Path-dependent Martingale Problems and Additive Functionals

Adrien Barrasso, Francesco Russo

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Adrien BARRASSO * Francesco RUSSO†

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Abstract. The paper introduces and investigates the natural extension to the path-dependent setup of the usual concept of canonical Markov class introduced by Dynkin and which is at the basis of the theory of Markov processes. That extension, indexed by starting paths rather than starting points will be called path-dependent canonical class. Associated with this is the generalization of the notions of semi-group and of additive functionals to the path-dependent framework. A typical example of such family is constituted by the laws $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$, where for fixed time s and fixed path η defined on $[0,s]$, $\mathbb{P}^{s,\eta}$ being the (unique) solution of a path-dependent martingale problem or more specifically a weak solution of a path-dependent SDE with jumps, with initial path η. In a companion paper we apply those results to study path-dependent analysis problems associated with BSDEs.

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1 Introduction

In this paper we extend some aspects of the theory of Markov processes to the (non-Markovian) path-dependent case. The crucial object of Markov canonical class introduced by Dynkin is replaced with the one of *path-dependent canonical class*. The associated notion of Markov semigroup is extended to the notion of *path-dependent system of projectors*. The classical Markovian concept of (Martingale) Additive Functional is generalized to the one of *path-dependent (Martingale) Additive Functional*. We then study some general path-dependent martingale problems with applications to weak solutions of path-dependent SDEs

*ENSTA ParisTech, Unité de Mathématiques appliquées, 828, boulevard des Maréchaux, F-91120 Palaiseau, France and Ecole Polytechnique, F-91128 Palaiseau, France. E-mail: adrien.barrasso@ensta-paristech.fr

†ENSTA ParisTech, Unité de Mathématiques appliquées, 828, boulevard des Maréchaux, F-91120 Palaiseau, France. E-mail: francesco.russo@ensta-paristech.fr

(possibly) with jumps and show that, under well-posedness, the solution of the martingale problem provides a path-dependent canonical class. The companion paper [3] will exploit these results to extend the links between BSDEs and (possibly Integro) PDEs obtained in [4], to a path-dependent framework.

The theory of Additive Functionals associated to a Markov process was initiated during the early '60s, see the historical papers [14], [18], [8] and see [12] for a complete theory in the homogeneous setup. The strong links between martingale problems and Markov processes were first observed for the study of weak solutions of SDEs in [20], and more generally in [15] or [16] for example. Weak solutions of path-dependent SDEs possibly with jumps were studied in [16], where the author shows their equivalence to some path-dependent martingale problems and proves existence and uniqueness of a solution under Lipschitz conditions. More recent results concerning path-dependent martingale problems may be found in [7]. However, at our knowledge, the structure of the set of solutions for different starting paths was not yet studied.

The setup of this paper is the canonical space $(\Omega, \mathcal{F})$ where $\Omega := \mathbb{D}(\mathbb{R}_+, E)$ is the Skorokhod space of cadlag functions from $\mathbb{R}_+$ into a Polish space E and $\mathcal{F}$ is its Borel σ-field. $X = (X_t)_{t\in\mathbb{R}_+}$ denotes the canonical process and the initial filtration $\mathbb{F}^o$ is defined by $\mathcal{F}^o_t := \sigma(X_r | r \in [0,t])$ for all $t \geq 0$.

A path-dependent canonical class will be a set of probability measures $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ defined on the canonical space and such that, for some fixed (s,η), $\mathbb{P}^{s,\eta}$ models a forward (path-dependent) dynamics in law, with imposed initial path η on the time interval $[0,s]$. As already mentioned, it constitutes the natural adaptation to the path-dependent world of the notion of canonical Markov class $(\mathbb{P}^{s,x})_{(s,x)\in\mathbb{R}_+\times E}$, where in general, $\mathbb{P}^{s,x}$ models the law of some Markov stochastic process, with imposed value x at time s. $\mathbb{F}^{s,\eta}$ is the augmented initial filtration fulfilling the usual conditions.

In substitution of a Markov semigroup associated with a Markov canonical class, we introduce a path-dependent system of projectors denoted $(P_s)_{s\in\mathbb{R}_+}$ and a one-to-one connection between them and path-dependent canonical classes. Each projector P_s acts on the space of bounded random variables. This brings us to introduce the notion of **weak generator** $(\mathcal{D}(A), A)$ of $(P_s)_{s\in\mathbb{R}_+}$ which will permit us in the companion paper [3] to define *mild* type solutions of path-dependent PDEs of the form

$$\left\{\begin{array}{l} D\Phi + \frac{1}{2}Tr(\sigma\sigma^\intercal\nabla^2\Phi) + \beta\nabla\Phi + f(\cdot,\cdot,\Phi,\sigma\sigma^\intercal\nabla\Phi) = 0 \text{ on } [0,T]\times\Omega \\ \Phi_T = \xi \text{ on } \Omega, \end{array}\right. \tag{1.1}$$

where D is the horizontal derivative and ∇ the vertical gradient in the sense of [13, 9] and β, σ are progressively measurable path-dependent coefficients.

As mentioned earlier, given a path-dependent canonical class we also introduce the notion of path-dependent Additive Functional (resp. path-dependent square integrable Martingale Additive Functional), which is a real-valued random-field $M := (M_{t,u})_{0\leq t\leq u<+\infty}$ such that for any $(s,\eta) \in \mathbb{R}_+ \times \Omega$, there exists a real cadlag $\mathbb{F}^{s,\eta}$-adapted process (resp. $\mathbb{F}^{s,\eta}$-square integrable martingale) $M^{s,\eta}$ called the **cadlag version of** M **under** $\mathbb{P}^{s,\eta}$, and verifying for all $s \leq t \leq u$ that

$M_{t,u} = M^{s,\eta}_u - M^{s,\eta}_t$ $\mathbb{P}^{s,\eta}$ a.s. Under some reasonable measurability assumptions on the path-dependent canonical class, we extend to our path-dependent setup some classical results of Markov processes theory concerning the quadratic covariation and the angular bracket of square integrable MAFs. As in the Markovian set-up, examples of path-dependent canonical classes arise from solutions of a (this time path-dependent) martingale problem as we explain below. Let χ be a set of cadlag processes adapted to the initial filtration $\mathbb{F}^o$. For some given $(s,\eta) \in \mathbb{R}_+ \times \Omega$, we say that a probability measure $\mathbb{P}^{s,\eta}$ on $(\Omega, \mathcal{F})$ **solves the martingale problem with respect to χ starting in** (s,η) if

- $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$;
- all elements of χ are on $[s, +\infty[$ $(\mathbb{P}^{s,\eta}, \mathbb{F}^o)$-martingales.

We show that merely under some well-posedness assumptions, the set of solutions for varying starting times and paths $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ defines a path-dependent canonical class. This in particularly holds for weak solutions of path-dependent SDEs possibly with jumps.

The paper is organized as follows. In Section 3, we introduce the notion of path-dependent canonical class in Definition 3.4 and of path-dependent system of projectors in Definition 3.8 and prove a one-to-one correspondence between those two concepts in Corollary 3.11. In Section 4, we introduce the notion of path-dependent Additive Functional, in short AF (resp. Martingale Additive Functional, in short MAF). We state in Proposition 4.6 and Corollary 4.9 that for a given square integrable path-dependent MAF $(M_{t,u})_{(t,u)\in\Delta}$, we can exhibit two non-decreasing path-dependent AFs with $\mathcal{L}^1$-terminal value, denoted respectively by $([M]_{t,u})_{(t,u)\in\Delta}$ and $(\langle M\rangle_{t,u})_{(t,u)\in\Delta}$, which will play respectively the role of a quadratic variation and an angular bracket of it. Then in Corollary 4.12, we state that the Radon-Nikodym derivative of the mentioned angular bracket of a square integrable path-dependent MAF with respect to a reference function V, is a progressively measurable process which does not depend on the probability. In Section 5, we introduce what we mean by path-dependent martingale problem with respect to a set of processes χ, to a time s and a starting path η, see Definition 5.4. Suppose that χ is a countable set of cadlag $\mathbb{F}^o$-adapted processes which are uniformly bounded on each interval $[0,T]$; in Proposition 5.12, we state that, whenever the martingale problem with respect to χ is well-posed, then the solution $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ defines a path-dependent canonical class. In Subsection 5.2, Definition 5.14 introduces the notion of weak generator of a path-dependent system of projectors, and Definition 5.15 that of martingale problem associated to a path-dependent operator $(D(A), A)$. Suppose now that for any (s,η) the martingale problem associated with $(D(A), A)$ is well-posed, and let $(P_s)_{s\in\mathbb{R}_+}$ be the system of projectors associated to the canonical class constituted by the solutions $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$. Then $(D(A), A)$ is a weak generator of $(P_s)_{s\in\mathbb{R}_+}$, and $(P_s)_{s\in\mathbb{R}_+}$ is the unique system of projectors such that this holds. In other words, $(P_s)_{s\in\mathbb{R}_+}$ can be analytically associated to $(D(A), A)$ without ambiguity. Finally, in Section 6, we consider path-dependent SDEs with jumps, whose coefficients are denoted by β, σ, w. If for any couple

(s,η), the SDE has a unique weak solution, then Theorem 6.7 ensures that the set of solutions $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ defines a path-dependent canonical class. Under the additional assumptions that β,σ,w are bounded and continuous in ω for fixed other variables, then Proposition 6.13 states that $(s,\eta)\longmapsto\mathbb{P}^{s,\eta}$ is continuous for the topology of weak convergence.

2 Preliminaries

In the whole paper we will use the following notions, notations and vocabulary.

A topological space E will always be considered as a measurable space with its Borel σ-field which shall be denoted $\mathcal{B}(E)$ and if S is another topological space equipped with its Borel σ-field, $\mathcal{B}(E,S)$ will denote the set of Borel functions from E to S. For some fixed $d\in\mathbb{N}^*$, $\mathcal{C}_c^\infty(\mathbb{R}^d)$ will denote the set of smooth functions with compact support. For fixed $d,k\in\mathbb{N}^*$, $\mathcal{C}^k(\mathbb{R}^d)$, (resp. $\mathcal{C}_b^k(\mathbb{R}^d)$) will denote the set of functions k times differentiable with continuous (resp. bounded continuous) derivatives.

Let $(\Omega,\mathcal{F})$, $(E,\mathcal{E})$ be two measurable spaces. A measurable mapping from $(\Omega,\mathcal{F})$ to $(E,\mathcal{E})$ shall often be called a **random variable** (with values in E), or in short r.v. If $\mathbb{T}$ is indices set, a family $(X_t)_{t\in\mathbb{T}}$ of r.v. with values in E, will be called a **random field** (indexed by $\mathbb{T}$ with values in E). In the particular case when $\mathbb{T}$ is a subinterval of $\mathbb{R}_+$, $(X_t)_{t\in\mathbb{T}}$ will be called a **stochastic process** (indexed by $\mathbb{T}$ with values in E). If the mapping $\begin{array}{ccc}(t,\omega) & \longmapsto & X_t(\omega)\\ (\mathbb{T}\times\Omega,\mathcal{B}(\mathbb{T})\otimes\mathcal{F}) & \longrightarrow & (E,\mathcal{E})\end{array}$ is measurable, then the process (or random field) $(X_t)_{t\in\mathbb{T}}$ will be said to be **measurable** (indexed by $\mathbb{T}$ with values in E).

On a fixed probability space $(\Omega,\mathcal{F},\mathbb{P})$, for any $p\geq 1$, $\mathcal{L}^p$ will denote the set of real-valued random variables with finite p-th moment. Two random fields (or stochastic processes) $(X_t)_{t\in\mathbb{T}}$, $(Y_t)_{t\in\mathbb{T}}$ indexed by the same set and with values in the same space will be said to be **modifications (or versions) of each other** if for every $t\in\mathbb{T}$, $\mathbb{P}(X_t=Y_t)=1$. A filtered probability space $\big(\Omega,\mathcal{F},\mathbb{F}:=(\mathcal{F}_t)_{t\in\mathbb{R}_+},\mathbb{P}\big)$ will be called called **stochastic basis** and will be said to **fulfill the usual conditions** if the filtration is right-continuous, if the probability space is complete and if $\mathcal{F}_0$ contains all the $\mathbb{P}$-negligible sets. Let us fix a stochastic basis $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$. If $Y=(Y_t)_{t\in\mathbb{R}_+}$ is a stochastic process and τ is a stopping time, we denote Y^τ the process $t\mapsto Y_{t\wedge\tau}$ which we call **stopped process** (by τ). If $\mathcal{C}$ is a set of processes, we will say that Y is locally in $\mathcal{C}$ (resp. locally verifies some property) if there exist an a.s. increasing sequence of stopping times $(\tau_n)_{n\geq 0}$ tending a.s. to infinity such that for every n, the stopped process Y^{τ_n} belongs to $\mathcal{C}$ (resp. verifies this property).

Given two martingales M,N, we denote by $[M]$ (resp. $[M,N]$) the **quadratic variation** of M (resp. **covariation** of M,N). If M,N are locally square integrable martingales, $\langle M,N\rangle$ (or simply $\langle M\rangle$ if $M=N$) will denote their (predictable) **angular bracket**. Two locally square integrable martingales vanishing at zero M,N will be said to be **strongly orthogonal** if $\langle M,N\rangle=0$.
If A is an adapted process with bounded variation then $Var(A)$ (resp. $Pos(A)$,

$Neg(A)$) will denote its total variation (resp. positive variation, negative variation), see Proposition 3.1, chap. 1 in [17]. In particular for almost all $\omega \in \Omega$, $t \mapsto Var_t(A(\omega))$ is the total variation function of the function $t \mapsto A_t(\omega)$.

3 Path-dependent canonical classes

We will introduce here an abstract context which is relevant for the study of path-dependent stochastic equations. The definitions and results which will be presented here are inspired from the theory of Markov processes and of additive functionals which one can find for example in [12].

The first definition refers to the canonical space that one can find in [16], see paragraph 12.63.

Notation 3.1. *In the whole section E will be a fixed Polish space, i.e. a separable complete metrizable topological space, that we call the* ***state space****.*

Ω will denote $\mathbb{D}(\mathbb{R}_+, E)$ the space of functions from $\mathbb{R}_+$ to E being right-continuous with left limits (e.g. cadlag). For every $t \in \mathbb{R}_+$ we denote the coordinate mapping $X_t : \omega \mapsto \omega(t)$ and we define on Ω the σ-field $\mathcal{F} := \sigma(X_r | r \in \mathbb{R}_+)$. On the measurable space $(\Omega, \mathcal{F})$, we introduce ***initial filtration*** *$\mathbb{F}^o := (\mathcal{F}^o_t)_{t \in \mathbb{R}_+}$, where $\mathcal{F}^o_t := \sigma(X_r | r \in [0,t])$, and the (right-continuous)* ***canonical filtration*** *$\mathbb{F} := (\mathcal{F}_t)_{t \in \mathbb{R}_+}$, where $\mathcal{F}_t := \bigcap_{s>t} \mathcal{F}^o_s$. $(\Omega, \mathcal{F}, \mathbb{F})$ will be called the* ***canonical space*** *(associated to E). On $\mathbb{R}_+ \times \Omega$, we will denote by $\mathcal{P}ro^o$ (resp. $\mathcal{P}re^o$) the $\mathbb{F}^o$-progressive (resp. $\mathbb{F}^o$-predictable) σ-field. Ω will be equipped with the Skorokhod topology which is Polish since E is Polish (see Theorem 5.6 in chapter 3 of [15]), and for which the Borel σ-field is $\mathcal{F}$, see Proposition 7.1 in chapter 3 of [15]. This in particular implies that $\mathcal{F}$ is separable, being the Borel σ-field of a separable metric space.*

$\mathcal{P}(\Omega)$ will denote the set of probability measures on Ω and will be equipped with the topology of weak convergence of measures which also makes it a Polish space since Ω is Polish (see Theorems 1.7 and 3.1 in [15] chapter 3). It will also be equipped with the associated Borel σ-field.

Notation 3.2. *For any $\omega \in \Omega$ and $t \in \mathbb{R}_+$, the path ω stopped at time t $r \mapsto \omega(r \wedge t)$ will be denoted ω^t.*

Remark 3.3. *In Sections 3,4 and Subsections 5.1, 5.2, all notions and results can easily be adapted to different canonical spaces Ω: for instance, $\mathcal{C}(\mathbb{R}_+, E)$, the space of continuous functions from $\mathbb{R}_+$ to E; $\mathcal{C}([0,T], E)$ (resp. $\mathbb{D}([0,T], E)$) the space of continuous (resp. cadlag) functions from $[0,T]$ to E, for some $T > 0$; fixing $x \in E$, $\mathcal{C}_x(\mathbb{R}_+, E)$ (resp. $\mathcal{C}_x([0,T], E)$) the space of continuous functions from $\mathbb{R}_+$ (resp. $[0,T]$) to E starting at x .*

Definition 3.4. *A* ***path-dependent canonical class*** *will be a family $(\mathbb{P}^{s,\eta})_{(s,\eta) \in \mathbb{R}_+ \times \Omega}$ of probability measures defined on the canonical space $(\Omega, \mathcal{F})$, which verifies the three following items.*

1. *For every $(s, \eta) \in \mathbb{R}_+ \times \Omega$, $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$;*

2. *for every* $s \in \mathbb{R}_+$ *and* $F \in \mathcal{F}$, *the mapping* $\begin{array}{ccc} \eta & \longmapsto & \mathbb{P}^{s,\eta}(F) \\ \Omega & \longrightarrow & [0,1] \end{array}$ *is* $\mathcal{F}^o_s$*-measurable;*

3. *for every* $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $t \geq s$ *and* $F \in \mathcal{F}$,

$$\mathbb{P}^{s,\eta}(F|\mathcal{F}^o_t)(\omega) = \mathbb{P}^{t,\omega}(F) \text{ for } \mathbb{P}^{s,\eta} \text{ almost all } \omega. \tag{3.1}$$

This implies in particular that for every $(s,\eta) \in \mathbb{R}_+ \times \Omega$ *and* $t \geq s$, *then* $(\mathbb{P}^{t,\omega})_{\omega\in\Omega}$ *is a regular conditional expectation of* $\mathbb{P}^{s,\eta}$ *by* $\mathcal{F}^o_t$, *see the Definition above Theorem 1.1.6 in [20] for instance.*

A path-dependent canonical class $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ *will be said to be* ***progressive*** *if for every* $F \in \mathcal{F}$, *the mapping* $(t,\omega) \longmapsto \mathbb{P}^{t,\omega}(F)$ *is* $\mathbb{F}^o$*-progressively measurable.*

In concrete examples, path-dependent canonical classes will always verify the following important hypothesis which is a reinforcement of (3.1).

Hypothesis 3.5. *For every* $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $t \geq s$ *and* $F \in \mathcal{F}$,

$$\mathbb{P}^{s,\eta}(F|\mathcal{F}_t)(\omega) = \mathbb{P}^{t,\omega}(F) \text{ for } \mathbb{P}^{s,\eta} \text{ almost all } \omega. \tag{3.2}$$

Remark 3.6. *By approximation through simple functions, one can easily show the following. Let* Z *be a random variable.*

- *Let* $s \geq 0$. *The functional* $\eta \longmapsto \mathbb{E}^{s,\eta}[Z]$ *is* $\mathcal{F}^o_s$*-measurable and for every* $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $t \geq s$, $\mathbb{E}^{s,\eta}[Z|\mathcal{F}^o_t](\omega) = \mathbb{E}^{t,\omega}[Z]$ *for* $\mathbb{P}^{s,\eta}$ *almost all* ω, *provided previous expectations are finite;*
- *if the path-dependent canonical class is progressive,* $(t,\omega) \longmapsto \mathbb{E}^{t,\omega}[Z]$ *is* $\mathbb{F}^o$*-progressively measurable, provided previous expectations are finite.*

Notation 3.7.

