2^2} n^{\frac{3}{4}} \ln n \right].$$

To summarise we recall the origin of the different terms in the previous bound: E_1 comes from neglecting the sums on the small blocks, E_2 is the error of order 3 in the expansion, E_3 comes from the approximation of the exponential by the product, finally E_4 is the error from the replacement of $e^{-\frac{1}{2}\lambda^2}$ by $e^{-\frac{1}{2} \frac{\lambda^2}{\|S_n\|_2^2} \sum v_k^2}$.

Denote by Y_1 a r.v. with $\mathcal{N}(0, 1)$ -distribution. To simplify, we write $R_n := \frac{w_n}{\|S_n\|_2}$. The bound (101) reads:

$$(102) \quad H_{\frac{S_n}{\|S_n\|_2}, Y_1}(\lambda) = \left| \int_X e^{i\lambda \frac{S_n(x)}{\|S_n\|_2}} d\mu(x) - e^{-\frac{1}{2}\lambda^2} \right|$$

$$\leq C \left[|\lambda| R_n n^{\frac{1}{4}} + |\lambda|^3 R_n^3 n (\ln n)^2 + \lambda^4 R_n^4 n^{\frac{3}{2}+\delta} + \lambda^2 R_n^2 n^{\frac{3}{4}-\delta} \right].$$

$$(103) \quad \leq C \left[|\lambda| R_n n^{\frac{1}{4}} + |\lambda|^3 R_n^3 n^{1+\delta} + \lambda^4 R_n^4 n^{\frac{3}{2}+\delta} + \lambda^2 R_n^2 n^{\frac{3}{4}+\delta} \right].$$

Notice that δ can be taken arbitrary small. A change of its value modifies the generic constant C in the previous inequalities. We have obtained an inequality of the form

$$H_{\frac{S_n}{\|S_n\|_2}, Y_1}(\lambda) \leq C \sum_{i=1}^4 |\lambda|^{\alpha_i} R_n^{\alpha_i} n^{\gamma_i},$$

where the exponents are given by the previous inequality. In view of (74), it follows

$$d\left(\frac{S_n}{\|S_n\|_2}, Y_1\right) \leq \frac{C}{U_n} + C \sum_{i=1}^4 \alpha_i^{-1} U_n^{\alpha_i} R_n^{\alpha_i} n^{\gamma_i}.$$

Now, we optimize the choice of $U = U_n$. Under the condition

$$(104) \quad R_n = \frac{w_n}{\|S_n\|_2} \in C[n^{-\frac{7}{12}}, n^{-\frac{5}{12}}],$$

the largest term in the bound (101) is the fourth one. One choses $U_n = n^{-\frac{1}{4}} R_n^{-\frac{2}{3}}$. For the validity of the bounds, recall that we have to check $|\zeta| v_{n,k} \leq 1$. Since $v_{n,k} \leq C n^{\frac{1}{4} + \frac{1}{2}\delta} w_n$, and in view of the previous choice of U_n , this condition reduces to $R_n \leq n^{-\frac{1}{4}-\delta}$, which follows from (104).

Finally, under Condition (104), we get

$$d\left(\frac{S_n}{\|S_n\|_2}, Y_1\right) \leq C n^{\frac{1}{4}+\delta} \left(\frac{w_n}{\|S_n\|_2}\right)^{\frac{2}{3}}.$$

It means that the distribution of $\frac{S_n}{\|S_n\|_2}$ is close to the normal distribution if

$$\max_{j=1}^n \|f_j\|_\infty \ll n^{-\frac{3}{8}} \|S_n\|_2. \quad \square$$

REFERENCES

- [Be10] Beck (J.): Randomness of the square root of 2 and the giant leap, part 1,2: *Periodica Mathematica Hungarica* 60(2) (2010) p.137-242 and 62(2) (2011) p. 127-246.
- [BrU17] Bromberg (M.) and Ulcigrai (C.): A temporal central limit theorem for real-valued cocycles over rotations, preprint ArXiv:1705.06484
- [CoIsLe17] Conze (J.-P.), Isola (S.) and Le Borgne (S.): Diffusive behaviour of ergodic sums over rotations, preprint 2017, ArXiv:1705.10550
- [CoGu12] Conze (J.-P.), Gutkin (E.): On recurrence and ergodicity for geodesic flows on non-compact periodic polygonal surfaces. *Ergodic Theory Dynam. Systems*, 32(2): 491-515, 2012.
- [DoSa16] Dolgopyat (D.) and O. Sarig (O.): Temporal distributional limit theorems for dynamical systems. *Journal of Statistical Physics* (2016), p. 1-34.
- [Fe] Feller (W.): *An introduction to probability theory and its application, Vol II, second edition.*
- [HaLi30] Hardy (G.H.) and Littlewood (J.E.): Some problems of diophantine approximation: a series of cosecants, *Bulletin of the Calcutta Mathematical Society*, no. 20 (1930), p. 251-266.
- [Hu09] Huvneers (F.): Subdiffusive behavior generated by irrational rotations, *Ergodic Theory Dynam. Systems* 29 (2009), no. 4, p. 1217-1233.
- [Ki98] Kitchens (B.): *Symbolic dynamics. One-sided, two-sided and countable state Markov shifts* Universitext. Springer-Verlag, Berlin, 1998.
- [Le98] Lezaud (P.): Chernoff-type bound for finite Markov chains, *Ann. Appl. Probab.* 8 (1998), no. 3, p. 849-867.

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