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CARTAN GEOMETRIES ON COMPLEX MANIFOLDS OF ALGEBRAIC DIMENSION ZERO

INDRANIL BISWAS, SORIN DUMITRESCU, AND BENJAMIN MCKAY

ABSTRACT. We show that compact complex manifolds of algebraic dimension zero bearing a holomorphic Cartan geometry of algebraic type have infinite fundamental group. This generalizes the main Theorem in [DM] where the same result was proved for the special cases of holomorphic affine connections and holomorphic conformal structures.

1. INTRODUCTION

Holomorphic affine connections, holomorphic projective connections and holomorphic conformal structures are among the important geometric examples of holomorphic Cartan geometries [Sh].

Contrary to the real setting, compact complex manifolds admitting a holomorphic Cartan geometry are very rare. All known examples of compact complex manifolds admitting a holomorphic Cartan geometry actually admit a *flat* holomorphic Cartan geometry. It should be mentioned that the model of the flat Cartan geometry (see Definition 2.1) could be different. Indeed, compact complex parallelizable manifolds [Wa] biholomorphic to quotients G/Γ , where G is complex semi-simple Lie group and $\Gamma \subset G$ a uniform lattice, are known to admit holomorphic affine connections, but not flat holomorphic affine connections [Du3].

Nevertheless, the following was conjectured in [DM]: *Compact complex simply connected manifolds admitting a holomorphic Cartan geometry with model (G, H) are biholomorphic to the model G/H .* (See [DM, Section 6].)

This question is open even in the very special case of holomorphic projective connections: accordingly to the conjecture, complex compact simply connected manifolds bearing a holomorphic projective connection should be complex projective spaces (endowed with the standard flat projective geometry). This is not known even for smooth complex projective varieties (except for the complex dimensions one, two [KO, KO1] and three [PR]). This conjecture's prediction is that compact complex simply connected manifolds X do not admit holomorphic affine connections. This is known to be true for Kähler manifolds, since in this case the Chern classes of the holomorphic tangent bundle TX must vanish [At] and using Yau's proof of Calabi's conjecture one can show that X admits a finite unramified cover which is a complex torus [IKO]. According to the conjecture, the only simply connected compact complex manifold which admits a holomorphic conformal structure is the smooth

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quadric hypersurface in projective space. This is not known even for smooth complex projective varieties (except for the complex dimensions one, two [KO2] and three [PR1]). It is known that the smooth quadric hypersurface admits precisely one holomorphic conformal structure (the standard one).

Examples of non-Kähler complex compact manifolds of algebraic dimension zero admitting holomorphic Riemannian metrics [Le, Gh] of constant sectional curvature (and consequently the associated Levi-Civita holomorphic affine connections, which are projectively flat) were constructed by Ghys in [Gh]. They also admit the corresponding flat holomorphic conformal structures. These examples are deformations of parallelizable manifolds covered by $SL(2, \mathbb{C})$.

In the direction of the conjecture, it was proved in [DM, Theorem 1], that *compact complex simply connected manifolds with algebraic dimension zero admit neither holomorphic affine connections, nor holomorphic conformal structures.*

The main results of this paper generalize the main theorem in [DM] (Theorem 1): we will prove that *compact complex manifolds of algebraic dimension zero bearing a holomorphic Cartan geometry of algebraic type (Theorem 3.2) or a holomorphic rigid geometric structure of algebraic type (Theorem 4.1) have infinite fundamental group.*

2. CARTAN GEOMETRIES

Let X be a complex manifold, G a complex connected Lie group and H a closed complex Lie subgroup of G . The Lie algebras of the Lie groups G and H will be denoted by \mathfrak{g} and \mathfrak{h} respectively.