- $\mathcal{B}_b(\Omega)$ *(resp.* $\mathcal{B}^+_b(\Omega)$*) will denote the space of measurable (resp. non-negative measurable) bounded r.v.*
- *Let* $s \geq 0$. $\mathcal{B}^s_b(\Omega)$ *will denote the space of* $\mathcal{F}^o_s$*-measurable bounded r.v.*

Definition 3.8.

1. *A linear map* $Q : \mathcal{B}_b(\Omega) \to \mathcal{B}_b(\Omega)$ *is said* **positivity preserving monotonic** *if for every* $\phi \in \mathcal{B}^+_b(\Omega)$ *then* $Q[\phi] \in \mathcal{B}^+_b(\Omega)$ *and for every increasing converging (in the pointwise sense) sequence* $f_n \underset{n}{\longrightarrow} f$ *we have that* $Q[f_n] \underset{n}{\longrightarrow} Q[f]$ *in the pointwise sense.*
2. *A family* $(P_s)_{s\in\mathbb{R}_+}$ *of positivity preserving monotonic linear operators on* $\mathcal{B}_b(\Omega)$ *will be called a* ***path-dependent system of projectors*** *if it verifies the three following properties.*

- *For all $s \in \mathbb{R}_+$, the restriction of P_s to $\mathcal{B}_b^s(\Omega)$ coincides with the identity;*
- *for all $s \in \mathbb{R}_+$, P_s maps $\mathcal{B}_b(\Omega)$ into $\mathcal{B}_b^s(\Omega)$;*
- *for all $s, t \in \mathbb{R}_+$ with $t \geq s$, $P_s \circ P_t = P_s$.*

Proposition 3.9. *Let $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ be a path-dependent canonical class. For every $s \in \mathbb{R}_+$, we define $P_s : \phi \longmapsto (\eta \mapsto \mathbb{E}^{s,\eta}[\phi])$. Then $(P_s)_{s\in\mathbb{R}_+}$ defines a path-dependent system of projectors.*

Proof. For every $s \geq 0$ each map P_s is linear, positivity preserving and monotonic using the usual properties of the expectation under a given probability. The rest follows taking into account Definitions 3.4, 3.8 and Remark 3.6.

□

Proposition 3.10. *Let $(P_s)_{s\in\mathbb{R}_+}$ be a path-dependent system of projectors. For any $(s,\eta) \in \mathbb{R}_+ \times \Omega$, we set*

$$\mathbb{P}^{s,\eta} : \begin{pmatrix} F & \longmapsto & P_s[\mathbb{1}_F](\eta) \\ \mathcal{F} & \longrightarrow & \mathbb{R} \end{pmatrix}. \tag{3.3}$$

Then for all (s,η), $\mathbb{P}^{s,\eta}$ defines a probability measure and $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ is a path-dependent canonical class.

Proof. We fix s and η. Since $\emptyset, \Omega \in \mathcal{F}_s^o$, then by the first item of Definition 3.8, $P_s[\mathbb{1}_\emptyset] = \mathbb{1}_\emptyset$ and $P_s[\mathbb{1}_\Omega] = \mathbb{1}_\Omega$, so $\mathbb{P}^{s,\eta}(\emptyset) = 0$ and $\mathbb{P}^{s,\eta}(\Omega) = 1$. For any $F \in \mathcal{F}$, since P_s is positivity preserving and $\mathbb{1}_\emptyset \leq \mathbb{1}_F \leq \mathbb{1}_\Omega$ then $\mathbb{1}_\emptyset \leq P_s[\mathbb{1}_F] \leq \mathbb{1}_\Omega$ so, $\mathbb{P}^{s,\eta}$ takes values in $[0,1]$. If $(F_n)_n$ is a sequence of pairwise disjoint elements of $\mathcal{F}$ then the increasing sequence $\sum_{k=0}^N \mathbb{1}_{F_k}$ converges pointwise to $\mathbb{1}_{\bigcup_n F_n}$. Since the P_s are linear and monotonic then $\sum_n P_s[\mathbb{1}_{F_n}] = P_s[\mathbb{1}_{\bigcup_n F_n}]$, hence $\sum_n \mathbb{P}^{s,\eta}(F_n) = \mathbb{P}^{s,\eta}\left(\bigcup_n F_n\right)$. So for every (s,η), $\mathbb{P}^{s,\eta}$, is σ-additive, positive, vanishing in $\emptyset$ and takes value 1 in Ω hence is a probability measure.
Then, for any (s,η) we have $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = P_s[\mathbb{1}_{\{\omega^s=\eta^s\}}](\eta) = \mathbb{1}_{\{\omega^s=\eta^s\}}(\eta) = 1$ since $\{\omega^s = \eta^s\} \in \mathcal{F}_s^o$, so item 1. of Definition 3.4 is satisfied. Concerning item 2., at fixed $s \in \mathbb{R}_+$ and $F \in \mathcal{F}$, we have $(\eta \mapsto \mathbb{P}^{s,\eta}(F)) = P_s[\mathbb{1}_F]$ which is $\mathcal{F}_s^o$-measurable since P_s has its range in $\mathcal{B}_b^s(\Omega)$, see Definition 3.8.

It remains to show item 3. We now fix $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $t \geq s$ and $F \in \mathcal{F}$ and show that (3.1) holds. Let $G \in \mathcal{F}_t^o$. We need to show that $\mathbb{E}^{s,\eta}[\mathbb{1}_G \mathbb{1}_F] = \mathbb{E}^{s,\eta}[\mathbb{1}_G(\zeta)\mathbb{E}^{t,\zeta}[\mathbb{1}_F]]$. We have

$$\begin{aligned} \mathbb{E}^{s,\eta}[\mathbb{1}_G \mathbb{1}_F] &= \mathbb{E}^{s,\eta}[\mathbb{E}^{t,\zeta}[\mathbb{1}_G(\omega)\mathbb{1}_F(\omega)]] \\ &= \mathbb{E}^{s,\eta}[\mathbb{E}^{t,\zeta}[\mathbb{1}_G(\zeta)\mathbb{1}_F(\omega)]] \\ &= \mathbb{E}^{s,\eta}[\mathbb{1}_G(\zeta)\mathbb{E}^{t,\zeta}[\mathbb{1}_F(\omega)]], \end{aligned}$$

where the first equality comes from the fact that $P_s = P_s \circ P_t$ and the second from the fact that $G \in \mathcal{F}_t^o$ and $\mathbb{P}^{t,\zeta}(\omega^t = \zeta^t) = 1$ so $\mathbb{1}_G = \mathbb{1}_G(\zeta)$ $\mathbb{P}^{t,\zeta}$ a.s. □

Corollary 3.11. *The mapping*

$$\Phi : (\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega} \longmapsto (Z \longmapsto (\eta \mapsto \mathbb{E}^{s,\eta}[Z]))_{s\in\mathbb{R}_+}, \tag{3.4}$$

is a bijection between the set of path-dependent canonical classes and the set of path-dependent system of projectors, whose reciprocal map is given by

$$\Phi^{-1} : (P_s)_{s\in\mathbb{R}_+} \longmapsto (F \mapsto P_s[\mathbb{1}_F](\eta))_{(s,\eta)\in\mathbb{R}_+\times\Omega}. \tag{3.5}$$

Proof. Φ is by Proposition 3.9 well-defined. Moreover it is injective since if $\mathbb{P}^1$ and $\mathbb{P}^2$ are two probabilities such that respective expectations of all the bounded r.v. are the same then $\mathbb{P}^1 = \mathbb{P}^2$. Then given a path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$, by Proposition 3.10 $(\mathbb{P}^{s,\eta} : F \mapsto P_s[\mathbb{1}_F](\eta))_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ is a path-dependent canonical class. It is then enough to show that the image through Φ of that path-dependent canonical class is indeed $(P_s)_{s\in\mathbb{R}_+}$. Let $(Q_s)_{s\in\mathbb{R}_+}$ denote its image by Φ, in order to conclude we are left to show that $Q_s = P_s$ for all s.
We fix s. For every $F \in \mathcal{F}, \eta \in \Omega$ we have $Q_s[\mathbb{1}_F](\eta) = \mathbb{P}^{s,\eta}(F) = P_s[\mathbb{1}_F](\eta)$ so Q_s and P_s coincide on the indicator functions, hence on the simple functions by linearity, and everywhere by monotonicity and the fact that every bounded Borel function is the limit of an increasing sequence of simple functions. □

Definition 3.12. *From now on, two elements mapped by the previous bijection will be said to be* ***associated****.*

Remark 3.13. *Path-dependent canonical classes naturally extend canonical Markov classes (see Definition C.5 in [4] for instance) as follows.*
Let $(\mathbb{P}^{s,x})_{(s,x)\in\mathbb{R}_+\times E}$ be a canonical Markov class with state space E and let $(P_{s,t})_{0\leq s\leq t}$ denote its transition kernel, see Definition C.3 in [4].
For all $(s,\eta) \in \mathbb{R}_+ \times \Omega$, let $\mathbb{P}^{s,\eta}$ be the unique probability measure on $(\Omega, \mathcal{F})$ such that $\mathbb{P}^{s,\eta}(\omega^s = \eta^s)$ and $\mathbb{P}^{s,\eta}$ coincides on $\sigma(X_r|r \geq s)$ with $\mathbb{P}^{s,\eta(s)}$. Then $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ is a path-dependent canonical class. Let $(P_s)_{s\in\mathbb{R}_+}$ denote the associated path-dependent system of projectors. Then for all bounded Borel $\phi : E \mapsto \mathbb{R}$, $\eta \in \Omega$ and $0 \leq s \leq t$ we have

$$P_s[\phi \circ X_t](\eta) = \mathbb{E}^{s,\eta}[\phi(X_t)] = \mathbb{E}^{s,\eta(s)}[\phi(X_t)] = P_{s,t}[\phi](\eta(s)). \tag{3.6}$$

Notation 3.14. *For the rest of this section, we are given a path-dependent canonical class $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ and $(P_s)_{s\in\mathbb{R}_+}$ denotes the associated path-dependent system of projectors.*

Definition 3.15. *Let $\mathbb{P}$ be a probability on $(\Omega, \mathcal{F})$. If $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, we call* ***$\mathbb{P}$-closure*** *of $\mathcal{G}$ the σ-field generated by $\mathcal{G}$ and the set of $\mathbb{P}$-negligible sets. We denote it $\mathcal{G}^{\mathbb{P}}$. In the particular case $\mathcal{G} = \mathcal{F}$, we call $\mathcal{F}^{\mathbb{P}}$* ***$\mathbb{P}$-completion*** *of $\mathcal{F}$.*

Remark 3.16. *Thanks to Remark 32.b) in Chapter II of [10], we have an equivalent definition of the $\mathbb{P}$-closure of some sub-σ-field $\mathcal{G}$ of $\mathcal{F}$ which can be*

characterized by the following property: $B \in \mathcal{G}^{\mathbb{P}}$ if and only if there exist $F \in \mathcal{G}$ such that $\mathbb{1}_B = \mathbb{1}_F$ $\mathbb{P}$ a.s.
Moreover, $\mathbb{P}$ can be extended to a probability on $\mathcal{G}^{\mathbb{P}}$ by setting $\mathbb{P}(B) := \mathbb{P}(F)$ for such events.

Notation 3.17. *For any $(s,\eta) \in \mathbb{R}_+ \times \Omega$ we will consider the stochastic basis $\left(\Omega, \mathcal{F}^{s,\eta}, \mathbb{F}^{s,\eta} := (\mathcal{F}_t^{s,\eta})_{t\in\mathbb{R}_+}, \mathbb{P}^{s,\eta}\right)$ where $\mathcal{F}^{s,\eta}$ is the $\mathbb{P}^{s,\eta}$-completion of $\mathcal{F}$, $\mathbb{P}^{s,\eta}$ is extended to $\mathcal{F}^{s,\eta}$ and $\mathcal{F}_t^{s,\eta}$ is the $\mathbb{P}^{s,\eta}$-closure of $\mathcal{F}_t$ for every $t \in \mathbb{R}_+$.*

We remark that, for any $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $(\Omega, \mathcal{F}^{s,\eta}, \mathbb{F}^{s,\eta}, \mathbb{P}^{s,\eta})$ is a stochastic basis fulfilling the usual conditions, see 1.4 in [17] Chapter I.

A direct consequence of Remark 32.b) in Chapter II of [10] is the following.

Proposition 3.18. *Let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, $\mathbb{P}$ a probability on $(\Omega,\mathcal{F})$ and $\mathcal{G}^{\mathbb{P}}$ the $\mathbb{P}$-closure of $\mathcal{G}$. Let $Z^{\mathbb{P}}$ be a real $\mathcal{G}^{\mathbb{P}}$-measurable random variable. There exists a $\mathcal{G}$-measurable random variable Z such that $Z = Z^{\mathbb{P}}$ $\mathbb{P}$-a.s.*

Proposition 3.18 yields the following.

Proposition 3.19. *Let $\mathbb{P}$ be a probability measure on $(\Omega,\mathcal{F})$, let $\mathbb{G} := (\mathcal{G}_t)_{t\in\mathbb{R}_+}$ be a filtration and $\mathbb{G}^{\mathbb{P}}$ denote $(\mathcal{G}_t^{\mathbb{P}})_{t\in\mathbb{R}_+}$. Let Z be a positive or $\mathcal{L}^1$-random variable and $t \in \mathbb{R}_+$. Then $\mathbb{E}[Z|\mathcal{G}_t] = \mathbb{E}[Z|\mathcal{G}_t^{\mathbb{P}}]$ $\mathbb{P}$ a.s. In particular, $(\mathbb{P},\mathbb{G})$-martingales are also $(\mathbb{P},\mathbb{G}^{\mathbb{P}})$-martingales.*

According to Proposition 3.19 for $\mathbb{P} = \mathbb{P}^{s,\eta}$, the related conditional expectations with respect to $\mathcal{F}_t^{s,\eta}$ coincide with conditional expectations with respect to $\mathcal{F}_t$. For that reason we will only use the notation $\mathbb{E}^{s,\eta}[\cdot|\mathcal{F}_t]$ omitting the (s,η)-superscript over $\mathcal{F}_t$.

In the next proposition, $\mathcal{F}_t^{o,s,\eta}$ will denote for any $(s,\eta) \in \mathbb{R}_+ \times \Omega$ and $t \geq s$ the $\mathbb{P}^{s,\eta}$-closure of $\mathcal{F}_t^o$.

Proposition 3.20. *Assume that Hypothesis 3.5 holds. For any $(s,\eta) \in \mathbb{R}_+ \times \Omega$ and $t \geq s$, $\mathcal{F}_t^{o,s,\eta} = \mathcal{F}_t^{s,\eta}$.*

Proof. We fix s,η,t. Since inclusion $\mathcal{F}_t^{o,s,\eta} \subset \mathcal{F}_t^{s,\eta}$ is obvious, we show the converse inclusion.
Let $F^{s,\eta} \in \mathcal{F}_t^{s,\eta}$. By Remark 3.16, there exists $F \in \mathcal{F}_t$, such that $\mathbb{1}_{F^{s,\eta}} = \mathbb{1}_F$ $\mathbb{P}^{s,\eta}$ a.s. It is therefore sufficient to prove the existence of some $F^o \in \mathcal{F}_t^o$ such that $\mathbb{1}_{F^o} = \mathbb{1}_F$ $\mathbb{P}^{s,\eta}$ a.s. (and therefore $\mathbb{1}_{F^o} = \mathbb{1}_{F^{s,\eta}}$ $\mathbb{P}^{s,\eta}$ a.s.) to conclude that $F^{s,\eta} \in \mathcal{F}_t^{o,s,\eta}$.

We set $Z : \begin{array}{ccc} \omega & \longmapsto & \mathbb{P}^{t,\omega}(F) \\ \Omega & \longrightarrow & [0,1] \end{array}$. By (3.2) and the fact that $F \in \mathcal{F}_t$, we have

$$Z(\omega) = \mathbb{P}^{t,\omega}(F) = \mathbb{E}^{s,\eta}[\mathbb{1}_F|\mathcal{F}_t](\omega) = \mathbb{1}_F(\omega) \quad \mathbb{P}^{s,\eta}\text{a.s.} \tag{3.7}$$

By Definition 3.4, Z is $\mathcal{F}_t^o$-measurable, so $F^o := Z^{-1}(\{1\})$ belongs to $\mathcal{F}_t^o$, and we will proceed showing that $\mathbb{1}_{F^o} = \mathbb{1}_F$ $\mathbb{P}^{s,\eta}$ a.s.

By construction, $\mathbb{1}_{F^o}(\omega) = 1$ iff $\mathbb{P}^{t,\omega}(F) = 1$ and $\mathbb{1}_{F^o}(\omega) = 0$ iff $\mathbb{P}^{t,\omega}(F) \in [0, 1[$. So

$$\begin{array}{ll} & \{\omega : \mathbb{1}_{F^o}(\omega) \neq \mathbb{1}_F(\omega)\} \\ = & \{\omega : \mathbb{1}_{F^o}(\omega) = 1 \text{ and } \mathbb{1}_F(\omega) = 0\} \bigcup \{\omega : \mathbb{1}_{F^o}(\omega) = 0 \text{ and } \mathbb{1}_F(\omega) = 1\} \\ = & \{\omega : \mathbb{P}^{t,\omega}(F) = 1 \text{ and } \mathbb{1}_F(\omega) = 0\} \bigcup \{\omega : \mathbb{P}^{t,\omega}(F) \in [0, 1[\text{ and } \mathbb{1}_F(\omega) = 1\} \\ \subset & \{\omega : \mathbb{P}^{t,\omega}(F) \neq \mathbb{1}_F(\omega)\}, \end{array} \tag{3.8}$$

where the latter set is $\mathbb{P}^{s,\eta}$-negligible by (3.7). □

Combining Propositions 3.18 and 3.20, we have the following.

Corollary 3.21. *Assume that Hypothesis 3.5 holds and let us fix $(s, \eta) \in \mathbb{R}_+ \times \Omega$ and $t \geq s$. Given an $\mathcal{F}_t^{s,\eta}$-measurable r.v. $Z^{s,\eta}$, there exists an $\mathcal{F}_t^o$-measurable r.v. Z^o such that $Z^{s,\eta} = Z^o$ $\mathbb{P}^{s,\eta}$ a.s.*

Definition 3.22. *If $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is a probability space and $\mathcal{G}$ is a sub-σ-field of $\tilde{\mathcal{F}}$, we say that $\mathcal{G}$ is $\mathbb{P}$**-trivial** if for any element G of $\mathcal{G}$, then $\mathbb{P}(G) \in \{0, 1\}$.*

Corollary 3.23. *Assume that Hypothesis 3.5 holds. For every $(s, \eta) \in \mathbb{R}_+ \times \Omega$, $\mathcal{F}_s^o$ and $\mathcal{F}_s$ are $\mathbb{P}^{s,\eta}$-trivial.*

Proof. We fix $(s, \eta) \in \mathbb{R}_+ \times \Omega$. We start by showing that $\mathcal{F}_s^o$ is $\mathbb{P}^{s,\eta}$-trivial. For every $B \in \mathcal{F}_s^o$ and ω we have $\mathbb{1}_B(\omega) = \mathbb{1}_B(\omega^s)$, and since $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$, we have $\mathbb{1}_B(\omega^s) = \mathbb{1}_B(\eta^s)$ $\mathbb{P}^{s,\eta}$ a.s. So $\mathbb{P}^{s,\eta}(B) = \mathbb{E}^{s,\eta}[\mathbb{1}_B(\omega)] = \mathbb{1}_B(\eta^s) \in \{0, 1\}$. Then, it is clear that adding $\mathbb{P}^{s,\eta}$-negligible sets does not change the fact of being $\mathbb{P}^{s,\eta}$-trivial, so $\mathcal{F}_s^{o,s,\eta}$ (which by Proposition 3.20 is equal to $\mathcal{F}_s^{s,\eta}$) is $\mathbb{P}^{s,\eta}$-trivial and therefore so is $\mathcal{F}_s \subset \mathcal{F}_s^{s,\eta}$.

