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JEAN-PIERRE CONZE AND STÉPHANE LE BORGNE

IRMAR - UMR 6625, F-35000 Rennes, France

ABSTRACT. Let $x \mapsto x + \alpha$ be a rotation on the circle and let φ be a step function. We denote by $\varphi_n(x)$ the corresponding ergodic sums $\sum_{j=0}^{n-1} \varphi(x + j\alpha)$. Under an assumption on α, for example when α has bounded partial quotients, and a Diophantine condition on the discontinuity points of φ, we show that $\varphi_n / \|\varphi_n\|_2$ is asymptotically Gaussian for n in a set of density 1. The method is based on decorrelation inequalities for the ergodic sums taken at times q_k, where the q_k's are the denominators of α.

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1. Introduction

Let us consider an irrational rotation $x \mapsto x + \alpha \bmod 1$ on $X = \mathbb{R}/\mathbb{Z}$. By the Denjoy-Koksma inequality, the ergodic sums $\varphi_L(x) = \sum_0^{L-1} \varphi(x+j\alpha)$ of a centered BV (bounded variation) function φ are uniformly bounded along the sequence (q_n) of denominators of α. But, besides, one has a stochastic behaviour at a certain scale along other sequences (L_n). In a sense the process defined by the above sums $S_L\varphi(x)$ presents more complexity than the ergodic sums under the action of hyperbolic maps for which a central limit theorem is often satisfied. We propose a quantitative analysis of this phenomenon.

Several papers have been devoted to this topic. M. Denker and R. Burton (1987), M. Lacey (1993), M. Weber (2000) and other authors proved the existence of functions whose ergodic sums over rotations satisfy a CLT after self-normalization. D. Volný and P. Liardet in 1997 showed that, when α has unbounded partial quotients, for a dense G_δ set of functions f in the class of absolutely continuous, or Lipschitz continuous or differentiable functions, the distributions of the random variables $c_n^{-1} \sum_{j=0}^{n-1} f \circ T^j$, $c_n \uparrow \infty$ and $c_n/n \to 0$, are dense in the set of all probability measures on the real line.

Most often in these works the functions that are dealt with are not explicit. Here we consider ergodic sums of simple functions such as step functions. Let us mention the following related papers. For $\psi := 1_{[0,\frac{1}{2}[} - 1_{[\frac{1}{2},0[}$, F. Huveneers [Hu09] studied the existence of sequences $(L_n)_{n\in\mathbb{N}}$ such that $S_{L_n}\psi$ after normalization is asymptotically normally distributed. In [CoIsLe17] it was shown that, when α has unbounded partial quotients, along some subsequences the ergodic sums of some step functions φ can be approximated by a Brownian motion.

Here we will use as in [Hu09] a method based on decorrelation inequalities which applies in particular to the bounded type case (bpq), i.e. when the sequence (a_n) of partial quotients of α is bounded. It relies on an abstract central limit theorem valid under some suitable decorrelation conditions. If φ is a step function, we give conditions which insure that for n in a set of density 1, the distribution of $S_n\varphi/\|S_n\varphi\|_2$ is close to a normal distribution.

Beside the remarkable recent "temporal" limit theorems for rotations (see [Be10], [BrUl17], [DoSa16]), this shows that, even if it is in a weak sense, a "spatial" asymptotic normal distribution can also be observed.

The results are presented in Section 2. There are based on the decorrelation of the ergodic sums taken at times q_k and on an abstract central limit theorem, whose proofs are given in Sections 4 and 5.

An important point is the control of the variance $\|S_n\varphi\|_2^2$ for n belonging to a set of density 1, at least in the case of α with bounded partial quotients. In the special case where α is the golden ratio, this information can be improved.

2. Variance of the ergodic sums

Notation The uniform measure on $\mathbb{T}^1$ identified with $X = [0,1[$ is denoted by μ. The arguments of the functions are taken modulo 1. For a 1-periodic function φ, we denote

by $V(\varphi)$ the *variation* of φ computed for its restriction to the interval $[0,1[$ and use the shorthand BV for "bounded variation". BV_0 denotes the space of BV functions φ on the circle such that $\mu(\varphi)=0$. By C we denote a numerical constant whose value may change from a line to the other or inside a line.

The number $\alpha=[0;a_1,a_2,\ldots]$ is an irrational number in $[0,1[$, with partial quotients (a_n) and denominators (q_n): Let $\mathcal{C}$ be the class of centered BV functions. If φ belongs to $\mathcal{C}$, its Fourier coefficients $c_r(\varphi)$ satisfy:

$$c_r(\varphi)=\frac{\gamma_r(\varphi)}{r}, r\neq 0, \text{ with } K(\varphi):=\sup_{r\neq 0}|\gamma_r(\varphi)|<+\infty. \tag{1}$$

The class $\mathcal{C}$ contains the step functions with a finite number of discontinuities.

The ergodic sums $\sum_{j=0}^{N-1}\varphi(x+j\alpha)$ will be denoted by $\varphi_N(x)$. Their Fourier series are

$$\varphi_N(x):=\sum_{j=0}^{N-1}\varphi(x+j\alpha)=\sum_{r\neq 0}\frac{\gamma_r(\varphi)}{r}\,e^{\pi i(N-1)r\alpha}\,\frac{\sin\pi Nr\alpha}{\sin\pi r\alpha}\,e^{2\pi irx}. \tag{2}$$

If φ belongs to $\mathcal{C}$, then so do the sums φ_N and we have

$$|c_r(\varphi_N)|\leq\frac{K(\varphi)}{|r|}\,\frac{|\sin\pi Nr\alpha|}{|\sin\pi r\alpha|}\leq\frac{N\,K(\varphi)}{|r|}.$$

If φ is a BV function, then so is φ_N and $\mathrm{V}(\varphi_N)\leq NV(\varphi)$.

2.1. **Reminders on continued fractions.**

For $u\in\mathbb{R}$, $\|u\|$ denotes its distance to the integers: $\|u\|:=\inf_{n\in\mathbb{Z}}|u-n|=\min(\{u\},1-\{u\})\in[0,\frac{1}{2}]$. Recall that

$$2\|x\|\leq|\sin\pi x|\leq\pi\|x\|,\ \forall x\in\mathbb{R}.$$

Let $\alpha\in[0,1[$ be an irrational number. Then, for each $n\geq 1$, we write $\alpha=\dfrac{p_n}{q_n}+\dfrac{\theta_n}{q_n}$, where p_n and q_n are the numerators and denominators of α. Recall that

$$\frac{1}{a_{n+1}+2}\leq\frac{q_n}{q_{n+1}+q_n}\leq q_n\|q_n\alpha\|=q_n|\theta_n|\leq\frac{q_n}{q_{n+1}}=\frac{q_n}{a_{n+1}q_n+q_{n-1}}\leq\frac{1}{a_{n+1}}, \tag{3}$$

$$\|k\alpha\|\geq\|q_{n-1}\alpha\|\geq\ \frac{1}{q_n+q_{n-1}}\geq\ \frac{1}{2q_n},\ \text{for } 1\leq k<q_n, \tag{4}$$

$$\|q_n\alpha\|=(-1)^n(q_n\alpha-p_n),\ \theta_n=(-1)^n\|q_n\alpha\|,\ \alpha=\frac{p_n}{q_n}+(-1)^n\frac{\|q_n\alpha\|}{q_n}. \tag{5}$$

For $n\geq 1$ we put

$$m=m(n):=\ell,\quad \text{if } n\in[q_\ell,\,q_{\ell+1}[. \tag{6}$$

If α has bounded partial quotients ($K=\sup a_n<\infty$), then there is a constant $\lambda>0$ such that $\dfrac{\ln n}{m(n)}\in[\lambda,\,\ln(K+1)]$.

Thereafter we will need the following assumption which is satisfied by a.e. α:

Hypothesis 1. *There are two constants $A \geq 1$, $p \geq 0$ such that*

$$a_n \leq A\, n^p, \forall n \geq 1. \tag{7}$$

Recall that every integer $n \geq 1$ can be represented as follows (Ostrowski's expansion):

$$n = \sum_{k=0}^{m(n)} b_k\, q_k, \text{with } 0 \leq b_0 \leq a_1 - 1,\ 0 \leq b_k \leq a_{k+1} \text{ for } 1 \leq k < m,\ 1 \leq b_m \leq a_{m+1}. \tag{8}$$

Indeed, if $n \in [q_m,\ q_{m+1}[$, we can write $n = b_m q_m + r$, with $1 \leq b_m \leq a_{m+1}$, $0 \leq r < q_m$. and by iteration, we get (8).

In this way, we can associate to every n its coding, which is a word $b_0...b_m$, with $b_j \in \{0, 1, ..., a_j\}$, $j = 0, ..., m$. Let us call "admissible" a finite word $b_0...b_m$, $b_j \in \{0, 1, ..., a_j\}$, with $b_m \neq 0$, such that for two consecutive letters b_j, b_{j+1}, we have $b_j b_{j+1} \neq u a_{j+1}$, with $u \neq 0$.

Let us show that the Ostrowski's expansion of an integer n is admissible. The proof is by induction. Let n be in $[q_m,\ q_{m+1}[$. We start the construction of the Ostrowski's expansion of n as above. Now the following steps of the algorithm yield the Ostrowski's expansion of $n - b_m q_m$ (excepted some zero's which might be added at the end). Since $n - b_m q_m \in [0,\ q_m[$, the Ostrowski's expansion of $n - b_m q_m$ is admissible. It remains to check that, if $b_m = a_{m+1}$, then $b_{m-1} = 0$. If $b_{m-1} \neq 0$, we would have $n \geq a_{m+1} q_m + u q_{m-1} \geq a_{m+1} q_m + q_{m-1} = q_{m+1}$, a contradiction.

Therefore, if we associate to an admissible word the integer $n = b_0 + b_1 q_1 + ... + b_m q_m$, there is a 1 to 1 correspondence between the Ostrowski's expansion of the integers n, when n runs in $\mathbb{N}$, and the set of (finite) admissible words starting with $b_0 \leq a_1 - 1$.

For n given by (8), putting $n_0 = b_0$, $n_k = \sum_{t=0}^{k} b_t\, q_t$, for $k \leq m(n)$ and

$$f_k(x) := \sum_{i=0}^{b_k - 1} \varphi_{q_k}(x + (n_{k-1} + i q_k)\alpha), \tag{9}$$

the ergodic sum φ_n reads:

$$\begin{aligned} \varphi_n(x) &= \sum_{k=0}^{m(n)} \sum_{j=n_{k-1}}^{n_k - 1} \varphi(x + j\alpha) = \sum_{k=0}^{m(n)} \sum_{j=0}^{b_k\, q_k - 1} \varphi(x + n_{k-1}\alpha + j\alpha) \\ &= \sum_{k=0}^{m(n)} \sum_{i=0}^{b_k - 1} \varphi_{q_k}(x + (n_{k-1} + i q_k)\alpha) = \sum_{k=0}^{m(n)} f_k(x). \end{aligned} \tag{10}$$

By convention, the expression $\sum_{i=0}^{b_k - 1} \varphi_{q_k}(x + (n_{k-1} + i q_k)\alpha)$ is taken to be 0, if $b_k = 0$.

If q is a denominator of α and φ is a BV function, one has (Denjoy-Koksma inequality) :

$$\|\varphi_q\|_\infty = \sup_x |\sum_{i=0}^{q-1} \varphi(x + i\alpha)| \leq V(\varphi). \tag{11}$$

One can also show that if φ satisfies (1) then $\|\varphi_{q_n}\|_2 \leq 2\pi\, K(\varphi)$. By Denjoy-Koksma inequality, we have $\|f_k\| \leq b_k V(\varphi) \leq a_{k+1} V(\varphi$.

2.2. **Lower bound for the variance.**

Let n be in $[q_{\ell-1},\, q_\ell[$. The variance at time n is

$$\begin{aligned}\|\varphi_n\|_2^2 &= 2\sum_{k>1} |c_k(\varphi)|^2 \frac{(\sin \pi nk\alpha)^2}{(\sin \pi k\alpha)^2} \geq 2\sum_{j=1}^{\ell} |c_{q_j}(\varphi)|^2 \frac{(\sin \pi nq_j\alpha)^2}{(\sin \pi q_j\alpha)^2} \\ &\geq \frac{8}{\pi^2}\sum_{j=1}^{\ell} |c_{q_j}(\varphi)|^2 \frac{\|nq_j\alpha\|^2}{\|q_j\alpha\|^2} \geq \frac{8}{\pi^2}\sum_{j=1}^{\ell} |\gamma_{q_j}(\varphi)|^2 a_{j+1}^2 \|nq_j\alpha\|^2.\end{aligned}$$

In the previous inequalities, we kept only indices that belong to the sequence (q_n) .

For $\varphi = \varphi^0 = \{.\} - \frac{1}{2}$, we have $|c_{q_j}(\varphi^0)| = \frac{1}{2\pi q_j}$; hence, for φ^0, the lower bound

$$\|\varphi_n^0\|_2^2 \geq \frac{2}{\pi^4}\sum_{j=1}^{\ell} a_{j+1}^2 \|nq_j\alpha\|^2 \geq \frac{2}{\pi^4}\,\delta^2 \sum_{j=1}^{\ell} 1_{\|nq_j\alpha\|\geq\delta}, \text{ for } 0<\delta<\frac{1}{2}. \tag{12}$$

More generally, with the numerical constant $c = \frac{8}{\pi^2}$,

$$\|\varphi_n\|_2^2 \geq c\,\delta^2 \sum_{j=1}^{\ell} |\gamma_{q_j}(\varphi)|^2\, a_{j+1}^2\, 1_{\|nq_j\alpha\|\geq\delta} \geq c\,\delta^2 \sum_{j=1}^{\ell} |\gamma_{q_j}(\varphi)|^2\, 1_{\|nq_j\alpha\|\geq\delta}. \tag{13}$$

Bounds for the mean variance

An upper bound for the variance and a lower bound for the mean of the variance are shown in [CoIsLe17]):

Proposition 2.1. *There are constants $C, c > 0$ such that, if φ satisfies (1), then*

$$\|\varphi_n\|_2^2 \leq CK(\varphi)^2 \sum_{j=0}^{\ell} a_{j+1}^2,\ \forall n \in [q_\ell, q_{\ell+1}[, \tag{14}$$

and the mean of the variances $\langle D\varphi\rangle_n := \frac{1}{n}\sum_{k=0}^{n-1} \|\varphi_k\|_2^2$ satisfies:

$$\langle D\varphi\rangle_n \geq c\sum_{j=0}^{\ell-1} |\gamma_{q_j}(\varphi)|^2\, a_{j+1}^2,\ \forall n \in [q_\ell, q_{\ell+1}[. \tag{15}$$

The proposition implies that, if $v_\ell \leq q_{\ell+1}$ is an integer such that $\|\varphi_{v_\ell}\|_2 = \max_{k<q_{\ell+1}} \|\varphi_k\|_2$, then $\|\varphi_{v_\ell}\|_2^2 \geq c\sum_{j=0}^{\ell-1} |\gamma_{q_j}|^2 a_{j+1}^2$. If the partial quotients of α are bounded, under a condition on the sequence (γ_k), using the results below, it can be deduced that the behaviour of $S_{v_\ell}/\|S_{v_\ell}\|_2$ for the indices giving the record variances is approximately Gaussian. But our goal is to obtain such a behaviour for a large set of integers n. To do it, we need to obtain a lower bound of the variance for such a large set.

Bounds for the variance for a large set of integers

We will assume that φ satisfies the condition:

$$\exists \eta, \eta_0 > 0 \text{ such that } \mathrm{Card}\{j \leq N : |\gamma_{q_j}(\varphi)| \geq \eta\} \geq \eta_0 N, \, \forall N \geq 1. \tag{16}$$

This condition is satisfied for example by the following functions:
$\varphi^0(\,\cdot\,) = \{\,\cdot\,\} - \frac{1}{2}$, $\varphi(\frac{r}{s}, \,\cdot\,) = 1_{[0,\frac{r}{s}[} - \frac{r}{s}$, for $r, s \in \mathbb{N}, 0 < r < s$,
$\varphi(\beta, \,\cdot\,) = 1_{[0,\beta[} - \beta$, for almost every β (by an argument of equirepartition).

