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A variational inequality and applications to quasistatic problems with Coulomb friction

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Abstract

The aim of this paper is to study an evolution variational inequality that generalizes some contact problems with Coulomb friction in small deformation elasticity. Using an incremental procedure, appropriate estimates and convergence properties of the discrete solutions, the existence of a continuous solution is proved. This abstract result is applied to quasistatic contact problems with a local Coulomb friction law for nonlinear Hencky and also for linearly elastic materials.

1 Introduction

This paper concerns the analysis of an evolution variational inequality that represents a generalization of some quasistatic elastic problems with point-wise Coulomb friction and relaxed unilateral contact.

Existence and approximation of solutions to the quasistatic elastic problems have been studied for various contact conditions. Based on the variational formulation proposed in [1], the quasistatic unilateral contact problems with local Coulomb friction have been studied in [2, 3, 4] and the normal compliance models have been investigated by several authors, see, e.g. [5, 6, 7] and references therein. Dynamic frictional contact problems with normal compliance laws for some viscoelastic bodies have been studied in [8, 5, 9, 10, 11].

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A comprehensive presentation of contact models for the quasistatic processes can be found in [12]. An unified approach, which can be applied to various quasistatic problems, including unilateral and bilateral contact with nonlocal friction, or normal compliance conditions, has been considered in [13], and different (quasi)static contact problems with nonlocal friction are analyzed in [14].

A static contact problem with relaxed unilateral conditions and pointwise Coulomb friction was studied in [15], based on abstract formulations and Ky Fan's fixed point theorem. Recently, an extension of these results to dynamic contact problems in viscoelasticity was treated in [16, 17].

This paper extends the results presented in [15] to a new evolution variational inequality involving a nonlinear operator and with applications to two-field formulations of some nonsmooth elastic quasistatic contact problems with friction.

The paper is organized as follows. In Section 2, a general evolution variational inequality is analyzed by an incremental method. Using the Ky Fan's theorem, the existence of incremental solutions is proved. Then several estimates and compactness arguments enable to pass to limits in order to establish the existence of a continuous solution. In Section 3, applications to quasistatic contact problems with local Coulomb friction, for nonlinear Hencky and linearly elastic bodies, are presented.

2 An implicit variational inequality

For simplicity and also in view of applications to contact mechanics, we shall confine attention to the case when Ω is an open, bounded, connected set $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with the boundary $\Gamma \in C^{1,1}$ and with Ξ an open part of Γ .

We denote $\Xi_T := \Xi \times (0, T)$, where $0 < T < +\infty$, and define the closed convex cones $L_-^2(\Xi)$, $L_-^2(\Xi_T)$ in the Hilbert spaces $L^2(\Xi)$, $L^2(\Xi_T)$, respectively, as follows:

$$L_-^2(\Xi) := \{\delta \in L^2(\Xi); \delta \leq 0 \text{ a.e. in } \Xi\},$$

$$L_-^2(\Xi_T) := \{\delta \in L^2(0, T; L^2(\Xi)); \delta \leq 0 \text{ a.e. in } \Xi_T\}.$$

Let $\underline{\kappa}, \bar{\kappa} : \mathbb{R} \rightarrow \mathbb{R}$ be two mappings with $\underline{\kappa}$ lower semicontinuous and $\bar{\kappa}$ upper semicontinuous, satisfying the following conditions:

$$\underline{\kappa}(s) \leq \bar{\kappa}(s) \leq 0 \quad \forall s \in \mathbb{R}, \tag{1}$$

$$\exists r_0 \geq 0 \text{ such that } |\underline{\kappa}(s)| \leq r_0 \quad \forall s \in \mathbb{R}. \tag{2}$$

For every $\zeta \in L^2(\Xi)$, define the following subset of $L^2_-(\Xi)$:

$$\Lambda(\zeta) = \{\eta \in L^2_-(\Xi); \underline{\kappa} \circ \zeta \leq \eta \leq \bar{\kappa} \circ \zeta \text{ a.e. in } \Xi\}, \quad (3)$$

which is clearly nonempty, because the bounding functions belong to the respective set, closed and convex.

Since $\text{meas}(\Xi) < \infty$ and $\underline{\kappa}, \bar{\kappa}$ satisfy (2), it is also readily seen that for all $\zeta \in L^2(\Xi)$ the set $\Lambda(\zeta)$ is bounded in norm in $L^2(\Xi)$ by $R_0 = r_0(\text{meas}(\Xi))^{1/2}$ and in $L^\infty(\Xi)$ by r_0 .

The following compactness theorem proved in [18] will be used in this paper.

Theorem 2.1. *Let \hat{X}, \hat{U} and \hat{Y} be three Banach spaces such that $\hat{X} \subset \hat{U} \subset \hat{Y}$ with compact embedding from \hat{X} into \hat{U} .*

(i) Let \mathcal{G} be bounded in $L^p(0, T; \hat{X})$, where $1 \leq p < \infty$, and $\partial\mathcal{G}/\partial t := \{f; f \in \mathcal{G}\}$ be bounded in $L^1(0, T; \hat{Y})$. Then \mathcal{G} is relatively compact in $L^p(0, T; \hat{U})$.

(ii) Let \mathcal{G} be bounded in $L^\infty(0, T; \hat{X})$ and $\partial\mathcal{G}/\partial t$ be bounded in $L^r(0, T; \hat{Y})$, where $r > 1$. Then \mathcal{G} is relatively compact in $C([0, T]; \hat{U})$.

If H is a Hilbert space, unless otherwise stated we shall denote by $\langle \cdot, \cdot \rangle_H$ its inner product and by $\|\cdot\|_H$ the corresponding norm.

Let $(V, \|\cdot\|, \langle \cdot, \cdot \rangle)$ and $(U, \|\cdot\|_U)$ be two Hilbert spaces such that $V \subset U$ with continuous and compact embedding.

Consider a functional $F : V \rightarrow \mathbb{R}$ differentiable on V and assume that its derivative $F' : V \rightarrow V$ is strongly monotone and Lipschitz continuous, that is there exist two constants $\alpha, \beta > 0$ for which

$$\alpha\|v - u\|^2 \leq \langle F'(v) - F'(u), v - u \rangle \quad (4)$$

and

$$\|F'(v) - F'(u)\| \leq \beta\|v - u\| \quad (5)$$

for all $u, v \in V$.

Using the relations

$$\begin{aligned} F(v) - F(u) &= \int_0^1 \langle F'(u + r(v - u)), v - u \rangle dr \\ &= \langle F'(u), v - u \rangle + \int_0^1 \langle F'(u + r(v - u)) - F'(u), v - u \rangle dr \end{aligned}$$

and (4), (5), it is easily seen that for all $u, v \in V$ it results

$$\begin{aligned} \langle F'(u), v - u \rangle + \frac{\alpha}{2} \|v - u\|^2 &\leq F(v) - F(u) \\ &\leq \langle F'(u), v - u \rangle + \frac{\beta}{2} \|v - u\|^2. \end{aligned} \quad (6)$$

We remark that since F satisfies (6), it follows that F is strictly convex and sequentially weakly lower semicontinuous on V .