□

4 Path-dependent Additive Functionals

In this section, we introduce the notion of Path-dependent Additive Functionals that we use in the paper. As already anticipated, this can be interpreted as a path-dependent extension of the notion of non-homogeneous Additive Functionals of a canonical Markov class developed in [5]. For that reason, several proofs of this section are very similar to those of [5] and are inspired from [12] Chapter XV, which treats the time-homogeneous case.

We keep on using Notation 3.1 and we fix a path-dependent canonical class $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ and assume the following for the whole section.

Hypothesis 4.1. $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ *is progressive and verifies Hypothesis 3.5.*

We will use the notation $\Delta := \{(t, u) \in \mathbb{R}_+^2 | t \leq u\}$.

Definition 4.2. *On $(\Omega, \mathcal{F})$, a **path-dependent Additive Functional** (in short path-dependent AF) will be a random-field $A := (A_{t,u})_{(t,u)\in\Delta}$ with values in $\mathbb{R}$ verifying the two following conditions.*

1. *For any* $(t,u) \in \Delta$, $A_{t,u}$ *is* $\mathcal{F}^o_u$*-measurable;*

2. *for any* $(s,\eta) \in \mathbb{R}_+ \times \Omega$, *there exists a real cadlag* $\mathbb{F}^{s,\eta}$*-adapted process* $A^{s,\eta}$ *(taken equal to zero on* $[0,s]$ *by convention) such that for any* $\eta \in \Omega$ *and* $s \leq t \leq u$,
$$A_{t,u} = A^{s,\eta}_u - A^{s,\eta}_t \quad \mathbb{P}^{s,\eta} \text{ a.s.}$$

We denote by A^t *the (*$\mathbb{F}^o$*-adapted) process* $u \mapsto A_{t,u}$ *indexed by* $[t,+\infty[$. *For any* $(s,\eta) \in [0,t] \times \Omega$, $A^{s,\eta}_{\cdot} - A^{s,\eta}_t$ *is a* $\mathbb{P}^{s,\eta}$*-version of* A^t *on* $[t,+\infty[$. $A^{s,\eta}$ *will be called the* ***cadlag version of*** A ***under*** $\mathbb{P}^{s,\eta}$.

A path-dependent Additive Functional will be called a ***path-dependent Martingale Additive Functional*** *(in short path-dependent MAF) if under any* $\mathbb{P}^{s,\eta}$ *its cadlag version is a martingale.*

More generally, a path-dependent AF will be said to verify a certain property (being non-decreasing, of bounded variation, square integrable, having $\mathcal{L}^1$*-terminal value) if under any* $\mathbb{P}^{s,\eta}$ *its cadlag version verifies it.*

Finally, given two increasing path-dependent AFs A *and* B, A *will be said to be* ***absolutely continuous with respect to*** B *if for any* $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $dA^{s,\eta} \ll dB^{s,\eta}$ *in the sense of stochastic measures. This means that* $dA^{s,\eta}(\omega)$ *is absolutely continuous with respect to* $dB^{s,\eta}(\omega)$ *for* $\mathbb{P}^{s,\eta}$ *almost all* ω.

Remark 4.3. *The set of path-dependent AFs (resp. path-dependent AFs with bounded variation, path-dependent AFs with* $\mathcal{L}^1$*-terminal value, path-dependent MAFs, square integrable path-dependent MAFs) is a linear space.*

Lemma 4.4. *Let* M *be an* $\mathbb{F}^o$*-adapted process such that for all* (s,η), *on* $[s,+\infty[$, M *is a* $(\mathbb{P}^{s,\eta},\mathbb{F}^o)$*-martingale.*
Then, for all (s,η), $M_{\cdot\vee s} - M_s$ *admits a* $\mathbb{P}^{s,\eta}$*-version which is a* $(\mathbb{P}^{s,\eta},\mathbb{F}^{s,\eta})$ *cadlag martingale* $M^{s,\eta}$ *vanishing in* $[0,s]$. *In particular* $M_{t,u}(\omega) := M_u(\omega) - M_t(\omega)$ *defines a path-dependent MAF with cadlag version* $M^{s,\eta}$ *under* $\mathbb{P}^{s,\eta}$.

Proof. By Propositions 3.19 and 3.20, M is also on $[s,+\infty[$ a $(\mathbb{P}^{s,\eta},\mathbb{F}^{s,\eta})$-martingale hence $M_{\cdot\vee s} - M_s$ is on $\mathbb{R}_+$ a $(\mathbb{P}^{s,\eta},\mathbb{F}^{s,\eta})$-martingale and vanishes on $[0,s]$. Since $\mathbb{F}^{s,\eta}$ satisfies the usual conditions, then $M_{\cdot\vee s} - M_s$ admits a cadlag $\mathbb{P}^{s,\eta}$-modification $M^{s,\eta}$ which also is a $(\mathbb{P}^{s,\eta},\mathbb{F}^{s,\eta})$-martingale vanishing in $[0,s]$. It clearly verifies that $M_{t,u} = M_u - M_t = M^{s,\eta}_u - M^{s,\eta}_t$ $\mathbb{P}^{s,\eta}$-a.s. for all $s \leq t \leq u$. □

Example 4.5. *Let* Z *be an* $\mathcal{F}$*-measurable bounded r.v. A typical example of process verifying the conditions of previous Lemma 4.4 is given by* $M^Z : (t,\omega) \longmapsto \mathbb{E}^{t,\omega}[Z]$, *see Remark 3.6.*

The following results state that, for a given square integrable path-dependent MAF $(M_{t,u})_{(t,u)\in\Delta}$ we can exhibit two non-decreasing path-dependent AFs with $\mathcal{L}^1$-terminal value, denoted respectively by $([M]_{t,u})_{(t,u)\in\Delta}$ and $(\langle M\rangle_{t,u})_{(t,u)\in\Delta}$, which will play respectively the role of a quadratic variation and an angular bracket of it. Moreover we will show that the Radon-Nikodym derivative of the mentioned angular bracket of a square integrable path-dependent MAF with

respect to a reference function V is a progressively measurable process which does not depend on the probability.
The proof of the proposition below is postponed to the appendix.

Proposition 4.6. *Let $(M_{t,u})_{(t,u)\in\Delta}$ be a square integrable path-dependent MAF, and for any $(s,\eta)\in\mathbb{R}_+\times\Omega$, $[M^{s,\eta}]$ denote the quadratic variation of its cadlag version $M^{s,\eta}$ under $\mathbb{P}^{s,\eta}$. Then there exists a non-decreasing path-dependent AF with $\mathcal{L}^1$-terminal value which we will call $([M]_{t,u})_{(t,u)\in\Delta}$ and which, for any $(s,\eta)\in\mathbb{R}_+\times\Omega$, has $[M^{s,\eta}]$ as cadlag version under $\mathbb{P}^{s,\eta}$.*

The next result can be seen as an extension of Theorem 15 Chapter XV in [12] to a path-dependent context and will be needed to show that the result above also holds for the angular bracket. Its proof is also postponed to the appendix.

Proposition 4.7. *Let $(B_{t,u})_{(t,u)\in\Delta}$ be a non-decreasing path-dependent AF with $\mathcal{L}^1$- terminal value. For any $(s,\eta)\in\mathbb{R}_+\times\Omega$, let $B^{s,\eta}$ be its cadlag version under $\mathbb{P}^{s,\eta}$ and let $A^{s,\eta}$ be the predictable dual projection of $B^{s,\eta}$ in $(\Omega,\mathcal{F}^{s,\eta},\mathbb{F}^{s,\eta},\mathbb{P}^{s,\eta})$. Then there exists a non-decreasing path-dependent AF with $\mathcal{L}^1$-terminal value $(A_{t,u})_{(t,u)\in\Delta}$ such that under any $\mathbb{P}^{s,\eta}$, the cadlag version of A is $A^{s,\eta}$.*

Remark 4.8.

1. *About the notion of dual predictable projection (also called compensator) related to some stochastic basis we refer to Theorem 3.17 in Chapter I of [17].*

2. *We recall that, whenever M,N are two local martingales, the angle bracket $\langle M,N\rangle$ is the dual predictable projection of $[M,N]$, see Proposition 4.50 b) in Chapter I of [17].*

Corollary 4.9. *Let $(M_{t,u})_{(t,u)\in\Delta}$, $(N_{t,u})_{(t,u)\in\Delta}$ be two square integrable path-dependent MAFs, let $M^{s,\eta}$ (respectively $N^{s,\eta}$) be the cadlag version of M (respectively N) under $\mathbb{P}^{s,\eta}$. Then there exists a bounded variation path-dependent AF with $\mathcal{L}^1$-terminal value, denoted $(\langle M,N\rangle_{t,u})_{(t,u)\in\Delta}$, such that under any $\mathbb{P}^{s,\eta}$, the cadlag version of $\langle M,N\rangle$ is $\langle M^{s,\eta},N^{s,\eta}\rangle$. If $M=N$ the path-dependent AF $\langle M,N\rangle$ will be denoted $\langle M\rangle$ and is non-decreasing.*

Proof. This can be proved as for Corollary 4.11 in [5], replacing parameter (s,x) with (s,η). □

The result below concerns the Radon-Nikodym derivative of a non-decreasing continuous path-dependent AF with respect to some reference measure dV. Its proof is postponed to the Appendix.

Proposition 4.10. *Let $V:\mathbb{R}_+\longrightarrow\mathbb{R}$ be a non-decreasing continuous function. Let A be a non-negative, non-decreasing path-dependent AF absolutely continuous with respect to V, and for any $(s,\eta)\in\mathbb{R}_+\times\Omega$ let $A^{s,\eta}$ be the cadlag version of A under $\mathbb{P}^{s,\eta}$. There exists an $\mathbb{F}^o$-progressively measurable process h such that for any $(s,\eta)\in\mathbb{R}_+\times\Omega$, $A^{s,\eta}=\int_s^{\cdot\vee s}h_r dV_r$, in the sense of indistinguishability.*

Proposition 4.11. *Let $(A_{t,u})_{(t,u)\in\Delta}$ be a path-dependent AF with bounded variation, taking $\mathcal{L}^1$-terminal value. Then there exists an increasing path-dependent AF that we denote $(Pos(A)_{t,u})_{(t,u)\in\Delta}$ (resp. $(Neg(A)_{t,u})_{(t,u)\in\Delta}$), which, for any $(s,\eta)\in\mathbb{R}_+\times\Omega$, has $Pos(A^{s,\eta})$ (resp. $Neg(A^{s,\eta})$)) as cadlag version under $\mathbb{P}^{s,\eta}$.*

Proof. This can be proved similarly as forProposition 4.14 in [5], replacing parameter (s,x) with (s,η). □

Corollary 4.12. *Let V be a continuous non-decreasing function. Let M and N be two square integrable path-dependent MAFs and let $M^{s,\eta}$ (respectively $N^{s,\eta}$) be the cadlag version of M (respectively N) under a fixed $\mathbb{P}^{s,\eta}$. Assume that $\langle N\rangle$ is absolutely continuous with respect to dV. There exists an $\mathbb{F}^o$-progressively measurable process k such that for any $(s,\eta)\in\mathbb{R}_+\times\Omega$, $\langle M^{s,\eta},N^{s,\eta}\rangle=\int_s^{\cdot\vee s}k_r dV_r$.*

Proof. The proof follows the same lines as the one of Proposition 4.17 in [5] replacing parameter (s,x) by (s,η) and Borel functions of (t,X_t) with $\mathbb{F}^o$-progressively measurable processes. We make use of Corollary 4.9, Propositions 4.11 and 4.10, respectively in substitution of Corollary 4.11 an Propositions 4.14 and 4.13. □

Corollary 4.13. *Let V be a continuous non-decreasing function. Let M (resp. N) be an $\mathbb{F}^o$-adapted process such that for all (s,η), M (resp. N) is on $[s,+\infty[$ a $(\mathbb{P}^{s,\eta},\mathbb{F}^o)$ square integrable martingale. For any (s,η), let $M^{s,\eta}$ (resp. $N^{s,\eta}$) denote its $\mathbb{P}^{s,\eta}$-cadlag version. Assume that for all (s,η), $d\langle N^{s,\eta}\rangle\ll dV$.*
Then there exists an $\mathbb{F}^o$-progressively measurable process k such that for any $(s,\eta)\in\mathbb{R}_+\times\Omega$, $\langle M^{s,\eta},N^{s,\eta}\rangle=\int_s^{\cdot\vee s}k_r dV_r$.

Proof. The mentioned cadlag versions exist because of Lemma 4.4. The statement follows by the same Lemma 4.4 and Corollary 4.12.

□

5 Path-dependent Martingale problems

5.1 Abstract Martingale Problems

In this section we show that, whenever a (path-dependent) martingale problem is well-posed, then its solution is a path-dependent canonical class verifying Hypothesis 3.5. This relies on the same mathematical tools than those used by D.S Stroock and S.R.S Varadhan in the context of Markovian diffusions in [20]. Indeed it was already known that the ideas of [20] could be used in any type of Markovian setup and not just for martingale problems associated to diffusions, see [15] for example. One of the interests of the following lines is to show that their scope goes beyond the Markovian framework. First we prove that $\eta\mapsto\mathbb{P}^{s,\eta}$ is measurable, using well-posedness arguments and the celebrated Kuratowsky Theorem. Then we show in Proposition 5.12 that the solution of the martingale problem verifies (3.2), which is the analogous formulation of Markov property, through the theory of regular conditional expectations and again the fact that the martingale problem is well-posed.

Notation 5.1. *For every $t \in \mathbb{R}_+$, $\Omega^t := \{\omega \in \Omega : \omega = \omega^t\}$ will denote the set of constant paths after time t. We also denote $\Lambda := \{(s,\eta) \in \mathbb{R}_+ \times \Omega : \eta \in \Omega^s\}$.*

Proposition 5.2.

1. *Λ is a closed subspace of $\mathbb{R}_+ \times \Omega$, hence a Polish space when equipped with the induced topology.*
2. *For any $t \in \mathbb{R}_+$, Ω^t is also a closed subspace of Ω.*

Proof. We will only show the first statement since the proof of the second one is similar but simpler. Let $(s_n, \eta_n)_n$ be a sequence in Λ. Let $(s,\eta) \in \mathbb{R}_+ \times \Omega$ and assume that $s_n \to s$ and that η_n tends to η for the Skorokhod topology. Then η_n tends to η Lebesgue a.e. Let $\epsilon > 0$. There is a subsequence (s_{n_k}) such that $|s_{n_k} - s| \leq \epsilon$, implying that for all k, η_{n_k} is constantly equal to $\eta_{n_k}(s_{n_k})$ on $[s+\epsilon, +\infty[$. Since η_n tends to η Lebesgue a.e., then necessarily, $\eta_{n_k}(s_{n_k})$ tends to some $c \in E$ and η takes value c a.e. on $[s+\epsilon, +\infty[$. This holds for every ϵ, and η is cadlag, so η is constantly equal to c on $[s, +\infty[$, implying that $(s,\eta) \in \Lambda$. □

From now on, Λ, introduced in Notation 5.1, is equipped with the trace topology.

Proposition 5.3. *The Borel σ-field $\mathcal{B}(\Lambda)$ is equal to the trace σ-field $\Lambda \cap \mathcal{P}ro^o$. For any $t \in \mathbb{R}_+$, the Borel σ-field $\mathcal{B}(\Omega^t)$ is equal to the trace σ-field $\Omega^t \cap \mathcal{F}^o_t$.*

Proof. Again we only show the first statement since the proof of the second one is similar. By definition of the topology on Λ, it is clear that $\mathcal{B}(\Lambda) = \Lambda \cap \mathcal{B}(\mathbb{R}_+ \times \Omega) = \Lambda \cap (\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F})$ contains $\Lambda \cap \mathcal{P}ro^o$. We show the converse inclusion. The sets $\Lambda \cap ([s,u] \times \{\omega(r) \in A\})$ for $s,u,r \in \mathbb{R}_+$ with $s \leq u$, $A \in \mathcal{B}(E)$ generate $\Lambda \cap (\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F})$ so it is enough to show that these sets belong to $\Lambda \cap \mathcal{P}ro^o$.

We fix $s \leq u$ and r in $\mathbb{R}_+$, and $A \in \mathcal{B}(E)$. We have

$$\begin{aligned} \Lambda \bigcap ([s,u] \times \{\omega(r) \in A\}) &= \left\{ (t,\omega) : \left\{ \begin{array}{l} t \in [s,u] \\ \omega = \omega^t \\ \omega(r) \in A \end{array} \right. \right\} \\ &= \left\{ (t,\omega) : \left\{ \begin{array}{l} t \in [s,u] \\ \omega = \omega^t \\ \omega(r \wedge t) \in A \end{array} \right. \right\} \\ &= \Lambda \bigcap \left\{ (t,\omega) : \left\{ \begin{array}{l} t \in [s,u] \\ \omega(r \wedge t) \in A. \end{array} \right. \right\} \end{aligned} \tag{5.1}$$

We are left to show that $\left\{ (t,\omega) : \left\{ \begin{array}{l} t \in [s,u] \\ \omega(r \wedge t) \in A \end{array} \right. \right\} \in \mathcal{P}ro^o$, or equivalently that

$$t \mapsto \mathbb{1}_{[s,u]}(t)\mathbb{1}_A(X_{r\wedge t}) \text{ is } \mathbb{F}^o - \text{progressively measurable.} \tag{5.2}$$

Now $t \mapsto X_{r\wedge t}$ is right-continuous and $\mathbb{F}^o$-adapted so it is an E-valued $\mathbb{F}^o$-progressively measurable process, see Theorem 15 in [10] Chapter IV. By composition with a Borel function, $t \mapsto \mathbb{1}_A(X_{r\wedge t})$ is a real-valued $\mathbb{F}^o$-progressively measurable process; (5.2) follows since $t \mapsto \mathbb{1}_{[s,u]}(t)$ is $\mathbb{F}^o$-progressively measurable and the product of the two $\mathbb{F}^o$-progressively measurable processes remains $\mathbb{F}^o$-progressively measurable. □

Definition 5.4. *Let $(s,\eta) \in \Lambda$ and χ be a set of $\mathbb{F}^o$-adapted processes. We say that a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$* ***solves the martingale problem with respect to*** *χ* ***starting in*** *(s,η) if*

- *$\mathbb{P}(\omega^s = \eta^s) = 1$,*
- *all elements of χ are on $[s, +\infty[$ $(\mathbb{P}, \mathbb{F}^o)$-martingales.*

Remark 5.5. *We insist on the following important fact. If $M \in \chi$ is cadlag and $\mathbb{P}$ solves the martingale problem associated to χ, then by Theorem 3 in [11] Chapter VI, M is also on $[s, +\infty[$ a $(\mathbb{P}, \mathbb{F})$-martingale.*

Notation 5.6. *For fixed $(s,\eta) \in \Lambda$ and χ, the set of probability measures solving the martingale problem with respect to χ starting in (s,η) will be denoted $MP^{s,\eta}(\chi)$.*

Definition 5.7. *Let us consider a set χ of processes. If for every $(s,\eta) \in \Lambda$, $MP^{s,\eta}(\chi)$ is reduced to a single element $\mathbb{P}^{s,\eta}$, we will say that the martingale problem associated to χ is* ***well-posed****. In this case we will always extend the mapping*

$$\begin{array}{ccc} (s,\eta) & \longmapsto & \mathbb{P}^{s,\eta} \\ \Lambda & \longrightarrow & \mathcal{P}(\Omega) \end{array} \tag{5.3}$$

to $\mathbb{R}_+ \times \Omega$ by setting for all $(s,\eta) \in \mathbb{R}_+ \times \Omega$, $\mathbb{P}^{s,\eta} := \mathbb{P}^{s,\eta^s}$.