For $j < \ell$, we estimate how many times for $n \leq q_\ell$, we have $\{nq_j\alpha\} \in I_\delta := [0, \delta] \cup [1-\delta, 1]$. Since $\|n\, q_j\alpha\| = \{n\, q_j\alpha\}$ or $1 - \{n\, q_j\alpha\}$, depending whether $\{n\, q_j\alpha\} \in [0, \frac{1}{2}[$ or $\in]\frac{1}{2}, 1[$, we have $1_{\|nq_j\alpha\|<\delta} = I_\delta(\{nq_j\alpha\})$.

Remark: We are looking for values of n such that $\sum_{j=1}^{\ell} 1_{I_\delta}(\{n\, q_j\, \alpha\})$ is small. Of course there are special values of n, like $n = q_t$, such that this quantity is big, as shown by Denjoy-Koksma inequality Let us check it directly.

Recall that for any irrational α, there are $\lambda \in]0, 1[$ and $C > 0$ such that

$$q_n/q_m \leq C\lambda^{m-n}, \; \forall m \geq n. \tag{17}$$

We have $\|q_t q_j\alpha\| \leq q_j \|q_t\alpha\| < q_j/q_{t+1}$, for $j \leq t$, and $\|q_t q_j\alpha\| \leq q_t\|q_j\alpha\| < q_t/q_{j+1}$, for $j > t$. By (17), this implies $\|q_t\, q_j\alpha\| \leq C\lambda^{|t-j|}$, $\forall j, t$; hence, for $\delta > 0$, $\|q_t\, q_j\alpha\| \leq \delta$ if $|t - j| \geq M := \ln(\delta^{-1})/\ln(C^{-1}\, \delta^{-1})$.

For the complementary I_δ^c of I_δ on the circle, it follows: $\sum_{j\geq 1} 1_{I_\delta^c}(\{q_t\, q_j\, \alpha\}) \leq M, \; \forall t \geq 1$. This gives a bound for $n = q_t$ when j varies, as expected.

Nevertheless, we will show that $\sum_{j=1}^{\ell} 1_{I_\delta}(\{n\, q_j\, \alpha\})$ is small for a large set of values of n.

Lemma 2.2. *For every $\delta \in]0, \frac{1}{2}[$ and every interval of integers $I = [N_1, N_2[$ of length L, we have*

$$\sum_{n=N_1}^{N_2-1} 1_{I_\delta}(\{n\, q_j\, \alpha\}) \leq 20\, (\delta + q_{j+1}^{-1})\, L, \forall j \text{ such that } q_{j+1} \leq 2L. \tag{18}$$

Proof. For a fixed j and $0 \leq N_1 < N_2$, let us describe the behaviour of the sequence $(\|n\, q_j\alpha\|, n = N_1, ..., N_2 - 1)$.

Recall that (modulo 1) we have $q_j\alpha = \theta_j$, with $\theta_j = (-1)^j\|q_j\alpha\|$ (see (5)). We treat the case j even (hence $\theta_j > 0$). The case j odd is analogous.

Therefore the problem is to count how many times, for j even, we have $\{n\, \theta_j\} < \delta$ or $1 - \delta < \{n\, \theta_j\}$.

We start with $n = n_1 := N_1$. Putting $w(j, 1) := \{n_1\, \theta_j\}$, we have $\{n\, \theta_j\} = w(j, 1) + (n - n_1)\theta_j$, for $n = n_1, n_1 + 1, ..., n_2 - 1$, where n_2 is such that $w(j, 1) + (n_2 - 1 - n_1)\, \theta_j < 1 < w(j, 1) + (n_2 - n_1)\, \theta_j$.

Putting $w(j, 2) := \{n_2\, \theta_j\}$, the previous inequality shows that $w(j, 2) = w(j, 1) + (n_2 - n_1)\, \theta_j - 1 < \theta_j$.

Starting now from $n = n_2$, we have $\{n\,\theta_j\} = w(j,2)+(n-n_2)\theta_j$ for $n = n_2, n_2+1, ..., n_3-1$, where n_3 is such that:

$$w(j,2) + (n_3 - 1 - n_2)\,\theta_j < 1 < w(j,2) + (n_3 - n_2)\,\theta_j.$$

Again we put $w(j,3) := \{n_3\,\theta_j\} = \{w(j,2)+(n_3-n_2)\,\theta_j\} = w(j,2)+(n_3-n_2)\,\theta_j - 1 < \theta_j$.

We iterate this construction and obtain a sequence $n_1 < n_2 < ...n_{R(j)-1} < n_{R(j)}$ (with $n_{R(j)-1} < N_2 \le n_{R(j)}$) such that

$$\{n\theta_j\} = w(j,i) + (n - n_i)\theta_j,\ \forall n \in [n_i, n_{i+1}[,$$
$$w(j,i) + (n_{i+1} - 1 - n_i)\,\theta_j < 1 < w(j,i) + (n_{i+1} - n_i)\,\theta_j,$$

where $(w(j,i), i = 1, ..., R(j))$ is defined recursively by $w(j,i+1) = \{w(j,i)+(n_{i+1}-n_i)\,\theta_j\}$ and satisfies $w(j,i) < \theta_j$, for every i.

Since $(n_{i+1} - n_i + 1)\,\theta_j \ge w(j,i) + (n_{i+1} - n_i)\,\theta_j > 1$ for $i \ne 1$ and $i \ne R(j)$, we have $n_{i+1} - n_i \ge \theta_j^{-1} - 1$, for each $i \ne 1, R(j)$. This implies $R(j) \le \dfrac{L}{\theta_j^{-1} - 1} + 2$.

For each i, the number of integers $n \in [n_i, n_{i+1} - 1[$ such that $\{n\theta_j\} \in [0, \delta[\cup]1-\delta, 1[$ is bounded by $2(1 + \delta\,\theta_j^{-1})$. (This number is less than 2 if $\delta < \theta_j$.)

Altogether, the number of integers $n \in I$ such that $\{n\theta_j\} \in [0, \delta[\cup]1-\delta, 1[$ is bounded by

$$2R(j)(1 + \delta\,\theta_j^{-1}) \le 2(\frac{L}{\theta_j^{-1} - 1} + 2)\,(1 + \delta\,\theta_j^{-1})$$
$$\le 4\,(L + \theta_j^{-1})\,(\delta + \theta_j)\ \le 4\,(L + 2q_{j+1})\,(\delta + q_{j+1}^{-1}).$$

If $q_{j+1} \le L$, the previous term at right is less than $20\,L\,(\delta + q_{j+1}^{-1})$. This shows (18). □

Lemma 2.3. *Suppose that φ satisfies Condition (16). Then there are positive constants c_1, c_2 (not depending on δ) such that, if I is an interval $[N_1, N]$ of length L and ℓ such that $q_\ell \le 2L$, for every $\delta \in]0, \frac{1}{2}[$, the subset of I*

$$V(I,\delta) := \{n \in I : \|\varphi_n\|_2 \ge c_1\,\delta\,\sqrt{\ell})\} \tag{19}$$

has a complementary in I satisfying:

$$\operatorname{Card} V(I,\delta)^c \le c_2\,(\delta + \ell^{-1})\,L. \tag{20}$$

Proof. Let $\zeta = \frac{1}{2}\eta_0$, where η_0 is the constant in (16), and let

$$A = A(I,\delta) := \{n \in I : \sum_{j=0}^{\ell-1} 1_{I_\delta}(\{n\,q_j\,\alpha\}) \le \zeta\,\ell\}.$$

Let us bound the density $L^{-1}\operatorname{Card} A(I,\delta)^c$ of the complementary of $A(I,\delta)$ in I by counting the number of values of n in I such that $\|nq_j\alpha\| < \delta$ in an array indexed by (j,n).

By summation of (18) for $j = 0$ to $\ell - 1$ and the definition of $A(I, \delta)$, the following inequalities are satisfied:

$$20\,(\delta\ell + \sum_{0\le j\le \ell-1} q_{j+1}^{-1})\,L \ge \sum_{0\le j\le \ell-1,\ n\in I} 1_{I_\delta}(\{n\,q_j\,\alpha\})$$
$$\ge \sum_{n\in A^c}\ \sum_{0\le j\le \ell-1} 1_{I_\delta}(\{n\,q_j\,\alpha\}) \ge \sum_{n\in A^c} \zeta\,\ell = \zeta\,\ell\,\mathrm{Card}\,A^c.$$

Let C be the finite constant $C := \sum_{j=0}^{\infty} q_j^{-1}$. Then the set $A = A(I, \delta)$ satisfies:

$$\mathrm{Card}\,A^c \le 20\,\zeta^{-1}\,(\delta + C\ell^{-1})\,L, \tag{21}$$

$$n \in A \Rightarrow \mathrm{Card}\{j \le \ell - 1 : \|nq_j\alpha\| \ge \delta\} \ge (1 - \zeta)\,\ell. \tag{22}$$

In view of (22) and Condition (16), we have, for $n \in A$:

$$\mathrm{Card}\big(\{j \le \ell - 1 : \|nq_j\alpha\| \ge \delta\} \bigcap \{j : |\gamma_{q_j}(\varphi)| \ge \eta\}\big) \ge (1 - (\zeta + 1 - \eta_0))\,\ell = \frac{1}{2}\eta_0\,\ell.$$

Putting $c_2 := 20\,\zeta^{-1}\max(1, C)$ and $c_1 = \frac{1}{2}c\,\eta^2\,\eta_0$ (c_1, c_2 do not depend on δ), this implies by (13):

$$\|\varphi_n\|_2^2 \ge \frac{1}{2}c\,\delta^2\,\eta^2\,\eta_0\,\ell = c_1^2\,\delta^2\,\ell,\ \forall n \in A,\ \text{and}\ \mathrm{Card}\,A \ge 1 - c_2\,(\delta + \ell^{-1})\,L; \tag{23}$$

hence $A = A(I, \delta) \subset V(I, \delta)$ and therefore $V(I, \delta)$ satisfies (26). □

Lemma 2.4. *Suppose that φ satisfies Condition (16). The subset W of $\mathbb{N}$ defined by*

$$W := \{n \in \mathbb{N} :\ \|\varphi_n\|_2 \ge c_1\,\sqrt{m(n)}/\sqrt{\ln m(n)}\} \tag{24}$$

satisfies

$$\lim_{N\to\infty} \frac{\mathrm{Card}\,\big(W \cap [0, N[\big)}{N} = 1. \tag{25}$$

Proof. By (17), there is a fixed integer u_0 such that $q_{m(N)-u} \le \frac{1}{2}q_{m(N)}$, for $u_0 \le u < m(N)$.

Let us take $u_N = \lfloor \frac{1}{2}m(N) \rfloor$ and let $N_1 := q_{m(N)-u_N}$. By this choice of N_1, the condition of Lemma (2.3) is satisfied by $[N_1, N]$ and $\ell = m(N)$ for N big enough, since

$$q_{m(N)} \le 2(q_{m(N)} - q_{m(N)-u_N}) \le 2(N - N_1).$$

If $n \in [N_1, N]$, then $m(N) - u_N = m(N_1) \le m(n) \le m(N)$.

Let $B = \{n \in [N_1, N] : \|\varphi_n\|_2 \ge c_1\sqrt{m(n)}/\sqrt{\ln m(n)}\}$.

If n is in the complementary B^c of B in $[N_1, N]$, then we have:

$$\|\varphi_n\|_2 \le c_2\sqrt{m(n)}/\sqrt{\ln m(n)} \le c_2\sqrt{m(N)}/\sqrt{\ln m(N_1)}.$$

Therefore, by Lemma (2.3) with $\delta = (\ln m(N_1))^{-\frac{1}{2}}$, we have

$$\mathrm{Card}\,B^c \le c_2\,((\ln m(N_1))^{-\frac{1}{2}} + m(N)^{-1})\,(N - N_1) \tag{26}$$

Now, $W^c \cap [0, N[\subset [0, N_1[\bigcup B^c$, hence: given $\varepsilon > 0$, for N big enough,

$$\begin{aligned}\frac{\text{Card}\left(W^c \cap [0, N[\right)}{N} &\leq \frac{N_1}{N} + c_2\left((\ln m(N_1))^{-\frac{1}{2}} + m(N)^{-1}\right) \\ &= \frac{q_{m(N)-u_N}}{N} + c_2\left((\ln(m(N) - u_N))^{-\frac{1}{2}} + m(N)^{-1}\right) \leq \varepsilon. \quad \square\end{aligned}$$

Remark: if α is bpq, then $m(n)$ is of order $\ln n$ and the density of W satisfies, for some constant $C > 0$,

$$\frac{\text{Card}\left(W \cap [0, N[\right)}{N} \geq 1 - C(\ln \ln N)^{-\frac{1}{2}}. \tag{27}$$

2.3. **A special case: the golden ratio.**

For a special class of quadratic irrationals, the previous results can be reinforced. Suppose that α is in the class $\mathbb{Q}_0$ of irrationals such that for some integer $a \geq 1$, $a_n = a, \forall n \geq 1$, or equivalently that α is the root > 1 of the equation $x^2 - ax - 1 = 0$. Then the variance $\|\varphi_n\|_2^2$ has the right order of magnitude for n in a big set of integers. More precisely, the following bounds hold for the ergodic sums of φ under the rotation by α:

Theorem 2.5. *If φ satisfies Condition (16), there are positive constants ζ, η_1, η_2, R such that, for N big enough:*

$$\#\{n \leq N : \ \eta_1 \ln n \leq \|\varphi_n\|_2^2 \leq \eta_2 \ln n\} \geq N\,(1 - R\,N^{-\zeta}). \tag{28}$$

Proof. For simplicity, we write the proof when $\alpha = [1; 11111...]$ is the golden ratio. Its partial quotients are equal to 1 and its denominators are given by the Fibonacci sequence $(q_n) : q_0 = 1$, $q_1 = 1$ and $q_{n+1} = q_n + q_{n-1}$ for $n \geq 1$. We have:

$$q_n = \frac{2+\alpha}{5}\alpha^n + \frac{3-\alpha}{5}\left(\frac{-1}{\alpha}\right)^n. \tag{29}$$

There exists $c > 0$ such that $\frac{1}{c}\alpha^{-n} \leq \|q_n\alpha\| \leq c\alpha^{-n}$. The following relations hold:

$$\alpha^n + (-\alpha)^{-n} \in \mathbb{Z}, \ \alpha^n = -(-\alpha)^{-n} \text{mod } 1, \ d(\mathbb{Z}, \alpha^n) = \alpha^{-n}, n \geq 1. \tag{30}$$

For every integer $n \in [q_\ell, \, q_{\ell+1}[$, its Ostrowski expansion (corresponding to the Fibonacci sequence) has the form

$$n = \sum_{j=0}^{\ell} b_j\, q_j, \ \text{with} \ \ 0 \leq b_j \leq 1 \text{ for } 1 \leq j < \ell, \ b_\ell = 1, \tag{31}$$

and is such that in the "word" $b_0...b_\ell$ two successive b_i's are not both equal to 1 ("admissible" word). As we have seen in Subsection 2.1, every admissible word $b_0...b_\ell$ starting with 0 is the Ostrowski expansion of an integer $n < q_{\ell+1}$.

Let us consider the subshift of finite type corresponding to admissible words

$$X := \{b = (b_i)_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} \text{ such that } b_i b_{i+1} = 0, \forall i\}$$

and the associated dynamical system (X, σ, μ), where σ is the left shift and μ the probability measure on X with maximal entropy $(\ln(\alpha))$.

Observe that, for $n \geq 3$, the measure of a cylinder of length n in X can take three values $\alpha^{-n+1}/(\alpha+2)$, $\alpha^{-n+2}/(\alpha+2)$ and $\alpha^{-n+3}/(\alpha+2)$. This implies that, if a subset E of X is a union of cylinders of length n, then

$$\frac{1}{3}\,\mu(E) \leq \#\{b \text{ cylinder of length } n :\ b \subset E\}\,\alpha^{-n} \leq 3\,\mu(E). \tag{32}$$

We will use it, via an inequality of large deviations, to get an estimate of the number of values of $n < q_\ell$ giving a big variance.