Let $(X, \|\cdot\|_X)$ be a Hilbert space such that $X \subset L^2(\Gamma)$ with continuous and compact embedding, and $l_0 : V \rightarrow X$, $l : V \rightarrow L^2(\Xi)$, $\phi : L^2_-(\Xi) \times V \rightarrow \mathbb{R}$ be three mappings satisfying the following conditions:

$$l_0 \text{ is linear and continuous,} \quad (7)$$

$$\begin{aligned} \exists k_1 > 0 \text{ such that } \forall v_1, v_2 \in V, \\ \|l(v_1) - l(v_2)\|_{L^2(\Xi)} \leq k_1 \|v_1 - v_2\|_U, \end{aligned} \quad (8)$$

$$\begin{aligned} \forall \gamma, \delta \in L^2_-(\Xi), \forall v, w \in V \text{ verifying } \gamma \in \Lambda(l(v)) \text{ and } \delta \in \Lambda(l(w)), \\ \langle \gamma - \delta, l_0(v - w) \rangle_{L^2(\Xi)} \leq 0. \end{aligned} \quad (9)$$

$$\forall \gamma \in L^2_-(\Xi), \forall \theta \geq 0, \forall v_1, v_2, v \in V,$$

$$\phi(\gamma, v_1 + v_2) \leq \phi(\gamma, v_1) + \phi(\gamma, v_2), \quad (10)$$

$$\phi(\gamma, \theta v) = \theta \phi(\gamma, v), \quad (11)$$

$$\forall v \in V, \quad \phi(0, v) = 0, \quad (12)$$

$$\exists k_2, k_3 > 0 \text{ such that } \forall \gamma, \delta \in L^2_-(\Xi), \forall v \in V,$$

$$|\phi(\gamma, v) - \phi(\delta, v)| \leq k_2 \|\gamma - \delta\|_{L^2(\Xi)} \|v\|_U, \quad (13)$$

$$|\phi(\gamma, v) - \phi(\delta, v)| \leq k_3 \|\gamma - \delta\|_{X'} \|v\|, \quad (14)$$

$$\exists k_4 > 0 \text{ such that } \|\gamma_1 - \gamma_2\|_{X'} \leq k_4 (\|u_1 - u_2\| + \|f_1 - f_2\|), \quad (15)$$

for all $\gamma_{1,2} \in L^2_-(\Xi)$, $u_{1,2}, f_{1,2}, d_{1,2} \in V$ verifying

$$\begin{aligned} (Q_1) \quad &\langle F'(u_1), v - u_1 \rangle - \langle \gamma_1, l_0(v - u_1) \rangle_{L^2(\Xi)} \\ &+ \phi(\gamma_1, v - d_1) - \phi(\gamma_1, u_1 - d_1) \geq \langle f_1, v - u_1 \rangle \quad \forall v \in V, \end{aligned}$$

$$\begin{aligned} (Q_2) \quad &\langle F'(u_2), v - u_2 \rangle - \langle \gamma_2, l_0(v - u_2) \rangle_{L^2(\Xi)} \\ &+ \phi(\gamma_2, v - d_2) - \phi(\gamma_2, u_2 - d_2) \geq \langle f_2, v - u_2 \rangle \quad \forall v \in V, \end{aligned}$$

and we assume that $k_3 k_4 < \alpha$. (16)

Let $f \in W^{1,2}(0, T; V)$, $u^0 \in V$, $\lambda^0 \in \Lambda(l(u^0))$ be given and satisfy the following compatibility condition:

$$\begin{aligned} \langle F'(u^0), v - u^0 \rangle - \langle \lambda^0, l_0(v - u^0) \rangle_{L^2(\Xi)} \\ + \phi(\lambda^0, v) - \phi(\lambda^0, u^0) \geq \langle f(0), v - u^0 \rangle \quad \forall v \in V. \end{aligned} \quad (17)$$

Consider the following problem.

Problem Q: Find $u \in W^{1,2}(0, T; V)$, $\lambda \in W^{1,2}(0, T; X')$ such that $u(0) = u^0$, $\lambda(0) = \lambda^0$, $\lambda(t) \in \Lambda(l(u(t)))$ for almost all $t \in (0, T)$, and

$$\begin{aligned} \langle F'(u), v - \dot{u} \rangle - \langle \lambda, l_0(v - \dot{u}) \rangle_{L^2(\Xi)} + \phi(\lambda, v) \\ - \phi(\lambda, \dot{u}) \geq \langle f, v - \dot{u} \rangle \quad \forall v \in V \quad \text{a.e. on } (0, T). \end{aligned} \quad (18)$$

2.1 Incremental formulations

For $n \in \mathbb{N}^*$, we set $\Delta t := T/n$, $t_i := i \Delta t$, $i = 0, 1, \dots, n$. If θ is a continuous function of $t \in [0, T]$ valued in some vector space, we use the notations $\theta^i := \theta(t_i)$ unless $\theta = u$, and if ϖ^i , $\forall i \in \{0, 1, \dots, n\}$, are elements of some vector space, then we set

$$\partial \varpi^i := \frac{\varpi^{i+1} - \varpi^i}{\Delta t}, \quad \Delta \varpi^i := \varpi^{i+1} - \varpi^i \quad \forall i \in \{0, 1, \dots, n-1\}.$$

We approximate the problem Q using the following sequence of incremental problems $(Q^{i,n})_{i=0,1,\dots,n-1}$.

Problem $Q^{i,n}$: Find $u^{i+1} \in V$, $\lambda^{i+1} \in \Lambda(l(u^{i+1}))$ such that

$$\begin{aligned} \langle F'(u^{i+1}), v - \partial u^i \rangle - \langle \lambda^{i+1}, l_0(v - \partial u^i) \rangle_{L^2(\Xi)} + \phi(\lambda^{i+1}, v) \\ - \phi(\lambda^{i+1}, \partial u^i) \geq \langle f^{i+1}, v - \partial u^i \rangle \quad \forall v \in V. \end{aligned} \quad (19)$$

It is easily seen that for all $i \in \{0, 1, \dots, n-1\}$ the problem $Q^{i,n}$ is equivalent to the following implicit variational inequality.

Problem $\hat{Q}^{i,n}$: Find $u^{i+1} \in V$, $\lambda^{i+1} \in \Lambda(l(u^{i+1}))$ such that

$$\begin{aligned} \langle F'(u^{i+1}), v - u^{i+1} \rangle - \langle \lambda^{i+1}, l_0(v - u^{i+1}) \rangle_{L^2(\Xi)} + \phi(\lambda^{i+1}, v - u^i) \\ - \phi(\lambda^{i+1}, u^{i+1} - u^i) \geq \langle f^{i+1}, v - u^{i+1} \rangle \quad \forall v \in V. \end{aligned} \quad (20)$$

Let us define the following functions:

$$\begin{aligned}
u_n(0) &= \hat{u}_n(0) = u^0, \quad \lambda_n(0) = \lambda^0, \quad f_n(0) = f^0 \quad \text{and} \\
\forall i \in \{0, 1, \dots, n-1\}, \quad \forall t \in (t_i, t_{i+1}], \\
u_n(t) &= u^{i+1}, \quad \lambda_n(t) = \lambda^{i+1}, \\
\hat{u}_n(t) &= u^i + (t - t_i)\partial u^i, \\
\hat{\lambda}_n(t) &= \lambda^i + (t - t_i)\partial \lambda^i, \\
f_n(t) &= f^{i+1}.
\end{aligned}$$

Then for all $n \in N^*$ each of the sequences of inequalities $(Q^{i,n})_{i=0,1,\dots,n-1}$, $(\hat{Q}^{i,n})_{i=0,1,\dots,n-1}$ is equivalent to the following incremental formulation.

Problem Q^n : Find $u_n \in L^2(0, T; V)$, $\lambda_n \in L^2(\Xi_T)$ such that $\lambda_n(t) \in \Lambda(l(u_n(t))) \quad \forall t \in (0, T)$ and

$$\begin{aligned}
\langle F'(u_n(t)), v - \frac{d}{dt}\hat{u}_n(t) \rangle - \langle \lambda_n(t), l_0(v - \frac{d}{dt}\hat{u}_n(t)) \rangle_{L^2(\Xi)} \\
+ \phi(\lambda_n(t), v) - \phi(\lambda_n(t), \frac{d}{dt}\hat{u}_n(t)) \quad (21) \\
\geq \langle f_n(t), v - \frac{d}{dt}\hat{u}_n(t) \rangle \quad \forall v \in V, \quad \text{a.e. on } (0, T).
\end{aligned}$$

First, we prove the existence of a solution to the incremental problem $\hat{Q}^{i,n}$ by a fixed point method.