Notation 5.8. *We fix a dense sequence $(x_n)_{n\geq 0}$ of elements of E.*
For any $s \in \mathbb{R}_+$, we will denote by Π_s the set of elements of $\mathcal{F}^o_s$ of type $\{\omega(t_1) \in B(x_{i_1}, r_1), \cdots, \omega(t_N) \in B(x_{i_N}, r_N)\}$ where $N \in \mathbb{N}$, $t_1, \cdots, t_N \in [0,s] \cap \mathbb{Q}$, $i_1, \cdots, i_N \in \mathbb{N}$, $r_1, \cdots, r_N \in \mathbb{Q}_+$ and where $B(x,r)$ denotes the open ball centered in x and of radius r.

It is easy to show that for any $s \in \mathbb{R}_+$, Π_s is a countable π-system generating $\mathcal{F}^o_s$, see [1] Definition 4.9 for the notions of π-system and λ-system.

Below we consider the set $\mathcal{A}_s$ of probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F})$ for which there exists $\eta \in \Omega$ such that $\mathbb{P}$ solves the martingale problem with respect to χ starting at (s,η).

Proposition 5.9. *We fix a countable set χ of cadlag $\mathbb{F}^o$-adapted processes which are uniformly bounded on each interval $[0,T]$, and some $s \in \mathbb{R}_+$. Let $\mathcal{A}_s := \bigcup_{\eta\in\Omega} MP^{s,\eta}(\chi)$. Then $\mathcal{A}_s$ is a Borel set of $\mathcal{P}(\Omega)$.*

For the proof of this proposition we need a technical lemma.

Lemma 5.10. *We fix $s \in \mathbb{R}_+$. An element $\mathbb{P}$ of $\mathcal{P}(\Omega)$ belongs to $\mathcal{A}_s$ if and only if it verifies the following conditions:*

1. $\mathbb{P}(F) \in \{0,1\}$ *for all* $F \in \Pi_s$*;*
2. $\mathbb{E}^{\mathbb{P}}[(M_u - M_t)\mathbb{1}_F] = 0$ *for all* $M \in \chi$, $t, u \in [s, +\infty[\cap\mathbb{Q}$ *such that* $t \le u$, $F \in \Pi_t$.

Proof. By definition of $\mathcal{A}_s$, an element $\mathbb{P}$ of $\mathcal{P}(\Omega)$ belongs to $\mathcal{A}_s$ iff

a) there exists $\eta \in \Omega$ such that $\mathbb{P}(\omega^s = \eta^s) = 1$;

b) for all $M \in \chi$, $(M_t)_{t \in [s,+\infty[}$ is a $(\mathbb{P}, \mathbb{F}^o)$-martingale.

Item a) above is equivalent to saying that $\mathcal{F}^o_s$ is $\mathbb{P}$-trivial which is equivalent to item 1. of the Lemma's statement by Dynkin's Lemma (see 4.11 in [1]), since Π_s is a π-system generating $\mathcal{F}^o_s$ and since the sets $F \in \mathcal{F}^o_s$ such that $\mathbb{P}(F) \in \{0,1\}$ form a λ-system.
On the other hand, it is clear that item b) above implies item 2. in the statement of the Lemma. Conversely, assume that $M \in \chi$ satisfies item 2. of the statement. We fix $s \le t \le u$. Let $(t_n)_n, (u_n)_n$ be two sequences of rational numbers which converge to respectively to t, u strictly from the right and such that $t_n \le u_n$ for all n. For every fixed n, we have $\mathbb{E}^{\mathbb{P}}[(M_{u_n} - M_{t_n})\mathbb{1}_G] = 0$ for all $G \in \Pi_t$. We then pass to the limit in n using the fact that M is right-continuous at fixed ω, and the dominated convergence theorem and taking into account the fact that M is bounded on compact intervals; this yields $\mathbb{E}^{\mathbb{P}}[(M_u - M_t)\mathbb{1}_G] = 0$ for all $G \in \Pi_t$. Since sets $G \in \mathcal{F}^o_t$ verifying this property form a λ-system and since Π_t is a π-system generating $\mathcal{F}^o_t$, then by Dynkin's lemma (see 4.11 in [1]), $\mathbb{E}^{\mathbb{P}}[(M_u - M_t)\mathbb{1}_G] = 0$ for all $G \in \mathcal{F}^o_t$. This implies that $(M_t)_{t \in [s,+\infty[}$ is a $(\mathbb{P}, \mathbb{F}^o)$-martingale which concludes the proof of Lemma 5.10. □

Proof of Proposition 5.9.
We fix $s \in \mathbb{R}_+$. We recall that for any bounded random variable ϕ, $\mathbb{P} \mapsto \mathbb{E}^{\mathbb{P}}[\phi]$ is Borel. In particular for all $F \in \Pi_s$, $\mathbb{P} \longmapsto \mathbb{P}(F)$ and for all $M \in \chi$, $t, u \in [s, +\infty[\cap\mathbb{Q}$, $F \in \Pi_t$, $\mathbb{P} \longmapsto \mathbb{E}^{\mathbb{P}}[(M_u - M_t)\mathbb{1}_F]$ are Borel maps. The result follows by Lemma 5.10, taking into account the fact Π_t is countable for any t, and χ and the rational number set $\mathbb{Q}$ are also countable. Indeed since $\{0\}$ and $\{0,1\}$ are Borel sets, $\mathcal{A}_s$ is Borel being a countable intersection of preimages of Borel sets by Borel functions. □

Proposition 5.11. *Let χ be a countable set of cadlag $\mathbb{F}^o$-adapted processes which are uniformly bounded on each interval $[0,T]$. We assume that the martingale problem associated to χ is well-posed, see Definition 5.7. Let $s \in \mathbb{R}_+$. Then* $\Phi_s : \begin{pmatrix} \eta & \longmapsto & \mathbb{P}^{s,\eta} \\ \Omega^s & \longrightarrow & \mathcal{P}(\Omega) \end{pmatrix}$ *is Borel. Moreover,* $\begin{pmatrix} (s,\eta) & \longmapsto & \mathbb{P}^{s,\eta} \\ \mathbb{R}_+ \times \Omega & \longrightarrow & \mathcal{P}(\Omega) \end{pmatrix}$ *is $\mathbb{F}^o$-adapted.*

Proof. We fix $s \in \mathbb{R}_+$ and set

$$\Phi_s : \begin{array}{ccc} \eta & \longmapsto & \mathbb{P}^{s,\eta} \\ \Omega^s & \longrightarrow & \mathcal{A}_s, \end{array} \tag{5.4}$$

where $\mathcal{A}_s$ is defined as in Proposition 5.9. Φ_s is surjective by construction. It is also injective. Indeed, if $\eta_1, \eta_2 \in \Omega^s$ are different, there exists $t \in [0, s]$ such that $\eta_1(t) \neq \eta_2(t)$ and we have $\mathbb{P}^{s,\eta_1}(\omega(t) = \eta_1(t)) = 1$ and $\mathbb{P}^{s,\eta_2}(\omega(t) = \eta_2(t)) = 1$ so clearly $\mathbb{P}^{s,\eta_1} \neq \mathbb{P}^{s,\eta_2}$.

We can therefore introduce the reciprocal mapping

$$\Phi_s^{-1} : \begin{array}{ccc} \mathbb{P}^{s,\eta} & \longmapsto & \eta \\ \mathcal{A}_s & \longrightarrow & \Omega^s, \end{array} \tag{5.5}$$

which is a bijection. We wish to show that it is Borel. Since the Borel σ-algebra of Ω^s is generated by the sets of type $\{\omega(r \wedge s) \in A\}$ where $r \in \mathbb{R}_+$ and $A \in \mathcal{B}(E)$, it is enough to show that $\Phi_s(\{\omega(r \wedge s) \in A\})$ is for any r, A a Borel subset of $\mathcal{P}(\Omega)$. We then have $\Phi_s(\{\omega(r \wedge s) \in A\}) = \mathcal{A}_s \cap \{\mathbb{P} : \mathbb{P}(\omega(r \wedge s) \in A) = 1\}$ which is Borel being the intersection of $\mathcal{A}_s$ which is Borel by Lemma 5.10, and of the preimage of $\{1\}$ by the Borel function $\mathbb{P} \mapsto \mathbb{P}(F)$ with $F = \{\omega(r \wedge s) \in A\}$. So Φ_s^{-1} is a Borel bijection which maps the Borel set $\mathcal{A}_s$ of the Polish space $\mathcal{P}(\Omega)$ into the Polish space Ω^s. By Kuratowsky theorem (see Corollary 3.3 in [19]), $\Phi_s : \begin{array}{ccc} \eta & \longmapsto & \mathbb{P}^{s,\eta} \\ \Omega^s & \longrightarrow & \mathcal{P}(\Omega) \end{array}$ is Borel.

Let us justify the second part of the statement. Since by Proposition 5.3, $\mathcal{B}(\Omega^s) = \Omega^s \cap \mathcal{F}_s^o$ for all s, it is clear that $\left(\begin{array}{ccc} \eta & \longmapsto & \eta^s \\ \Omega & \longrightarrow & \Omega^s \end{array}\right)$ is $(\mathcal{F}_s^o, \mathcal{B}(\Omega^s))$-measurable and therefore that $\left(\begin{array}{ccc} \eta & \longmapsto & \mathbb{P}^{s,\eta} \\ \Omega & \longrightarrow & \mathcal{P}(\Omega) \end{array}\right)$ is $\mathcal{F}_s^o$-measurable.

□

Proposition 5.12. *Let χ be a countable set of cadlag $\mathbb{F}^o$-adapted processes which are uniformly bounded on each interval $[0, T]$, and assume that the martingale problem associated to χ is well-posed, see Definition 5.7. Then $(\mathbb{P}^{s,\eta})_{(s,\eta) \in \mathbb{R}_+ \times \Omega}$ is a path-dependent canonical class verifying Hypothesis 3.5 .*

Proof. The first two items of Definition 3.4 are directly implied by Proposition 5.11 and the fact that $\mathbb{P}^{s,\eta} \in MP^{s,\eta}(\chi)$ hence $\mathbb{P}^{s,\eta}(\omega^s = \eta^s)$ for all (s, η). It remains to show the validity of Hypothesis 3.5.

We fix $(s, \eta) \in \mathbb{R}_+ \times \Omega$ and $t \geq s$. Since Ω is Polish and $\mathcal{F}_t$ is a sub σ-field of its Borel σ-field, there exists a regular conditional expectation of $\mathbb{P}^{s,\eta}$ by $\mathcal{F}_t$ (see Theorem 1.1.6 in [20]), meaning a set of probability measures $(\mathbb{Q}^{t,\zeta})_{\zeta \in \Omega}$ on $(\Omega, \mathcal{F})$ such that

1. for any $F \in \mathcal{F}$, $\zeta \mapsto \mathbb{Q}^{t,\zeta}(F)$ is $\mathcal{F}_t$-measurable;
2. for any $F \in \mathcal{F}$, $\mathbb{P}^{s,\eta}(F|\mathcal{F}_t)(\zeta) = \mathbb{Q}^{t,\zeta}(F)$ $\mathbb{P}^{s,\eta}$ a.s.

We will now show that for $\mathbb{P}^{s,\eta}$ almost all ζ, we have

$$\mathbb{Q}^{t,\zeta} = \mathbb{P}^{t,\zeta}, \tag{5.6}$$

so that item 2. above will imply Hypothesis 3.5. In order to show that equality, we will show that for $\mathbb{P}^{s,\eta}$ almost all ζ, $\mathbb{Q}^{t,\zeta}$ solves the Martingale problem associated to χ starting in (t,ζ) and conclude (5.6) since $MP^{t,\zeta}$ is a singleton, taking into account the fact the corresponding martingale problem is well-posed.

For any $F \in \mathcal{F}_t^o$, by item 2. above we have $\mathbb{Q}^{t,\zeta}(F) = \mathbb{1}_F(\zeta)\ \mathbb{P}^{s,\eta}$ a.s. Since Π_t is countable, there exists a $\mathbb{P}^{s,\eta}$-null set N_1 such that for all $\zeta \in N_1^c$ we have $\mathbb{Q}^{t,\zeta}(F) = \mathbb{1}_F(\zeta)$ for all $F \in \Pi_t$. Then since Π_t is a π-system generating $\mathcal{F}_t^o$ and since sets verifying the previous relation define a λ-system, we have by Dynkin's lemma (see 4.11 in [1]) that for all $\zeta \in N_1^c$, $\mathbb{Q}^{t,\zeta}(F) = \mathbb{1}_F(\zeta)$ for all $F \in \mathcal{F}_t^o$. Now for every fixed $\zeta \in N_1^c$, since $\{\omega : \omega^t = \zeta^t\} \in \mathcal{F}_t^o$, we have $\mathbb{Q}^{t,\zeta}(\omega^t = \zeta^t) = \mathbb{1}_{\{\omega:\omega^t=\zeta^t\}}(\zeta) = 1$, which is the first item of Definition 5.4 related to $MP^{t,\zeta}(\chi)$.

We then show that for $\mathbb{P}^{s,\eta}$-almost all ζ, the elements of χ are $(\mathbb{Q}^{t,\zeta}, \mathbb{F}^o)$-martingales, which constitutes the second item of Definition 5.4.
For any $t_1 \leq t_2$ in $[t,+\infty[$, $M \in \chi$ and $F \in \mathcal{F}_{t_1}^o$, we have

$$\begin{array}{rcl} \mathbb{E}^{\mathbb{Q}^{t,\zeta}}[(M_{t_2} - M_{t_1})\mathbb{1}_F] & = & \mathbb{E}^{s,\eta}[(M_{t_2} - M_{t_1})\mathbb{1}_F|\mathcal{F}_t](\zeta) \\ & = & \mathbb{E}^{s,\eta}[\mathbb{E}^{s,\eta}[(M_{t_2} - M_{t_1})\mathbb{1}_F|\mathcal{F}_{t_1}]|\mathcal{F}_t](\zeta) \\ & = & \mathbb{E}^{s,\eta}[\mathbb{E}^{s,\eta}[(M_{t_2} - M_{t_1})|\mathcal{F}_{t_1}]\mathbb{1}_F|\mathcal{F}_t](\zeta) \\ & = & 0, \end{array} \tag{5.7}$$

for $\mathbb{P}^{s,\eta}$ almost all ζ by Remark 5.5 since M is a $(\mathbb{P}^{s,\eta}, \mathbb{F})$-martingale on$[s,+\infty[$ and $F \in \mathcal{F}_{t_1}^o \subset \mathcal{F}_{t_1}$. Since χ and the set of rational numbers are countable and taking into account the fact that for any $r \geq 0$, $\mathcal{F}_r^o$ is countably generated, there exists a $\mathbb{P}^{s,\eta}$-null set N_2 such that for any $\zeta \in N_2^c$, we have for any $t_1 \leq t_2$ in $[t,+\infty[\cap\mathbb{Q}$, $M \in \chi$, $F \in \mathcal{F}_{t_1}^o$, that $\mathbb{E}^{\mathbb{Q}^{t,\zeta}}[(M_{t_2} - M_{t_1})\mathbb{1}_F] = 0$.

Let $\zeta \in N_2^c$. We will now show that this still holds for any $t_1 \leq t_2$ in $[t,+\infty[$, $M \in \chi$, $F \in \mathcal{F}_{t_1}^o$. We consider rational valued sequences $(t_1^n)_n$ (resp. $(t_2^n)_n$) which converge to t_1 (resp. to t_2) strictly from the right and such that $t_1^n \leq t_2^n$ for all n. For all n, $\mathbb{E}^{\mathbb{Q}^{t,\zeta}}[(M_{t_2^n} - M_{t_1^n})\mathbb{1}_F] = 0$; since M is right-continuous and bounded on finite intervals, by dominated convergence, we can pass to the limit in n and we obtain $\mathbb{E}^{\mathbb{Q}^{t,\zeta}}[(M_{t_2} - M_{t_1})\mathbb{1}_F] = 0$. Therefore if $\zeta \notin N_1 \bigcup N_2$ which is $\mathbb{P}^{s,\eta}$-negligible, then $\mathbb{Q}^{t,\zeta}(\omega^t = \zeta^t) = 1$ and all the elements of χ are $(\mathbb{Q}^{t,\zeta}, \mathbb{F}^o)$-martingales. This means that $\mathbb{Q}^{t,\zeta} = \mathbb{P}^{t,\zeta}$ by well-posedness and concludes the proof of Proposition 5.12. □

5.2 Martingale problem associated to an operator and weak generators

This section links the notion of martingale problem with respect to a natural notion of (weak) generator. In this section Notation 3.1 will be again in force. Let $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ be a path-dependent canonical class and the corresponding path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$, see Definition 3.12. Let $V : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ be a non-decreasing cadlag function.

In the sequel of this section, we are given a couple $(\mathcal{D}(A), A)$ verifying the following.

Hypothesis 5.13.

1. $\mathcal{D}(A)$ *is a linear subspace of the space of* $\mathbb{F}^o$*-progressively measurable processes;*

2. A *is a linear mapping from* $\mathcal{D}(A)$ *into the space of* $\mathbb{F}^o$*-progressively measurable processes;*

3. *for all* $\Phi \in \mathcal{D}(A)$*,* $\omega \in \Omega$*,* $t \geq 0$*,* $\int_0^t |A\Phi_r(\omega)|dV_r < +\infty$*;*

4. *for all* $\Phi \in \mathcal{D}(A)$*,* $(s, \eta) \in \mathbb{R}_+ \times \Omega$ *and* $t \in [s, +\infty[$*, we have* $\mathbb{E}^{s,\eta}\left[\int_s^t |A(\Phi)_r|dV_r\right] < +\infty$ *and* $\mathbb{E}^{s,\eta}[|\Phi_t|] < +\infty$*.*

Inspired from the classical literature (see 13.28 in [17]) we introduce a notion of weak generator.