Let $0 \leq \kappa_0 \leq \ell$. The choice of κ_0 will be specified later. Let $0 \leq j \leq \ell$.

We have $\|q_i q_j \alpha\| \leq q_j/q_i \leq \alpha^{j-i}$, for $i \geq j$, and $\|q_i q_j \alpha\| \leq q_i/q_j \leq \alpha^{i-j}$, for $i \leq j$. It follows:

$$\begin{aligned} &|\|\sum_{i=1}^{\ell} b_i q_i q_j \alpha\| - \|\sum_{i=j-\kappa_0}^{j+\kappa_0} b_i q_i q_j \alpha\|| \leq \|\sum_{i=j+\kappa_0+1}^{\ell} b_i q_i q_j \alpha\| + \|\sum_{i=1}^{j-\kappa_0-1} b_i q_i q_j \alpha\| \\ &\leq \sum_{i=j+\kappa_0+1}^{\ell} \frac{1}{\alpha^{i-j}} + \sum_{i=1}^{j-C-1} \frac{1}{\alpha^{j-i}} \leq \frac{2}{\alpha^{\kappa_0+1}-\alpha^{\kappa_0}}. \end{aligned} \tag{33}$$

Using (29) we get the following expression for $q_i q_j$:

$$q_i q_j = \frac{1}{5}\alpha^{i+j+2} + \frac{2-\alpha}{5}\left(\frac{-1}{\alpha}\right)^{i+j} + \frac{(-1)^i}{5}\left(\alpha^{j-i} + (-\alpha)^{i-j}\right).$$

It follows that, modulo 1, $\displaystyle\sum_{i=j-\kappa_0}^{j+\kappa_0} b_i q_i q_j \alpha = A + B$, with

$$A := \sum_{i=j-\kappa_0}^{j+\kappa_0} \frac{b_i}{5}\alpha^{i+j+3} + \sum_{i=j-\kappa_0}^{j+\kappa_0} b_i \frac{2-\alpha}{5}(-1)^{i+j}\alpha^{1-i-j},$$

$$\begin{aligned} B &:= \frac{(-1)^j}{5}\sum_{k=-\kappa_0+1}^{\kappa_0+1}\left(b_{j+1-k}(-1)^{k+1} + b_{k+j-1}\right)\alpha^k \\ &= \frac{(-1)^j}{5}\sum_{k=-\kappa_0+1}^{0}\left(b_{j+1-k}(-1)^{k+1} + b_{k+j-1}\right)\alpha^k + \frac{(-1)^j}{5}\sum_{k=1}^{\kappa_0+1}\left(b_{j+1-k}(-1)^{k+1} + b_{k+j-1}\right)\alpha^k. \end{aligned}$$

Since (from (30))

$$\sum_{k=1}^{\kappa_0+1}\left(b_{j+1-k}(-1)^{k+1} + b_{k+j-1}\right)\alpha^k = \sum_{k=1}^{\kappa_0+1}\left(b_{j+1-k} + b_{k+j-1}(-1)^{k+1}\right)\alpha^{-k} \bmod 1,$$

modulo $\frac{1}{5}$ we have $B = \gamma_{b,\kappa_0}(j)$, with

$$\gamma_{b,\kappa_0}(j) = \frac{(-1)^j}{5}\left(\sum_{k=-\kappa_0+1}^{0}\left(b_{j+1-k}(-1)^{k+1} + b_{k+j-1}\right)\alpha^k + \sum_{k=1}^{\kappa_0+1}\left(b_{j+1-k} + b_{k+j-1}(-1)^{k+1}\right)\alpha^{-k}\right),$$

and

$$\sum_{i=j-\kappa_0}^{j+\kappa_0} b_i q_i q_j \alpha = A + B = A + \gamma_{b,\kappa_0}(j).$$

These computations and (33) show that, there is a constant κ_0' such that, for κ_0 big enough and $j \geq \kappa_0$,

$$d(\sum_{i=0}^{\ell} b_i q_i q_j \alpha - \gamma_{b,\kappa_0}(j), \mathbb{Z}/5) \leq {\kappa_0}'(2\kappa_0+1)\alpha^{-\kappa_0} \leq \delta. \tag{34}$$

For every $\delta \in]0, \frac{1}{10}[$ and every interval $[N_1, N_2]$ of length L, we have by a slight extension of Lemma 2.2:

$$\#\{n \in [N_1, N_2] :\ d(nq_j\alpha, \mathbb{Z}/5) \leq \delta\} \leq 100\,(\delta + q_{j+1}^{-1})L, \forall j \text{ such that } q_{j+1} \leq 2L. \tag{35}$$

Consider the set $A_\delta \subset X$ defined by

$$A_\delta := \{b :\ d(\gamma_{b,\kappa_0}(\kappa_0), \mathbb{Z}/5) \geq 2\delta\}.$$

From (35) (for $j = \kappa_0, N_1 = 0, N_2 = q_{\kappa_0} - 1$), it follows

$$\#\{b \text{ cylinder of length } \kappa_0 :\ d(\sum_{i=0}^{\kappa_0-1} b_i q_i q_{\kappa_0}\alpha, \mathbb{Z}/5) \leq 3\delta\} \leq 600\,\delta q_{\kappa_0},$$

which implies, in terms of measure for the subshift by (32):

$$\mu\{b :\ d(\sum_{i=0}^{\kappa_0-1} b_i q_i q_{\kappa_0}\alpha, \mathbb{Z}/5) \leq 3\delta\} \leq 1800\,\delta. \tag{36}$$

But if $b \in X$ is such that $d(\sum_{i=0}^{\kappa_0-1} b_i q_i q_{\kappa_0}\alpha, \mathbb{Z}/5) > 3\delta$, then by (34) $b \in A_\delta$; hence by (36):

$$\mu(A_\delta) \geq 1 - 1800\,\delta. \tag{37}$$

For μ-almost $b \in X$ we have $\ell^{-1}\sum_{j=0}^{\ell} \mathbf{1}_{A_\delta}(\sigma^j(b)) \to \mu(A_\delta)$. A more precise information is given by an inequality of large deviations for irreducible Markov chains with finite state space (see [Le98], Theorem 3.3.; here we apply it for the Markov chain deduced from X with state space the set of words of length $2\kappa_0 + 1$ in X): for $\varepsilon \in]0,1[$, there are two positive constants $R = R(\varepsilon)$, $r = r(\varepsilon)$ such that

$$\mu\{\frac{1}{\ell}\sum_{j=0}^{\ell} \mathbf{1}_{A_\delta}(\sigma^j(b)) \leq \mu(A_\delta)(1-\varepsilon)\} \leq Re^{-r\ell}.$$

According to the definition of A_δ, it means that

$$\mu\,(b :\ \#\{j \leq \ell :\ d(\gamma_{b,\kappa_0}(j), \mathbb{Z}/5) \geq 2\delta\} \leq \mu(A_\delta)(1-\varepsilon)\,\ell) \leq Re^{-r\ell}$$

and (by (34) satisfied for $j \geq \kappa_0$) it implies

$$\mu(b :\ \#\{j \leq \ell :\ d(\sum_{i=0}^{\ell} b_i q_i q_j \alpha, \mathbb{Z}/5) \geq \delta\} \leq \mu(A_\delta)(1-\varepsilon)\,\ell - \kappa_0) \leq Re^{-r\ell},$$

so that the number of words b of length ℓ such that

$$\#\{j \leq \ell :\ d(\sum_{i=0}^{\ell} b_i q_i q_j \alpha, \mathbb{Z}/5) \leq \delta\} \leq \mu(A_\delta)(1-\varepsilon)\ell - \kappa_0$$

is smaller than $3Re^{-r\ell}\alpha^\ell$ by (32).

If $\ell > \kappa_0/(\mu(A_\delta)(1-\varepsilon)\varepsilon)$ the number words b of length ℓ such that

$$\#\{j \leq \ell :\ d(\sum_{i=0}^{\ell} b_i q_i q_j \alpha, \mathbb{Z}/5) \leq \delta\} \leq \mu(A_\delta)(1-\varepsilon)^2\ell$$

is smaller than $3Re^{-r\ell}\alpha^\ell$.

The $q_{\ell+1}$ first integer numbers n are represented by words of length ℓ. The proportion of them for which $\|\varphi_n\|_2^2 \geq 2\pi^{-4}\delta^2\mu(A_\delta)(1-\varepsilon)^2\ell$ is thus larger than $1 - 3Re^{-r\ell}$. We have, for some $\zeta > 0$,

$$q_{\ell+1}^{-1}\,\#\{n < q_{\ell+1} :\ \|\varphi_n\|_2^2 \geq 2\pi^{-4}\delta^2\mu(A_\delta)(1-\varepsilon)^2\ell\} \geq 1 - 3Re^{-r\ell} \geq 1 - R\,q_{\ell+1}^{-\zeta}.$$

More generally, let $\varphi \in \mathcal{C}$ satisfy Condition (16), i.e.,

$$\liminf \frac{1}{\ell}\#\{j \leq \ell :\ |\gamma_{q_j}(\varphi)| \geq \eta\} > \eta_0 > 0.$$

For a lower bound of the variance we use (13) and (37). We can choose δ and ε such that $\eta_0 + \mu(A_\delta)(1-\varepsilon)^2 > 1$, for ℓ large enough, the number of integers $n < q_{\ell+1}$ such that

$$\frac{\|\varphi_n\|_2^2}{\ell} \geq c\eta^2\delta^2(\eta_0 + \mu(A_\delta)(1-\varepsilon)^2 - 1),$$

is larger than $q_{\ell+1}(1 - R\,q_{\ell+1}^{-\zeta})$. This proves Theorem 2.5. □

3. **A central limit theorem and its application to rotations**

3.1. **Decorrelation and CLT.**

Decorrelation between partial ergodic sums

Proposition 3.1. *If ψ and φ are BV centered functions, under Hypothesis 1 there are constants $C, \theta_1, \theta_2, \theta_3$ such that the following decorrelation inequalities hold for every $1 \leq n \leq m \leq \ell$:*

$$|\int_X \psi\,\varphi_{b_n q_n}\,d\mu| \leq C\,\mathrm{V}(\psi)\ \mathrm{V}(\varphi)\,\frac{n^{\theta_1}}{q_n}\,b_n, \tag{38}$$

$$|\int_X \psi\,\varphi_{b_n q_n}\varphi_{b_m q_m}\,d\mu| \leq C\,\mathrm{V}(\psi)\,\mathrm{V}(\varphi)^2\,\frac{m^{\theta_2}}{q_n}\,b_n b_m, \tag{39}$$

$$|\int_X \psi\,\varphi_{b_n q_n}\varphi_{b_m q_m}\varphi_{b_\ell q_\ell}\,d\mu| \leq C\,\mathrm{V}(\psi)\,\mathrm{V}(\varphi)^3\,\frac{\ell^{\theta_3}}{q_n}\,b_n b_m b_\ell. \tag{40}$$

The proposition is proved in Section 4. From the proposition we will deduce a convergence toward a gaussian distribution under a variance condition. The method is like in [Hu09], with here a quantitative form, bounding the distance to the normal distribution.

Recall that, if X, Y are two r.r.v.'s defined on the same probability space, their mutual (Kolmogorov) distance in distribution is defined by: $d(X,Y) = \sup_{x\in\mathbb{R}} |\mathbb{P}(X \le x) - \mathbb{P}(Y \le x)|$. Below Y_1 denotes a r.v. with a normal distribution $\mathcal{N}(0,1)$.

Proposition 3.2. *Let $(q_k)_{1\le k\le n}$ be an increasing sequence of positive integers satisfying the lacunary condition: there exists $\rho < 1$ such that*

$$q_k/q_m \le C\,\rho^{m-k},\ 1 \le k < m \le n. \tag{41}$$

Let $(f_k)_{1\le k\le n}$ be real centered BV functions and u_k constants such that $\|f_k\|_\infty \le u_k$ and

$$\mathrm{V}(f_k) \le C\,u_k\,q_k,\ 1 \le k \le n. \tag{42}$$

Suppose that there are finite constants $C > 0$ and $\theta \in \mathbb{R}$ such that the following conditions hold for every centered BV function:

$$|\int_X \psi\, f_k\, d\mu| \le C\,\mathrm{V}(\psi)\,u_k\,\frac{k^\theta}{q_k},\ 1 \le k \le n, \tag{43}$$

$$|\int_X \psi\, f_k\, f_m\, d\mu| \le C\,V(\psi)\,u_k\,u_m\,\frac{m^\theta}{q_k},\ 1 \le\ k \le m \le n, \tag{44}$$

$$|\int_X \psi\, f_k\, f_m\, f_t\, d\mu| \le C\,V(\psi)\,u_k\,u_m\,u_t\,\frac{t^\theta}{q_k},\ 1 \le\ k \le m \le t \le n. \tag{45}$$

Then, for every $\delta > 0$, there is a constant $C(\delta) > 0$ such that

$$d(\frac{f_1+\cdots+f_n}{\|f_1+\cdots+f_n\|_2}, Y_1) \le C(\delta)\,(\frac{\max_{j=1}^n u_j}{\|f_1+\cdots+f_n\|_2})^{\frac{2}{3}}\,n^{\frac{1}{4}+\delta}. \tag{46}$$

The proposition is proved in Section 5.

We apply this abstract proposition to an irrational rotation: the q_k's will be the denominators of α and the functions f_k (defined by (9)) will be translated ergodic sums of a BV function φ.

More precisely, let $\varphi \in \mathcal{C}$ and φ_n the ergodic sum $\sum_{j=0}^{n-1}\varphi(x+j\alpha)$. We use the decomposition $\varphi_n = \sum_{k\le m(n)} f_k$ given by (10), with the notation $m(n) = \ell$, where ℓ is such that $n \in [q_\ell,\, q_{\ell+1}[$. (The notation n in Proposition 3.2 has been replaced by ℓ and in the application we have $\ell = m(n)$).

As already observed, we have $\|f_k\| \le b_k V(\varphi) \le a_{k+1}V(\varphi)$. Up to a fixed factor, the constant u_k in the statement of the proposition can be taken to be $a_{k+1} \le k^p$, for some constant $p > 0$, by Hypothesis 1.

Therefore, in (46), we have $\max_{j=1}^{m(n)} u_j/\|f_1+\cdots+f_{m(n)}\|_2 \le m(n)^p/\|\varphi_n\|_2$. It follows:

$$d(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1) \le C(\delta)\,m(n)^{\frac{1}{4}+\frac{2}{3}p+\delta}\,\|\varphi_n\|_2^{-\frac{2}{3}}.$$

If we restrict to a set of values of n such that the b_k's in the Ostrowski expansion of n are uniformly bounded, or (without restriction on n) if we take α bpq, we get the bound:

$$d(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1) \leq C(\delta)\, m(n)^{\frac{1}{4}+\delta}\, \|\varphi_n\|_2^{-\frac{2}{3}}.$$

Therefore, in order to have an effective rate in the previous formula, we need to bound from below the variance $\|\varphi_n\|_2^2$.

For example, when α is bpq, we have the implication: if $\|\varphi_n\|_2^2 \geq cm(n)$ for a constant $c > 0$, then $d(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1) \leq c^{\frac{1}{3}}\, C(\delta)\, m(n)^{-\frac{1}{12}+\delta}$.

As we have seen, there are two methods to obtain a lower bound for the variance. This leads to the following applications to the CLT.

1) From (14) and (15) one can get an information on the density of a set of integers n for which $\|\varphi_n\|_2^2$ is large.

For a given BV function φ, define $c_0 \geq 0$ by

$$c_0 = c_0(\varphi, \alpha) := \liminf_{\ell \geq 1} \frac{c \sum_{j=0}^{\ell-1} |\gamma_{q_j}(\varphi)|^2\, a_{j+1}^2}{C\, K(\varphi)^2 \sum_{j=0}^{\ell} a_{j+1}^2}, \tag{47}$$

where c and C are the constants in (14) and (15). The a_k's depend on α and so does c_0.