Let $\Phi^i : L_-^2(\Xi) \rightarrow 2^{L_-^2(\Xi)} \setminus \{\emptyset\}$ be the set-valued mapping defined by

$$\text{for all } \gamma \in L_-^2(\Xi) \quad \Phi^i(\gamma) = \Lambda(l(u_\gamma)), \quad (22)$$

where u_γ is the solution of the following variational inequality of the second kind: find $u_\gamma \in V$ such that

$$\begin{aligned}
\langle F'(u_\gamma), v - u_\gamma \rangle - \langle \gamma, l_0(v - u_\gamma) \rangle_{L^2(\Xi)} + \phi(\gamma, v - u^i) \quad (23) \\
- \phi(\gamma, u_\gamma - u^i) \geq \langle f^{i+1}, v - u_\gamma \rangle \quad \forall v \in V.
\end{aligned}$$

It is easily seen that λ is a fixed point of Φ^i , i.e. $\lambda \in \Phi^i(\lambda)$, iff $(u^{i+1}, \lambda^{i+1}) = (u_\lambda, \lambda)$ is a solution of the problem $\hat{Q}^{i,n}$.

We shall prove the existence of a fixed point of the multifunction Φ^i by using a corollary of the Ky Fan's fixed point theorem [19], proved in [15] in the particular case of a reflexive Banach space.

Definition 2.1. Let Y be a reflexive Banach space, D a weakly closed set in Y , and $\Phi : D \rightarrow 2^Y \setminus \{\emptyset\}$ be a multivalued function. Φ is called sequentially weakly upper semicontinuous if $z_p \rightharpoonup z$, $y_p \in \Phi(z_p)$ and $y_p \rightharpoonup y$ imply $y \in \Phi(z)$.

Proposition 2.1. ([15]) Let Y be a reflexive Banach space, D a convex, closed and bounded set in Y , and $\Phi : D \rightarrow 2^D \setminus \{\emptyset\}$ a sequentially weakly upper semicontinuous multivalued function such that $\Phi(z)$ is convex for every $z \in D$. Then Φ has a fixed point.

Note that since Y is a reflexive Banach space and D is convex, closed and bounded, there is no assumption that Y is separable, see [15, 20].

Theorem 2.2. Assume that (1 - 5), (7 - 14) hold. Then there exists $\lambda \in L^2_-(\Xi)$ such that $\lambda \in \Phi^i(\lambda)$ and $(u^{i+1}, \lambda^{i+1}) = (u_\lambda, \lambda)$ is a solution of the problem $\hat{Q}^{i,n}$.

Proof. We apply Proposition 2.1 to $\Phi = \Phi^i$, $Y = L^2(\Xi)$ and $D = L^2_-(\Xi) \cap \{\zeta \in L^2(\Xi); \|\zeta\|_{L^2(\Xi)} \leq R_0\}$.

The set $D \subset L^2(\Xi)$ is clearly convex, closed and bounded.

By (4), (5), (7), (8), (10 - 14), for every $\gamma \in D$ the classical variational inequality (23) has a unique solution u_γ .

Since for each $\zeta \in L^2(\Xi)$ the set $\Lambda(\zeta)$ is nonempty, convex, closed, and bounded by R_0 , it follows that $\Phi^i(\gamma) = \Lambda(l(u_\gamma))$ is a nonempty, convex and closed subset of D for every $\gamma \in D$.

In order to prove that the multifunction Φ^i is sequentially weakly upper semicontinuous, let $\gamma_p \rightharpoonup \gamma$ in $L^2(\Xi)$, $\gamma_p \in D$, $\eta_p \in \Phi^i(\gamma_p) \forall p \in \mathbb{N}$, $\eta_p \rightharpoonup \eta$ in $L^2(\Xi)$ and let us verify that $\eta \in \Phi^i(\gamma)$.

Let $u_{\gamma_p} \in V$ be the solution of the variational inequality

$$\begin{aligned} \langle F'(u_{\gamma_p}), v - u_{\gamma_p} \rangle - \langle \gamma_p, l_0(v - u_{\gamma_p}) \rangle_{L^2(\Xi)} + \phi(\gamma_p, v - u^i) \\ - \phi(\gamma_p, u_{\gamma_p} - u^i) \geq \langle f^{i+1}, v - u_{\gamma_p} \rangle \quad \forall v \in V. \end{aligned} \quad (24)$$

Taking $v = 0$ in (24) and using (10), we obtain

$$\langle F'(u_{\gamma_p}) - F'(0), u_{\gamma_p} \rangle \leq \langle \gamma_p, l_0(u_{\gamma_p}) \rangle_{L^2(\Xi)} + \phi(\gamma_p, -u_{\gamma_p}) + \langle f^{i+1} - F'(0), u_{\gamma_p} \rangle,$$

so that, by (4), (7), (12), (14),

$$\alpha \|u_{\gamma_p}\|^2 \leq (\|\gamma_p\|_{X'} \|l_0\| + k_3 \|\gamma_p\|_{X'} + \|f^{i+1}\| + \|F'(0)\|) \|u_{\gamma_p}\|,$$

which implies

$$\|u_{\gamma_p}\| \leq \alpha^{-1} (\|l_0\| + k_3) \|\gamma_p\|_{X'} + \|f^{i+1}\| + \|F'(0)\|.$$

Thus

$$\|u_{\gamma_p}\| \leq C_1(\|\gamma_p\|_{X'} + \|f^{i+1}\| + \|F'(0)\|) \quad \forall p \in \mathbb{N}, \quad (25)$$

where $C_1 = \alpha^{-1} \max(\|l_0\| + k_3, 1)$.

As the sequence $(\gamma_p)_p$ is bounded in $L^2(\Xi)$, it follows that $(u_{\gamma_p})_p$ is bounded in V which implies that there exists a subsequence, still denoted by $(u_{\gamma_p})_p$, and an element $u \in V$ such that

$$u_{\gamma_p} \rightharpoonup u \quad \text{in } V. \quad (26)$$

By (6) and (10), the inequality (24) implies

$$\begin{aligned} F(v) - F(u_{\gamma_p}) - \langle \gamma_p, l_0(v - u_{\gamma_p}) \rangle_{L^2(\Xi)} + \phi(\gamma_p, v - u_{\gamma_p}) \\ \geq \langle f^{i+1}, v - u_{\gamma_p} \rangle + \frac{\alpha}{2} \|v - u_{\gamma_p}\|^2 \quad \forall v \in V, \end{aligned} \quad (27)$$

and taking $v = u$, we obtain

$$\begin{aligned} F(u) - F(u_{\gamma_p}) - \langle \gamma_p, l_0(u - u_{\gamma_p}) \rangle_{L^2(\Xi)} + \phi(\gamma_p, u - u_{\gamma_p}) \\ \geq \langle f^{i+1}, u - u_{\gamma_p} \rangle + \frac{\alpha}{2} \|u - u_{\gamma_p}\|^2. \end{aligned}$$