Definition 5.14. *We say that* $(\mathcal{D}(A), A)$ *is a* ***weak generator*** *of a path-dependent system of projectors* $(P_s)_{s\in\mathbb{R}_+}$ *if for all* $\Phi \in \mathcal{D}(A)$*,* $(s, \eta) \in \mathbb{R}_+ \times \Omega$ *and* $t \in [s, +\infty[$*, we have*

$$P_s[\Phi_t](\eta) = \Phi_s(\eta) + \int_s^t P_s[A(\Phi)_r](\eta)dV_r. \tag{5.8}$$

Definition 5.15. *We will call* ***martingale problem associated to*** $(\mathcal{D}(A), A)$ *the martingale problem (in the sense of Definition 5.4) associated to the set of processes* χ *constituted by the processes* $\Phi - \int_0^\cdot A(\Phi)_r dV_r$*,* $\Phi \in \mathcal{D}(A)$*. It will be said to be* ***well-posed*** *if it is well-posed in the sense of Definition 5.7.*

Proposition 5.16. $(\mathcal{D}(A), A)$ *is a weak generator of* $(P_s)_{s\in\mathbb{R}_+}$ *iff* $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ *solves the martingale problem associated to* $(\mathcal{D}(A), A)$*.*

Moreover, if $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ *solves the well-posed martingale problem associated to* $(\mathcal{D}(A), A)$ *then* $(P_s)_{s\in\mathbb{R}_+}$ *is the unique path-dependent system of projectors for which* $(\mathcal{D}(A), A)$ *is a weak generator.*

Proof. We start assuming that $(\mathcal{D}(A), A)$ is a weak generator of $(P_s)_{s\in\mathbb{R}_+}$. Let $\Phi \in \mathcal{D}(A)$, $s \leq t \leq u$. $\mathbb{P}^{s,\eta}$ a.s. we have

$$\begin{array}{ll} & \mathbb{E}^{s,\eta}[\Phi_u - \Phi_t - \int_t^u A(\Phi)_r dV_r|\mathcal{F}_t^o](\omega) \\ = & \mathbb{E}^{t,\omega}[\Phi_u - \Phi_t - \int_t^u A(\Phi)_r dV_r] \\ = & P_t[\Phi_u](\omega) - \Phi_t(\omega) - \int_t^u P_t[A(\Phi_r)](\omega)dV_r \\ = & 0, \end{array} \tag{5.9}$$

where the first equality holds by Remark 3.6, the second one by Fubini's theorem and the third one because $(\mathcal{D}(A), A)$ is assumed to be a weak generator of $(P_s)_{s\in\mathbb{R}_+}$. By definition of path-dependent canonical class, we have $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$. By (5.9), for all $\Phi \in \mathcal{D}(A)$, $\Phi - \int_s^\cdot A(\Phi)_r dV_r$ is a $(\mathbb{P}^{s,\eta}, \mathbb{F}^o)$-martingale, and therefore $\mathbb{P}^{s,\eta}$ solves the martingale problem associated to $(\mathcal{D}(A), A)$ starting in (s, η).

Conversely, let us assume that $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ solves the martingale problem associated to $(\mathcal{D}(A), A)$. Let $\Phi \in \mathcal{D}(A)$ and $(s, \eta) \in \mathbb{R}_+ \times \Omega$ be fixed. By

Definitions 5.15 and 5.7, $M[\Phi] := \Phi - \int_0^{\cdot} A(\Phi)_r dV_r$, is a $(\mathbb{P}^{s,\eta}, \mathbb{F}^o)$-martingale on $[s, +\infty[$. Moreover, since $\mathbb{P}^{s,\eta}(\omega^s = \eta^s) = 1$ and being Φ_s $\mathcal{F}^o_s$-measurable, we obtain $\Phi_s = \Phi_s(\eta)$ $\mathbb{P}^{s,\eta}$ a.s. Therefore, for any $t \geq s$, $\Phi_t - \Phi_s(\eta) - \int_s^t A(\Phi)_r dV_r = M[\Phi]_t - M[\Phi]_s$ a.s.; so taking the $\mathbb{P}^{s,\eta}$ expectation, by Fubini's Theorem and Definition 3.12 it yields

$$\begin{array}{ll} & P_s[\Phi_t](\eta) - \Phi_s(\eta) - \int_s^t P_s[A(\Phi)_r](\eta) dV_r \\ = & \mathbb{E}^{s,\eta}\left[\Phi_t - \Phi_s(\eta) - \int_s^t A(\Phi)_r dV_r\right] \\ = & \mathbb{E}^{s,\eta}\left[M[\Phi]_t - M[\Phi]_s\right] \\ = & 0, \end{array} \tag{5.10}$$

hence that $(\mathcal{D}(A), A)$ is a weak generator of $(P_s)_{s\in\mathbb{R}_+}$.

Finally assume moreover that the martingale problem is well-posed and that $(\mathcal{D}(A), A)$ is a weak generator of another path-dependent system of projectors $(Q_s)_{s\in\mathbb{R}_+}$ with associated path-dependent canonical class $(\mathbb{Q}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$. Then by the first statement of the present proposition, $(\mathbb{Q}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ solves the martingale problem associated to $(\mathcal{D}(A), A)$. Since that martingale problem is well-posed we have $(\mathbb{Q}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega} = (\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ and by Proposition 3.11, $(Q_s)_{s\in\mathbb{R}_+} = (P_s)_{s\in\mathbb{R}_+}$. □

Remark 5.17. *When the conditions of previous proposition are verified, one can therefore associate analytically to $(\mathcal{D}(A), A)$ a unique path-dependent system of projectors $(P_s)_{s\in\mathbb{R}_+}$ through Definition 5.14.*

Combining Proposition 5.16 and Lemma 4.4 yields the following.

Corollary 5.18. *Assume that $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$ is progressive and fulfills Hypothesis 3.5. Suppose that $(\mathcal{D}(A), A)$ is a weak generator of $(P_s)_{s\in\mathbb{R}_+}$. Let $\Phi \in \mathcal{D}(A)$, and fix (s, η). Then $\Phi - \int_0^{\cdot} A(\Phi)_r dV_r$ admits on $[s, +\infty[$ a $\mathbb{P}^{s,\eta}$ version $M[\Phi]^{s,\eta}$ which is a $(\mathbb{P}^{s,\eta}, \mathbb{F}^{s,\eta})$-cadlag martingale. In particular, the random field defined by $M[\Phi]_{t,u}(\omega) := \Phi_u(\omega) - \Phi_t(\omega) - \int_t^u A\Phi_r(\omega) dV_r$ defines a MAF with cadlag version $M[\Phi]^{s,\eta}$ under $\mathbb{P}^{s,\eta}$.*

We insist on the fact that in previous corollary, Φ is not necessarily cadlag. That result will be crucial in the companion paper [3].

6 Weak solutions of path-dependent SDEs

We will now focus on a more specific type of martingale problem which will be associated to a path-dependent Stochastic Differential Equation with jumps. In this section we will refer to notions of [17] Chapters II, III, VI and [16] Chapter XIV.5.

We fix $m \in \mathbb{N}^*$, $E = \mathbb{R}^m$, the associated canonical space, see Definition 3.1, and a finite positive measure F on $\mathcal{B}(\mathbb{R}^m)$ not charging 0.

Definition 6.1. $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, W, p)$ *will be called a* ***space of driving processes*** *if* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ *is a stochastic basis fulfilling the usual conditions,* W

is an m-dimensional Brownian motion and p is a Poisson measure of intensity $q(dt, dx) := dt \otimes F(dx)$, and W, p are optional for the underlying filtration.

We now fix the following objects defined on the canonical space.

- β, an $\mathbb{R}^m$-valued $\mathbb{F}^o$-predictable process;
- σ, a $\mathbb{M}_m(\mathbb{R})$-valued $\mathbb{F}^o$-predictable process;
- w, an $\mathbb{R}^m$-valued $\mathcal{P}re^o \otimes \mathcal{B}(\mathbb{R}^m)$-measurable function on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^m$,

where $\mathbb{M}_m(\mathbb{R})$ denotes the set of real-valued square matrices of size m.

Definition 6.2. *Let $(s, \eta) \in \mathbb{R}_+ \times \Omega$. We call a* ***weak solution of the SDE with coefficients*** *β, σ, w* ***and starting in*** *(s, η) any probability measure $\mathbb{P}^{s,\eta}$ on $(\Omega, \mathcal{F})$ such that there exists a space of driving processes $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, W, p)$, on it an m-dimensional $\tilde{\mathbb{F}}$-adapted cadlag process $\tilde{X}$ such that $\mathbb{P}^{s,\eta} = \tilde{\mathbb{P}} \circ \tilde{X}^{-1}$ and such that the following holds.*

Let $\tilde{\beta} := \beta.(\tilde{X}(\cdot))$, $\tilde{\sigma} := \sigma.(\tilde{X}(\cdot))$ and $\tilde{w} := w(\cdot, \tilde{X}(\cdot), \cdot)$. We have the following.

- *for all $t \in [0, s]$, $\tilde{X}_t = \eta(t)$ $\tilde{\mathbb{P}}$ a.s.;*
- *$\int_s^\cdot \left(\|\tilde{\beta}_r\| + \|\tilde{\sigma}_r\|^2 + \int_{\mathbb{R}^m}(\|\tilde{w}(r, \cdot, y)\| + \|\tilde{w}(r, \cdot, y)\|^2)F(dy)\right) dr$ takes finite values $\tilde{\mathbb{P}}$ a.s.;*
- *$\tilde{X}^i_t = \eta_i(s) + \int_s^t \tilde{\beta}^i_r dr + \sum_{j \leq m} \int_s^t \tilde{\sigma}^{i,j}_r dW^j_r + \tilde{w}^i \star (p - q)_t$ $\tilde{\mathbb{P}}$ a.s. for all $t \geq s$, $i \leq m$,*

where $\star$ is the integration against random measures, see [17] Chapter II.2.d for instance.

Remark 6.3. *Previous Definition 6.2 corresponds to Definition 14.73 in [16]. However, in the second item we have required that*

$$\int_s^\cdot \int_{\mathbb{R}^m} (\|\tilde{w}(r, \cdot, y)\| + \|\tilde{w}(r, \cdot, y)\|^2)F(dy)dr$$

takes finite values a.s. so that $\tilde{w} \star (p - q)$ is a well-defined purely discontinuous locally square integrable martingale with angle bracket the $\mathbb{M}_m(\mathbb{R})$-valued process $\int_s^{\cdot \vee s} \int_{\mathbb{R}^m} \tilde{w}\tilde{w}^\intercal(r, \cdot, y)F(dy)dr$, (see Definition 1.27, Proposition 1.28 and Theorem 1.33 in [16] chapter II) and we will not need to use any truncation function.

With this definition, if $\mathbb{P}^{s,\eta}$ is a weak solution of the SDE starting at some (s, η), then under $\mathbb{P}^{s,\eta}$, $(X_t)_{t \geq s}$ is a special semimartingale.

Definition 6.4. *Let $s \in \mathbb{R}_+$ and $(Y_t)_{t \geq s}$ be a cadlag special semimartingale defined on the canonical space with (unique) decomposition $Y = Y_s + B + M^c + M^d$ where B is predictable with bounded variation, M^c a continuous local martingale, M^d a purely discontinuous local martingale, all three vanishing at the initial time $t = s$. We will call* ***characteristics*** *of Y the triplet (B, C, ν) where $C = \langle M^c \rangle$ and ν is the predictable compensator of the measure of the jumps of Y.*

There are several known equivalent characterizations of weak solutions of path-dependent SDEs with jumps which we will now state in our setup.

Notation 6.5. *For every $f \in \mathcal{C}^2_b(\mathbb{R}^m)$ and $t \geq 0$, we denote by $A_t f$ the r.v.*

$$\beta_t \cdot \nabla f(X_t) + \frac{1}{2} Tr(\sigma_t \sigma_t^\intercal \nabla^2 f(X_t)) + \int_{\mathbb{R}^m} (f(X_t + w(t,\cdot,y)) - f(X_t) - \nabla f(X_t) \cdot w(t,\cdot,y)) F(dy). \tag{6.1}$$

Proposition 6.6. *Let $(s,\eta) \in \mathbb{R}_+ \times \Omega$ be fixed and let $\mathbb{P} \in \mathcal{P}(\Omega)$. There is equivalence between the following properties.*

1. *$\mathbb{P}$ is a weak solution of the SDE with coefficients β, σ, w;*
2. *$\mathbb{P}(\omega^s = \eta^s) = 1$ and $(X_t)_{t \geq s}$ is under $\mathbb{P}$ a special semimartingale with characteristics*
   - *$B = \int_s^\cdot \beta_r dr$;*
   - *$C = \int_s^\cdot (\sigma\sigma^\intercal)_r dr$;*
   - *$\nu : (\omega, G) \mapsto \int_s^{+\infty} \int_E \mathbb{1}_G(r, w(\omega,r,y)) \mathbb{1}_{\{w(\omega,r,y) \neq 0\}} F(dy) dr$;*
3. *$\mathbb{P}$ solves $MP^{s,\eta}(\chi)$ where χ is constituted of processes $f(X_\cdot) - \int_0^\cdot A_r f dr$ for all $f \in \mathcal{C}^2_b(\mathbb{R}^m)$.*
4. *$\mathbb{P}$ solves $MP^{s,\eta}(\chi')$ where χ' is constituted of processes $f(X_\cdot) - \int_0^\cdot A_r f dr$ for all functions $f : x \mapsto cos(\theta \cdot x)$ and $f : x \mapsto sin(\theta \cdot x)$ with $\theta \in \mathbb{Q}^m$.*

Proof. Equivalence between items 1. and 2. is a consequence of Theorem 14.80 in [16]. The equivalence between items 2., 3. and 4. if θ was ranging in $\mathbb{R}^m$ is shown in Theorem 2.42 of [17] chapter II. Observe that 4. is stated for $\theta \in \mathbb{R}^m$; however the proof of the implication (4. $\Longrightarrow$ 2.) in Theorem 2.42 of [17] chapter II only uses the values of θ in $\mathbb{Q}^m$. □

Theorem 6.7. *Assume that for any $(s,\eta) \in \mathbb{R}_+ \times \Omega$, the SDE with coefficients β, σ, w and starting in (s,η) admits a unique weak solution $\mathbb{P}^{s,\eta}$. Then $(\mathbb{P}^{s,\eta})_{(s,\eta) \in \mathbb{R}_+ \times \Omega}$ is a path-dependent canonical class verifying Hypothesis 3.5.*

Proof. By Proposition 6.6, $\mathbb{P}^{s,\eta}$ is for each (s,η) the unique solution of $MP^{s,\eta}(\chi)$ where χ is constituted of the processes $f(X_\cdot) - \int_s^\cdot A_r f dr$ for all functions $f : x \mapsto cos(\theta \cdot x)$ or $f : x \mapsto sin(\theta \cdot x)$ with $\theta \in \mathbb{Q}^m$. Since χ is a countable set of cadlag $\mathbb{F}^o$-adapted processes which are bounded on bounded intervals, we can conclude by Proposition 5.12. □

We recall two classical examples of conditions on the coefficients for which it is known that there is existence and uniqueness of a weak solution for the path-dependent SDE, hence for which the above theorem applies, see Theorem 14.95 and Corollary 14.82 in [16].

Example 6.8. *We suppose β, σ, w to be bounded. Moreover we suppose that for all $n \in \mathbb{N}^*$ there exist $K_2^n \in L^1_{loc}(\mathbb{R}_+)$ and a Borel function $K_3^n : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R}$ such that $\int_{\mathbb{R}^m} K_3^n(\cdot, y)F(dy) \in L^1_{loc}(\mathbb{R}_+)$ verifying the following.*

For all $x \in \mathbb{R}^m$, $t \geq 0$ and $\omega, \omega' \in \Omega$ such that $\underset{r\leq t}{sup} \|\omega(r)\| \leq n$ and $\underset{r\leq t}{sup} \|\omega'(r)\| \leq n$, we have

- $\|\sigma_t(\omega) - \sigma_t(\omega')\| \leq K_2^n(t) \underset{r\leq t}{sup} \|\omega(r) - \omega'(r)\|^2$;
- $\|w(t, \omega, x) - w(t, \omega', x)\| \leq K_3^n(t, x) \underset{r\leq t}{sup} \|\omega(r) - \omega'(r)\|^2$.

Finally we suppose that one of the two following hypotheses is fulfilled.

1. *There exists $K_1^n \in L^1_{loc}(\mathbb{R}_+)$ such that for all $t \geq 0$ and $\omega \in \Omega$, $\|\beta_t(\omega) - \beta_t(\omega')\| \leq K_1^n(t) \underset{r\leq t}{sup} \|\omega(r) - \omega'(r)\|$;*
2. *there exists $c > 0$ such that for all $x \in \mathbb{R}^m$, $t \geq 0$ and $\omega \in \Omega$, $x^\intercal \sigma_t(\omega)\sigma_t(\omega)^\intercal x \geq c\|x\|^2$;*

If the assumptions of Theorem 6.7 are fulfilled and β, σ (resp. w) are bounded and continuous in ω for fixed t (resp. fixed t, y), then $(s, \eta) \longmapsto \mathbb{P}^{s,\eta}$ is continuous for the topology of weak convergence, and in particular, the path-dependent canonical class is progressive hence all results of Section 4 can be applied with respect to $(\mathbb{P}^{s,\eta})_{(s,\eta)\in\mathbb{R}_+\times\Omega}$.

Proposition 6.9. *Assume that that β, σ, w are bounded. Let $(s_n, \eta_n)_n$ be a sequence in Λ which converges to some (s, η). For every $n \in \mathbb{N}$, let $\mathbb{P}^n$ be a weak solution starting in (s_n, η_n) of the SDE with coefficients β, σ, w. Then $(\mathbb{P}^n)_{n\geq 0}$ is tight.*

We recall some notations from [17] Chapter VI which we will use in the proof of Proposition 6.9.

Notation 6.10. *For any $\omega \in \Omega$ and interval $\mathcal{I}$ of $\mathbb{R}_+$, we denote $W(\omega, \mathcal{I}) := \sup_{s,t\in\mathcal{I}} \|\omega(t) - \omega(s)\|$. For any $\omega \in \Omega$, $N \in \mathbb{N}^\star$ and $\theta > 0$, we write*
$W_N(\omega, \theta) := \sup_{0\leq t\leq t+\theta\leq N} W(\omega, [t, t+\theta]) = \sup_{s,t\in[0,N]:\, |t-s|\leq\theta} \|\omega(t) - \omega(s)\|$.
For any $\omega \in \Omega$, $N \in \mathbb{N}^\star$ and $\theta > 0$, we denote
$W'_N(\omega, \theta) := \inf \left\{ \max_{i\leq r} W(\omega, [t_{i-1}, t_i[) : \quad 0 = t_0 < \cdots < t_r = N; \quad \forall 1 \leq i \leq r : t_i - t_{i-1} \geq \theta \right\}$.

We will also recall the classical general tightness criterion in $\mathcal{P}(\Omega)$ which one can find for example in Theorem 3.21 of [17] Chapter VI.

Theorem 6.11. *Let $(\mathbb{P}^n)_{n\geq 0}$ be a sequence of elements of $\mathcal{P}(\Omega)$, then it is tight iff it verifies the two following conditions.*

$$\begin{cases} \forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \exists K > 0 \quad \forall n \in \mathbb{N} : \quad \mathbb{P}^n \left(\sup_{t\leq N} \|\omega(t)\| > K \right) \leq \epsilon \\ \forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \forall \alpha > 0 \quad \exists \theta \quad \forall n \in \mathbb{N} : \quad \mathbb{P}^n(W'_N(\omega, \theta) < \alpha) \geq 1 - \epsilon. \end{cases} \tag{6.2}$$

Finally we will also need to introduce a definition.

Definition 6.12. *A sequence of probability measures on* $(\Omega, \mathcal{F})$ *is called* $\mathcal{C}$*-tight if it is tight and if each of its limiting points has all its support in* $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^m)$.