Definition 2.1. A *holomorphic Cartan geometry* (P, ω) on X with model (G, H) is a holomorphic principal (right) H -bundle $\pi : P \rightarrow X$ endowed with a holomorphic \mathfrak{g} -valued 1-form ω satisfying:

- (1) $\omega_p : T_p P \rightarrow \mathfrak{g}$ is a complex linear isomorphism for all $p \in P$,
- (2) the restriction of ω to every fiber of π coincides with the left invariant Maurer-Cartan form of H , and
- (3) $(R_h)^* \omega = \text{Ad}(h)^{-1} \omega$, for all $h \in H$, where R_h is the right action of h on P and Ad is the adjoint representation of G on \mathfrak{g} .

Definition 2.2. The *kernel* N of a model (G, H) is the largest subgroup of H normal in G ; the kernel is clearly a closed complex normal subgroup, and we denote its Lie algebra by \mathfrak{n} . A model is *effective* if its kernel is $N = \{1\}$. A Cartan geometry is *effective* if its model is effective. Any Cartan geometry (P, ω) has *induced effective* Cartan geometry $(\bar{P}, \bar{\omega})$ where $\bar{P} = P/N$ and $\bar{\omega}$ is the $\mathfrak{g}/\mathfrak{n}$ -valued 1-form on \bar{P} which pulls back via the obvious quotient map $P \rightarrow \bar{P}$ to $\omega + \mathfrak{n}$.

Definition 2.3. If $\text{Ad}(H/N) \subset GL(\mathfrak{g}/\mathfrak{n})$ is an algebraic subgroup, then the holomorphic Cartan geometries with model (G, H) are said to be *algebraic type*.

Holomorphic affine connections, holomorphic projective connections, and holomorphic conformal structures are examples of holomorphic Cartan geometries of algebraic type [Pe, Sh].

Notice that Definition 2.1 implies, in particular, that the complex dimension of the homogeneous model space G/H is the same as the complex dimension d of the manifold X . Let us give some important examples.

A holomorphic affine connection corresponds to the model $(\mathbb{C}^d \rtimes \mathrm{GL}(d, \mathbb{C}), \mathrm{GL}(d, \mathbb{C}))$ of the complex affine space acted on by the group $\mathbb{C}^d \rtimes \mathrm{GL}(d, \mathbb{C})$ of complex affine transformations.

A holomorphic projective connection is a Cartan geometry with model $(\mathrm{PGL}(d+1, \mathbb{C}), Q)$, where $\mathrm{PGL}(d+1, \mathbb{C})$ is the complex projective group acting on the complex projective space $\mathbb{C}P^d$ and $Q \subset \mathrm{PGL}(d+1, \mathbb{C})$ is the maximal parabolic subgroup that fixes a given point in $\mathbb{C}P^d$.

Holomorphic conformal structures are modelled on the quadric $z_0^2 + z_1^2 + \dots + z_{d+1}^2 = 0$ in $\mathbb{C}P^{d+1}$. Here G is the subgroup $\mathbb{P}O(d+2, \mathbb{C})$ of the complex projective group $\mathrm{PGL}(d+2, \mathbb{C})$ preserving the quadric and H is the stabilizer of a given point in the quadric. For more details about those Cartan geometries and for many other examples one can see [Sh].

Definition 2.4. A (local) biholomorphism between two open subsets U and V in X is a (local) *automorphism* of a Cartan geometry (P, ω) if it lifts to a principal H -bundle isomorphism between $\pi^{-1}(U)$ and $\pi^{-1}(V)$ that preserves ω .

A (local) holomorphic vector field on X is a (local) *Killing field* of (P, ω) if its (local) flow acts by (local) automorphisms.

The Cartan geometry (P, ω) is called *locally homogeneous* on an open subset $U \subset X$ if $T_u X$ is spanned by local Killing fields of (P, ω) at every point $u \in U$.

The curvature of (P, ω) is defined as a 2-form Ω on P with values in \mathfrak{g} , given by the following formula

$$\Omega(Y, Z) = d\omega(Y, Z) - [\omega(Y), \omega(Z)]_{\mathfrak{g}}$$

for all local holomorphic vector fields Y, Z on P . When at least one of Y and Z is a vertical vector field (meaning lies in the kernel of $d\pi$), then $\Omega(Y, Z) = 0$ (see [Sh, 5.3.10]). From this it follows that Ω is basic: it may be considered as a holomorphic H -equivariant map

$$P \longrightarrow V := \mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$$

because of the following construction.