As F is sequentially weakly lower semicontinuous, using the previous relation, (7), (13) and the compact embeddings $X \subset L^2(\Xi)$, $V \subset U$, we have

$$\begin{aligned} \limsup_{p \rightarrow \infty} \frac{\alpha}{2} \|u - u_{\gamma_p}\|^2 \\ \leq F(u) + \limsup_{p \rightarrow \infty} (-F(u_{\gamma_p})) + \lim_{p \rightarrow \infty} |\langle \gamma_p, l_0(u - u_{\gamma_p}) \rangle_{L^2(\Xi)}| \\ + \lim_{p \rightarrow \infty} \phi(\gamma_p, u - u_{\gamma_p}) - \lim_{p \rightarrow \infty} \langle f^{i+1}, u - u_{\gamma_p} \rangle \\ \leq F(u) - \liminf_{p \rightarrow \infty} F(u_{\gamma_p}) + \lim_{p \rightarrow \infty} \|\gamma_p\|_{L^2(\Xi)} \|l_0(u - u_{\gamma_p})\|_{L^2(\Xi)} \\ + \lim_{p \rightarrow \infty} k_2 \|\gamma_p\|_{L^2(\Xi)} \|u - u_{\gamma_p}\|_U - \lim_{p \rightarrow \infty} \langle f^{i+1}, u - u_{\gamma_p} \rangle \\ = F(u) - \liminf_{p \rightarrow \infty} F(u_{\gamma_p}) \leq 0, \end{aligned}$$

which proves that

$$u_{\gamma_p} \rightarrow u \quad \text{in } V. \quad (28)$$

Passing to the limit in (24), it follows that u is a solution of (23) and since its solution is unique we obtain that $u_\gamma = u = \lim_{p \rightarrow \infty} u_{\gamma_p}$.

Now, the relation $\eta^p \in \Phi^i(\gamma_p)$ implies

$$\eta^p \in \Lambda(l(u_{\gamma_p}))$$

that is

$$\underline{\kappa} \circ l_p \leq \eta^p \leq \bar{\kappa} \circ l_p \quad \text{a.e. in } \Xi, \quad (29)$$

for all $p \in \mathbb{N}$, where $l_p := l(u_{\gamma_p})$. The relations (29) are equivalent to

$$\int_{\omega} \underline{\kappa} \circ l_p \leq \int_{\omega} \eta^p \leq \int_{\omega} \bar{\kappa} \circ l_p,$$

for every measurable subset $\omega \subset \Xi$ and for all $p \in \mathbb{N}$.

Using (28), (8), the semi-continuity of $\underline{\kappa}$ and $\bar{\kappa}$, the relations (1), (2), the convergence property $\int_{\omega} \eta^p \rightarrow \int_{\omega} \eta$, and passing to limits according to Fatou's lemma, we obtain

$$\int_{\omega} \underline{\kappa} \circ l(u_{\gamma}) \leq \int_{\omega} \eta \leq \int_{\omega} \bar{\kappa} \circ l(u_{\gamma}), \quad (30)$$

for every measurable subset $\omega \subset \Xi$, which implies $\eta \in \Phi^i(\gamma)$.

By Proposition 2.1 there exists a fixed point λ of Φ^i and (u_{λ}, λ) is clearly a solution to the problem $\hat{Q}^{i,n}$. \square

Remark 2.1. *This existence result insures also that there exists (u^0, λ^0) satisfying the compatibility condition (17).*

2.2 Existence of a solution to the continuous problem

We now establish some useful estimates independent of n for the solutions of the incremental formulations $\hat{Q}^{i,n}$ and Q^n .

Lemma 2.1. *Under the above hypotheses, for all $n \in N^*$ and all $i \in \{0, 1, \dots, n-1\}$ the following estimates hold:*

$$\|u^{i+1}\| \leq C_1(\|\lambda^{i+1}\|_{X'} + \|f^{i+1}\| + \|F'(0)\|), \quad (31)$$

$$\|\Delta u^i\| \leq \frac{k_3}{\alpha} \|\Delta \lambda^i\|_{X'} + \frac{1}{\alpha} \|\Delta f^i\|, \quad (32)$$

$$\|\Delta \lambda^i\|_{X'} \leq k_4(\|\Delta u^i\| + \|\Delta f^i\|), \quad (33)$$

$$\|\Delta u^i\| \leq C_2 \|\Delta f^i\|, \quad (34)$$

$$\|\Delta \lambda^i\|_{X'} \leq C_3 \|\Delta f^i\|, \quad (35)$$

where $C_2 = \frac{k_3 k_4 + 1}{\alpha - k_3 k_4}$, $C_3 = \frac{(\alpha + 1) k_4}{\alpha - k_3 k_4}$.

Proof. By similar arguments to those that enabled to prove (25), using (20) the estimate (31) follows.

If we take $v = u^i$ in (20) then

$$\begin{aligned} & \langle F'(u^{i+1}), u^i - u^{i+1} \rangle - \langle \lambda^{i+1}, l_0(u^i - u^{i+1}) \rangle_{L^2(\Xi)} \\ & - \phi(\lambda^{i+1}, u^{i+1} - u^i) \geq \langle f^{i+1}, u^i - u^{i+1} \rangle, \end{aligned} \quad (36)$$

and taking $v = u^{i+1}$ in (20), corresponding to $i - 1$ if $i \geq 1$, or in (17) if $i = 0$, we have

$$\begin{aligned} & \langle F'(u^i), u^{i+1} - u^i \rangle - \langle \lambda^i, l_0(u^{i+1} - u^i) \rangle_{L^2(\Xi)} + \phi(\lambda^i, u^{i+1} - u^i) \\ & - \phi(\lambda^i, u^i - u^{i-1}) \geq \langle f^i, u^{i+1} - u^i \rangle. \end{aligned} \quad (37)$$

By (10), the inequalities (36) and (37) imply

$$\begin{aligned} & \langle F'(u^{i+1}) - F'(u^i), u^{i+1} - u^i \rangle \leq \langle \lambda^{i+1} - \lambda^i, l_0(u^{i+1} - u^i) \rangle_{L^2(\Xi)} \\ & + \phi(\lambda^i, u^{i+1} - u^i) - \phi(\lambda^{i+1}, u^{i+1} - u^i) + \langle f^{i+1} - f^i, u^{i+1} - u^i \rangle. \end{aligned} \quad (38)$$

As (u^{i+1}, λ^{i+1}) and (u^i, λ^i) are solutions of $\hat{Q}^{i+1,n}$ and $\hat{Q}^{i,n}$, respectively, we have $\lambda^{i+1} \in \Lambda(l(u^{i+1}))$, $\lambda^i \in \Lambda(l(u^i))$, so that by (9)

$$\langle \lambda^{i+1} - \lambda^i, l_0(u^{i+1} - u^i) \rangle_{L^2(\Xi)} \leq 0.$$

Using this relation, (4) and (14) in (38), we have

$$\alpha \|u^{i+1} - u^i\|^2 \leq k_3 \|\lambda^{i+1} - \lambda^i\|_{X'} \|u^{i+1} - u^i\| + \|f^{i+1} - f^i\| \|u^{i+1} - u^i\|,$$

from which (32) follows.

From (15), we obtain (33) and by (32), (33) the estimates (34), (35) can be easily verified. \square

Based on the previous lemma and the fact that $f \in W^{1,2}(0, T; V)$ is absolutely continuous, after possibly being redefined on a set of measure zero, the following estimates can be established by a straightforward computation, see, e.g. [21], [22].

