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Plane partitions and their pedestal polynomials

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Abstract

Given a partially ordered set $\mathcal{S}$, we define, for a linear extension P of $\mathcal{S}$, a multivariate polynomial, counting certain reverse partitions on $\mathcal{S}$ called P-pedestals. We establish a remarkable property of this polynomial: it does not depend on the choice of P. For $\mathcal{S}$ a Young diagram, we show that this polynomial generalizes the hook polynomial.

1 Introduction

Let $\mathcal{S}$ be a partially ordered set. In this work we associate to $\mathcal{S}$ a multivariate polynomial $\mathfrak{h}$. When $\mathcal{S}$ is a Young diagram, the principal specialization of $\mathfrak{h}$ coincides with the hook polynomial, see section 4.

Our construction of $\mathfrak{h}$ begins with defining a polynomial $\mathfrak{h}_P$, where P is an arbitrary linear extension of $\mathcal{S}$. Then we show that in fact $\mathfrak{h}_P$ does not depend on P. The proof uses the equality (4) (precise definitions are given in sections 2 and 3), which is implied by the bijection between the set of reverse partitions on $\mathcal{S}$ and the product of the set of P-pedestals on $\mathcal{S}$ and the set of Young diagrams with at most $|\mathcal{S}|$ rows.

To facilitate the exposition we take for $\mathcal{S}$ the set of nodes of a Young diagram λ. In this situation, linear extensions of $\mathcal{S}$ correspond to standard Young tableaux of shape λ, see Definition 1. Our results and proofs work in the same way for general $\mathcal{S}$ (and linear extensions of $\mathcal{S}$ instead of standard Young tableaux).

After this work was completed, we were informed that the coefficients of our polynomial are values of the "β-function", in the terminology of Stanley's book [3], theorem 3.13.1. Our approach uses very different ideas, in particular, the factorization property (4).

2 Main result

Let $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$, $\lambda_1 \geq \dots \geq \lambda_l > 0$, be a partition of n, $\lambda_1 + \dots + \lambda_l = n$. We identify λ with its Young diagram, that is, the set of nodes

$$\alpha = (i, j) \quad \text{with} \quad j = 1, \dots, \lambda_i \quad \text{for each} \quad i = 1, \dots, l \ .$$

A *standard Young tableau of shape* λ is a bijection $Q\colon \lambda \to \{1, \dots, n\}$ such that the function $Q(i, j)$ increases in i and j. We denote the set of these standard Young tableaux by $\mathfrak{st}_\lambda$.

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Definition 1. *Let $\preccurlyeq$ be the minimal transitive partial order on λ, containing $(i,j) \prec (i+1,j)$ and $(i,j) \prec (i,j+1)$ for all possible (i,j). A standard Young tableau of shape λ can be identified with a linear extension of $\preccurlyeq$, that is, a linear order, compatible with the partial order $\preccurlyeq$. We denote by $\preccurlyeq_P$ the linear order associated to a standard tableau P.*

Let $\mathbb{Z}_{\geqslant 0}$ be the set of non-negative integers. A *reverse plane partition of shape* λ is a function $\mathfrak{Q}\colon \lambda \to \mathbb{Z}_{\geqslant 0}$, non-decreasing in i and j. It is *column-strict* if it increases in j. We visualize reverse plane partitions by placing the number $\mathfrak{Q}(i,j)$ in the node (i,j) for each $(i,j) \in \lambda$. We denote by $|\mathfrak{Q}|$ the volume of $\mathfrak{Q}$, $|\mathfrak{Q}| = \sum_{(i,j)\in\lambda} \mathfrak{Q}(i,j)$.