Theorem 3.3. *Let α be an irrational number satisfying Hypothesis 1. Let φ be a BV centered function such that $c_0(\varphi, \alpha) > 0$. Then there is a set $E_0(\ell)$ in $[1, q_\ell]$ with $\frac{1}{q_\ell}\#E_0(\ell) \geq \frac{1}{2}c_0(\varphi, \alpha)$ such that, for $n \in E_0(\ell)$, the ergodic sums φ_n for the rotation by α satisfy:*

$$\|\varphi_n\|_2^2 \geq \frac{1}{2} c_0\, C\, K(\varphi)^2 \sum_{j=1}^{\ell} a_j^2.$$

Moreover, for every $\delta > 0$, there is a constant $C_0(\delta) > 0$ such that

$$d(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1) \leq C_0(\delta)\, (\frac{\max_{j=1}^{\ell} a_j}{(\sum_{j=1}^{\ell} a_j^2)^{\frac{1}{2}}})^{\frac{2}{3}}\, \ell^{\frac{1}{4}+\delta},\ \forall n \in E_0(\ell).$$

In particular, if $\max_{j=1}^{\ell} a_j / \min_{j=1}^{\ell} a_j, \leq C\, \ell^{\gamma}$, with $\gamma < \frac{1}{8}$, then

$$d(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1) \leq C_0(\delta)\, \ell^{\frac{8\gamma-1}{12}+\delta} << 1,\ \forall n \in E_0(\ell).$$

If α is bpq, the above bound holds with $\gamma = 0$.

Proof. Let $E_0(\ell)$ be the set $\{k \leq q_\ell : \|\varphi_k\|_2^2 \geq \frac{1}{2}c_0(\varphi, \alpha)\, C\, K(\varphi)^2 \sum_{j=0}^{\ell} a_{j+1}^2\}$. For ℓ big enough, its density satisfies $\frac{1}{q_\ell}\#E_0(\ell) \geq \frac{1}{2}c_0(\varphi, \alpha)$.

Indeed, let c_1 be such that $c_0 - \frac{1}{4}c_0^2 < c_1 < c_0$. For ℓ big enough, by (14), (15) and the definition of $c_0(\varphi,\alpha)$, we have for $E_0(\ell)$ denoted simply by E_0:

$$c_1\, C\, K(\varphi)^2 \sum_{j=0}^{\ell} a_{j+1}^2 \le c \sum_{j=0}^{\ell-1} |\gamma_{q_j}(\varphi)|^2\, a_{j+1}^2 \le \frac{1}{q_\ell} \sum_{k=1}^{q_\ell} \|\varphi_k\|_2^2,$$

$$\frac{1}{q_\ell} \sum_{k=1}^{q_\ell} \|\varphi_k\|_2^2 \le \frac{\#E_0}{q_\ell}\, C\, K(\varphi)^2 \sum_{j=0}^{\ell} a_{j+1}^2 + \frac{\#E_0^c}{q_\ell}\, \frac{c_0}{2} C\, K(\varphi)^2 \sum_{j=0}^{\ell} a_{j+1}^2;$$

hence: $c_1 \le \frac{1}{q_\ell}\#E_0 + \frac{1}{q_\ell}\frac{c_0}{2}\,\#E_0^c = \frac{1}{q_\ell}\#E_0 + \frac{1}{q_\ell}\frac{c_0}{2}\,(q_\ell - \#E_0)$, which implies: $\frac{\#E_0}{q_\ell} \ge (c_1 - \frac{1}{2}c_0)/(1-\frac{1}{2}c_0) > \frac{1}{2}c_0$.

Now the conclusion of the theorem follow from the discussion after Proposition 3.2. □

The drawback of this first method is that it yields only a subset of integers of positive density.

2) Another method to show a CLT is to use 2.4 or, for an irrational α like the golden ratio, Theorem 2.5.

We state the result in an increasing order of generality for α and decreasing strength of the conclusion.

Theorem 3.4. *Let φ be a function in $\mathcal{C}$ satisfying Condition (16).*

1) Let α be in $\mathbb{Q}_0$ (for example the golden ratio). For a constant $b > 0$, let

$$V_b := \{n \ge 1 :\ \|\varphi_n\|_2 \ge b\sqrt{\log n}\}.$$

Then, if b is small enough, the density of V_b satisfies:

$$\mathrm{Card}(V_b \bigcap [1,N]) \ge N\,(1 - R\,N^{-\zeta}). \tag{48}$$

and the following CLT with rate holds along V_b: for $\delta_0 \in]0, \frac{1}{2}[$, there is a constant $K(\delta_0)$ such that for $n \in V_b$,

$$d(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1) \le K(\delta_0)\ \log n^{-\frac{1}{12}+\delta_0}. \tag{49}$$

2) Suppose that α is bpq. For a constant $b > 0$, let

$$W_b := \{n \ge 1 :\ \|\varphi_n\|_2 \ge b\,(\log\log n)^{-\frac{1}{2}}\sqrt{\log n}\}.$$

Then, if b is small enough, W_b has density 1 in the integers and the following CLT with rate holds along W: for $\delta_0 \in]0, \frac{1}{2}[$, there is a constant $K(\delta_0)$ such that (49) holds for $n \in W_b$.

3) Suppose that α is such that $a_n \le Cn^p$. For a constant $b > 0$, let

$$Z_b := \{n \in \mathbb{N} :\ \|\varphi_n\|_2 \ge b\sqrt{m(n)}/\sqrt{\ln m(n)}\} \tag{50}$$

satisfies

$$\lim_{N\to\infty} \frac{\text{Card}\left(W \cap [0,N[\right)}{N} = 1. \tag{51}$$

If b is small enough, Z_b has density 1 in the integers and for $\delta_0 \in]0, \frac{1}{2}[$, there is a constant $K(\delta_0)$ such that, if $p < \frac{1}{8}$, the following CLT with rate holds along Z_b:

$$d(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1) \leq K(\delta_0)\, m(n)^{-\frac{1}{12}+\frac{2}{3}p+\delta_0},\ \forall n \in Z_b. \tag{52}$$

Proof. We use Proposition 3.2 with f_n defined by (9) (see comments after the statement of Proposition 3.2).

For a lower bound of the variance we use Theorem 2.5 in case 1) and Lemma 2.4 in case 2).

For 3), we have, if $n \in Z_b$,

$$d(\frac{\varphi_n}{\|\varphi_n\|_2}, Y_1) \leq C(\delta)\,(\frac{m(n)^p}{\|\varphi_n\|_2})^{\frac{2}{3}}\, m(n)^{\frac{1}{4}+\delta} \leq C(\delta)\,(m(n)^{p-\frac{1}{2}}\,(\ln m(n))^{\frac{1}{2}})^{\frac{2}{3}}\, m(n)^{\frac{1}{4}+\delta}$$
$$\leq C(\delta)\, m(n)^{-\frac{1}{12}+\frac{2}{3}p+\delta}\,(\ln m(n))^{\frac{1}{3}} << 1.$$

The factor $(\ln m(n))^{\frac{1}{3}}$ can be absorbed in the factor $m(n)^{\delta}$ by taking δ smaller. □

Remark 1. The previous CLT are written with a self-normalisation. In the first case (i.e., when the a_k's are fixed), let us consider the sequence of variables $(\varphi_n/\sqrt{n})$, with a fixed normalisation. Then, for $n \in V_b$, the accumulation points of the sequence of its distributions are Gaussian non degenerated with variances belonging to a compact interval.

3.2. **Application to step functions, examples and counter-example.**

If φ belongs to the class $\mathcal{C}$ of centered BV functions, with Fourier series $\sum_{r\neq 0} \frac{\gamma_r(\varphi)}{r}\, e^{2\pi i r.}$, to be able to apply Theorems 3.3 and 3.4 we have to check Condition (16), that is:

$$\exists \eta, \eta_0 > 0 \text{ such that } \text{Card}\{j \leq N : |\gamma_{q_j}(\varphi)| \geq \eta\} \geq \eta_0\, N,\ \forall N \geq 1.$$

This is clear for φ^0 below. For step functions, the behaviour of the variance is linked to the diophantine properties of the discontinuities with respect to α.

Example 1 Consider the function $\varphi^0(x) = \{x\} - \frac{1}{2} = \frac{-1}{2\pi i} \sum_{r\neq 0} \frac{1}{r}\, e^{2\pi i r x}$. In this example, the numbers γ_r are all equal and no discussion on the relations between the γ_k's and the a_k's is necessary.

Example 2 Let $\varphi := 1_{[0,\frac{1}{2}[} - 1_{[\frac{1}{2},1[} = \sum_r \frac{2}{\pi i(2r+1)}\, e^{2\pi i(2r+1).}$ (here $\gamma_{q_k} = 0$ if q_k is even, $= \frac{2}{\pi i}$ if q_k is odd). Clearly, (16) is satisfied.

Let us consider now more general step functions. Let $\varphi := \sum_{j\in J} v_j\,(1_{I_j} - \mu(I_j))$, with $I_j = [u_j, w_j[$. Its Fourier coefficients are

$$c_r = \sum_{j\in J} v_j \frac{e^{-2\pi i r w_j} - e^{-2\pi i r u_j}}{2\pi i r} = \sum_{j\in J} \frac{v_j}{\pi r}\, e^{-\pi i r(u_j+w_j)}\, \sin(\pi r(u_j - w_j)),\ r \neq 0.$$

$$(53)\quad q_\ell \le n < q_{\ell+1} \Rightarrow \langle D\varphi\rangle_n \ge C \sum_{k=1}^{\ell} |\sum_{j\in J} v_j\, e^{-2\pi i q_k u_j}\, (e^{2\pi i q_k (u_j - w_j)} - 1)|^2 a_{k+1}^2.$$

If the a_k's are bounded, then an argument of equirepartition for almost every choice of the parameters $(u_j,\ w_j)$ implies the bound $\langle D\varphi\rangle_n \ge c \ln n$ and Condition (16). Let us show it for particular cases.

Example 3 Consider the following functions which depend only on one parameter: $\varphi^\beta = 1_{[0,\beta[} - \beta = \sum_{r\neq 0} \frac{1}{\pi r} e^{-\pi i r\beta} \sin(\pi r\beta)\ e^{2\pi i r.}$. We have

$$\langle D\varphi^\beta\rangle_n \ge c_1 \sum_{k=0}^{\ell-1} a_{k+1}^2 \sin^2(\pi q_k\beta) \ge 4C_1 \sum_{k=0}^{\ell-1} a_{k+1}^2 \|q_k\beta\|^2\,, \quad \forall n \in [q_\ell, q_{\ell+1}[\,.$$

Since (q_k) is a strictly increasing sequence of integers, for almost every β in $\mathbb{T}$, the sequence $(q_k\beta)$ is uniformly distributed modulo 1 in $\mathbb{T}^1$. We have:

$$\lim_N \frac{1}{N} \sum_{k=1}^{N} \sin^2(\pi q_k\beta) = \frac{1}{2},\ \text{for a.e. } \beta.$$

Recall that a way to proof it is to use Weyl equirepartition criterium and the law of large numbers for orthogonal bounded random variables.

More generally, it can be shown that, if $1 \le a_n \le n^\gamma$, with $\gamma < \frac{1}{6}$, then:

$$\lim_N \frac{\sum_{k=1}^N a_k^2 \sin^2(\pi q_k\beta)}{\sum_{k=1}^N a_k^2} = \frac{1}{2},\ \text{for almost every } \beta.$$

It is an easy consequence of the following lemma:

Lemma 3.5. *Let (u_n) be a sequence of real numbers such that $1 \le u_n \le n^\gamma$, with $0 \le \gamma < \frac{1}{3}$. If (X_n) is a sequence of bounded orthogonal random variables on a probability space (Ω, μ), then*

$$(54)\quad \lim_N \frac{\sum_{k=1}^N u_k X_k}{\sum_{k=1}^N u_k} = 0,\ \textit{for a.e. } \omega.$$

Proof. By orthogonality, we have $\dfrac{\int_\Omega |\sum_{k=1}^N u_k X_k|^2 d\mu}{(\sum_{k=1}^N u_k)^2} \le N^{2\gamma-1}$.

Setting $R_N(\omega) := \frac{\sum_{k=1}^N u_k X_k(\omega)}{\sum_{k=1}^N u_k}$, it implies $\sum_{n=1}^\infty \|R_{n^p}\|_2^2 < +\infty$, if $p(1-2\gamma) > 1$, hence $\lim_n R_{n^p}(\omega) = 0$, for a.e. ω, if $\gamma < \frac{1-\frac{1}{p}}{2}$. Therefore, it suffices to show that:

$$\lim_n \sup_{0\le \ell < n^{p+1} - n^p} |R_{n^p+\ell}(\omega) - R_{n^p}(\omega)| = 0.$$

We have $|R_{n^p+\ell}(\omega) - R_{n^p}(\omega)|$

$$
\begin{aligned}
&= |\frac{(\sum_{k=1}^{n^p} u_k)(\sum_{k=n^p+1}^{n^p+\ell} u_k\, X_k) - (\sum_{k=n^p+1}^{n^p+\ell} u_k)(\sum_{k=1}^{n^p} u_k\, X_k)}{(\sum_{k=1}^{n^p+\ell} u_k)(\sum_{k=1}^{n^p} u_k)}| \\
&= |\frac{\sum_{k=n^p+1}^{n^p+\ell} u_k\, X_k}{\sum_{k=1}^{n^p+\ell} u_k} - \frac{(\sum_{k=n^p+1}^{n^p+\ell} u_k)(\sum_{k=1}^{n^p} u_k\, X_k)}{(\sum_{k=1}^{n^p+\ell} u_k)(\sum_{k=1}^{n^p} u_k)}| \\
&= |\frac{\sum_{k=n^p+1}^{n^p+\ell} u_k\, X_k}{\sum_{k=1}^{n^p+\ell} u_k}| + |\frac{(\sum_{k=n^p+1}^{n^p+\ell} u_k)(\sum_{k=1}^{n^p} u_k\, X_k)}{(\sum_{k=1}^{n^p+\ell} u_k)(\sum_{k=1}^{n^p} u_k)}| \\
&\leq C\frac{\sum_{k=n^p+1}^{n^p+\ell} u_k}{\sum_{k=1}^{n^p+\ell} u_k} + C\frac{\sum_{k=n^p+1}^{n^p+\ell} u_k}{\sum_{k=1}^{n^p} u_k} \leq 2C\frac{\sum_{k=n^p+1}^{(n+1)^p} u_k}{\sum_{k=1}^{n^p} u_k}.
\end{aligned}
$$

Therefore, $\sup_{0\leq\ell<n^{p+1}-n^p} |R_{n^p+\ell}(\omega) - R_{n^p}(\omega)| \leq Cpn^{p\gamma-1} \to 0$, if $p\gamma - 1 < 0$.

Taking p=3, It follows that (54) holds if $\gamma < \min(\frac{1}{p}, \frac{1-\frac{1}{p}}{2}) = \frac{1}{3}$. □

Therefore, using Theorem 3.3, we can conclude that for a.e. β, if α is such that $a_n \leq n^\gamma$, with $\gamma < \frac{1}{8}$, then $(\varphi^\beta_{n_k}/\|\varphi^\beta_{n_k}\|_2)$ converges in distribution toward $\mathcal{N}(0,1)$ along a subsequence (n_k) with positive upper density.

Observe that, if α is not bpq, there are many β's which do not satisfy the previous equipartition property. Let $\beta = \sum_{n\geq 0} b_n q_n \alpha \bmod 1$, $b_n \in \mathbb{Z}$, be the so-called Ostrowski expansion of β, where q_n are the denominators of α. It can be shown that, if $\sum_{n\geq 0} \frac{|b_n|}{a_{n+1}} < \infty$, then $\lim_k \|q_k\beta\| = 0$. There is an uncountable set of β's satisfying the previous condition if α is not bpq, but ergodicity of the cocycle holds if β is not in the countable set $\mathbb{Z}\alpha + \mathbb{Z}$.