Lemma 2.2. For all $n \in N^*$

$$\|u_n(t)\| \leq C_1(\|\lambda_n(t)\|_{X'} + \|f_n(t)\| + \|F'(0)\|) \quad \forall t \in [0, T], \quad (39)$$

$$\|u_n(t) - \hat{u}_n(t)\| \leq \frac{T}{n} \left\| \frac{d}{dt} \hat{u}_n(t) \right\| \leq C_2 \left\| f_n(t) - f_n\left(t - \frac{T}{n}\right) \right\| \quad (40)$$

$$\leq C_2 \int_{t-\frac{T}{n}}^{\min\{t+\frac{T}{n}, T\}} \|\dot{f}(\tau)\| d\tau \quad \forall t \in [0, T],$$

$$\|\lambda_n(t) - \hat{\lambda}_n(t)\|_{X'} \leq \frac{T}{n} \left\| \frac{d}{dt} \hat{\lambda}_n(t) \right\|_{X'} \quad (41)$$

$$\leq C_3 \left\| f_n(t) - f_n\left(t - \frac{T}{n}\right) \right\| \quad \forall t \in [0, T],$$

$$\|u_n - \hat{u}_n\|_{L^2(0, T; V)} = \frac{T}{n\sqrt{3}} \left\| \frac{d}{dt} \hat{u}_n \right\|_{L^2(0, T; V)} \quad (42)$$

$$\leq C_2 \frac{T}{n\sqrt{3}} \|\dot{f}\|_{L^2(0, T; V)},$$

$$\|\lambda_n - \hat{\lambda}_n\|_{L^2(0, T; X')} = \frac{T}{n\sqrt{3}} \left\| \frac{d}{dt} \hat{\lambda}_n \right\|_{L^2(0, T; X')} \quad (43)$$

$$\leq C_3 \frac{T}{n\sqrt{3}} \|\dot{f}\|_{L^2(0, T; V)}.$$

Using Lemma 2.2, it follows that $(\hat{u}_n)_n$ is bounded in $W^{1,2}(0, T; V)$, $(\hat{\lambda}_n)_n$ is bounded in $W^{1,2}(0, T; X') \cap L^\infty(\Xi_T)$, and since all these functions are absolutely continuous, after possibly being redefined on a set of measure zero, we have the following convergence results.

Lemma 2.3. There exist subsequences of $(u_n, \hat{u}_n)_n$ and $(\lambda_n, \hat{\lambda}_n)_n$, denoted by $(u_{n_p}, \hat{u}_{n_p})_p$ and $(\lambda_{n_p}, \hat{\lambda}_{n_p})_p$, and two elements $u \in W^{1,2}(0, T; V)$, $\lambda \in W^{1,2}(0, T; X') \cap L^2(\Xi_T)$ such that

$$u_{n_p}(t) \rightharpoonup u(t) \text{ in } V \quad \forall t \in [0, T], \quad (44)$$

$$\hat{u}_{n_p} \rightharpoonup u \text{ in } W^{1,2}(0, T; V), \quad (45)$$

$$\lambda_{n_p}(t) \rightharpoonup \lambda(t) \text{ in } X' \quad \forall t \in [0, T], \quad (46)$$

$$\lambda_{n_p}, \hat{\lambda}_{n_p} \rightharpoonup \lambda \text{ in } L^2(0, T; L^2(\Xi)), \quad (47)$$

$$\hat{\lambda}_{n_p} \rightharpoonup \lambda \text{ in } W^{1,2}(0, T; X'). \quad (48)$$

Lemma 2.4. *For the subsequences $(\hat{u}_{n_p})_p$, $(\lambda_{n_p})_p$, the following relation holds:*

$$\liminf_{p \rightarrow \infty} \int_0^T \phi(\lambda_{n_p}(t), \frac{d}{dt} \hat{u}_{n_p}(t)) dt \geq \int_0^T \phi(\lambda(t), \frac{d}{dt} \hat{u}(t)) dt. \quad (49)$$

Proof. According to Theorem 2.1 with $\mathcal{G} = (\hat{\lambda}_{n_p})_p$, $\hat{X} = L^2(\Xi)$, $\hat{U} = H^{\iota-1/2}(\Xi)$, $\hat{Y} = X'$, $p = 2$, $0 < \iota < \frac{1}{2}$, and to (47), (48), we obtain that

$$\lambda_{n_p}, \hat{\lambda}_{n_p} \rightarrow \lambda \text{ in } L^2(0, T; X'). \quad (50)$$

By (14) it follows that

$$\begin{aligned} & \left| \int_0^T (\phi(\lambda_{n_p}(t), \frac{d}{dt} \hat{u}_{n_p}(t)) - \phi(\lambda(t), \frac{d}{dt} \hat{u}_{n_p}(t))) dt \right| \\ & \leq k_3 \int_0^T \|\lambda_{n_p}(t) - \lambda(t)\|_{X'} \|\frac{d}{dt} \hat{u}_{n_p}(t)\| dt \\ & \leq k_3 \|\lambda_{n_p} - \lambda\|_{L^2(0, T; X')} \|\frac{d}{dt} \hat{u}_{n_p}\|_{L^2(0, T; V)}, \end{aligned}$$

which implies

$$\lim_{p \rightarrow \infty} \int_0^T (\phi(\lambda_{n_p}(t), \frac{d}{dt} \hat{u}_{n_p}(t)) - \phi(\lambda(t), \frac{d}{dt} \hat{u}_{n_p}(t))) dt = 0. \quad (51)$$

Since by (10), (11), (14), $\phi(\lambda(t), \cdot)$ is convex lower semicontinuous on V for a.e. $t \in [0, T]$, the mapping $\int_0^T \phi(\lambda(t), \cdot) dt$ is convex lower semicontinuous on $L^2(0, T; V)$ (see, e.g. [23]), so that

$$\liminf_{p \rightarrow \infty} \int_0^T \phi(\lambda(t), \frac{d}{dt} \hat{u}_{n_p}(t)) dt \geq \int_0^T \phi(\lambda(t), \frac{d}{dt} \hat{u}(t)) dt. \quad (52)$$

From (51) and (52), (49) follows. \square

Now, we prove the main strong convergence and existence result.

Theorem 2.3. *Under the assumptions (1 - 5), (7 - 16), every convergent subsequence of Lemma 2.3, $(u_{n_p}, \hat{u}_{n_p})_p$, $(\lambda_{n_p}, \hat{\lambda}_{n_p})_p$, and their limits $u \in W^{1,2}(0, T; V)$, $\lambda \in W^{1,2}(0, T; X') \cap L^2(\Xi_T)$ have the following strong convergence properties*

$$u_{n_p}(t) \rightarrow u(t) \text{ in } V \quad \forall t \in [0, T], \quad (53)$$

$$\lambda_{n_p}(t) \rightarrow \lambda(t) \text{ in } X' \quad \forall t \in [0, T], \quad (54)$$

and (u, λ) is a solution to the problem Q .

Proof. In order to prove (53), we use the same method as the one that enabled to obtain (28). By (10) the sequence $(\hat{Q}^{i,n})_{i=0,1,\dots,n-1}$ implies the following inequality: for every $t \in [0, T]$

$$\begin{aligned} & \langle F'(u_n(t)), v - u_n(t) \rangle - \langle \lambda_n(t), l_0(v - u_n(t)) \rangle_{L^2(\Xi)} \\ & + \phi(\lambda_n(t), v - u_n(t)) \geq \langle f_n(t), v - u_n(t) \rangle \quad \forall v \in V \end{aligned} \quad (55)$$

and taking $v = u$, by (6) we derive

$$\begin{aligned} & F(u(t)) - F(u_{n_p}(t)) - \langle \lambda_{n_p}(t), l_0(u(t) - u_{n_p}(t)) \rangle_{L^2(\Xi)} \\ & + \phi(\lambda_{n_p}(t), u(t) - u_{n_p}(t)) \geq \langle f_{n_p}(t), u(t) - u_{n_p}(t) \rangle + \frac{\alpha}{2} \|u(t) - u_{n_p}(t)\|^2 \quad \forall p \in \mathbb{N}. \end{aligned} \quad (56)$$