Proof of Proposition 6.9.
We fix a converging sequence $(s_n, \eta_n) \underset{n}{\longrightarrow} (s, \eta)$ in Λ, and for every n, a weak solution $\mathbb{P}^n$ of the SDE with coefficients β, σ, w starting in (s_n, η_n). In order to show that $(\mathbb{P}^n)_{n\geq 0}$ is tight, we will use Theorem 6.11. The main idea consists in combining the fact that the canonical process X under $\mathbb{P}^n$ is deterministic on $[0, s_n]$, where it coincides with η_n with the fact that on $[s_n, +\infty[$ it is a semimartingale with known characteristics. So we will split the study of the modulus of continuity of path ω on these two intervals $[0, s_n]$ and $[s_n, +\infty[$.

Since η_n tends to η, the set $\{\eta_n : n \geq 0\}$ is relatively compact in Ω so by Theorem 1.14.b in [17] Chapter VI we have

$$\left\{ \begin{array}{l} \forall N \in \mathbb{N}^* \quad \exists K_1 > 0 \quad \forall n \in \mathbb{N} : \quad \sup\limits_{t\in[0,N]} \|\eta_n(t)\| \leq K_1 \\ \forall N \in \mathbb{N}^* \quad \forall \alpha > 0 \quad \exists \theta_1 > 0 \quad \forall n \in \mathbb{N} : \quad W'_N(\eta_n, \theta_1) < \alpha. \end{array} \right. \tag{6.3}$$

For fixed $n \in \mathbb{N}$, we now introduce the process
$X^n : \omega \longmapsto \eta_n(s_n)\mathbb{1}_{[0,s_n[} + \omega\mathbb{1}_{[s_n,+\infty[}$, we denote by $\mathbb{Q}^n := \mathbb{P}^n \circ (X^n)^{-1} \in \mathcal{P}(\Omega)$ its law under $\mathbb{P}^n$ and we now show that $(\mathbb{Q}^n)_{n\geq 0}$ is tight.

By Proposition 6.6, under $\mathbb{P}^n$, $(X_t)_{t\in[s_n,+\infty[}$ is a special semimartingale with initial value $\eta_n(s_n)$ and characteristics (see Definition 6.4) $\int_{s_n}^{\cdot} \beta_r dr$, $\int_{s_n}^{\cdot} (\sigma\sigma^\intercal)_r dr$ and $(\omega, A) \mapsto \int_{s_n}^{+\infty} \int_{\mathbb{R}^m} \mathbb{1}_A(r, w(r, \omega, y))\mathbb{1}_{\{w(r,\omega,y)\neq 0\}}F(dy)dr$. Therefore, since X^n is constant on $[0, s_n[$ and since on $[s_n, +\infty[$ its law under $\mathbb{P}^n$ coincides with the one of X, we can say that $\mathbb{Q}^n$ is the law of a special semi-martingale (starting at time $t = 0$) with initial value $\eta_n(s_n)$, and characteristics $\int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r)\beta_r dr$, $\int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r)(\sigma\sigma^\intercal)_r dr$ and
$(\omega, G) \mapsto \int_0^{+\infty} \mathbb{1}_{[s_n,+\infty[}(r) \int_{\mathbb{R}^m} \mathbb{1}_G(r, w(r, \omega, y))\mathbb{1}_{\{w(r,\omega,y)\neq 0\}}F(dy)dr$.
Theorem 4.18 in [17] chapter VI implies that $(\mathbb{Q}^n)_{n\geq 0}$ is tight if and only if the properties below hold true.

1. $(\mathbb{Q}^n \circ X_0^{-1})_{n\geq 0}$ is tight;
2. the following sequences are $\mathcal{C}$-tight (under $(\mathbb{Q}^n)_{n\geq 0}$):
   (a) $(B^n := \int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r)\beta_r dr)_{n\geq 0}$;
   (b) $\left(\tilde{C}^n := \int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r) \left((\sigma\sigma^\intercal)_r + \int_{\mathbb{R}^m} (ww^\intercal)(r, \cdot, y)F(dy)\right) dr\right)_{n\geq 0}$;
   (c) $\left(G^n_p := \int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r) \int_{\mathbb{R}^m} \mathbb{1}_{\{w(r,\omega,y)\neq 0\}}((p\|w(r, \cdot, y)\| - 1)^+) \wedge 1F(dy)dr\right)_{n\geq 0}$ for all $p \in \mathbb{N}$;
3. for all $N > 0$, $\epsilon > 0$,

$$\lim_{a\to\infty} \sup_n \mathbb{Q}^n \left(\int_{s_n}^N \int_{\mathbb{R}^m} \mathbb{1}_{\{\|w(r,\cdot,y)\|>a\}} F(dy)dr > \epsilon \right) = 0. \tag{6.4}$$

Item 3. trivially holds since w is bounded. At this point $\eta_n(s_n)$ is a bounded sequence according to the first line of (6.3) and the fact that the sequence $(s_n)_{n\geq 0}$ is bounded, so $(\mathbb{Q}^n \circ X_0^{-1})_{n\geq 0} = (\delta_{\eta_n(s_n)})_{n\geq 0}$ is obviously tight. We are left to show item 2. By Proposition 3.36 in [17] chapter VI, items 2. (a) and 2. (b) hold if $(Var(B^n))_{n\geq 0} = (\int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r)\|\beta_r\|dr)_{n\geq 0}$ and $(Tr(\tilde{C}^n))_{n\geq 0} = \left(\int_0^{\cdot} \mathbb{1}_{[s_n,+\infty[}(r)\left(Tr(\sigma\sigma_r^\intercal) + \int_{\mathbb{R}^m} Tr(ww^\intercal(r,\cdot,y))F(dy)\right)dr\right)_{n\geq 0}$ are $\mathcal{C}$-tight. Finally, β,σ,w,F being bounded, there exists some strictly positive constant K such that all the processes given below are increasing:

- $t \mapsto Kt - Var(B^n)_t, \quad n \geq 0;$
- $t \mapsto Kt - Tr(\tilde{C}^n_t), \quad n \geq 0;$
- $t \mapsto Kt - (G^n_p)_t, \quad n,p \geq 0.$

In the terminology of [17] chapter VI, this means that the increasing processes $Var(B^n), \quad n \geq 0$, $Tr(\tilde{C}^n), \quad n \geq 0$, $G^n_p \quad n,p \geq 0$ are strongly dominated by the increasing function $t \mapsto Kt$. The singleton $t \mapsto Kt$ being trivially $\mathcal{C}$-tight, Proposition 3.35 in [17] chapter VI implies that the dominated sequences of processes $(Var(B^n))_{n\geq 0}$, $(Tr(\tilde{C}^n))_{n\geq 0}$ and $(G^n_p)_{n\geq 0}$ for all p are $\mathcal{C}$-tight. Finally $(\mathbb{Q}^n)_{n\geq 0}$ is tight.

Now by Theorem 6.11 this implies that

$$\begin{cases} \forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \exists K_2 > 0 \quad \forall n \in \mathbb{N}: \quad \mathbb{Q}^n\left(\sup_{t\leq N} \|\omega(t)\| > K_2\right) \leq \epsilon \\ \forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \forall \alpha > 0 \quad \exists \theta_2 \quad \forall n \in \mathbb{N}: \quad \mathbb{Q}^n(W'_N(\omega,\theta_2) < \alpha) \geq 1-\epsilon. \end{cases} \tag{6.5}$$

Combining the first line of (6.3) and the first line of (6.5) and by construction of $\mathbb{Q}^n$, taking $K = K_1 + K_2$ for instance, we have

$$\forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \exists K > 0 \quad \forall n \in \mathbb{N}: \quad \mathbb{P}^n\left(\sup_{t\leq N} \|\omega(t)\| > K\right) \leq \epsilon. \tag{6.6}$$

Our aim is now to show that

$$\forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \forall \alpha > 0 \quad \exists \theta \quad \forall n \in \mathbb{N}: \quad \mathbb{P}^n(W'_N(\omega,\theta) < \alpha) \geq 1-\epsilon; \tag{6.7}$$

this combined with (6.6) will imply by Theorem 6.11 that $(\mathbb{P}^n)_{n\geq 0}$ is tight.

In what follows, if $\eta,\omega \in \Omega$ and $s \in \mathbb{R}_+$, $\eta \otimes_s \omega$ will denote the path $\eta\mathbb{1}_{[0,s[} + \omega\mathbb{1}_{[s,+\infty[}$, which still belongs to Ω.

By construction of $\mathbb{Q}^n$, for every n, $\mathbb{P}^n$ is the law of $\eta_n \otimes_{s_n} \omega$ under $\mathbb{Q}^n$. Therefore, (6.7) is equivalent to

$$\forall N \in \mathbb{N}^* \quad \forall \epsilon > 0 \quad \forall \alpha > 0 \quad \exists \theta \quad \forall n \in \mathbb{N}: \quad \mathbb{Q}^n(W'_N(\eta_n \otimes_{s_n} \omega,\theta) < \alpha) \geq 1-\epsilon, \tag{6.8}$$

and this is what we will now show to conclude the proof of Proposition 6.9. So we prove (6.8).

We fix some $N \in \mathbb{N}^*$, $\alpha > 0$ and $\epsilon > 0$. Combining the second lines of (6.3) and of (6.5), there exists $\theta > 0$ such that for all $n \geq 0$,

$$\begin{cases} W'_N(\eta_n, \theta) < \frac{\alpha}{4} \\ \mathbb{Q}^n(W'_N(\omega, \theta) < \frac{\alpha}{4}) \geq 1 - \epsilon. \end{cases} \tag{6.9}$$

We show below that, for every n

$$\{\omega | W'_N(\omega, \theta) < \frac{\alpha}{4}\} \subset \{\omega | W'_N(\eta_n \otimes_{s_n} \omega, \theta) < \alpha\}. \tag{6.10}$$

This together with (6.9) will imply that for all n,

$$\mathbb{Q}^n(W'_N(\eta_n \otimes_{s_n} \omega, \theta) < \alpha) \geq \mathbb{Q}^n(W'_N(\omega, \theta) < \frac{\alpha}{4}) \geq 1 - \epsilon,$$

hence that (6.8) is verified.

We fix n. To establish (6.10) let ω such that $W'_N(\omega, \theta) < \frac{\alpha}{4}$; we need to show that

$$W'_N(\eta_n \otimes_{s_n} \omega, \theta) < \alpha. \tag{6.11}$$

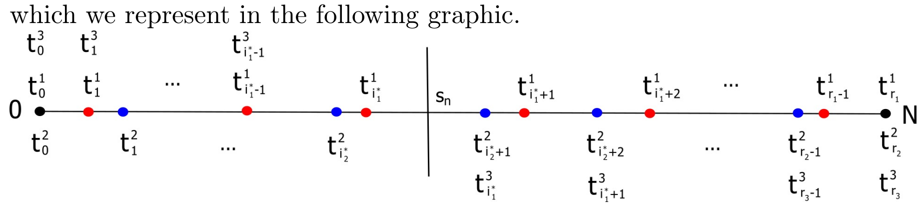
By the first line of (6.9) and the definition of W'_N (see Notation 6.10), there exist two subdivisons of $[0, N]$ $0 = t^1_0 < \cdots < t^1_{r_1} = N$, $0 = t^2_0 < \cdots < t^2_{r_2} = N$ with increments $t^j_i - t^j_{i-1} \geq \theta$ for all $1 \leq i \leq r_j$ and $j = 1, 2$, such that

$$\begin{cases} W(\eta_n, [t^1_{i-1}, t^1_i[) & \leq & \frac{\alpha}{4} \text{ for all } 1 \leq i \leq r_1 \\ W(\omega, [t^2_{i-1}, t^2_i[) & \leq & \frac{\alpha}{4} \text{ for all } 1 \leq i \leq r_2. \end{cases} \tag{6.12}$$

We set $i^*_j := \max\ \{i : t^j_i \leq s_n\}$ for $j = 1, 2$ and introduce the third subdivision

$$(t^3_0, \cdots, t^3_{r_3}) := (t^1_0, \cdots, t^1_{i^*_1 - 1}, t^2_{i^*_2 + 1}, \cdots, t^2_{r_2}), \tag{6.13}$$

which we represent in the following graphic.



As for the other two, the subdivision of $[0, N]$ above verifies $t^3_i - t^3_{i-1} \geq \theta$ for all i. Indeed, $t^3_i - t^3_{i-1}$ is either equal to $t^1_i - t^1_{i-1} \geq \theta$, or to $t^2_j - t^2_{j-1} \geq \theta$ for some j, or to $t^2_{i^*_2+1} - t^1_{i^*_1-1} \geq t^1_{i^*_1} - t^1_{i^*_1-1} \geq \theta$ where the first inequality follows by the fact that $t^1_{i^*_1-1} \leq t^1_{i^*_1} \leq s_n < t^2_{i^*_2+1}$.

Now by definition of $W'_N(\eta_n \otimes_{s_n} \omega, \theta)$, in order to show (6.11) and conclude this proof, it is enough to show that

$$W(\eta_n \otimes_{s_n} \omega, [t^3_{i-1}, t^3_i[) < \alpha, \tag{6.14}$$

for all $1 \leq i \leq r_3$.

If $i \leq i_1^*-1$, then $[t^3_{i-1}, t^3_i[= [t^1_{i-1}, t^1_i[\subset [0, s_n[$ where $\eta_n \otimes_{s_n} \omega$ coincides with η_n so $W(\eta_n \otimes_{s_n} \omega, [t^3_{i-1}, t^3_i[) = W(\eta_n, [t^1_{i-1}, t^1_i[) \leq \frac{\alpha}{4} < \alpha$ by the first line of (6.12). Similarly, if $i \geq i_1^* + 1$, then $[t^3_{i-1}, t^3_i[= [t^2_{i-i_1^*+i_2^*}, t^2_{i-i_1^*+i_2^*+1}[\subset [s_n, +\infty[$ where $\eta_n \otimes_{s_n} \omega$ coincides with ω so $W(\eta_n \otimes_{s_n} \omega, [t^3_{i-1}, t^3_i[) = W(\omega, [t^2_{i-i_1^*+i_2^*}, t^2_{i-i_1^*+i_2^*+1}[) \leq \frac{\alpha}{4} < \alpha$ by the second line of (6.12). Finally, we consider the specific case $i = i_1^*$ meaning that $[t^3_{i-1}, t^3_i[= [t^1_{i_1^*-1}, t^2_{i_2^*+1}[$ contains s_n. We have

$$\begin{array}{rcl} W(\eta_n \otimes_{s_n} \omega, [t^1_{i_1^*-1}, t^2_{i_2^*+1}[) & \leq & W(\eta_n \otimes_{s_n} \omega, [t^1_{i_1^*-1}, t^1_{i_1^*}[) \\ & + & W(\eta_n \otimes_{s_n} \omega, [t^1_{i_1^*}, s_n[) + W(\eta_n \otimes_{s_n} \omega, [s_n, t^2_{i_2^*+1}[) \\ & \leq & W(\eta_n, [t^1_{i_1^*-1}, t^1_{i_1^*}[) + W(\eta_n, [t^1_{i_1^*}, s_n[) + W(\omega, [s_n, t^2_{i_2^*+1}[) \\ & \leq & W(\eta_n, [t^1_{i_1^*-1}, t^1_{i_1^*}[) + W(\eta_n, [t^1_{i_1^*}, t^1_{i_1^*+1}[) \\ & + & W(\omega, [t^2_{i_2^*}, t^2_{i_2^*+1}[) \\ & \leq & \frac{\alpha}{4} + \frac{\alpha}{4} + \frac{\alpha}{4} \\ & < & \alpha, \end{array} \tag{6.15}$$

by (6.12). So (6.14) is verified for all i and the proof is complete.

□

Proposition 6.13. *Assume that β, σ (resp. w) are bounded and that for Lebesgue almost all t (resp. $dt \otimes dF$ almost all (t, y)), $\beta(t, \cdot), \sigma(t, \cdot)$ (resp. $w(t, \cdot, y)$) are continuous. Assume that for any $(s, \eta) \in \mathbb{R}_+ \times \Omega$ there exists a unique weak solution $\mathbb{P}^{s,\eta}$ of the SDE of coefficients β, σ, w starting in (s, η). Then $\begin{array}{ccc} (s,\eta) & \longmapsto & \mathbb{P}^{s,\eta} \\ \Lambda & \longrightarrow & \mathcal{P}(\Omega) \end{array}$ is continuous. Moreover the path-dependent canonical class $(\mathbb{P}^{s,\eta})_{(s,\eta) \in \mathbb{R}_+ \times \Omega}$ is progressive.*

Remark 6.14. *Taking Theorem 6.7 into account, the family of probabilities $(\mathbb{P}^{s,\eta})_{(s,\eta) \in \mathbb{R}_+ \times \Omega}$ of Proposition 6.13 constitutes a progressive path-dependent canonical class verifying Hypothesis 3.5. It therefore verifies Hypothesis 4.1 and all results of Section 4 apply.*

Proof. of Proposition 6.13.
We consider a convergent sequence $(s_n, \eta_n) \underset{n}{\longrightarrow} (s, \eta)$ in Λ. Since β, σ are bounded, by Proposition 6.9 $(\mathbb{P}^{s_n,\eta_n})_{n \in \mathbb{N}}$ is tight, hence relatively compact by Prokhorov's theorem. We consider a subsequence $\mathbb{P}^{s_{n_k},\eta_{n_k}} \underset{k}{\longrightarrow} \mathbb{Q}$ and we show below that $\mathbb{Q}$ is a weak solution of the SDE with coefficients β, σ, w, starting at (s, η). Since that problem has a unique solution, we will have $\mathbb{Q} = \mathbb{P}^{s,\eta}$. This will imply that $\mathbb{P}^{s_n,\eta_n} \underset{n}{\longrightarrow} \mathbb{P}^{s,\eta}$, hence the announced continuity.

We will indeed verify item 3. of Proposition 6.6. For the convenience of the reader, we will omit the extraction of the subsequence in the notations.

We start by showing

$$\mathbb{Q}(\omega^s = \eta^s) = 1. \tag{6.16}$$

The set

$$D := \{t \in \mathbb{R}_+ : \mathbb{Q}(X_t \neq X_{t^-}) > 0\} \cup \left\{t \in [0, s] : \eta(t) \neq \eta(t^-)\right\}, \tag{6.17}$$

is countable because first η is a cadlag function and second because of Proposition 3.12 in [17] Chapter VI which states that, for every probability $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, the set $D_0 := \{t \in \mathbb{R}_+ : \mathbb{Q}(X_t \neq X_{t^-}) > 0\}$ is countable. If $t \notin D_0$ then

$$\mathbb{P}^{s_n,\eta_n} \circ X_t^{-1} \underset{n}{\Longrightarrow} \mathbb{Q} \circ X_t^{-1}, \tag{6.18}$$

by Proposition 3.14 ibidem. Since η_n converges to η in the Skorohod topology, if $t \notin D$ (t is a continuity point of η), then it follows that $\eta_n(t) \underset{n}{\longrightarrow} \eta(t)$, see Proposition 2.3 of [17] Chapter VI.