Consider the holomorphic isomorphism of vector bundles

$$\omega : TP \longrightarrow P \times \mathfrak{g} \tag{2.1}$$

Since Ω is basic, using this isomorphism, $K(p) = (\omega_p^{-1})^* \Omega$ can be seen as an element of $V = \mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$. This curvature map $K : P \longrightarrow V$ is H -equivariant for the action of H on V defined as follows: the adjoint action of H on \mathfrak{g} preserves the subalgebra \mathfrak{h} , so H acts on $\mathfrak{g}/\mathfrak{h}$; the actions of H on $\mathfrak{g}/\mathfrak{h}$ and \mathfrak{g} together produce an action of H on V (see [Sh, 5.3.23]).

Again using the isomorphism in (2.1) one can define the r -jet of K as a H -equivariant map

$$P \longrightarrow W^r,$$

where $W^r = V \oplus \text{Hom}(\mathfrak{g}, V) \oplus \dots \oplus \text{Hom}(\otimes^r \mathfrak{g}, V)$. This map associates to each $p \in P$ the r -jet of the curvature K at p defined as the element

$$(K(p), D^1(K)(p), \dots, D^r(K)(p)) \in W^r$$

with

$$D^k(K)(p)(X_1 \otimes \dots \otimes X_k) = (\widetilde{X}_1 \cdot \dots \cdot \widetilde{X}_k \cdot K)(p),$$

where \widetilde{X}_i is the unique ω -constant vector field on P with value $X_i \in T_p P = \mathfrak{g}$. Notice that

$$D^k K(p \cdot h^{-1}) = h \circ D^k K(p) \circ \text{Ad}^k(h^{-1})$$

where Ad^k is the tensor representation on $\otimes^k \mathfrak{g}$ of $\text{Ad}(H)$ [Me, Proposition 3.1].

The following Theorem 2.5 is an adaptation to the framework of Cartan geometries of Gromov's result about the automorphism group of an analytic rigid geometric structure (see Definition 4.2) [Gr, Section 3.5 C]. The main tool is the adaptation, to the context of Cartan geometries, of Gromov's result of integrability in local automorphisms of jets of local biholomorphisms preserving the r -jet of the curvature (see [Me] and [Pe]).

Theorem 2.5. *Let X be a compact complex simply connected manifold endowed with a holomorphic effective Cartan geometry (P, ω) with model (G, H) of algebraic type. Let L be the connected component of the automorphism group of (P, ω) . Then the stabilizer L_{x_0} of a point $x_0 \in X$ is isomorphic as a complex Lie group to a complex algebraic subgroup of H . In particular, if L has a global fixed point in X , then L is isomorphic as a complex Lie group to a complex algebraic subgroup of H .*

Proof. The L -action on X lifts to an L -action on P by bundle automorphisms preserving ω [Me] Prop. 3.7. The stabilizer L_{x_0} of a point $x_0 \in X$ acts on P preserving the fiber $\pi^{-1}(x_0)$ (and the left invariant Maurer-Cartan form on the fiber). Pick a point $\hat{x}_0 \in P$ mapping to $x_0 \in X$. Each element γ of L_{x_0} takes the point \hat{x}_0 to some point $\hat{x}_0 h^{-1}$ for some element $h = i(\gamma) \in H$. This defines a homomorphism $i: L_{x_0} \rightarrow H$. The choice of point \hat{x}_0 affects i by inner conjugation in H .

The homomorphism i is injective: the kernel of i consists of automorphisms of the bundle P acting trivially on X and preserving ω , and hence preserving the constant vector fields. By H -equivariance, and the fact that X is connected, the kernel of i acts trivially on P , so i is injective.