Using that F is sequentially weakly lower semicontinuous, (7), (13), the compact embeddings $X \subset L^2(\Xi)$, $V \subset U$ and that for all $t \in [0, T]$ $(\lambda_{n_p}(t))_p$ is bounded in $L^2(\Xi)$ by R_0 , the previous relation implies

$$\begin{aligned} & \limsup_{p \rightarrow \infty} \frac{\alpha}{2} \|u(t) - u_{n_p}(t)\|^2 \\ & \leq F(u(t)) + \limsup_{p \rightarrow \infty} (-F(u_{n_p}(t))) + \lim_{p \rightarrow \infty} |\langle \lambda_{n_p}(t), l_0(u(t) - u_{n_p}(t)) \rangle_{L^2(\Xi)}| \\ & + \lim_{p \rightarrow \infty} \phi(\lambda_{n_p}(t), u(t) - u_{n_p}(t)) - \lim_{p \rightarrow \infty} \langle f_{n_p}(t), u(t) - u_{n_p}(t) \rangle \\ & \leq F(u(t)) - \liminf_{p \rightarrow \infty} F(u_{n_p}(t)) + \lim_{p \rightarrow \infty} \|\lambda_{n_p}(t)\|_{L^2(\Xi)} \|l_0(u(t) - u_{n_p}(t))\|_{L^2(\Xi)} \\ & + \lim_{p \rightarrow \infty} k_2 \|\lambda_{n_p}(t)\|_{L^2(\Xi)} \|u(t) - u_{n_p}(t)\|_U - \lim_{p \rightarrow \infty} \langle f_{n_p}(t), u(t) - u_{n_p}(t) \rangle \\ & = F(u(t)) - \liminf_{p \rightarrow \infty} F(u_{n_p}(t)) \leq 0, \end{aligned}$$

which proves (53).

By Theorem 2.1 with $\mathcal{G} = (\hat{\lambda}_{n_p})_p$, $\hat{X} = L^2(\Xi)$, $\hat{U} = H^{\iota-1/2}(\Xi)$, $\hat{Y} = X'$, $r = 2$, $0 < \iota < \frac{1}{2}$, it follows that

$$\hat{\lambda}_{n_p} \rightarrow \lambda \quad \text{in } C([0, T]; X'), \quad (57)$$

so that by (41) we obtain (54).

It remains to prove that (u, λ) is a solution of the problem Q .

First, since $\lambda_{n_p}(t) \in \Lambda(l(u_{n_p}(t)))$ for all $t \in (0, T)$, we have

$$\int_{\omega} \underline{\kappa} \circ l(u_{n_p}) \leq \int_{\omega} \lambda_{n_p} \leq \int_{\omega} \bar{\kappa} \circ l(u_{n_p}), \quad (58)$$

for every measurable subset $\omega \subset \Xi_T$ and for all $p \in \mathbb{N}$.

Using (53), (8), the semi-continuity of $\underline{\kappa}$ and $\bar{\kappa}$, the relations (1), (2), (47), which implies the convergence property $\int_{\omega} \lambda_{n_p} \rightarrow \int_{\omega} \lambda$, and passing to limits according to Fatou's lemma, we obtain

$$\int_{\omega} \underline{\kappa} \circ l(u) \leq \int_{\omega} \lambda \leq \int_{\omega} \bar{\kappa} \circ l(u), \quad (59)$$

for every measurable subset $\omega \subset \Xi_T$, which implies $\lambda(t) \in \Lambda(l(u(t)))$ for almost all $t \in (0, T)$.

Second, integrating both sides in (21) over $[0, T]$ and passing to the limit, by the relations (53), (54), (45), (49), it follows that for all $v \in L^2(0, T; V)$

$$\begin{aligned} & \int_0^T \langle F'(u(t)), v(t) - \dot{u}(t) \rangle dt - \int_0^T \langle \lambda(t), l_0(v(t) - \dot{u}(t)) \rangle_{L^2(\Xi)} dt \\ & + \int_0^T \phi(\lambda(t), v(t)) dt - \int_0^T \phi(\lambda(t), \dot{u}(t)) dt \geq \int_0^T \langle f(t), v(t) - \dot{u}(t) \rangle dt. \end{aligned}$$

By Lebesgue's theorem, it follows that (u, λ) is a solution of the variational inequality (18). \square

3 Applications to two quasistatic contact problems

Consider an elastic body occupying the set $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$, where $\Gamma_1, \Gamma_2, \Gamma_3$ are open, disjoint parts of Γ and $\text{meas}(\Gamma_1) > 0$. Assume the small deformation hypothesis and that the inertial effects are negligible.

We denote by $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ the displacement field, by $\boldsymbol{\varepsilon}$ the infinitesimal strain tensor and by $\boldsymbol{\sigma}$ the stress tensor, with the components $u = (u_i)$, $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ and $\boldsymbol{\sigma} = (\sigma_{ij})$, respectively. We use the classical decompositions $\mathbf{u} = u_N \mathbf{n} + \mathbf{u}_T$, $u_N = \mathbf{u} \cdot \mathbf{n}$, $\boldsymbol{\sigma} \mathbf{n} = \sigma_N \mathbf{n} + \boldsymbol{\sigma}_T$, $\sigma_N = (\boldsymbol{\sigma} \mathbf{n}) \cdot \mathbf{n}$, where \mathbf{n} is the outward normal unit vector to Γ with the components $n = (n_i)$. The usual summation convention will be used for $i, j, k, l = 1, \dots, d$.

Consider the Hilbert space \mathbf{V} and the closed convex sets $L_-^2(\Gamma_3)$, $\Lambda_1(\zeta)$ as follows:

$$\begin{aligned} \mathbf{V} &= \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^d); \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_1\}, \\ L_-^2(\Gamma_3) &:= \{\delta \in L^2(\Gamma_3); \delta \leq 0 \text{ a.e. in } \Gamma_3\}, \\ \Lambda_1(\zeta) &= \{\eta \in L_-^2(\Gamma_3); \underline{\kappa} \circ \zeta \leq \eta \leq \bar{\kappa} \circ \zeta \text{ a.e. in } \Gamma_3\} \quad \forall \zeta \in L^2(\Gamma_3). \end{aligned}$$

Assume that in Ω a body force $\varphi_1 \in W^{1,2}(0, T; L^2(\Omega; \mathbb{R}^d))$ is prescribed, on Γ_1 the displacement vector equals zero and on Γ_2 a traction $\varphi_2 \in W^{1,2}(0, T; L^2(\Gamma_2; \mathbb{R}^d))$ is applied.

On Γ_3 , the contact between the body and a support is possible with the initial gap denoted by g_0 and the gap corresponding to the solution \mathbf{u} denoted by $[u_N] := u_N - g_0$. We assume that there exists $\mathbf{g} \in \mathbf{V}$ such that $g_N = g_0$ on Γ_3 . Since the displacements, their derivatives and the gap are assumed small, we obtain the following unilateral contact condition at time t : $[u_N] \leq 0$ on Γ_3 .

On the potential contact surface Γ_3 , the displacements and the stress vector will satisfy some contact conditions having the following form:

$$\underline{\kappa}([u_N]) \leq \sigma_N \leq \bar{\kappa}([u_N]).$$

Assume that, for all $\gamma, \delta \in L^2_-(\Gamma_3)$ and all $\mathbf{v}, \mathbf{w} \in \mathbf{V}$ such that $\gamma \in \Lambda_1([v_N]), \delta \in \Lambda_1([w_N])$,

$$\langle \gamma - \delta, v_N - w_N \rangle_{L^2(\Gamma_3)} \leq 0. \quad (60)$$

Different choices for $\underline{\kappa}, \bar{\kappa}$ will give various contact and friction conditions as can be seen in the following examples.