Example 5. Let φ be the step function: $\varphi = \varphi(\beta,\gamma,\cdot) = 1_{[0,\ \beta]} - 1_{[\gamma,\ \beta+\gamma]}$. The Fourier coefficients are $c_r(\varphi) = \frac{2i}{\pi}\frac{1}{r}e^{-\pi i r(\beta+\gamma)}\sin(\pi r\beta)\sin(\pi r\gamma)$. We have

$$\|\widehat{\varphi}_{q_k}\|_2^2 \;=\; \frac{4}{\pi^2}\sum_{r\neq 0}\frac{1}{r^2}\,|\sin(\pi r q_k\beta)|^2\,|\sin(\pi r q_k\gamma)|^2.$$

As above, since (q_k) is a strictly increasing sequence of integers, for almost every (β,γ) in $\mathbb{T}^2$, the sequence $(q_k\beta, q_k\gamma)$ is uniformly distributed in $\mathbb{T}^2$. We have for a.e. (β,γ):

$$
\begin{aligned}
\lim_n \frac{1}{n}\sum_{k=1}^n \|\widehat{\varphi}_{q_k}\|_2^2 \;&=\; \frac{4}{\pi^2}\sum_{r\neq 0}\lim_n\frac{1}{n}\sum_{k=1}^n\frac{|\sin(\pi r q_k\beta)|^2\,|\sin(\pi r q_k\gamma)|^2}{r^2} \\
&=\; \frac{4}{\pi^2}\sum_{r\neq 0}\int\int\frac{|\sin(\pi r y)|^2\,|\sin(\pi r z)|^2}{r^2}\,dy\,dz = \frac{1}{3}.
\end{aligned}
$$

In both last examples, the uniform distribution argument shows that Condition (16) is satisfied for almost every choice of the parameters.

A counter-example For a parameter $\gamma > 0$, let the sequence $(a_n)_{n\geq 1}$ be defined by

$$a_n = \lfloor n^\gamma \rfloor \text{ if } n \in \{2^k \ : \ k \geq 0\}, \ = 1 \text{ if } n \notin \{2^k \ : \ k \geq 0\}.$$

Let α be the number which has $(a_n)_n$ for sequence of partial quotients. In order not to have to discuss the relations between the q_n and the γ_{q_n}, let us take for φ the function φ^0 defined above for which the numbers $\gamma_k = \frac{-1}{2\pi i}$, $\forall k$. For a given ℓ, let n_ℓ be the largest of the integers $n < q_{\ell+1}$ such that

$$\mathrm{Var}(\varphi_n) = \max_{k<q_{\ell+1}} \mathrm{Var}(\varphi_k).$$

We have

$$\mathrm{Var}(\varphi_{n_\ell}) \geq c \sum_{k=0}^{\ell-1} |\gamma_{q_k}|^2 a_{k+1}^2 \geq c \sum_{j=0}^{\lfloor \log_2(\ell) \rfloor} 2^{2\gamma j} \geq c\ell^{2\gamma}.$$

In the sum $\varphi_{n_\ell}(x) = \sum_{k=0}^{\ell} \sum_{j=0}^{b_k\, q_k-1} \varphi(x + N_{k-1}\alpha + j\alpha)$ defined in (10), we can isolate the indices k for which $k+1$ is a power of 2 (for the other indices $a_{k+1} = 1$) and write $\varphi_{n_\ell}(x) = U_\ell + V_\ell$ with

$$U_\ell = \sum_{p=1}^{\lfloor \log_2(\ell) \rfloor} \sum_{j=0}^{b_{2^p}\, q_{2^p}-1} \varphi(x + N_{2^p-1}\alpha + j\alpha),$$
$$V_\ell = \sum_{k\in[0,\ell]\cap\{a_{k+1}=1\}} \sum_{j=0}^{b_k\, q_k-1} \varphi(x + N_{k-1}\alpha + j\alpha).$$

We will see in (77) that the variance of a sum where b_k equals 0 or 1 is bounded as follows

$$\mathrm{Var}\Big(\sum_{k\in[0,\ell]\cap\{a_{k+1}=1\}} \sum_{j=0}^{b_k\, q_k-1} \varphi(x + N_{k-1}\alpha + j\alpha)\Big) \leq C\ell \log(\ell).$$

On the other side, we also have

$$\Big| \sum_{p=1}^{\lfloor \log_2(\ell) \rfloor} \sum_{j=0}^{b_{2^p-1}\, q_{2^p-1}-1} \varphi(x + N_{2^p-2}\alpha + j\alpha)\Big| \leq \sum_{p=1}^{\lfloor \log_2(\ell) \rfloor} a_{2^p} V(\varphi) \leq C \sum_{p=1}^{\lfloor \log_2(\ell) \rfloor} 2^{\gamma p} \leq C\ell^\gamma.$$

The previous bounds imply that

$$\frac{\varphi_{n_\ell}}{\|\varphi_{n_\ell}\|_2} = \frac{U_\ell}{\|\varphi_{n_\ell}\|_2} + \frac{V_\ell}{\|\varphi_{n_\ell}\|_2} \tag{55}$$

with $\|\varphi_{n_\ell}\|_2 \geq c\ell^\gamma$, $\|U_\ell\|_\infty \leq C\ell^\gamma$, $\|V_\ell\|_2 \leq (C\ell\log(\ell))^{1/2}$. Thus, if $\gamma > 1/2$, one has

$$\left\| \frac{U_\ell}{\|\varphi_{n_\ell}\|_2} \right\|_\infty \leq \frac{C}{c} \text{ and } \left\| \frac{V_\ell}{\|\varphi_{n_\ell}\|_2} \right\|_2 \to 0,$$

and the limit points of the distributions of the $\frac{\varphi_{n_\ell}}{\|\varphi_{n_\ell}\|_2}$ have all their supports included in $[-\frac{C}{c}, \frac{C}{c}]$. It implies that none is gaussian.

4. **Proof of the decorrelation (Proposition 3.1)**

To prove the decorrelation properties 3.1, we first truncate the Fourier series of the ergodic sums φ_q. For functions in $\mathcal{C}$, we easily control the remainders and it suffices to treat the case of trigonometric polynomials.

4.1. **Some preliminary inequalities.**

Lemma 4.1. *For each irrational α and $n \geq 1$, we have*

$$q_{n+1}/q_{n+k} \leq C\,\rho^k,\ \forall k \geq 1,\ \textit{with}\ \rho = (\sqrt{5}-1)/2, C = (3\sqrt{5}+5)/2. \tag{56}$$

Proof. For $n \geq 1$ fixed, set $r_0 = q_n$, $r_1 = q_{n+1}$, $r_{k+1} = r_k + r_{k-1}$, for $k \geq 1$. It follows immediately by induction that $q_{n+k} \geq r_k$, $\forall k \geq 0$.

Denote $c = \frac{1}{\sqrt{5}}$ and let $\lambda_1 = \frac{\sqrt{5}}{2} + \frac{1}{2}$, $\lambda_2 = -\frac{\sqrt{5}}{2} + \frac{1}{2}$ be the two roots of the polynomial $\lambda^2 - \lambda - 1$. Since $\lambda_j^{\ell+1} = \lambda_j^\ell + \lambda_j^{\ell-1}$ for each $j = 1, 2$ and $\ell \geq 1$, we obtain by induction that:

$$q_{n+k} \geq r_k = c\,\lambda_1^k\,(q_{n+1} - \lambda_2 q_n) - c\,\lambda_2^k\,(q_{n+1} - \lambda_1 q_n),\ k \geq 0, n \geq 1. \tag{57}$$

Take $k \geq 1$. Since $\lambda_2 < 0$ and $\left|1 - \frac{\lambda_1 q_n}{q_{n+1}}\right| < \lambda_1$ (as $\lambda_1 > 1$), from (57), we obtain

$$q_{n+k} \geq c\lambda_1^k \left(1 - \lambda_1\,(\frac{|\lambda_2|}{\lambda_1})^k\right) q_{n+1} = c\lambda_1^k q_{n+1} \left(1 - |\lambda_2|\,(\frac{|\lambda_2|}{\lambda_1})^{k-1}\right) \geq c\lambda_1^k q_{n+1}(1 - |\lambda_2|).$$

It follows that, for $k \geq 1$, we have $q_{n+k} \geq c_1\,\lambda_1^k\,q_{n+1}$, with $c_1 := \frac{3\sqrt{5}-5}{10}$, which gives (56). □

For the frequently used quantities $\dfrac{q_{n+1}}{q_n}$ and $\dfrac{q_{n+1}}{q_n}\,\ln q_{n+1}$, we introduce the notation:

$$a'_n := \frac{q_{n+1}}{q_n},\ c_n := \frac{q_{n+1}}{q_n}\,\ln q_{n+1}. \tag{58}$$

Observe that $a'_n \leq a_{n+1} + 1$.

We will prove the decorrelation inequalities of Lemma 4.3 under Hypothesis 1 on α, that is: there are two constants $A \geq 1$, $p \geq 0$ such that $a_n \leq A\,n^p, \forall n \geq 1$.

It follows from it: for constants B, C, $b_n \leq B\,n^p$ and, since $q_n \leq B^n\,(n!)^p$,

$$\ln q_n \leq C\,n\,\ln n,\ c_n \leq C\,n^{p+1}\,\ln n. \tag{59}$$

The case when α is bpq corresponds to $p = 0$ in the previous inequalities and we have then $\ln q_n \leq C\,n$.

Lemma 4.2. *If q_n, q_ℓ are denominators of α, we have with an absolute constant C:*

$$\sum_{|j| \geq q_\ell} \frac{1}{j^2} \frac{\|q_n j\alpha\|^2}{\|j\alpha\|^2} \leq C\,\frac{q_n}{q_\ell},\ \textit{for}\ n \leq \ell. \tag{60}$$

Proof. Observe first that, if f is a nonnegative BV function with integral $\mu(f)$ and if q is a denominator of α, then:

$$\sum_{j=q}^{\infty} \frac{f(j\alpha)}{k^2} \leq \frac{2\mu(f)}{q} + \frac{2V(f)}{q^2}.$$

Indeed, by Denjoy-Koksma inequality applied to $f-\mu(f)$, $\sum_{j=q}^{\infty}\frac{f(j\alpha)}{j^2}$ is less than

$$\sum_{i=1}^{\infty}\frac{1}{(iq)^2}\sum_{r=0}^{q-1}f((iq+r)\alpha)\le\frac{1}{q^2}(\sum_{i=1}^{\infty}\frac{1}{i^2})\,(q\,\mu(f)+V(f))=\frac{\pi^2}{6}(\frac{\mu(f)}{q}+\frac{V(f)}{q^2}).$$

Taking for $f(x)$ respectively $1_{[0,\frac{1}{L}]}(|x|)$ and $\frac{1}{x^2}1_{[\frac{1}{L},\frac{1}{2}[}(|x|)$, we obtain with $L=1/q_\ell$:

$$\sum_{j:\,\|j\alpha\|\le 1/q_n,\,j\ge q_\ell}\frac{1}{j^2}\le C\,\frac{q_n+q_\ell}{q_nq_\ell^2},\qquad\sum_{j:\,\|j\alpha\|\ge 1/q_n,\,j\ge q_\ell}\frac{1}{j^2}\frac{1}{\|j\alpha\|^2}\le C\,q_n\,\frac{q_n+q_\ell}{q_\ell^2}.\tag{61}$$

Now, (60) follows, since for $n\le\ell$:

$$\begin{aligned}\frac{1}{2}\sum_{|j|\ge q_\ell}\frac{1}{j^2}\frac{\|q_nj\alpha\|^2}{\|j\alpha\|^2}&\le {q_n}^2\sum_{\|j\alpha\|\le 1/q_n,\,j\ge q_\ell}\frac{1}{j^2}+\sum_{\|j\alpha\|>1/q_n,\,j\ge q_\ell}\frac{1}{j^2}\frac{1}{\|j\alpha\|^2}\\&\le\frac{{q_n}^2}{q_nq_\ell}+\frac{{q_n}^2}{q_\ell^2}+\frac{q_n}{q_\ell}+\frac{{q_n}^2}{q_\ell^2}\le 4\frac{q_n}{q_\ell}.\quad\square\end{aligned}$$

4.2. **Truncation.**

The Fourier coefficients of the ergodic sum φ_q satisfy:

$$|\widehat{\varphi_{q_n}}(j)|=\frac{|\gamma_j(\varphi)|}{|j|}\,\frac{|\sin\pi q_nj\alpha|}{|\sin\pi j\alpha|}\le\frac{\pi}{2}\,\frac{K(\varphi)|}{|j|}\,\frac{\|q_nj\alpha\|}{\|j\alpha\|}.$$

Recall also (cf. (11)) that

$$\|\varphi_{q_n}\|_\infty=\sup_x|\sum_{\ell=0}^{q_n-1}\varphi(x+\ell\alpha)|\le V(\varphi)\text{ et }\|\varphi_{q_n}\|_2\le 2\pi\,K(\varphi).$$

Let S_Lf denote the partial sum of order $L\ge 1$ of the Fourier series of a function f and let $R_Lf=f-S_Lf$ be the remainder. We will take $L=q_\ell$ and bound the truncation error for the ergodic sums $\varphi_{b_n\,q_n}$.

We use the bound (60) which gives, for $q_n\le L=q_\ell$,

$$\begin{aligned}\|R_L\varphi_{b_nq_n}\|_2&=[\sum_{|j|\ge L}|\widehat{\varphi_{b_nq_n}}(j)|^2]^{\frac{1}{2}}=[\sum_{|j|\ge L}\frac{|\gamma_j(\varphi)|^2}{j^2}\,\frac{\|b_nq_nj\alpha\|^2}{\|j\alpha\|^2}]^{\frac{1}{2}}\\&\le b_n[\sum_{|j|\ge L}\frac{|\gamma_j(\varphi)|^2}{j^2}\,\frac{\|q_nj\alpha\|^2}{\|j\alpha\|^2}]^{\frac{1}{2}}\le C\,K(\varphi)\,b_n\,(\frac{q_n}{L})^{\frac{1}{2}}.\end{aligned}\tag{62}$$

For every bounded ψ, we have, for $q_n\le q_m\le q_\ell$ and $L=q_\ell$:

$$\begin{aligned}&|\int\psi\,[\varphi_{b_nq_n}\varphi_{b_mq_m}-S_L\varphi_{b_nq_n}\,S_L\varphi_{b_mq_m}]\,d\mu|\\&\qquad\le\|\psi\|_\infty[\|\varphi_{b_nq_n}\|_2\,\|R_L\varphi_{b_mq_m}\|_2+\|R_L\varphi_{b_nq_n}\|_2\,\|S_L\varphi_{b_mq_m}\|_2]\end{aligned}$$

and, as $\|\varphi_{b_n q_n}\|_2 \leq C b_n K(\varphi)$, we have by (62):

$$\begin{aligned}
&|\int \psi\,[\varphi_{b_n q_n}\varphi_{b_m q_m} - S_L\varphi_{b_n q_n}\, S_L\varphi_{b_m q_m}]\,d\mu| \\
&\quad \leq \|\psi\|_\infty[\|\varphi_{b_n q_n}\|_2\,\|R_L\varphi_{b_m q_m}\|_2 + \|R_L\varphi_{b_n q_n}\|_2\,\|\varphi_{b_m q_m}\|_2] \\
&\quad \leq C\,K(\varphi)^2\|\psi\|_\infty\, b_n b_m\,[(\frac{q_n}{L})^{\frac{1}{2}} + (\frac{q_m}{L})^{\frac{1}{2}}].
\end{aligned}$$

This gives the following truncation error, where the constant $C_1 = C_1(\psi,\varphi)$ is equal to $K(\varphi)^2\|\psi\|_\infty$, up to a universal factor,

$$|\int \psi\,[\varphi_{b_n\,q_n}\varphi_{b_m\,q_m} - S_{q_\ell}\varphi_{b_n\,q_n}\, S_{q_\ell}\varphi_{b_m q_m}]\,d\mu| \leq C_1\, b_n b_m\,(\frac{q_m}{q_\ell})^{\frac{1}{2}} \quad \text{for } q_n \leq q_m \leq q_\ell. \tag{63}$$

Using the Fejer kernel, we get

$$\|S_{q_r}\varphi_{b_n\,q_n}\|_\infty \leq \|\varphi_{b_n\,q_n}\|_\infty + \frac{1}{q_r}\sum_{|j|<q_r} |\widehat{j\varphi_{b_n q_n}}(j)| \leq CK(\varphi)\frac{1}{q_r}\sum_{j=1}^{q_r-1}\frac{1}{||j\alpha||},$$

and, by (11) and (69),

$$\|S_{q_r}\varphi_{b_n\,q_n}\|_\infty \leq b_n V(\varphi) + CK(\varphi)\ln(q_r).$$

With this inequality, by computations similar to the ones used to get (63), we have:

$$|\int \psi\,[\varphi_{b_n\,q_n}\varphi_{b_m\,q_m}\varphi_{b_\ell\,q_\ell} - S_{q_r}\varphi_{b_n\,q_n}\, S_{q_r}\varphi_{b_m q_m}\, S_{q_r}\varphi_{b_\ell q_\ell}]\,d\mu| \leq C\,b_n b_m b_\ell\,(\frac{q_\ell}{q_r})^{\frac{1}{2}}\ln(q_r), \tag{64}$$

when $q_n \leq q_m \leq q_\ell \leq q_r$.