Let $\epsilon > 0$, $t \in [0, s-\epsilon] \cap D^c$ be fixed. Since s_n tends to s, we can suppose without loss of generality that $s_n \geq s - \epsilon$ for all n, so that $\mathbb{P}^{s_n,\eta_n} \circ X_t^{-1} = \delta_{\eta_n(t)}$. By (6.18) this sequence converges to $\mathbb{Q} \circ X_t^{-1}$ which is therefore necessarily equal to $\delta_{\eta(t)}$ since $\eta_n(t)$ tends to $\eta(t)$ being $t \notin D$. This means that

$$\mathbb{Q}(\omega(t) = \eta(t)) = 1, \tag{6.19}$$

for all $t \in [0, s-\epsilon] \cap D^c$. Since $\epsilon > 0$ is arbitrary, (6.19) holds for all $t \in [0, s[\cap D^c$; and since ω is right-continuous and D is countable, (6.19) holds for all $t \in [0, s[$. We will now show that (6.19) also holds for $t = s$. We first note that

$$\eta_n(s_n) \underset{n}{\longrightarrow} \eta(s). \tag{6.20}$$

Indeed, without restriction of generality we can consider that $s_n \leq s+1$, so since $(s_n, \eta_n(s_n)) \in \Lambda$, η_n is constantly equal to $\eta_n(s_n)$ on $[s_n, +\infty[$ which contains $[s+1, +\infty[$. On the other hand η is constantly equal to $\eta(s)$ on $[s, +\infty[$ which also contains $[s+1, +\infty[$, and η_n tends to η almost everywhere on that interval, because it converges in the Skorokhod sense. So necessarily (6.20) holds.
We fix now some $f \in \mathcal{C}_c^\infty(\mathbb{R}^m)$. For all n, since $\mathbb{P}^{s_n,\eta_n}$ is a weak solution of the SDE starting at (s_n, η_n) and by Proposition 6.6, it follows that $f(\omega(\cdot)) - f(\eta_n(s_n)) - \int_{s_n}^{\cdot} A_r f(\omega) dr$ (see Notation 6.5) is a martingale on $[s_n, +\infty[$ under $\mathbb{P}^{s_n,\eta_n}$ vanishing in s_n. We consider a sequence $(t_p)_{p \in \mathbb{N}}$ in D^c converging to t strictly from the right. For all n, p we have

$$\begin{array}{rcl} \mathbb{E}^{s_n,\eta_n}[f(\omega(t_p))] & = & f(\eta_n(s_n)) + \mathbb{E}^{s_n,\eta_n}\left[\int_{s_n}^{t_p} A_r f(\omega) dr\right] \\ & = & f(\eta_n(s_n)) + \mathbb{E}^{s_n,\eta_n}\left[\int_{s}^{t_p} A_r f(\omega) dr\right] + \int_{s_n}^{s} \mathbb{E}^{s_n,\eta_n}[A_r f(\omega)] dr, \end{array} \tag{6.21}$$

where the second equality holds by Fubini's theorem since $A_r f(\omega)$ is uniformly bounded for r varying on bounded intervals. We now pass to the limit in n. Since $t_p \notin D$, taking into account (6.18), we have $\mathbb{P}^{s_n,\eta_n} \circ X_{t_p}^{-1} \underset{n}{\Longrightarrow} \mathbb{Q} \circ X_{t_p}^{-1}$; moreover f is bounded and continuous, so

$$\mathbb{E}^{s_n,\eta_n}[f(\omega(t_p))] \underset{n}{\longrightarrow} \mathbb{E}^{\mathbb{Q}}[f(\omega(t_p))]. \tag{6.22}$$

Since β, σ, w are bounded and $\beta(r, \cdot), \sigma(r, \cdot)$ (resp. $w(r, \cdot, y)$) are continuous for Lebesgue almost all r (resp. $dt \otimes dF$ almost all (r, y)) and since $f \in \mathcal{C}_c^\infty$, then

$\Phi : \omega \longmapsto \int_s^{t_p} A_r f(\omega) dr$ is a bounded continuous functional for the Skorokhod topology, so

$$\mathbb{E}^{s_n,\eta_n}\left[\int_s^{t_p} A_r f(\omega) dr\right] \underset{n}{\longrightarrow} \mathbb{E}^{\mathbb{Q}}\left[\int_s^{t_p} A_r f(\omega) dr\right]. \tag{6.23}$$

Finally since s_n tends to s and $A_r f$ is uniformly bounded for r varying on bounded intervals, we have

$$\int_{s_n}^{s} \mathbb{E}^{s_n,\eta_n}[A_r f(\omega)] dr \underset{n}{\longrightarrow} 0. \tag{6.24}$$

Combining relations (6.21), (6.20), (6.22), (6.23), (6.24), for all p, we get

$$\mathbb{E}^{\mathbb{Q}}[f(\omega(t_p))] = f(\eta(s)) + \mathbb{E}^{\mathbb{Q}}\left[\int_s^{t_p} A_r f(\omega) dr\right]. \tag{6.25}$$

We now pass to the limit in p. Since t_p tends to s from the right and ω is right-continuous, the left-hand side of (6.25) tends to $\mathbb{E}^{\mathbb{Q}}[f(\omega(s))]$. By dominated convergence, the second term in the right-hand side of (6.25) tends to 0. This yields $\mathbb{E}^{\mathbb{Q}}[f(\omega(s))] = f(\eta(s))$ and this holds for all $f \in \mathcal{C}_c^{\infty}(\mathbb{R}^m)$, which implies that $\mathbb{Q} \circ X_s^{-1} = \delta_{\eta(s)}$. So we have shown that (6.19) for $t = s$ and finally (6.16) since ω and η are cadlag.

We will proceed showing that $\mathbb{Q}$ solves weakly the SDE with respect to β, σ, w starting in (s, η). By Proposition 6.6 this holds iff for any $f \in \mathcal{C}_b^2(\mathbb{R}^m)$, $f(X_\cdot) - \int_s^{\cdot} A_r f dr$ is a $(\mathbb{Q}, (\mathcal{F}_t)_{t \in [s,+\infty[})$-martingale. We fix such an f, some $t \leq u$ in $]s, +\infty[\cap D^c$, $N \in \mathbb{N}^*$, $t_1 \leq \cdots \leq t_N \in [s,t] \cap D^c$ and $\phi_1, \cdots, \phi_N \in \mathcal{C}_b(\mathbb{R}^m, \mathbb{R})$. Taking into account Proposition 6.6, since $s < t$, for n large enough, we can suppose that $f(X_\cdot) - \int_t^{\cdot} A_r f dr$ is under every $\mathbb{P}^{s_n,\eta_n}$ a martingale on the interval $[t, +\infty[$. Therefore, for all n, we have

$$\mathbb{E}^{s_n,\eta_n}\left[\left(f(\omega(u)) - f(\omega(t)) - \int_t^u A_r f(\omega) dr\right) \underset{1 \leq i \leq N}{\Pi} \phi_i(\omega(t_i))\right] = 0. \tag{6.26}$$

We wish to pass to the limit in n. By Theorem 12.5 in [6], for any $r \in \mathbb{R}_+$, the mapping X_r is continuous on the set $C_r := \{\omega \in \Omega : \omega(r) = \omega(r^-)\}$. By construction of D and since $t, u, t_1, \cdots, t_N \notin D$, then $C_t, C_u, C_{t_1}, \cdots, C_{t_N}$ are of full $\mathbb{Q}$-measure hence that $\Phi := (X, X_u, X_t, X_{t_1}, \cdots, X_{t_N})$ is continuous on a set of full $\mathbb{Q}$-measure. By the mapping theorem (see Theorem 2.7 in [6] for instance), since $\mathbb{P}^{s_n,\eta_n} \underset{n}{\Longrightarrow} \mathbb{Q}$ and Φ is continuous on a set of full $\mathbb{Q}$-measure, then $\mathbb{P}^{s_n,\eta_n} \circ \Phi^{-1} \underset{n}{\Longrightarrow} \mathbb{Q} \circ \Phi^{-1}$, meaning $\mathbb{P}^{s_n,\eta_n} \circ (X, X_u, X_t, X_{t_1}, \cdots, X_{t_N})^{-1} \underset{n}{\Longrightarrow} \mathbb{Q} \circ (X, X_u, X_t, X_{t_1}, \cdots, X_{t_N})^{-1}$. Since $\omega \mapsto \int_t^u A_r f(\omega) dr, f, \phi_1, \cdots, \phi_N$ are bounded continuous functions, the previous convergence in law allows to pass to the limit in n in (6.26) so that for any $t \leq u \in]s, +\infty[\cap D^c$ and $t_1, \cdots, t_N \in [s,t] \cap D^c$

$$\mathbb{E}^{\mathbb{Q}}\left[\left(f(\omega(u)) - f(\omega(t)) - \int_t^u A_r f(\omega) dr\right) \underset{1 \leq i \leq N}{\Pi} \phi_i(\omega(t_i))\right] = 0. \tag{6.27}$$

Equality (6.27) still holds if $t = s$ and if some of the values $t, u, t_1, \cdots, t_N$ belong to D. Indeed to show this statement we approximate from the right such values by sequences of times not belonging to D and strictly greater than s and we then use the right-continuity of ω and the dominated convergence theorem.

By use of the functional monotone class theorem (see Theorem 21 in [10] Chapter I), we have

$$\mathbb{E}^{\mathbb{Q}}\left[\left(f(\omega(u)) - f(\omega(t)) - \int_t^u A_r f(\omega)dr\right)\mathbb{1}_G\right] = 0, \qquad (6.28)$$

for any $s \leq t \leq u$ and $G \in \sigma(X_r | r \in [s,t])$. Since $\mathbb{Q}(\omega^s = \eta^s) = 1$ then $\mathcal{F}_s^o$ is $\mathbb{Q}$-trivial, so equality (6.28) holds for all $G = G_s \cap G_t^s$ where $G_s \in \mathcal{F}_s^o$ and $G_t^s \in \sigma(X_r | r \in [s,t])$. Events of such type form a π-system generating $\mathcal{F}_t^o$ so by Dynkin's Lemma, (6.28) holds for all $G \in \mathcal{F}_t^o$. For all $s \leq t \leq u$, then we have

$$\mathbb{E}^{\mathbb{Q}}\left[\left(f(\omega(u)) - f(\omega(t)) - \int_t^u A_r f(\omega)_r dr\right)\middle| \mathcal{F}_t^o\right] = 0. \qquad (6.29)$$

So $f(X) - \int_s^\cdot A_r f_r dr$ is a $(\mathbb{Q}, (\mathcal{F}_t^o)_{t\in[s,+\infty[})$-martingale hence a $(\mathbb{Q}, (\mathcal{F}_t)_{t\in[s,+\infty[})$-martingale by Theorem 3 in [11] Chapter VI, that process being right-continuous. This implies that $\mathbb{Q}$ is a weak solution of the SDE with coefficients β, σ, w starting in (s,η). As anticipated, since the SDE is well-posed for every (s,η), we have $\mathbb{Q} = \mathbb{P}^{s,\eta}$ and the proof of the first statement is complete.
The second statement follows from the fact that a continuous function is Borel and that $\mathcal{B}(\Lambda) = \Lambda \cap \mathcal{P}ro^o$, see Proposition 5.3. □

Appendices

A Proofs of Section 4

Proof of Proposition 4.6.
In the whole proof $t < u$ will be fixed. We consider a sequence of subdivisions of $[t,u]$: $t = t_1^k < t_2^k < \cdots < t_k^k = u$ such that $\min_{i<k}(t_{i+1}^k - t_i^k) \underset{k\to\infty}{\longrightarrow} 0$. Let $(s,\eta) \in [0,t] \times \Omega$ with corresponding probability $\mathbb{P}^{s,\eta}$. For any k, we have $\sum_{i<k}\left(M_{t_i^k, t_{i+1}^k}\right)^2 = \sum_{i<k}(M^{s,\eta}_{t_{i+1}^k} - M^{s,\eta}_{t_i^k})^2$ $\mathbb{P}^{s,\eta}$ a.s., so by definition of quadratic variation we know that $\sum_{i<k}\left(M_{t_i^k, t_{i+1}^k}\right)^2 \overset{\mathbb{P}^{s,\eta}}{\underset{k\to\infty}{\longrightarrow}} [M^{s,\eta}]_u - [M^{s,\eta}]_t$. In the sequel we will construct an $\mathcal{F}_u^o$-measurable random variable $[M]_{t,u}$ such that for any $(s,\eta) \in [0,t] \times \Omega$, $\sum_{i<k}\left(M_{t_i^k, t_{i+1}^k}\right)^2 \overset{\mathbb{P}^{s,\eta}}{\underset{k\to\infty}{\longrightarrow}} [M]_{t,u}$. In that case $[M]_{t,u}$ will then be $\mathbb{P}^{s,\eta}$ a.s. equal to $[M^{s,\eta}]_u - [M^{s,\eta}]_t$.

Let $\eta \in \Omega$. $[M^{t,\eta}]$ is $\mathbb{F}^{t,\eta}$-adapted, so $[M^{t,\eta}]_u - [M^{t,\eta}]_t$ is $\mathcal{F}_u^{t,\eta}$-measurable and by Corollary 3.21, there is an $\mathcal{F}_u^o$-measurable variable which depends on (t,u,η), that we denote $\omega \mapsto a_{t,u}(\eta,\omega)$ such that $a_{t,u}(\eta,\omega) = [M^{t,\eta}]_u - [M^{t,\eta}]_t$, $\mathbb{P}^{t,\eta}$ a.s. We will show below that there is a jointly $\mathcal{F}_t^o \otimes \mathcal{F}_u^o$ -measurable version of $(\eta,\omega) \mapsto a_{t,u}(\eta,\omega)$.

For every integer $n \geq 0$, we set $a^n_{t,u}(\eta,\omega) := n \wedge a_{t,u}(\eta,\omega)$ which is in particular limit in probability of $n \wedge \sum_{i \leq k} \left(M_{t_i^k, t_{i+1}^k}\right)^2$ under $\mathbb{P}^{t,\eta}$. For any integers k, n and any $\eta \in \Omega$, we define the finite positive measures $\mathbb{Q}^{k,n,\eta}$, $\mathbb{Q}^{n,\eta}$ and $\mathbb{Q}^\eta$ on $(\Omega, \mathcal{F}_u^o)$ by

1. $\mathbb{Q}^{k,n,\eta}(F) := \mathbb{E}^{t,\eta}\left[\mathbb{1}_F\left(n \wedge \sum_{i<k}\left(M_{t_i^k, t_{i+1}^k}\right)^2\right)\right]$;
2. $\mathbb{Q}^{n,\eta}(F) := \mathbb{E}^{t,\eta}[\mathbb{1}_F\left(a^n_{t,u}(\eta,\omega)\right)]$;
3. $\mathbb{Q}^{\eta}(F) := \mathbb{E}^{t,\eta}[\mathbb{1}_F\left(a_{t,u}(\eta,\omega)\right)]$.

When k and n are fixed integers and F is a fixed event, by Remark 3.6, $\eta \longmapsto \mathbb{E}^{t,\eta}\left[F\left(n \wedge \sum_{i<k}\left(M_{t_i^k, t_{i+1}^k}\right)^2\right)\right]$, is $\mathcal{F}_t^o$ -measurable.

Then $n \wedge \sum_{i<k}\left(M_{t_i^k, t_{i+1}^k}\right)^2 \underset{k\to\infty}{\overset{\mathbb{P}^{t,\eta}}{\longrightarrow}} a^n_{t,u}(\eta,\omega)$, and this sequence is uniformly bounded by the constant n, so the convergence takes place in L^1, therefore $\eta \longmapsto \mathbb{Q}^{n,\eta}(F)$ is also $\mathcal{F}_t^o$-measurable as the pointwise limit in k of the functions $\eta \longmapsto \mathbb{Q}^{k,n,\eta}(F)$. Similarly, $a^n_{t,u}(\eta,\omega) \underset{n\to\infty}{\overset{\mathbb{P}^{t,\eta}-a.s.}{\longrightarrow}} a_t(\eta,\omega)$ and is non-decreasing, so by monotone convergence theorem, the function $\eta \longmapsto \mathbb{Q}^\eta(F)$ is $\mathcal{F}_t^o$-measurable being a pointwise limit in n of the functions $\eta \longmapsto \mathbb{Q}^{n,\eta}(F)$.

We make then use of Theorem 58 Chapter V in [11]: the property above, the separability of $\mathcal{F}$ and the fact that for any η, $\mathbb{Q}^\eta \ll \mathbb{P}^{t,\eta}$ by item 3. above, imply the existence of a jointly measurable (for $\mathcal{F}_t^o \otimes \mathcal{F}_u^o$) version of $(\eta,\omega) \mapsto a_{t,u}(\eta,\omega)$. That version will still be denoted by the same symbol. We recall that for any η, $a_{t,u}(\eta,\cdot)$ is the Radon-Nykodim density of $\mathbb{Q}^\eta$ with respect to $\mathbb{P}^{t,\eta}$.

We can now set $[M]_{t,u}(\omega) := a_{t,u}(\omega,\omega)$, which is a well-defined $\mathcal{F}_u^o$-measurable random variable. Since $a_{t,u}$ is $\mathcal{F}_t^o$-measurable in the first variable and for any η $\mathbb{P}^{t,\eta}(\omega^t = \eta^t) = 1$ we have the equalities

$$[M]_{t,u}(\omega) = a_{t,u}(\omega,\omega) = a_{t,u}(\eta,\omega) = [M^{t,\eta}]_u(\omega) - [M^{t,\eta}]_t(\omega)\ \mathbb{P}^{t,\eta}\text{a.s.} \quad \text{(A.1)}$$

We can then show that

$$[M]_{t,u} = [M^{s,\eta}]_u - [M^{s,\eta}]_t \quad \mathbb{P}^{s,\eta} \text{ a.s.}, \quad \text{(A.2)}$$

holds for every $(s,\eta) \in [0,t] \times \Omega$, and not just in the case $s = t$ that we have just established in (A.1). This can be done reasoning as in the proof of Proposition 4.4 in [5], replacing the use of the Markov property with item 3. of Definition 3.4.

So we have built an $\mathcal{F}_u^o$-measurable variable $[M]_{t,u}$ such that under any $\mathbb{P}^{s,\eta}$ with $s \leq t$, $[M^{s,\eta}]_u - [M^{s,\eta}]_t = [M]_{t,u}$ a.s. and this concludes the proof. $\square$

Proof of Proposition 4.7.
We start defining $A_{t,t} = 0$ for every $t \geq 0$. We then recall a property of the predictable dual projection which we will have to extend slightly. Let us fix (s, η) and the corresponding stochastic basis $(\Omega, \mathcal{F}^{s,\eta}, \mathbb{F}^{s,\eta}, \mathbb{P}^{s,\eta})$. For any $F \in \mathcal{F}^{s,\eta}$, let $N^{s,\eta,F}$ be the cadlag version of the martingale $r \longmapsto \mathbb{E}^{s,\eta}[\mathbb{1}_F|\mathcal{F}_r]$. Then for any $0 \leq t \leq u$, the predictable projection of the process $r \mapsto \mathbb{1}_F \mathbb{1}_{[t,u[}(r)$ is $r \mapsto N^{s,\eta,F}_{r^-} \mathbb{1}_{[t,u[}(r)$, see the proof of Theorem 43 Chapter VI in [11]. Therefore by definition of the dual predictable projection (see Definition 73 Chapter VI in [11]), for any $0 \leq t \leq u$ and $F \in \mathcal{F}^{s,\eta}$ we have $\mathbb{E}^{s,\eta}\left[\mathbb{1}_F(A^{s,\eta}_{u^-} - A^{s,\eta}_t)\right] = \mathbb{E}^{s,\eta}\left[\int_t^{u^-} N^{s,\eta,F}_{r^-} dB^{s,\eta}_r\right]$. Then, at fixed t, u, F, since for every $\epsilon > 0$ we have $\mathbb{E}^{s,\eta}\left[\mathbb{1}_F(A^{s,\eta}_{(u+\epsilon)^-} - A^{s,\eta}_t)\right] = \mathbb{E}^{s,\eta}\left[\int_t^{(u+\epsilon)^-} N^{s,\eta,F}_{r^-} dB^{s,\eta}_r\right]$, letting ϵ tend to zero we obtain by dominated convergence theorem that

$$\mathbb{E}^{s,\eta}\left[\mathbb{1}_F(A^{s,\eta}_u - A^{s,\eta}_t)\right] = \mathbb{E}^{s,\eta}\left[\int_t^u N^{s,\eta,F}_{r^-} dB^{s,\eta}_r\right], \tag{A.3}$$

taking into account the right-continuity of $A^{s,\eta}, B^{s,\eta}$ and the fact that they are both non-decreasing processes with $\mathcal{L}^1$ -terminal value.