Example 1. (Friction conditions with controlled normal stress)

Let $M \geq 0$ be a constant and define

$$\underline{\kappa}(s) = \underline{\kappa}_M(s) = \begin{cases} 0 & \text{if } s < 0, \\ -M & \text{if } s \geq 0, \end{cases} \quad \bar{\kappa}(s) = \bar{\kappa}_M(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ -M & \text{if } s > 0. \end{cases}$$

The classical Signorini's conditions correspond formally to $M = +\infty$.

Example 2. (Normal compliance conditions)

Various normal compliance conditions and friction laws can be obtained if one considers $\underline{\kappa} = \bar{\kappa} = \kappa$, where $\kappa : \mathbb{R} \rightarrow \mathbb{R}$ is some negative, decreasing, and bounded Lipschitz continuous function, so that σ_N is given by the relation $\sigma_N = \kappa([u_N])$.

It is easily seen that these two examples verify the condition (60).

Let $\mathcal{F} \geq 0$ be the coefficient of friction, assumed to be a Lipschitz continuous function on Γ , which ensures to belong to the set of the multipliers on $H^{1/2}(\Gamma)$ denoted by \mathcal{M} , see, e.g. [2], [4]. Therefore the mapping $H^{1/2}(\Gamma) \ni v \mapsto \mathcal{F}v \in H^{1/2}(\Gamma)$ is bounded with norm $\|\mathcal{F}\|_{\mathcal{M}}$.

In order to describe the frictional contact conditions on Γ_3 , we define

$$\begin{aligned} \forall \mathbf{l} \in \mathbf{V}, \quad \mathbf{S}_l &:= \left\{ \mathbf{v} \in \mathbf{V}; \int_{\Omega} \boldsymbol{\sigma}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\psi}) dx = \langle \mathbf{l}, \boldsymbol{\psi} \rangle_{\mathbf{V}} \right. \\ &\quad \left. \forall \boldsymbol{\psi} \in \mathbf{V} \text{ such that } \boldsymbol{\psi} = \mathbf{0} \text{ a.e. on } \Gamma_3 \right\}, \\ \mathbf{L} \in \mathbf{V}, \quad \langle \mathbf{L}, \mathbf{w} \rangle_{\mathbf{V}} &= \langle \boldsymbol{\varphi}_1, \mathbf{w} \rangle_{L^2(\Omega; \mathbb{R}^d)} + \langle \boldsymbol{\varphi}_2, \mathbf{w} \rangle_{L^2(\Gamma_2; \mathbb{R}^d)} \quad \forall \mathbf{w} \in \mathbf{V}, \\ \forall \mathbf{v} \in \mathbf{S}_L, \quad \langle \sigma_N(\mathbf{v}), w \rangle_{\Gamma} &= \int_{\Omega} \boldsymbol{\sigma}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\bar{\mathbf{w}}) dx - \langle \mathbf{L}, \bar{\mathbf{w}} \rangle_{\mathbf{V}} \quad \forall w \in H^{1/2}(\Gamma), \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\Gamma}$ denotes the duality pairing on $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$, $\bar{\mathbf{w}} \in \mathbf{V}$ satisfies $\bar{\mathbf{w}}_T = \mathbf{0}$ a.e. on Γ_3 , $\bar{\mathbf{w}}_N = w$ a.e. on Γ_3 . It is easy to verify that for all $\mathbf{v} \in \mathbf{S}_L$ $\sigma_N(\mathbf{v})$ depends only on the values of w on Γ_3 and not on the choices of $\bar{\mathbf{w}}$ having the above properties.

3.1 A contact problem with Coulomb friction for a nonlinear Hencky material

Assume that the elastic body satisfies the following nonlinear Hencky-Mises constitutive equation (see [24], [25]):

$$\boldsymbol{\sigma}(\mathbf{u}) = \hat{\boldsymbol{\sigma}}(\mathbf{u}) = \left(k - \frac{2}{3} \mu(\hat{\gamma}(\mathbf{u})) \right) (\text{tr } \boldsymbol{\varepsilon}(\mathbf{u})) \mathbf{I} + 2 \mu(\hat{\gamma}(\mathbf{u})) \boldsymbol{\varepsilon}(\mathbf{u}),$$

where k is the constant bulk modulus, μ is a continuously differentiable function in $[0, +\infty)$ satisfying

$$0 < \mu_0 \leq \mu(r) \leq \frac{3}{2} k, \quad 0 < \mu_1 \leq \mu(r) + 2 \frac{\partial \mu(r)}{\partial r} r \leq \mu_2, \quad \forall r \geq 0, \quad (61)$$

and, for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$,

$$\hat{\gamma}(\mathbf{u}) := \hat{\gamma}(\mathbf{u}, \mathbf{u}), \quad \hat{\gamma}(\mathbf{u}, \mathbf{v}) = -\frac{2}{3} \vartheta(\mathbf{u}) \vartheta(\mathbf{v}) + 2 \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}), \quad \vartheta(\mathbf{u}) := \text{tr } \boldsymbol{\varepsilon}(\mathbf{u}) = \text{div } \mathbf{u}.$$

Consider the following quasistatic contact problem with Coulomb friction.

Problem P_1^c : Find \mathbf{u} such that $\mathbf{u}(0) = \mathbf{u}_0$ and, for all $t \in (0, T)$,

$$\text{div } \boldsymbol{\sigma}(\mathbf{u}) = -\boldsymbol{\varphi}_1 \quad \text{in } \Omega, \quad (62)$$

$$\boldsymbol{\sigma}(\mathbf{u}) = \hat{\boldsymbol{\sigma}}(\mathbf{u}) \quad \text{in } \Omega, \quad (63)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1, \quad \boldsymbol{\sigma} \mathbf{n} = \boldsymbol{\varphi}_2 \quad \text{on } \Gamma_2, \quad (64)$$

$$\underline{\kappa}([u_N]) \leq \sigma_N \leq \bar{\kappa}([u_N]) \quad \text{on } \Gamma_3, \quad (65)$$

$$|\boldsymbol{\sigma}_T| \leq \mathcal{F} |\sigma_N| \quad \text{and} \quad (66)$$

$$\dot{\mathbf{u}}_T \neq \mathbf{0} \Rightarrow \boldsymbol{\sigma}_T = -\mathcal{F} |\sigma_N| \frac{\dot{\mathbf{u}}_T}{|\dot{\mathbf{u}}_T|} \quad \text{on } \Gamma_3.$$

Let $F_1 : \mathbf{V} \rightarrow \mathbb{R}$ be defined by

$$F_1(\mathbf{v}) = \frac{1}{2} k \int_{\Omega} \vartheta^2(\mathbf{v}) dx + \frac{1}{2} \int_{\Omega} \left(\int_0^{\hat{\gamma}(\mathbf{v})} \mu(r) dr \right) dx \quad \forall \mathbf{v} \in \mathbf{V}, \quad (67)$$

and $J : L^2_-(\Gamma_3) \times \mathbf{V} \rightarrow \mathbb{R}$ be defined by

$$J(\gamma, \mathbf{v}) = - \int_{\Gamma_3} \mathcal{F} \gamma |\mathbf{v}_T| ds \quad \forall \gamma \in L^2_-(\Gamma_3), \forall \mathbf{v} \in \mathbf{V}. \quad (68)$$

One can verify, see, e.g. [24], Ch. 8, that F_1 is differentiable on \mathbf{V} and for all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$

$$\langle F'_1(\mathbf{u}), \mathbf{v} \rangle_{\mathbf{V}} = \int_{\Omega} \left[\left(k - \frac{2}{3} \mu(\hat{\gamma}(\mathbf{u})) \right) \vartheta(\mathbf{u}) \vartheta(\mathbf{v}) + 2 \mu(\hat{\gamma}(\mathbf{u})) \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \right] dx. \quad (69)$$

Let $\mathbf{u}_0 \in \mathbf{V}$, $\lambda^0 \in \Lambda_1([u_{0N}])$ satisfy the following compatibility condition:

$$\begin{aligned} \langle F'_1(\mathbf{u}_0), \mathbf{v} - \mathbf{u}_0 \rangle_{\mathbf{V}} - \langle \lambda^0, v_N - u_{0N} \rangle_{L^2(\Gamma_3)} \\ + J(\lambda^0, \mathbf{v}) - J(\lambda^0, \mathbf{u}_0) \geq \langle \mathbf{L}(0), \mathbf{v} - \mathbf{u}_0 \rangle_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}. \end{aligned} \quad (70)$$

We have the following variational formulation of problem P_1^c .