Lemma 4.3. *If $a_{k+1} \leq Ak^p, \forall k \geq 1$ and $n \leq m \leq \ell$, we have for every $\Lambda \geq 1$:*

$$\sum_{j=1}^{\infty}\frac{\|q_n j\alpha\|}{j^2\,\|j\alpha\|} \leq C\,\frac{n^{p+2}\,\ln n}{q_{n+1}}, \tag{65}$$

$$\sum_{1\leq j,k<q_\Lambda,\, j\neq k}\frac{\|q_n j\alpha\|\|q_m k\alpha\|}{|k-j|\,k\,j\,\|j\alpha\|\,\|k\alpha\|} \leq \frac{C}{q_{n+1}}\,\Lambda^{2p+4}(\ln\Lambda)^2, \tag{66}$$

$$\sum_{-q_\Lambda<i,j,k<q_\Lambda,\, i+j+k\neq 0}\frac{\|q_n i\alpha\|\,\|q_m j\alpha\|\,\|q_\ell k\alpha\|}{|i+j+k|\,i\,j\,k\,\|i\alpha\|\,\|j\alpha\|\,\|k\alpha\|} \leq \frac{C}{q_{n+1}}\,\Lambda^{3p+8}. \tag{67}$$

Now we apply this lemma which is proved in the next subsection. Let us mention that Hardy and Littlewood in [HaLi30] considered similar quantities for α bpq. One of their motivations was to study the asymptotic of the number of integral points contained in homothetic of triangles.

Proof of Proposition 3.1

By (65) we have, with $K = C\,V(\psi)V(\varphi)$,

$$|\int \psi\,\varphi_{b_nq_n}\,d\mu| \le \sum_{j\neq 0} |\widehat{\varphi_{b_nq_n}}(j)|\,|\hat{\psi}(-j)| \le K\sum_{j\ge 1}\frac{\|b_nq_nj\alpha\|}{j^2\,\|j\alpha\|} \le \frac{Kb_n}{q_{n+1}}\,n^{p+2}\,\ln n.$$

This implies (38). We prove now (39).

With $L = q_\Lambda$, we have:

$$\int \psi\,S_L\varphi_{b_nq_n}\,S_L\varphi_{b_mq_m}\,d\mu = \sum_{|j|,|k|\le L, j\neq k} \widehat{\varphi_{b_nq_n}}(j)\,\widehat{\varphi_{b_mq_m}}(k)\,\hat{\psi}(j-k).$$

In what follows, the constant C is equal to $V(\psi)V(\varphi)^2$ up to a factor not depending on ψ and φ, which may change.

Recall that, by (56), there is a constant B such that $m \le B\ln q_m, \forall m \ge 1$.

By (66), it holds

$$\begin{aligned}|\int \psi\,S_L\varphi_{b_nq_n}\,S_L\varphi_{b_mq_m}\,d\mu| &\le \sum_{|j|,|k|\le L, j\neq k} |\widehat{\varphi_{b_nq_n}}(j)|\,|\widehat{\varphi_{b_mq_m}}(k)|\,|\hat{\psi}(j-k)|\\ &\le C\sum_{1\le j,k\le L}\frac{\|b_nq_nj\alpha\|\,\|b_mq_m\,k\alpha\|}{|k-j|\,k\,j\,\|j\alpha\|\,\|k\alpha\|} \le \frac{Cb_nb_m}{q_{n+1}}\,\Lambda^{2p+4}(\ln\Lambda)^2.\end{aligned}$$

Putting it together with (63) and replacing q_{n+1} by q_n, we get, for $n \le m \le \Lambda$,

$$\begin{aligned}|\int_X \psi\,\varphi_{b_nq_n}\varphi_{b_mq_m}\,d\mu| &\le Cb_nb_m\,[\frac{\Lambda^{2p+4}(\ln\Lambda)^2}{q_n} + (\frac{q_m}{q_\Lambda})^{\frac12}]\\ &\le Cb_nb_m\,[\frac{\Lambda^{2p+4}(\ln\Lambda)^2}{q_n} + \rho^{\frac{\Lambda-m}{2}}]. \qquad (68)\end{aligned}$$

Let us take $\Lambda - m$ of order $2(\ln\frac{1}{\rho})^{-1}\,\ln q_n$, i.e., such that the second term in the RHS in (68) is of order $1/q_n$. The first term is then less than

$$\frac{C_1}{q_n}\,\max(\,(\ln q_n)^{2p+5}, m^{2p+5}) \le \frac{C_2}{q_n}\,\max(\,(\ln q_n)^{2p+5}, (\ln q_m)^{2p+5}) \le \frac{C_2}{q_n}\,(\ln q_m)^{2p+5}.$$

As $\ln q_n \le Cn\ln n$ by Hypothesis 1, we have $\max((\ln q_n)^\theta, m^\theta) \le C\,m^{\theta+1}$.

This shows (39) with $\theta = 2p+5$. □

In the same way, (40) follows from (64) and (67).

4.3. **Proof of Lemma 4.3.**

As already mentioned, Lemma 4.3 is a consequence of the good repartition of the numbers $\|k\alpha\|$ when k varies between 1 and q_n. We will use this information through two inequalities given in the following lemma, which will be used several times.

Lemma 4.4. *We have*

$$\sum_{j=q_t}^{q_{t+1}-1} \frac{1}{\|j\alpha\|} \leq \sum_{j=1}^{q_{t+1}-1} \frac{1}{\|j\alpha\|} \leq Cq_{t+1} \ln q_{t+1}, \ \forall t \geq 0, \tag{69}$$

$$\sum_{1\leq j<q_{r+1}} \frac{1}{j\,\|j\alpha\|} \leq C \sum_{t=0}^{r} \frac{q_{t+1}}{q_t} \ln q_{t+1} = C \sum_{t=0}^{r} c_t, \ \forall r \geq 0. \tag{70}$$

Proof. For $1 \leq j < q_{t+1}$, one has $\|j\alpha\| \geq \frac{1}{q_{t+1}+q_t} \geq \frac{1}{2q_{t+1}}$. There is exactly one value of $j\alpha \bmod 1$, for $1 \leq j < q_{t+1}$, in each interval $[\frac{\ell}{q_{t+1}}, \frac{\ell+1}{q_{t+1}}[$, $\ell = 1, ..., q_{t+1}-1$. This implies:

$$\sum_{j=1}^{q_{t+1}-1} \frac{1}{\|j\alpha\|} \leq 2q_{t+1} + \sum_{\ell=1}^{q_{t+1}-1} \frac{1}{\ell/q_{t+1}} \leq Cq_{t+1} \ln q_{t+1}.$$

From (69), applied for $t = 1, ..., r$, we deduce (70):

$$\sum_{1\leq j<q_{r+1}} \frac{1}{j\,\|j\alpha\|} = \sum_{t=0}^{r} \sum_{q_t\leq j<q_{t+1}} \frac{1}{j\,\|j\alpha\|} \leq \sum_{t=0}^{r} \frac{1}{q_t} \sum_{q_t\leq j<q_{t+1}} \frac{1}{\|j\alpha\|} \leq C \sum_{t=0}^{r} \frac{q_{t+1}}{q_t} \ln q_{t+1}. \quad \square$$

4.3.1. *Bound (65) in Lemma 4.3.*

We want to bound $\sum_{j=1}^{\infty} \frac{\|q_n j\alpha\|}{j^2\,\|j\alpha\|}$. For $n < \ell$, we write $\sum_{j=1}^{q_\ell-1} \frac{\|q_n j\alpha\|}{j^2\,\|j\alpha\|} = (A) + (B)$, with

$$\begin{aligned}(A) &:= \sum_{j=1}^{q_{n+1}-1} \frac{1}{j} \frac{\|q_n j\alpha\|}{j\,\|j\alpha\|} \leq \frac{1}{q_{n+1}} \sum_{j=1}^{q_{n+1}-1} \frac{1}{j} \frac{1}{\|j\alpha\|} \\ &\leq \frac{1}{q_{n+1}} \sum_{k=0}^{n} \frac{1}{q_k} \sum_{j=q_k}^{q_{k+1}-1} \frac{1}{\|j\alpha\|} \leq C \frac{1}{q_{n+1}} \sum_{k=0}^{n} \frac{q_{k+1}}{q_k} \ln q_{k+1}, \text{ by (70);}\end{aligned}$$

$$\begin{aligned}(B) &:= \sum_{j=q_{n+1}}^{q_\ell-1} \frac{\|q_n j\alpha\|}{j^2\,\|j\alpha\|} \leq \sum_{k=n+1}^{\ell-1} \sum_{j=q_k}^{q_{k+1}-1} \frac{1}{j^2\,\|j\alpha\|} \\ &\leq \sum_{k=n+1}^{\ell-1} \frac{1}{q_k^2} \sum_{j=q_k}^{q_{k+1}-1} \frac{1}{\|j\alpha\|} \leq C \sum_{k=n+1}^{\ell-1} \frac{1}{q_k} \frac{q_{k+1}}{q_k} \ln q_{k+1}, \text{ by (69).}\end{aligned}$$

By (56), we know that $\frac{q_{n+1}}{q_k} \leq C\rho^{k-n}$, with $\rho < 1$, for $k \geq n+1$. By hypothesis, $a_{k+1} \leq Ak^p$. It follows with the notation (58):

$$(A) \leq \frac{C}{q_{n+1}} \sum_{k=0}^{n} c_k \leq \frac{C}{q_{n+1}} n^{p+2} \ln n$$

and, for (B), with a bound which doesn't depend on $\ell \geq n$:

$$\sum_{k=n+1}^{\ell-1} \frac{q_{n+1}}{q_k} \frac{q_{k+1}}{q_k} \ln q_{k+1} \leq C \sum_{j=0}^{\infty} \rho^j \, (j+n+1)^{p+1} \, \ln(j+n+1) \leq C' n^{p+1} \ln n. \quad \square$$

4.3.2. *Bounding (66) of the lemma 4.3.* We want to bound the sum:

$$\sum_{1 \leq j,k < q_\Lambda,\, j \neq k} \frac{\|q_n j\alpha\| \|q_m k\alpha\|}{|k-j| \, k \, j \, \|j\alpha\| \, \|k\alpha\|}$$

We cover the square of integers $[1, q_\Lambda[\times [1, q_\Lambda[$ by rectangles $R_{r,s} = [q_r, q_{r+1}[\times [q_s, q_{s+1}[$ for r and s varying between 0 and $\Lambda - 1$ and then we bound the sum on each of these rectangles.

We use the inequalities
- for $j < q_{k+1}$, $\|jq_k\alpha\|/j \leq \|q_k\alpha\| \leq 1/q_{k+1}$,
- for $j \geq q_{k+1}$ the (trivial) inequality $\|jq_k\alpha\| \leq 1$
- and the inequality (which is direct consequence of $\|(k-j)\alpha\| \leq \|j\alpha\| + \|k\alpha\|$)

$$\frac{1}{|k-j| \, \|j\alpha\| \, \|k\alpha\|} \leq \frac{1}{|k-j| \, \|(k-j)\alpha\| \, \|j\alpha\|} + \frac{1}{|k-j| \, \|(k-j)\alpha\| \, \|k\alpha\|}.$$

On one hand, we have, distinguishing different cases according to the positions of r and s with respect to $n+1$ and $m+1$:

$$\frac{\|q_n j\alpha\| \, \|q_m k\alpha\|}{|k-j| \, k \, j \, \|j\alpha\| \, \|k\alpha\|} \leq \frac{1}{q_{\max(r,n+1)} q_{\max(s,m+1)}} \frac{1}{|k-j| \|j\alpha\| \|k\alpha\|}.$$

On the other hand, by (69) and (70), we have

$$\begin{aligned}
&\sum_{(j,k) \in R_{r,s}} \frac{1}{|k-j| \|j\alpha\| \|k\alpha\|} \\
&\qquad \leq \sum_{R_{r,s}} \left(\frac{1}{|k-j| \, \|(k-j)\alpha\| \, \|j\alpha\|} + \frac{1}{|k-j| \, \|(k-j)\alpha\| \, \|k\alpha\|} \right) \\
&\qquad \leq q_{\max(r,s)+1} \ln(q_{\max(r,s)+1}) \sum_{t=0}^{\max(r,s)} c_t.
\end{aligned}$$

It follows

$$\begin{aligned}
\sum_{(j,k) \in R_{r,s}} \frac{\|q_n j\alpha\| \|q_m k\alpha\|}{|k-j| \, k \, j \, \|j\alpha\| \, \|k\alpha\|} &\leq \frac{q_{\max(r,s)+1}}{q_{\max(r,n+1)} q_{\max(s,m+1)}} \ln(q_{\max(r,s)+1}) \sum_{t=0}^{\max(r,s)} c_t \\
&\leq \frac{1}{q_{n+1}} \ln(q_{\Lambda+1}) \sum_{t=0}^{\Lambda} c_t \max_{k=1,\ldots,\Lambda} a'_k.
\end{aligned}$$

The square $[1,q_\Lambda[\times[1,q_\Lambda[$ is covered by Λ^2 rectangles $R_{r,s}$ and the sums on these rectangles are bounded by the same quantity. It follows

$$\sum_{1\le j,k<q_\Lambda,\, j\neq k} \frac{\|q_n j\alpha\|\|q_m k\alpha\|}{|k-j|\,k\,j\,\|j\alpha\|\,\|k\alpha\|} \le \Lambda^2 \frac{C}{q_{n+1}} \ln(q_{\Lambda+1}) \sum_{t=0}^{\Lambda} c_t \max_{k=1,\dots,\Lambda} a'_k,$$

and, with Hypothesis 1,

$$\sum_{1\le j,k<q_\Lambda,\, j\neq k} \frac{\|q_n j\alpha\|\|q_m k\alpha\|}{|k-j|\,k\,j\,\|j\alpha\|\,\|k\alpha\|} \le \frac{C}{q_{n+1}} \Lambda^{2p+4} \ln(\Lambda)^2. \quad \square$$

4.3.3. *Bounding (67) in Lemma 4.3.*

We want to bound the sum

$$\sum_{-q_\Lambda<i,j,k<q_\Lambda,\, i+j+k\neq 0} \frac{\|q_n i\alpha\|\|q_m j\alpha\|\|q_\ell k\alpha\|}{|i+j+k|\,i\,j\,k\,\|i\alpha\|\,\|j\alpha\|\,\|k\alpha\|}. \tag{71}$$

Here we consider sums with three indices i,j,k. Though we do not write it explicitly these sums are to be understood on non zero indices i,j,k such that $i+j+k\neq 0$. We cover the set of indices by sets of the form

$$R_{\pm r,\pm s,\pm t} = \{(i,j,k) \ / \ \pm i \in [q_r, q_{r+1}[, \pm j \in [q_s, q_{s+1}[, \pm k \in [q_t, q_{t+1}[\}$$

Distinguishing different cases according to the positions of r, s and t with respect to $n+1$, we get: if $(i,j,k)\in R_{\pm r,\pm s,\pm t}$,

$$\begin{aligned} &\frac{\|q_n i\alpha\|\|q_m j\alpha\|\|q_\ell k\alpha\|}{|i+j+k|\,i\,j\,k\,\|i\alpha\|\,\|j\alpha\|\,\|k\alpha\|} \\ &\qquad \le \frac{1}{q_{\max(r,n+1)} q_{\max(s,n+1)} q_{\max(t,n+1)}} \frac{1}{|i+j+k|\,\|i\alpha\|\,\|j\alpha\|\,\|k\alpha\|}. \end{aligned} \tag{72}$$