Let $\bar{\mathcal{S}}_\lambda$ be the set of reverse plane partitions of shape λ, and $\mathcal{S}_\lambda \subset \bar{\mathcal{S}}_\lambda$ the subset of reverse column-strict plane partitions of shape λ. Recall that the Schur function s_λ is the formal power series in infinitely many variables $\mathbf{x} = (x_0, x_1, x_2, ...)$, given by

$$s_\lambda(\mathbf{x}) = \sum_{\mathfrak{Q}\in\mathcal{S}_\lambda} \prod_{\alpha\in\lambda} x_{\mathfrak{Q}(\alpha)} \ .$$

We need the similarly defined formal power series $\bar{s}_\lambda$,

$$\bar{s}_\lambda(\mathbf{x}) = \sum_{\mathfrak{Q}\in\bar{\mathcal{S}}_\lambda} \prod_{\alpha\in\lambda} x_{\mathfrak{Q}(\alpha)} \ .$$

The Schur function $s_\lambda(\mathbf{x})$, unlike our 'wrong' Schur function, $\bar{s}_\lambda(\mathbf{x})$, is symmetric (see, e.g. [4]).

To define the pedestal polynomial we proceed as follows. Let $P, Q \in \mathfrak{st}_\lambda$ be two standard Young tableaux of shape λ. We are going to compare the corresponding linear orders $\preccurlyeq_P$ and $\preccurlyeq_Q$ on the set of the nodes of λ. Let $\alpha_1 = (1,1) \prec_Q \alpha_2 \prec_Q \cdots \prec_Q \alpha_n$ be the list of all the nodes of λ, enumerated according to the order $\preccurlyeq_Q$. We say that a node α_k is a (P,Q)-disagreement node, if $\alpha_{k+1} \prec_P \alpha_k$ (while, of course, $\alpha_k \prec_Q \alpha_{k+1}$). We define the reverse plane partition $q_{P,Q}$ of shape λ by

$$q_{P,Q}(\alpha_k) = \sharp\{l \ : \ l < k \ , \ \alpha_l \text{ is a } (P,Q)\text{-disagreement node}\} \ . \tag{1}$$

Indeed, the function $q_{P,Q}$ is non-decreasing with respect to the order $\preccurlyeq_Q$, hence $q_{P,Q}$ is a reverse plane partition.

The reverse plane partition $q_{P,Q}$ thus defined is called P-pedestal of Q, see [2]. Let ${}^\lambda M$ be the matrix, whose rows and columns are indexed by the standard tableaux of shape λ, with the entries

$$({}^\lambda M)_{P,Q} := \prod_{\alpha\in\lambda} x_{q_{P,Q}(\alpha)} \ . \tag{2}$$

We finally define the polynomial $\mathfrak{h}_P(\mathbf{x})$ by

$$\mathfrak{h}_P(\mathbf{x}) = \sum_{Q\in\mathfrak{st}_\lambda} ({}^\lambda M)_{P,Q} \ . \tag{3}$$

The set (as Q runs over $\mathfrak{st}_\lambda$) of P-pedestals depends on P. However, the polynomial $\mathfrak{h}_P$ has the following surprising property.

Theorem 2. *The function $\mathfrak{h}_P(\mathbf{x})$ does not depend on P from $\mathfrak{st}_\lambda$.*

Definition 3. *We call the function*

$$\mathfrak{h}_\lambda(\mathbf{x}) = \mathfrak{h}_P(\mathbf{x}) \ ,$$

where P is any standard Young tableau P of shape λ, the ***pedestal polynomial****.*

Our theorem states that the pedestal polynomial is well-defined. For example, $\mathfrak{h}_{(3,2)}(\mathbf{x}) = x_0^5 + x_0^4 x_1 + x_0^3 x_1^2 + x_0^2 x_1^3 + x_0^2 x_1^2 x_2$.

3 Proof of Theorem 2

Let R_n be the component of degree n of the ring of formal power series in variables $\mathbf{x} = (x_0, x_1, x_2, ...)$ with integer coefficients. The element $u(\mathbf{x})$ of R_n is a sum,

$$u(\mathbf{x}) = \sum_{i_1 \le i_2 \le ... \le i_n} a_{i_1 i_2 ... i_n} \mathbf{x}_{i_1, i_2, ..., i_n} \ , \quad \mathbf{x}_{i_1, i_2, ..., i_n} := x_{i_1} x_{i_2} ... x_{i_n} \ ,$$

where $i_1, i_2, ..., i_n$ are non-negative integers, and the coefficients $a_{i_1 i_2 ... i_n}$ are integer. For example, the functions $s_\lambda(\mathbf{x})$ and $\bar{s}_\lambda(\mathbf{x})$ belong to R_n.