Problem P_1^v : Find $\mathbf{u} \in W^{1,2}(0, T; \mathbf{V})$, $\lambda \in W^{1,2}(0, T; H^{-1/2}(\Gamma))$ such that $\mathbf{u}(0) = \mathbf{u}_0$, $\lambda(0) = \lambda^0$, $\lambda(t) \in \Lambda_1([u_N(t)])$ for almost all $t \in (0, T)$, and

$$\begin{aligned} \langle F'_1(\mathbf{u}), \mathbf{v} - \dot{\mathbf{u}} \rangle_{\mathbf{V}} - \langle \lambda, v_N - \dot{u}_N \rangle_{L^2(\Gamma_3)} + J(\lambda, \mathbf{v}) \\ - J(\lambda, \dot{\mathbf{u}}) \geq \langle \mathbf{L}, \mathbf{v} - \dot{\mathbf{u}} \rangle_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V} \quad \text{a.e. on } (0, T). \end{aligned} \quad (71)$$

The formal equivalence between the variational problem P_1^v and the classical problem (62)–(66) can be easily proved by using Green's formula. The Lagrange multiplier $\lambda \in L^2(\Gamma_3)$ satisfies the relation $\sigma_N = \lambda$ in $H^{-1/2}(\Gamma)$ that is

$$\langle \sigma_N(\mathbf{u}), w \rangle_{\Gamma} = \langle \lambda, w \rangle_{L^2(\Gamma_3)} \quad \forall w \in H^{1/2}(\Gamma).$$

Taking $\Xi = \Gamma_3$, $\Lambda = \Lambda_1$, $V = \mathbf{V}$, $U = H^{\iota}(\Omega; \mathbb{R}^d)$, $1 > \iota > \frac{1}{2}$, $X = H^{1/2}(\Gamma)$, $F = F_1$, $\phi = J$, $f = \mathbf{L}$, and $l_0(\mathbf{v}) = v_N$, $l(\mathbf{v}) = [v_N] = v_N - g_0$ $\forall \mathbf{v} \in \mathbf{V}$, it results that the problem P_1^v is a particular case of problem Q .

As it is straightforward to verify the assumptions (1 - 5), (7 - 15), and also (16) if $\|\mathcal{F}\|_{\mathcal{M}}$ is sufficiently small, by Theorem 2.3 we obtain the following existence result.

Proposition 3.1. *Under the previous assumptions and if $\|\mathcal{F}\|_{\mathcal{M}}$ is sufficiently small there exists a solution to problem P_1^v .*

3.2 A contact problem with local friction for a linearly elastic body

Let \mathcal{A} denote the elasticity tensor, with the components $\mathcal{A} = (\mathcal{A}_{ijkl})$ satisfying the following classical symmetry and ellipticity conditions: $\mathcal{A}_{ijkl} = \mathcal{A}_{jikl} = \mathcal{A}_{klij} \in L^\infty(\Omega) \forall i, j, k, l = 1, \dots, d, \exists \alpha_{\mathcal{A}} > 0$ such that $\mathcal{A}_{ijkl}\tau_{ij}\tau_{kl} \geq \alpha_{\mathcal{A}}\tau_{ij}\tau_{ij} \forall \tau = (\tau_{ij})$ satisfying $\tau_{ij} = \tau_{ji}, \forall i, j = 1, \dots, d$.

Consider the following elastic contact problem with Coulomb friction.

Problem P_2^c : Find \mathbf{u} such that $\mathbf{u}(0) = \mathbf{u}_0$, satisfying

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}) \text{ in } \Omega, \quad (72)$$

and (62), (64 - 66) for all $t \in (0, T)$.

Let us define the bilinear and symmetric mapping $a : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$ by

$$a(\mathbf{v}, \mathbf{w}) = \int_{\Omega} \mathcal{A}\boldsymbol{\varepsilon}(\mathbf{v}) \cdot \boldsymbol{\varepsilon}(\mathbf{w}) dx = \int_{\Omega} \mathcal{A}_{ijkl}\varepsilon_{ij}(\mathbf{v})\varepsilon_{kl}(\mathbf{w})dx \quad \forall \mathbf{v}, \mathbf{w} \in \mathbf{V}. \quad (73)$$

The form a is continuous on $\mathbf{V} \times \mathbf{V}$ and, since $\text{meas}(\Gamma_1) > 0$, by Korn's inequality is also \mathbf{V} -elliptic.

Let $\mathbf{u}_0 \in \mathbf{V}$, $\lambda^0 \in \Lambda_1([u_{0N}])$ satisfy the following compatibility condition:

$$\begin{aligned} a(\mathbf{u}_0, \mathbf{v} - \mathbf{u}_0) - \langle \lambda^0, v_N - u_{0N} \rangle_{L^2(\Gamma_3)} \\ + J(\lambda^0, \mathbf{v}) - J(\lambda^0, \mathbf{u}_0) \geq \langle \mathbf{L}(0), \mathbf{v} - \mathbf{u}_0 \rangle_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V}. \end{aligned} \quad (74)$$

We have the following variational formulation of problem P_2^c .

Problem P_2^v : Find $\mathbf{u} \in W^{1,2}(0, T; \mathbf{V})$, $\lambda \in W^{1,2}(0, T; H^{-1/2}(\Gamma))$ such that $\mathbf{u}(0) = \mathbf{u}_0$, $\lambda(0) = \lambda^0$, $\lambda(t) \in \Lambda_1([u_N(t)])$ for almost all $t \in (0, T)$, and

$$\begin{aligned} a(\mathbf{u}, \mathbf{v} - \dot{\mathbf{u}}) - \langle \lambda, v_N - \dot{u}_N \rangle_{L^2(\Gamma_3)} + J(\lambda, \mathbf{v}) \\ - J(\lambda, \dot{\mathbf{u}}) \geq \langle \mathbf{L}, \mathbf{v} - \dot{\mathbf{u}} \rangle_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V} \quad \text{a.e. on } (0, T). \end{aligned} \quad (75)$$

The Lagrange multiplier $\lambda \in L^2(\Gamma_3)$ satisfies again the relation $\sigma_N = \lambda$ in $H^{-1/2}(\Gamma)$.

Taking $\Xi, \Lambda, V, U, X, \phi, f, l_0, l$ as in 3.1 and $F(\mathbf{v}) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) \forall \mathbf{v} \in \mathbf{V}$, we see that the problem P_2^v is a particular case of problem Q so that by using again Theorem 2.3 one obtains the following existence result.

Proposition 3.2. *Under the previous assumptions and if $\|\mathcal{F}\|_{\mathcal{M}}$ is sufficiently small there exists a solution to problem P_2^v .*

Finally, we remark that viscoelastic or viscoplastic bulk behaviors can also be studied by using similar methods to those presented here.

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