We have

$$\begin{aligned} \frac{1}{\|i\alpha\|\,\|j\alpha\|\,\|k\alpha\|} \le \frac{1}{\|(i+j+k)\alpha\|\,\|j\alpha\|\,\|k\alpha\|} + \frac{1}{\|(i+j+k)\alpha\|\,\|i\alpha\|\,\|k\alpha\|} \\ + \frac{1}{\|(i+j+k)\alpha\|\,\|i\alpha\|\,\|j\alpha\|}. \end{aligned}$$

We then use (69) and (70) three times, sum over $R_{\pm r,\pm s,\pm t}$ and get:

$$\begin{aligned} &\sum_{(i,j,k)\in R_{\pm r,\pm s,\pm t}} \frac{1}{|i+j+k|\,\|i\alpha\|\,\|j\alpha\|\,\|k\alpha\|} \\ &\qquad \le \Big(\sum_{v=0}^{3\max(r,s,t)} c_v\Big) \ln^2(q_{\max(r,s,t)+1}) \left(q_{s+1}q_{t+1} + q_{r+1}q_{t+1} + q_{r+1}q_{s+1}\right). \end{aligned}$$

By (72) we then have:

$$\sum_{R_{\pm r,\pm s,\pm t}} \frac{\|q_n i\alpha\|\|q_m j\alpha\|\|q_\ell k\alpha\|}{|i+j+k|\, i\, j\, k\, \|i\alpha\|\, \|j\alpha\|\, \|k\alpha\|}$$
$$\leq C \frac{(\sum_{v=0}^{3\max(r,s,t)} c_v)\ln^2(q_{\max(r,s,t)})}{q_{\max(r,n+1)} q_{\max(s,n+1)} q_{\max(t,n+1)}} (q_{s+1}q_{t+1} + q_{r+1}q_{t+1} + q_{r+1}q_{s+1})$$
$$\leq \frac{C}{q_{n+1}} (\sum_{v=0}^{3\Lambda} c_v) \ln^2(q_{\Lambda+1}) (\max_{k=1,\dots,\Lambda} a'_k)^2.$$

One needs $8\Lambda^3$ boxes $R_{\pm r,\pm s,\pm t}$ to cover the set $\{-q_\Lambda < i,j,k < q_\Lambda,\ i+j+k \neq 0\}$. This implies that there exists C such that

$$\sum_{-q_\Lambda<i,j,k<q_\Lambda,\, i+j+k\neq 0} \frac{\|q_n i\alpha\|\|q_m j\alpha\|\|q_\ell k\alpha\|}{|i+j+k|\, i\, j\, k\, \|i\alpha\|\, \|j\alpha\|\, \|k\alpha\|} \leq \frac{C}{q_{n+1}} \Lambda^{3p+8}. \quad \square$$

5. **Proof of the CLT (Proposition 3.2)**

The difference $H_{X,Y}(\lambda) := |\mathbb{E}(e^{i\lambda X}) - \mathbb{E}(e^{i\lambda Y})|$ can be used to get an upper bound of the distance $d(X,Y)$ thanks to the following inequality ([Fe], Chapter XVI, Inequality (3.13)): if X has a vanishing expectation, then, for every $U > 0$,

$$d(X,Y) \leq \frac{1}{\pi} \int_{-U}^{U} H_{X,Y}(\lambda) \frac{d\lambda}{\lambda} + \frac{24}{\pi} \frac{1}{\sigma\sqrt{2\pi}} \frac{1}{U}. \tag{73}$$

Using (73), we get an upper bound of the distance between the distribution of X and the normal law by bounding $|\mathbb{E}(e^{i\lambda X}) - e^{-\frac{1}{2}\sigma^2\lambda^2}|$.

We will use the following remarks:

$$\mathrm{V}(fg) \leq \|f\|_\infty \mathrm{V}(g) + \|g\|_\infty \mathrm{V}(f),\ \forall f,g \text{ BV}, \tag{74}$$

$$\text{if } g \in \mathcal{C}^1(\mathbb{R},\mathbb{R}) \text{ and } u \text{ is BV, then } \mathrm{V}(g\circ u) \leq \|g'\|_\infty \mathrm{V}(u). \tag{75}$$

Let $w_k := \max_{j=1}^k u_j$, where u_j is larger than $\|f_j\|_\infty$ (see the proposition 3.2).

Since $\mathrm{V}(f_k) \leq C\|f_k\|_\infty\, q_k$, (43) implies

$$|\int_X f_k f_m\, d\mu| \leq C\, \frac{q_k}{q_m} m^\theta\, w_m^2, \text{ for } k \leq m. \tag{76}$$

5.1. **Bounding the moments.**

Lemma 5.1. *There is C_1 such that, for all $m, \ell \geq 1$,*

$$\|\sum_{k=m}^{m+\ell} f_k\|_2^2 \leq C_1\, \ln(m+\ell) \sum_{j\in[m,m+\ell]} u_j^2 \ \leq C_1\, w_{m+\ell}^2\, \ell\, \ln(m+\ell). \tag{77}$$

Proof. The computation is as in [Hu09]. Let $r_0 := \lfloor \frac{\theta}{\ln(1/\rho)} \ln(m+\ell) \rfloor + 1$, so that $(m+\ell)^\theta \rho^{r_0} \leq 1$. We have $\| \sum_{k=m}^{m+\ell} f_k \|_2^2 \leq 2\,(A) + 2\,(B)$,with

$$(A) = \sum_{0\leq r<r_0} \sum_{j\in[m,m+\ell]\cap[m-r,m+\ell-r]} |\int f_j\, f_{j+r} d\mu|,$$

$$(B) = \sum_{r_0\leq r<m+\ell} \sum_{j\in[m,m+\ell]\cap[m-r,m+\ell-r]} |\int f_j\, f_{j+r} d\mu|.$$

From (76) and (41), then by the choice of r_0, it follows:

$$\begin{aligned}(B) &\leq C \sum_{r_0\leq r<m+\ell} \sum_{j\in[m,m+\ell]\cap[m-r,m+\ell-r]} \rho^r (m+\ell)^\theta u_j u_{j+r} \\ &\leq \frac{1}{2} C \sum_{r_0\leq r<m+\ell} \sum_{j\in[m,m+\ell]\cap[m-r,m+\ell-r]} \rho^{r-r_0} [u_j^2 + u_{j+r}^2] \\ &\leq C \sum_{r_0\leq r<m+\ell} \sum_{j\in[m,m+\ell]} \rho^{r-r_0}\, u_j^2 \leq \frac{C}{1-\rho} \sum_{j\in[m,m+\ell]} u_j^2.\end{aligned}$$

For (A) we have:

$$\begin{aligned}(A) &= \sum_{0\leq r<r_0} \sum_{j\in[m,m+\ell]\cap[m-r,m+\ell-r]} |\int f_j\, f_{j+r} d\mu|, \\ &\leq \frac{1}{2} \sum_{0\leq r<r_0} \sum_{j\in[m,m+\ell]\cap[m-r,m+\ell-r]} [u_j^2 + u_{j+r}^2] \\ &\leq \sum_{0\leq r<r_0} \sum_{j\in[m,m+\ell]} u_j^2 \leq \frac{\theta}{\ln(1/\rho)} \ln(m+\ell) \sum_{j\in[m,m+\ell]} u_j^2.\end{aligned}$$

Therefore we get,

$$\| \sum_{k=m}^{m+\ell} f_k \|_2^2 \leq (\frac{2\theta}{\ln(1/\rho)} \ln(m+\ell) + \frac{2C}{1-\rho}) \sum_{j\in[m,m+\ell]} u_j^2. \quad \square$$

We will also need a bound for the terms of degree 3.

Lemma 5.2. *Under the assumption of Proposition 3.2, there exists $C > 0$ such that*

$$\int_X (\sum_{k=m}^{m+\ell} f_k)^3 \leq C\,\ell\, w_{m+\ell}^3 \ln^2(m+\ell),\ \forall m, n \geq 1. \tag{78}$$

Proof. It suffices to show the result for the sum $\sum_{m\leq s\leq t\leq u\leq m+\ell} |\int_X f_s f_t f_u\, d\mu|$.

Replacing f_k by $w_{m+\ell}^{-1} f_k$, we will deduce the bound (78) from the inequalities (42), (43), (44) when $w_k \leq 1$, for $1 \leq k \leq m+\ell$. By (42) and (74), we have

$$|\int_X f_s f_t f_u\, d\mu| \leq C \ \text{ and } \ \mathrm{V}(f_s f_t) \leq C(q_s + q_t) \leq 3C q_t.$$

From (43) and (41), we obtain $|\int_X (f_s f_t).f_u\, d\mu| \le C\frac{q_t}{q_u}u^\theta \le C\rho^{(u-t)}u^\theta$ and $|\int_X f_s.(f_t f_u)\, d\mu| \le C\rho^{(t-s)}u^\theta$.

Set $\kappa = \frac{\theta+3}{\ln(1/\rho)}\ln(\ell+m)$. If one of the two differences $t-s$ or $u-t$ is larger than κ then

$$|\int_X f_s f_t f_u\, d\mu| \le C\rho^\kappa u^\theta \le (\ell+m)^{-\theta-3}u^\theta \le C(\ell+m)^{-3}.$$

This gives

$$\sum_{m\le s\le t\le u\le m+\ell:\max(t-s,u-t)>\kappa} |\int_X f_s f_t f_u\, d\mu| \le C\ell^3(\ell+m)^{-3} \le C.$$

We have also

$$\sum_{m\le s\le t\le u\le m+\ell:\max(t-s,u-t)\le\kappa} |\int_X f_s f_t f_u\, d\mu| \le C\ell\kappa^2,$$

hence the result. □

5.2. **Proof of Proposition 3.2.**

We divide the proof in several steps.

5.2.1. *Defining blocks.* We split the sum $S_n := f_1 + \cdots + f_n$ into several blocks, small ones and large ones. The small ones will be removed, providing gaps and allowing to take advantage of the decorrelation properties assumed in the statement of the proposition.

Let δ be a parameter (*which will be chosen close to 0*) such that $0 < \delta < \frac{1}{2}$. We set

$$n_1 = n_1(n) := \lfloor n^{\frac{1}{2}+\delta}\rfloor, n_2 = n_2(n) := \lfloor n^\delta\rfloor,\ n \ge 1, \tag{79}$$

$$\nu = \nu(n) := n_1 + n_2,\ p(n) := \lfloor n/\nu(n)\rfloor + 1,\ n \ge 1. \tag{80}$$

We have:

$$p(n) = n^{\frac{1}{2}-\delta} + h_n\ \sim n^{\frac{1}{2}-\delta},\ \text{with } |h_n| \le 1. \tag{81}$$

For $0 \le k < p(n)$, we put (with $f_j = 0$, if $n < j \le n+\nu$)

$$F_{n,k} = f_{k\,\nu+1} + \cdots + f_{k\,\nu+n_1},\quad G_{n,k} = f_{k\,\nu+n_1+1} + \cdots + f_{(k+1)\,\nu}, \tag{82}$$

$$v_k = v_{n,k} := (\int_X F_{n,k}^2\, d\mu)^{\frac{1}{2}},\ S'_n = \sum_{k=0}^{p(n)-1} F_{n,k}. \tag{83}$$

The sums $F_{n,k}, G_{n,k}$ have respectively $n_1 \sim n^{\frac{1}{2}+\delta}, n_2 \sim n^\delta$ terms and S_n reads

$$S_n = \sum_{k=0}^{p(n)-1} (F_{n,k} + G_{n,k}). \tag{84}$$

The following inequalities are implied by (77):

$$v_{n,k} = \|F_{n,k}\|_2 \le Cn^{\frac{1}{4}+\frac{1}{2}\delta}(\ln n)^{1/2}\, w_n,\ \|G_{n,k}\|_2 \le C\, n^{\frac{\delta}{2}}(\ln n)^{1/2}\, w_n,\ 1 \le k \le p(n). \tag{85}$$

Lemma 5.3.

$$(86)\qquad |\|S_n\|_2^2 - \sum_{k=0}^{p(n)-1} v_k^2| = |\|S_n\|_2^2 - \sum_{k=0}^{p(n)-1} \|F_{n,k}\|_2^2| \leq C\, n^{\frac{3}{4}} \ln n\, w_n^2,$$

$$(87)\qquad \|S_n - S'_n\|_2^2 = \|\sum_{k=0}^{p(n)-1} G_k\|^2 \leq C\, n^{\frac{1}{2}-\delta} \ln n\, w_n^2.$$

Proof. We write simply$F_k = F_{n,k}$, $G_k = G_{n,k}$. The following inequality follows easily from (76) and (41):

$$|\int_X (\sum_{u=a}^{b} f_u)\,(\sum_{t=c}^{d} f_t)\, d\mu| \leq \frac{C\rho}{(1-\rho)^2}\, \rho^{c-b}\, d^\theta\, w_d^2,\ \forall a \leq b < c \leq d.$$

Therefore, we have, with $C_1 = \frac{C\rho}{(1-\rho)^2} \sum_{i\geq 0} \rho^{i\nu}$,

$$\sum_{0\leq j<k<p(n)} |\int F_j\, F_k\, d\mu| \leq \frac{C\rho}{(1-\rho)^2}\, n^\theta\, w_n^2 \sum_{0\leq j<k<p(n)} \rho^{k\nu+1-(j\nu+n_1)}$$
$$\leq \frac{C\rho}{(1-\rho)^2}\, n^\theta\, w_n^2\, \rho^{n_2} \sum_{0\leq j<k<p(n)} \rho^{(k-1)\nu-j\nu}$$
$$\leq \frac{C\rho}{(1-\rho)^2}\, \rho^{n_2}\, n^\theta\, w_n^2\, p(n) \sum_{i\geq 0} \rho^{i\nu} \leq C_1\, n^{\frac{1}{2}-\delta+\theta}\, w_n^2\, \rho^{n^\delta}.$$

The same bound holds for

$$\sum_{0\leq j<k-1<p(n)} |\int G_j\, F_k\, d\mu|, \sum_{0\leq j<k<p(n)} |\int F_j\, G_k\, d\mu|, \sum_{0\leq j<k<p(n)} |\int G_j\, G_k\, d\mu|.$$

The LHS of (86) is less than

$$\sum_{k=0}^{p(n)-1} [\int G_k^2\, d\mu + |\int G_k\, F_k\, d\mu| + |\int G_k\, F_{k+1}\, d\mu|]$$
$$+ 2 \sum_{k=0}^{p(n)-1} \sum_{0\leq j<k<p(n)} [|\int F_j\, (F_k + G_k)\, d\mu| + |\int G_j\, G_k\, d\mu|]$$
$$+ 2 \sum_{k=0}^{p(n)-1} \sum_{0\leq j<k-1<p(n)} |\int G_j F_k\, d\mu|.$$

The first term is bounded by $C\, n^{\frac{1}{2}-\delta}\, n^\delta \ln n\, w_n^2 = C\, n^{\frac{1}{2}} \ln n\, w_n^2$, the second one and the third one by $C\, n^{\frac{1}{2}-\delta}\, n^{\frac{1}{4}+\delta} \ln n\, w_n^2 = C\, n^{\frac{3}{4}} \ln n\, w_n^2$. For the other terms, a gap of length n^δ is available and we use the previous bounds which are of the form $C_1\, n^{\frac{1}{2}-\delta+\theta}\, w_n^2\, \rho^{n^\delta}$.

Therefore the LHS of (86) is less than: $C\, n^{\frac{1}{2}-\delta} \ln n\, w_n^2 + 2C\, n^{\frac{3}{4}} \ln n\, w_n^2 + 3C_1\, n^{\frac{1}{2}-\delta+\theta}\, w_n^2\, \rho^{n^\delta}$. The biggest term in this sum is $2C\, n^{\frac{3}{4}} \ln n\, w_n^2$, which gives (86).