For any $F \in \mathcal{F}$, we introduce the process $N^F : (t, \omega) \longmapsto \mathbb{P}^{t,\omega}(F)$. N^F takes values in $[0, 1]$ for every (t, ω), and by Definition 3.4, it is an $\mathbb{F}^o$-progressively measurable process such that for any $(s, \eta) \in \mathbb{R}_+ \times \Omega$, $N^{s,\eta,F}$ is a $\mathbb{P}^{s,\eta}$ cadlag version of N^F on $[s, +\infty[$.

For the rest of the proof, $0 \leq t < u$ are fixed. Following the same proof than that of Lemma 4.9 in [5] but with parameter (s, x) replaced with (s, η), we obtain the following.

Lemma A.1. *Let $F \in \mathcal{F}$. There exists an $\mathcal{F}_u$-measurable random variable which we will call $\int_t^u N^F_{r^-} dB_r$ such that for any $(s, \eta) \in [0, t] \times \Omega$, $\int_t^u N^F_{r^-} dB_r = \int_t^u N^{s,\eta,F}_{r^-} dB^{s,\eta}_r$ $\mathbb{P}^{s,\eta}$ a.s.*

Remark A.2. *By definition, the r.v. $\int_t^u N^F_{r^-} dB_r$ will not depend on (s, η).*

We continue now the proof of Proposition 4.7 by showing that for given $0 \leq t < u$ there is an $\mathcal{F}^o_u$-measurable r.v. $A_{t,u}$ such that for every $(s, \eta) \in [0, t] \times \Omega$, $(A^{s,\eta}_u - A^{s,\eta}_t) = A_{t,u}$ $\mathbb{P}^{s,\eta}$ a.s.

Similarly to what we did with the quadratic variation in Proposition 4.6, we start noticing that for any $\eta \in \Omega$, being $(A^{t,\eta}_u - A^{t,\eta}_t)$ $\mathcal{F}^{t,\eta}_u$-measurable, there exists by Corollary 3.21 an $\mathcal{F}^o_u$-measurable r.v. $\omega \mapsto a_{t,u}(\eta, \omega)$ such that

$$a_{t,u}(\eta, \omega) = A^{t,\eta}_u(\omega) - A^{t,\eta}_t(\omega) \quad \mathbb{P}^{t,\eta} \text{ a.s.} \tag{A.4}$$

As in the proof of Proposition 4.6, we show below the existence of a jointly-measurable version of $(\eta, \omega) \mapsto a_{t,u}(\eta, \omega)$.

For every $\eta \in \Omega$ we define on $\mathcal{F}^o_u$ the positive measure

$$\mathbb{Q}^\eta : F \longmapsto \mathbb{E}^{t,\eta}\left[\mathbb{1}_F a_{t,u}(\eta, \omega)\right] = \mathbb{E}^{t,\eta}\left[\mathbb{1}_F(A^{t,\eta}_u - A^{t,\eta}_t)\right]. \tag{A.5}$$

By Lemma A.1 and (A.3), for every $F \in \mathcal{F}^o_u$ we have

$$\mathbb{Q}^\eta(F) = \mathbb{E}^{t,\eta}\left[\int_t^u N^F_{r-}\, dB_r\right], \tag{A.6}$$

where we recall that $\int_t^u N^F_{r-} dB_r$ does not depend on η. So by Remark 3.6, $\eta \longmapsto \mathbb{Q}^\eta(F)$ is $\mathcal{F}^o_t$-measurable for any F. Moreover, by (A.5) for any η, $\mathbb{Q}^\eta \ll \mathbb{P}^{t,\eta}$. Again by Theorem 58 Chapter V in [11], there exists a version $(\eta,\omega) \mapsto a_{t,u}(\eta,\omega)$ -measurable for $\mathcal{F}^o_t \otimes \mathcal{F}^o_u$ of the related Radon-Nikodym densities.

We can now set $A_{t,u}(\omega) := a_{t,u}(\omega,\omega)$ which is then an $\mathcal{F}^o_u$-measurable r.v. It yields for any $\eta \in \Omega$

$$A_{t,u}(\omega) = a_{t,u}(\omega,\omega) = a_{t,u}(\eta,\omega) = A^{t,\eta}_u(\omega) - A^{t,\eta}_t(\omega) \quad \mathbb{P}^{t,\eta} \text{ a.s.} \tag{A.7}$$

Indeed the second equality holds given that $a_{t,u}$ is $\mathcal{F}^o_t$-measurable with respect to the first variable, taking into account that $\mathbb{P}^{t,\eta}(\omega^t = \eta^t) = 1$; the third equality follows by (A.4).

We now set $s < t$ and $\eta \in \Omega$. We want to show that we still have $A_{t,u} = A^{s,\eta}_u - A^{s,\eta}_t$ $\mathbb{P}^{s,\eta}$ a.s. So we consider $F \in \mathcal{F}^o_u$; we compute

$$\begin{array}{lll} & \mathbb{E}^{s,\eta}\left[\mathbb{1}_F(A^{s,\eta}_u - A^{s,\eta}_t)\right] & = \mathbb{E}^{s,\eta}\left[\int_t^u N^F_{r-} dB_r\right] \\ = & \mathbb{E}^{s,\eta}\left[\mathbb{E}^{s,\eta}\left[\int_t^u N^F_{r-} dB_r|\mathcal{F}_t\right]\right] & = \mathbb{E}^{s,\eta}\left[\mathbb{E}^{t,\omega}\left[\int_t^u N^F_{r-} dB_r\right]\right] \\ = & \mathbb{E}^{s,\eta}\left[\mathbb{E}^{t,\omega}\left[\mathbb{1}_F A_{t,u}\right]\right] & = \mathbb{E}^{s,\eta}\left[\mathbb{E}^{s,\eta}\left[\mathbb{1}_F A_{t,u}|\mathcal{F}_t\right]\right] \\ = & \mathbb{E}^{s,\eta}\left[\mathbb{1}_F A_{t,u}\right]. & \end{array} \tag{A.8}$$

Indeed, the first equality comes from (A.3) and Lemma A.1; concerning the fourth equality we recall that, by (A.5), (A.6) and (A.7), we have $\mathbb{E}^{t,\omega}\left[\int_t^u N^F_{r-} dB_r\right] = \mathbb{E}^{t,\omega}\left[\mathbb{1}_F A_{t,u}\right]$ for all ω. The third and fifth equalities come from Remark 3.6.

Since adding $\mathbb{P}^{s,\eta}$-null sets does not change the validity of (A.8), by Proposition 3.20 for any $F \in \mathcal{F}^{s,\eta}_u$ we have $\mathbb{E}^{s,\eta}\left[\mathbb{1}_F(A^{s,\eta}_u - A^{s,\eta}_t)\right] = \mathbb{E}^{s,\eta}\left[\mathbb{1}_F A_{t,u}\right]$.

Finally, since both $A^{s,\eta}_u - A^{s,\eta}_t$ and $A_{t,u}$ are $\mathcal{F}^{s,\eta}_u$-measurable, we can conclude that $A^{s,\eta}_u - A^{s,\eta}_t = A_{t,u}$ $\mathbb{P}^{s,\eta}$ a.s.

We emphasize that this holds for any $t \leq u$ and $(s,\eta) \in [0,t] \times \Omega$, $(A_{t,u})_{(t,u)\in\Delta}$ is the desired path-dependent AF, which ends the proof of Proposition 4.7. □

Proof of Proposition 4.10.

We set

$$C_{t,u} = A_{t,u} + (V_u - V_t) + (u - t), \tag{A.9}$$

which is a path-dependent AF with cadlag versions $C^{s,\eta}_t = A^{s,\eta}_t + V_t + t$ and we start by showing the statement for A and C instead of A and V.

The reason of the introduction of the intermediary function C is that for any $u > t$ we have $\frac{A^{s,\eta}_u - A^{s,\eta}_t}{C^{s,\eta}_u - C^{s,\eta}_t} \in [0,1]$; that property will be used extensively in connections with the application of dominated convergence theorem.

Since $A^{s,\eta}$ is non-decreasing for any $(s,\eta) \in \mathbb{R}_+ \times \Omega$, A can be taken positive (in the sense that $A_{t,u}(\omega) \geq 0$ for any $(t,u) \in \Delta$ and $\omega \in \Omega$) by considering A^+

(defined by $(A^+)_{t,u}(\omega) := A_{t,u}(\omega)^+$) instead of A. On $\mathbb{R}_+$ we set

$$\begin{aligned} K_t &= \liminf_{n\to\infty} \frac{A_{t,t+\frac{1}{n}}}{A_{t,t+\frac{1}{n}} + \frac{1}{n} + (V_{t+\frac{1}{n}} - V_t)} \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\min_{n\leq p\leq m} \frac{A_{t,t+\frac{1}{p}}}{A_{t,t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}. \end{aligned} \tag{A.10}$$

This liminf always exists and belongs to $[0,1]$ since the sequence belongs to $[0,1]$.
For any $(s,\eta) \in \mathbb{R}_+ \times \Omega$, since for all $t \geq s$ and $n \geq 0$,
$A_{t,t+\frac{1}{n}} = A^{s,\eta}_{t+\frac{1}{n}} - A^{s,\eta}_t$ $\mathbb{P}^{s,\eta}$ a.s., then $K^{s,\eta}$ defined by $K^{s,\eta}_t := \liminf_{n\to\infty} \frac{A^{s,\eta}_{t+\frac{1}{n}} - A^{s,\eta}_t}{C^{s,\eta}_{t+\frac{1}{n}} - C^{s,\eta}_t}$ is a $\mathbb{P}^{s,\eta}$-version of K, for $t \in [s,+\infty[$.
By Lebesgue Differentiation theorem (see Theorem 12 Chapter XV in [12] for a version of the theorem with a general atomless measure), for any (s,η), for $\mathbb{P}^{s,\eta}$-almost all ω, since $dC^{s,\eta}(\omega)$ is absolutely continuous with respect to $dA^{s,\eta}(\omega)$, $K^{s,\eta}(\omega)$ is a density of $dA^{s,\eta}(\omega)$ with respect to $dC^{s,\eta}(\omega)$.
For any $t \geq 0$, K_t is measurable with respect to $\bigcap_{n\geq 0} \mathcal{F}^o_{t+\frac{1}{n}} = \mathcal{F}_t$, by definition of the canonical filtration. For any $(t,\omega) \in \mathbb{R}_+ \times \Omega$, we now set

$$k_t(\omega) := \mathbb{E}^{t,\omega}[K_t]. \tag{A.11}$$

Remark 3.6 implies that k is an $\mathbb{F}^o$-adapted process. The path-dependent canonical class verifies Hypothesis 3.5, and K_t is $\mathcal{F}_t$-measurable then for any $(s,\eta) \in [t,+\infty[\times\Omega$, $K_t(\omega) = \mathbb{E}^{s,\eta}[K_t|\mathcal{F}_t](\omega) = \mathbb{E}^{t,\omega}[K_t] = k_t(\omega)$ $\mathbb{P}^{s,\eta}$-a.s.: hence k is on $[s,+\infty[$ a $\mathbb{P}^{s,\eta}$-version of K, and therefore of $K^{s,\eta}$.

The next main object of this proof is to show that k is an $\mathbb{F}^o$-progressively measurable process. For any integers (n,m), we define

$$k^{n,m} : (t,\eta) \mapsto \mathbb{E}^{t,\eta}\left[\min_{n\leq p\leq m} \frac{A_{t,t+\frac{1}{p}}}{A_{t,t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right],$$

and for all n,

$$k^{n} : (t,\eta) \mapsto \mathbb{E}^{t,\eta}\left[\inf_{p\geq n} \frac{A_{t,t+\frac{1}{p}}}{A_{t,t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right]. \tag{A.12}$$

We start showing that

$$\tilde{k}^{n,m} : \begin{array}{ccc} ((s,\eta),t) & \longmapsto & \mathbb{E}^{s,\eta}\left[\min_{n\leq p\leq m} \frac{A_{t,t+\frac{1}{p}}}{A_{t,t+\frac{1}{p}} + \frac{1}{p} + (V_{t+\frac{1}{p}} - V_t)}\right]\mathbb{1}_{s\leq t}, \\ (\mathbb{R}_+\times\Omega)\times\mathbb{R}_+ & \longrightarrow & [0,1], \end{array} \tag{A.13}$$

is measurable with respect to $\mathcal{P}ro^o \otimes \mathcal{B}(\mathbb{R}_+)$. In order to do so, we will show that it is measurable in the first variable (s,η), and right-continuous in the second

variable t, and conclude with Lemma 4.12 in [5].
We fix $t \in \mathbb{R}_+$. Since the path-dependent canonical class is progressive, by Remark 3.6, the map

$$\begin{array}{ccc} (s,\eta) & \longmapsto & \mathbb{E}^{s,\eta}\left[\min\limits_{n\leq p\leq m} \frac{A_{t,t+\frac{1}{p}}}{A_{t,t+\frac{1}{p}}+\frac{1}{p}+(V_{t+\frac{1}{p}}-V_t)}\right] \\ \mathbb{R}_+\times\Omega & \longrightarrow & [0,1], \end{array} \tag{A.14}$$

is measurable with respect to $\mathcal{P}ro^o$. The map $(s,\eta) \longmapsto \mathbb{1}_{[t,+\infty[}(s)$ is also trivially measurable with respect to $\mathcal{P}ro^o$; therefore the product of the latter map and (A.14), that we denote by $\tilde{k}(\cdot,\cdot,t)$ is also measurable with respect to $\mathcal{P}ro^o$. Moreover, if we fix $(s,\eta) \in \mathbb{R}_+\times\Omega$, reasoning exactly as in the proof of Proposition 4.13 in [5] we see that $t \mapsto \tilde{k}^{n,m}(s,\eta,t)$ is right-continuous, which by Lemma 4.12 in [5] implies the joint measurability of $\tilde{k}^{n,m}$.

Since $k^{n,m}(t,\eta) = \tilde{k}^{n,m}(t,t,\eta)$, and since $(t,\eta) \mapsto (t,\eta,t)$ is obviously $(\mathcal{P}ro^o, \mathcal{P}ro^o\otimes\mathcal{B}(\mathbb{R}_+))$-measurable, then by composition we can deduce that for any n,m, $k^{n,m}$ is an $\mathbb{F}^o$-progressively measurable process. By the dominated convergence theorem, $k^{n,m}$ tends pointwise to k^n when m goes to infinity, so k^n also is an $\mathbb{F}^o$-progressively measurable process for every n. Finally, since $K_t = \lim\limits_{n\to\infty}\inf\limits_{p\geq n} \frac{A_{t,t+\frac{1}{p}}}{A_{t,t+\frac{1}{p}}+\frac{1}{p}+(V_{t+\frac{1}{p}}-V_t)}$, taking the expectation and again by the dominated convergence theorem, k^n (defined in (A.12)) tends pointwise to k (defined in (A.11)), when n goes to infinity, so k is an $\mathbb{F}^o$-progressively measurable process. Considering that $(t,u,\omega) \mapsto V_u - V_t$ also trivially defines a non-negative non-decreasing path-dependent AF absolutely continuous with respect to C, defined in (A.9), we proceed similarly as at the beginning of the proof, replacing the path-dependent AF A with V.

Let the process K' be defined by $K'_t = \liminf\limits_{n\to\infty} \frac{V_{t+\frac{1}{n}}-V_t}{A_{t,t+\frac{1}{n}}+\frac{1}{n}+(V_{t+\frac{1}{n}}-V_t)}$, and for any (s,η), let $K'^{s,\eta}$ be defined on $[s,+\infty[$ by $K'^{s,\eta}_t = \liminf\limits_{n\to\infty} \frac{V_{t+\frac{1}{n}}-V_t}{A^{s,\eta}_{t+\frac{1}{n}}-A^{s,\eta}_t+\frac{1}{n}+(V_{t+\frac{1}{n}}-V_t)}$.
Then, for any (s,η), $K'^{s,\eta}$ on $[s,+\infty[$ is a $\mathbb{P}^{s,\eta}$-version of K', and it constitutes a density of dV with respect to $dC^{s,\eta}(\omega)$ on $[s,+\infty[$, for almost all ω. One shows then the existence of an $\mathbb{F}^o$-progressively measurable process k' such that for any (s,η), k' is a $\mathbb{P}^{s,\eta}$-version of K' and of $K'^{s,\eta}$ on $[s,+\infty[$.

By the considerations after (A.10), for any (s,η), under $\mathbb{P}^{s,\eta}$, we can write $\left\{\begin{array}{rcl} A^{s,\eta} &=& \int_s^{\cdot\vee s} K^{s,\eta}_r dC^{s,\eta}_r \\ V_{\cdot\vee s}-V_s &=& \int_s^{\cdot\vee s} K'^{s,\eta}_r dC^{s,\eta}_r. \end{array}\right.$ Now since $dA^{s,\eta} \ll dV$, we have for $\mathbb{P}^{s,\eta}$ almost all ω that the set $\{r\in[s,+\infty[: |K'^{s,\eta}_r(\omega)=0\}$ is negligible with respect to dV so also for $dA^{s,\eta}(\omega)$ and therefore we can write

$$\begin{array}{rcl} A^{s,\eta} &=& \int_s^{\cdot\vee s} K^{s,\eta}_r dC^{s,\eta}_r \\ &=& \int_s^{\cdot\vee s} \frac{K^{s,\eta}_r}{K'^{s,\eta}_r}\mathbb{1}_{\{K'^{s,\eta}_r\neq 0\}} K'^{s,\eta}_r dC^{s,\eta}_r + \int_s^{\cdot\vee s}\mathbb{1}_{\{K'^{s,\eta}_r=0\}} dA^{s,\eta}_r \\ &=& \int_s^{\cdot\vee s} \frac{K^{s,\eta}_r}{K'^{s,\eta}_r}\mathbb{1}_{\{K'^{s,\eta}_r\neq 0\}} dV_r, \end{array}$$

where we use the convention that for any two functions ϕ, ψ then $\frac{\phi}{\psi}\mathbb{1}_{\psi\neq 0}$ is defined by $\frac{\phi}{\psi}\mathbb{1}_{\{\psi\neq 0\}}(x) = \begin{cases} \frac{\phi(x)}{\psi(x)} \text{ if } \psi(x)\neq 0 \\ 0 \text{ if } \psi(x) = 0. \end{cases}$

We now set $h := \frac{k}{k'}\mathbb{1}_{\{k'_r\neq 0\}}$ which is an $\mathbb{F}^o$-progressively measurable process, and clearly for any (s,η), h is a $\mathbb{P}^{s,\eta}$-version of $H^{s,\eta} := \frac{K^{s,\eta}}{K'^{s,\eta}}\mathbb{1}_{\{K'^{s,\eta}\neq 0\}}$ on $[s,+\infty[$. So by Lemma 5.12 in [2], $H^{s,\eta} = h\ dV \otimes d\mathbb{P}^{s,\eta}$ a.e. on $[s,+\infty[$ and finally we have shown that under any $\mathbb{P}^{s,\eta}$, $A^{s,\eta} = \int_s^{\cdot\vee s} h_r dV_r$. □

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