The function $\bar{s}_{(n)}$, corresponding to the one-row partition $\lambda = (n)$,

$$\bar{s}_{(n)}(\mathbf{x}) = \sum_{i_1 \le i_2 \le ... \le i_n} \mathbf{x}_{i_1, i_2, ..., i_n} \in R_n$$

will play a special role in our argument.

We define the $*$-product on monomials by

$$\mathbf{x}_{i_1, i_2, ..., i_n} * \mathbf{x}_{j_1, j_2, ..., j_n} := \mathbf{x}_{i_1 + j_1, i_2 + j_2, ..., i_n + j_n}$$

and extend it by linearity to the ring structure on R_n. The ring $(R_n, *)$ is isomorphic to a subring of the ring $\mathbb{C}[[y_1, ..., y_n]]$ of formal power series in n variables, via the monomorphism $\varphi_n : R_n \to \mathbb{C}[[y_1, ..., y_n]]$, defined by

$$\varphi_n(\mathbf{x}_{i_1, i_2, ..., i_n}) = y_1^{i_1} ... y_n^{i_n} \ .$$

In particular, $(R_n, *)$ inherits from $\mathbb{C}[[y_1, ..., y_n]]$ the property of having no zero divisors.

Fix a standard Young tableau $P \in \mathfrak{st}_\lambda$. We will prove now the identity

$$\bar{s}_\lambda(\mathbf{x}) = \mathfrak{h}_P(\mathbf{x}) * \bar{s}_{(n)}(\mathbf{x}) \ , \tag{4}$$

which implies, due to the absence of zero divisors in $(R_n, *)$, the assertion of the theorem, since the first and the last terms in the identity do not depend on P.

The bijective proof of the identity (4) follows from [2]. Relations (46), (48) and (50) of that paper describe bijections b_{St} , b_{St}^{-1} between the set of reverse plane partitions of shape λ and the product of the set of P-pedestals and the set of Young diagrams with at most n rows. Let $\mathfrak{Q}$ be

a reverse plane partition and $b_{St}(\mathfrak{Q}) = (q, \mu)$, where q is a P-pedestal and μ a Young diagram. The construction of b_{St} (see below) implies that the monomial, corresponding to $\mathfrak{Q}$ in $\bar{s}_\lambda(\mathbf{x})$ is the $*$-product of the monomial corresponding to q in $\mathfrak{h}_P(\mathbf{x})$ and the monomial corresponding to μ in $\bar{s}_{(n)}(\mathbf{x})$, and the proof of (4) follows. A particular case of these bijections was used earlier by D. Knuth in [1].

The bijections b_{St}, b_{St}^{-1} are defined as follows.

Let $\mathfrak{Q}$ be a reverse plane partition of shape λ. We define the partition $\Pi(\mathfrak{Q})$ of $|\mathfrak{Q}|$ by just listing all the entries of the two-dimensional array of values of $\mathfrak{Q}$ in the non-increasing order. Note that $\Pi(\mathfrak{Q})$ has at most n rows.

To every reverse plane partition $\mathfrak{Q}$ we associate the standard Young tableau $Q(\mathfrak{Q}) \in \mathfrak{st}_\lambda$ as follows. Define the linear order $\preccurlyeq_{\mathfrak{Q}}$ on the nodes of λ by

$$\alpha' \prec_{\mathfrak{Q}} \alpha'' \text{ if } \mathfrak{Q}(\alpha') < \mathfrak{Q}(\alpha'') \text{ or if } \mathfrak{Q}(\alpha') = \mathfrak{Q}(\alpha'') \text{ and } \alpha' \prec_P \alpha'' .$$