An analogous computation shows that the LHS of (87) behaves like $\sum_{k=0}^{p(n)-1}\int G_k^2\,d\mu$ which gives the bound $C\,n^{\frac{1}{2}-\delta}\ln n\,w_n^2$; hence (87). □

5.2.2. *Approximation of the characteristic function of the sum (after removing the small blocks) by a product.*

Let $\mathrm{I}_{n,-1}(\lambda):=1$ and, for $0\le k\le p(n)-1$ and $\zeta\in\mathbb{R}$,

$$\mathrm{J}_n(\zeta):=\int_X e^{i\zeta\,S_n}\,d\mu,\quad \mathrm{I}_{n,k}(\zeta):=\int_X e^{i\zeta\,(F_{n,0}+\cdots+F_{n,k})}\,d\mu. \tag{88}$$

Lemma 5.4. *For $0\le k<p(n)$, we have*

$$|\mathrm{I}_{n,k}(\zeta)-\big(1-\frac{\zeta^2}{2}\,v_{n,k}^2\big)\;\mathrm{I}_{n,k-1}(\zeta)|\le C\left(|\zeta|^3\,w_n^3\,n^{1/2+\delta}\;\ln^2(n)+\zeta^4 w_n^4 n^{1+2\delta}\;\ln(n)^2\right). \tag{89}$$

Proof. Let $k\ge 1$. For $u\in\mathbb{R}$ we use the inequality:

$$e^{iu}=1+iu-\frac{1}{2}u^2-\frac{i}{6}u^3+u^4r(u),\ \text{with}\ |r(u)|\le\frac{1}{24}. \tag{90}$$

By (90), one has:

$$\mathrm{I}_{n,k}(\zeta)=\int_X e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}\,[1+i\zeta F_{n,k}-\frac{\zeta^2}{2}\,F_{n,k}^2-\frac{i}{6}\zeta^3F_{n,k}^3+\zeta^4F_{n,k}^4\,r(\zeta F_{n,k})]\,d\mu.$$

Recall the inequality (41) : $q_1+q_2+\ldots+q_n\le Cq_{n+1},\forall n\ge 1$. Using (75) and hypothesis (42), one has $\mathrm{V}(e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})})\le C|\zeta|\,w_n\,q_{(k-2)\,\nu+n_1}$. Using (75) and hypothesis (42), one obtains

$$\begin{aligned}
&|\int_X e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}\,F_{n,k}\,d\mu|\le\sum_{j=1}^{n_1}|\int_X e^{i\zeta\,(F_{n,0}+\cdots+F_{n,k-1})}\,f_{(k-1)\,\nu+j}\,d\mu|\\
&\le C|\zeta|\,w_n\sum_{j=1}^{n_1}\frac{q_{(k-2)\,\nu+n_1}}{q_{(k-1)\,\nu+j}}\,((k-1)\,\nu+j)^\theta\,w_{((k-2)\,\nu+n_1)}\\
&\le C|\zeta|\,w_n^2\,n^\theta\sum_{j=1}^{n_1}\frac{q_{(k-2)\,\nu+n_1}}{q_{(k-1)\,\nu+j}}\\
&\le C|\zeta|\,w_n^2\,n^\theta\sum_{j=1}^{n_1}\rho^{\nu+j-n_1}\le\frac{C\rho}{1-\rho}\,|\zeta|\,w_n^2\,n^\theta\,\rho^{n_2}.
\end{aligned}\tag{91}$$

Similarly, we apply (44) to bound $\int(e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}-\mathrm{I}_{n,k-1})\,f_kf_{k'}\,d\mu$ and get:

$$|\int_X e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}\,F_{n,k}^2\,d\mu-\mathrm{I}_{n,k-1}\big(\int_X F_{n,k}^2\,d\mu\big)\le C|\zeta|\,n^{\theta+\frac{1}{2}+\delta}e^{\rho n_2}. \tag{92}$$

Likewise inequality (45) implies

$$\begin{aligned}
&|\int_X(e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}-\mathbb{E}(e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})})\,F_{n,k}^3\,d\mu|\\
&\le CV(e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})})n^{3\delta}w_n^3\rho^{n^\delta}.
\end{aligned}$$

One deduces

$$|\int_X e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}\, F_{n,k}^3\, d\mu| \leq CV(\varphi)n^{1+3\delta}w_n^3\rho^{n^\delta} + |\int_X F_{n,k}^3\, d\mu|,$$

then by the lemma 5.2,

$$|\int_X e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}\, F_{n,k}^3\, d\mu| \leq C(V(\varphi)n^{1+3\delta}w_n^3\rho^{n^\delta} + n^{1/2+\delta}\, w_n^3\, \ln^2(n)). \tag{93}$$

At last we have

$$|\int_X e^{i\zeta(F_{n,0}+\cdots+F_{n,k-1})}\, \zeta^4 F_{n,k}^4\, r(\zeta F_{n,k})\, d\mu| \leq \zeta^4 \int_X F_{n,k}^4\, d\mu \leq \zeta^4 w_n^4 n^{1+2\delta}\, \ln(n)^2 \tag{94}$$

From (91), (92), (93) and (94) one deduces

$$\begin{aligned}&|\mathrm{I}_{n,k}(\zeta) - \big(1 - \frac{\zeta^2}{2}\, v_{n,k}^2\big)\, \mathrm{I}_{n,k-1}(\zeta)|\\ &\leq Cw_n^2\rho^{-n^\delta}(|\zeta|^2\, n^\theta + |\zeta|^3\, n^{\theta+\frac{1}{2}+\delta}) + |\zeta|^3\, w_n^3\, n^{1/2+\delta}\, \ln^2(n) + \zeta^4 w_n^4 n^{1+2\delta}\, \ln(n)^2.\end{aligned}$$

In the sum above, we keep only the last two terms, since we can take n big enough such that the two other terms are smaller than the last ones. □

If X and Y are two real square integrable random variables, then $|\mathbb{E}(e^{iX}) - \mathbb{E}(e^{iY})| \leq \|X - Y\|_2$. Therefore, using (87) in claim 1, we have

$$|\mathrm{J}_n(\zeta) - \mathrm{I}_{n,p(n)}(\zeta)| \leq\ |\zeta|\, \|S_n - S'_n\|_2 \leq C\, |\zeta|\, w_n\, n^{\frac{1}{4}-\frac{1}{2}\delta}(\ln n)^{1/2}, \tag{95}$$

then, by (95) and (89) of Lemma 5.4, we get

$$\begin{aligned}|\mathrm{J}_n(\zeta) - \prod_{k=1}^{p(n)}(1 - \frac{1}{2}\zeta^2 v_k^2)| \leq\ & |\mathrm{J}_n(\zeta) - \mathrm{I}_{n,p(n)}(\zeta)| + \sum_{k=0}^{p(n)-1} |\mathrm{I}_{n,k}(\zeta) - (1 - \frac{\zeta^2}{2}\, v_k^2)\, \mathrm{I}_{n,k-1}(\zeta)|\\ &\leq C\, [|\zeta|\, w_n\, n^{\frac{1}{4}-\frac{1}{2}\delta}(\ln n)^{1/2} + n^{\frac{1}{2}-\delta}|\zeta|^3\, w_n^3\, n^{1/2+\delta}\, \ln^2(n) + n^{\frac{1}{2}-\delta}\zeta^4 w_n^4 n^{1+2\delta}\, \ln(n)^2]\\ &\leq C\, [|\zeta|\, w_n\, n^{\frac{1}{4}} + |\zeta|^3\, w_n^3\, n \ln^2(n) + \zeta^4 w_n^4 n^{\frac{3}{2}+\delta}\, \ln(n)^2]. \qquad (96)\end{aligned}$$

5.2.3. *Approximation of the exponential by a product.*

Below, ζ will be such that $|\zeta|\, v_k \leq 1$. This condition implies: $0 \leq 1 - \frac{1}{2}\zeta^2 v_k^2 \leq 1$.

Lemma 5.5. *If $(\rho_k)_{k\in J}$ is a finite family of real numbers in $[0, 1[$, then*

$$0 \leq e^{-\sum_{k\in J}\rho_k} - \prod_{k\in J}(1 - \rho_k) \leq \sum_{k\in J}\rho_k^2,\ \textit{if } 0 \leq \rho_k \leq \frac{1}{2}, \forall k. \tag{97}$$

Proof. We have $\ln(1 - u) = -u - u^2\, v(u)$, with $\frac{1}{2} \leq v(u) \leq 1$, for $0 \leq u \leq \frac{1}{2}$.

Writing $1 - \rho_k = e^{-\rho_k - \varepsilon_k}$, with $\varepsilon_k = -(\ln(1 - \rho_k) + \rho_k)$, the previous inequality implies $0 \leq \varepsilon_k \leq \rho_k^2$, if $0 \leq \rho_k \leq \frac{1}{2}$. We have: $e^{-\sum_J \rho_k} - \prod_J(1 - \rho_k) = e^{-\sum_J \rho_k}\big(1 - e^{-\sum_J \varepsilon_k}\big)$.

Since $0 \leq \sum \varepsilon_k \leq \sum \rho_k^2$, if $0 \leq \rho_k \leq \frac{1}{2}$, and $1 - e^{-\sum \varepsilon_k} \leq \sum \varepsilon_k$, if $\sum_k \varepsilon_k \geq 0$, the following inequalities hold and imply (97):

$$0 \leq e^{-\sum \rho_k} - \prod(1 - \rho_k) \leq e^{-\sum \rho_k} \sum \varepsilon_k \leq \sum \varepsilon_k \leq \sum \rho_k^2,\ \text{if } \rho_k \leq \frac{1}{2}, \forall k. \quad \square$$

We apply (97) with $\rho_k = \frac{1}{2}\zeta^2 v_k^2$. The condition $\zeta^2 v_k^2 \leq 1$ is satisfied by the assumption on ζ. In view of (85) it follows:

$$|e^{\frac{1}{2}\zeta^2 \sum v_k^2} - \prod_{k=0}^{p(n)-1} (1 - \frac{1}{2}\zeta^2 v_k^2)| \leq \frac{1}{4}\zeta^4 \sum_{k=0}^{p(n)-1} v_k^4 \leq C\zeta^4 w_n^4 n^{\frac{3}{2}+\delta}. \tag{98}$$

5.2.4. *Conclusion.* From (96) and (98), it follows:

$$|J_n(\zeta) - e^{-\frac{1}{2}\zeta^2 \sum_{k=0}^{p(n)-1} v_k^2}| \leq C\,[|\zeta|\, w_n\, n^{\frac{1}{4}} + |\zeta|^3\, w_n^3\, n \ln^2(n) + \zeta^4\, w_n^4 n^{\frac{3}{2}+\delta} \ln^2(n)]. \tag{99}$$

Replacing ζ by $\frac{\lambda}{\|S_n\|_2}$, we get:

$$\begin{aligned} &|\int_X e^{i\lambda \frac{S_n(x)}{\|S_n\|_2}}\, d\mu(x) - e^{-\frac{1}{2}\frac{\lambda^2}{\|S_n\|_2^2}\sum_k v_k^2}| \\ &\quad \leq C\,[|\lambda| \frac{w_n}{\|S_n\|_2}\, n^{\frac{1}{4}} + |\lambda|^3 \frac{w_n^3}{\|S_n\|_2^3}\, n \ln^2(n) + \lambda^4 \frac{w_n^4}{\|S_n\|_2^4}\, n^{\frac{3}{2}+\delta} \ln^2(n)]. \end{aligned}$$

Since $|e^{-a} - e^{-b}| \leq |a-b|$, for any $a, b \geq 0$, we have, by (86):

$$|e^{-\frac{1}{2}\lambda^2} - e^{-\frac{1}{2}\frac{\lambda^2}{\|S_n\|_2^2}\sum_{k=0}^{p(n)-1} v_k^2}| \leq \frac{1}{2}\frac{\lambda^2}{\|S_n\|_2^2}\, |\|S_n\|_2^2 - \sum_{k=0}^{p(n)-1} v_k^2| \leq C\lambda^2 \frac{w_n^2}{\|S_n\|_2^2} n^{\frac{3}{4}} \ln n.$$

Finally we get the bound

$$\begin{aligned} &|\int_X e^{i\lambda \frac{S_n(x)}{\|S_n\|_2}}\, d\mu(x) - e^{-\frac{1}{2}\lambda^2}| \leq E_1 + E_2 + E_3 + E_4 \\ &\quad = C\,[|\lambda| \frac{w_n}{\|S_n\|_2}\, n^{\frac{1}{4}} + |\lambda|^3 \frac{w_n^3}{\|S_n\|_2^3}\, n \ln^2(n) + \lambda^4 \frac{w_n^4}{\|S_n\|_2^4}\, n^{\frac{3}{2}+\delta} \ln^2(n) + \lambda^2 \frac{w_n^2}{\|S_n\|_2^2} n^{\frac{3}{4}} \ln n]. \end{aligned} \tag{100}$$

To summarise we recall the origin of the different terms in the previous bound: E_1 comes from neglecting the sums on the small blocks, E_2 is the error of order 3 in the expansion, E_3 comes from the approximation of the exponential by the product, finally E_4 is the error from the replacement of $e^{-\frac{1}{2}\lambda^2}$ by $e^{-\frac{1}{2}\frac{\lambda^2}{\|S_n\|_2^2}\sum v_k^2}$.

Denote by Y_1 a r.v. with $\mathcal{N}(0,1)$-distribution. To simplify, we write $R_n := \frac{w_n}{\|S_n\|_2}$. The bound (100) reads:

$$\begin{aligned} H_{\frac{S_n}{\|S_n\|_2}, Y_1}(\lambda) &= |\int_X e^{i\lambda \frac{S_n(x)}{\|S_n\|_2}}\, d\mu(x) - e^{-\frac{1}{2}\lambda^2}| \\ &\leq C\,[|\lambda|\, R_n\, n^{\frac{1}{4}} + |\lambda|^3\, R_n^3\, n(\ln n)^2 + \lambda^4\, R_n^4\, n^{\frac{3}{2}+\delta} + \lambda^2\, R_n^2\, n^{\frac{3}{4}-\delta}]. \end{aligned} \tag{101}$$

$$\leq C\,[|\lambda|\, R_n\, n^{\frac{1}{4}} + |\lambda|^3\, R_n^3\, n^{1+\delta} + \lambda^4\, R_n^4\, n^{\frac{3}{2}+\delta} + \lambda^2\, R_n^2\, n^{\frac{3}{4}+\delta}]. \tag{102}$$

Notice that δ can be taken arbitrary small. A change of its value modifies the generic constant C in the previous inequalities. We have obtained an inequality of the form

$$H_{\frac{S_n}{\|S_n\|_2},Y_1}(\lambda) \leq C \sum_{i=1}^{4} |\lambda|^{\alpha_i} R_n^{\alpha_i} n^{\gamma_i},$$

where the exponents are given by the previous inequality. In view of (73), it follows

$$d(\frac{S_n}{\|S_n\|_2}, Y_1) \leq \frac{C}{U_n} + C \sum_{i=1}^{4} \alpha_i^{-1} U_n^{\alpha_i} R_n^{\alpha_i} n^{\gamma_i}.$$

Now, we optimize the choice of $U = U_n$. Under the condition

$$R_n = \frac{w_n}{\|S_n\|_2} \in C[n^{-\frac{7}{12}}, n^{-\frac{5}{12}}], \tag{103}$$

the largest term in the bound (100) is the fourth one. One choses $U_n = n^{-\frac{1}{4}} R_n^{-\frac{2}{3}}$. For the validity of the bounds, recall that we have to check $|\zeta| v_{n,k} \leq 1$. Since $v_{n,k} \leq C n^{\frac{1}{4}+\frac{1}{2}\delta} w_n$, and in view of the previous choice of U_n, this condition reduces to $R_n \leq n^{-\frac{1}{4}-\delta}$, which follows from (103).

Finally, under Condition (103), we get

$$d(\frac{S_n}{\|S_n\|_2}, Y_1) \leq C n^{\frac{1}{4}+\delta} (\frac{w_n}{\|S_n\|_2})^{\frac{2}{3}}.$$

It means that the distribution of $\frac{S_n}{\|S_n\|_2}$ is close to the normal distribution if

$$\max_{j=1}^{n} \|f_j\|_\infty << n^{-\frac{3}{8}} \|S_n\|_2. \quad \square$$

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Univ Rennes, CNRS, IRMAR - UMR 6625, F-35000 Rennes, France

E-mail address: conze@univ-rennes1.fr

E-mail address: stephane.leborgne@univ-rennes1.fr