Then $Q(\mathfrak{Q})$ is defined by the relation: $\preccurlyeq_{Q(\mathfrak{Q})} = \preccurlyeq_{\mathfrak{Q}}$. Now the bijection b_{St} is defined by

$$b_{St}(\mathfrak{Q}) = \left(q_{P,Q(\mathfrak{Q})}, \Pi\left(\mathfrak{Q} - q_{P,Q(\mathfrak{Q})}\right)\right) ,$$

where the reverse plane partition $\mathfrak{Q} - q_{P,Q(\mathfrak{Q})}$ is given by

$$\left(\mathfrak{Q} - q_{P,Q(\mathfrak{Q})}\right)(\alpha) = \mathfrak{Q}(\alpha) - q_{P,Q(\mathfrak{Q})}(\alpha), \ \alpha \in \lambda .$$

To construct the inverse bijection, b_{St}^{-1}, we first associate to every standard Young tableau $Q \in \mathfrak{st}_\lambda$ and every partition $p = (\Lambda_1, \Lambda_2, \ldots, \Lambda_n)$ the reverse plane partition $\mathfrak{Q}_{Q,p}$ of shape λ, by

$$\mathfrak{Q}_{Q,p}(i,j) = \Lambda_{Q(i,j)} .$$

Then

$$b_{St}^{-1}\left(q_{P,Q}, p\right) = q_{P,Q} + \mathfrak{Q}_{Q,p} .$$

The proof is finished. □

4 Discussion

The identity (4) is of independent interest. The principal specialization, $x_i \mapsto x^i$, turns the 'wrong' Schur function $\bar{s}_\lambda(\mathbf{x})$ into the generating function $\sigma_\lambda(x)$ for the number of reverse plane partitions of shape λ, given by the Mac-Mahon–Stanley formula

$$\sigma_\lambda(x) = \frac{1}{\prod_{\alpha\in\lambda}(1 - x^{h_\alpha})} ,$$

where h_α is the hook length of a node α of λ. The term $\bar{s}_{(n)}(\mathbf{x})$ turns into the generating function of Young diagrams with at most n rows:

$$\sigma_{(n)}(x) = \frac{1}{(1-x)(1-x^2)\ldots(1-x^n)} .$$

Finally, the function $\mathfrak{h}_\lambda(\mathbf{x})$ turns into the generating polynomial $\pi_\lambda(x)$ of the sequence $\{p_{\lambda,k}\}$ with $p_{\lambda,k}$ the number of P-pedestals of volume $k \geq 0$ (for any $P \in \mathfrak{st}_\lambda$). We obtain

$$\frac{1}{\prod_{\alpha\in\lambda}(1-x^{h_\alpha})} = \frac{\pi_\lambda(x)}{(1-x)(1-x^2)\ldots(1-x^n)} \ . \tag{5}$$

It follows from (3) that the function

$$\pi_\lambda(x) = \sum_{Q\in\mathfrak{st}_\lambda} x^{|q_{P,Q}|} \tag{6}$$

does not depend on the choice of P while the contribution of an individual standard Young tableau Q does.

The formula (5) can be found in [4], but there the polynomial $\pi_\lambda(x)$ is given by any of two other expressions:

$$\pi_\lambda(x) = x^{-l(\lambda)} \sum_{Q\in\mathfrak{st}_\lambda} x^{\mathrm{maj}(Q)} \ \text{ and } \ \pi_\lambda(x) = x^{-l(\lambda)} \sum_{Q\in\mathfrak{st}_\lambda} x^{\mathrm{comaj}(Q)} \ ,$$

where $l(\lambda) = \sum_{(i,j)\in\lambda}(i-1)$. It is interesting to note that in general neither of the two functions on $\mathfrak{st}_\lambda$, $\mathrm{maj}(\cdot) - l(\lambda)$ and $\mathrm{comaj}(\cdot) - l(\lambda)$, nor their partners for the transposed to λ Young diagram, belong to our family $\left\{\mathrm{vol}\left(q_{P,*}\right) : P \in \mathfrak{st}_\lambda\right\}$. For example, take $\lambda = (3,2,1)$.

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