Since X is simply connected, all local Killing fields of (P, ω) vanishing at x_0 extend to global Killing fields, whose flows belong to L_{x_0} [No, Am, Me, Pe]. Let $I^r \subset H$ be the stabilizer of the the r -jet of the curvature at \hat{x}_0 , i.e. the stabilizer in W^r of

$$(K(\hat{x}_0), D^1(K)(\hat{x}_0), \dots, D^r(K)(\hat{x}_0)).$$

Note that $i(L_{x_0})$ is a subgroup of I^r , as it acts on P preserving the Cartan geometry. For r large enough, the subgroup $i(L_{x_0}) \subset H$ contains the identity component in I^r [Me] (Proposition 3.8 and Theorem 3.11) (see also [Pe]). Since I^r is an algebraic group, it has finitely many connected components. By GAGA, and smoothness of algebraic groups, the analytic components, algebraic components, and topological path components of any complex algebraic group are the same. This implies that $L_{x_0} \cong i(L_{x_0})$ has finite index in I^r and so is an algebraic subgroup. \square

We will also need the following result which is a straightforward consequence of Gromov’s stratification result for orbits of local automorphisms of a rigid geometric structure [Gr, Section 3.4] adapted to the case of Cartan geometries by Melnick [Me, Theorem 4.1] and Pecastaing [Pe, Theorem 1.3]:

Lemma 2.6. *The L -action on X admits at least one closed (and hence compact) orbit.*

Proof. Since X is simply connected, the extendibility result of local Killing fields [No, Am, Me, Pe] shows that the L -orbits in X coincide with the orbits of local Killing fields of (P, ω) . The version of Gromov’s stratification result for analytic Cartan geometries [Me, Theorem 4.1] furnishes a stratification by L -invariant subsets

$$X = X_0 \cup X_1 \cup \dots \cup X_l$$

such that, for all i , $\cup_{j \geq i} X_j$ is an analytic subset in X and X_i is an open dense set in $\cup_{j \geq i} X_j$. Moreover, in each X_i , the L -orbits are the connected components of the fibers of some holomorphic submersion $f_i : X_i \rightarrow V_i$, where V_i is a complex manifold.

The L -orbits of elements in X_l are compact sets. \square

3. CARTAN GEOMETRIES ON MANIFOLDS OF ALGEBRAIC DIMENSION ZERO

In this section we prove the main result of the article (Theorem 3.2). We prove first the following Lemma 3.1 which is an adaptation for Cartan geometries of Proposition 2 in [DM] about rigid geometric structures.

Recall that a compact complex manifold is of algebraic dimension zero if it does not admit nonconstant meromorphic functions [Ue]. Those manifolds are far from being algebraic. Indeed, a manifold is bimeromorphic with an algebraic manifold if and only if its field of meromorphic functions separates points [Mo, Ue].

Lemma 3.1. *Let X be a compact complex simply connected manifold of algebraic dimension zero endowed with a holomorphic effective Cartan geometry (P, ω) . Then the connected component of the automorphism group of (P, ω) is a connected complex abelian Lie group L acting on X with an open dense orbit. The open dense orbit is the complement of an anticanonical divisor. Moreover, this open dense orbit is biholomorphic to L .*

Proof. By Theorem 1.2 in [Du] (see also [Du1]) the Cartan geometry (P, ω) is locally homogeneous on an open dense set in X . Recall that on simply connected manifolds, local

Killing vector fields of Cartan geometries extend to all of the manifold X [Am, No, Gr, Pe]. Therefore, it follows that there exists a Lie algebra \mathfrak{a} of globally defined Killing vector fields on X which span TX at the generic point. Let us fix a basis $\{X_1, \dots, X_k\}$ of \mathfrak{a} and consider the generalized Cartan geometry $(P, \omega, \mathfrak{a})$ (in the sense of Definition 4.11 in [Pe]) which is a juxtaposition of (P, ω) with the family of vector fields $\{X_1, \dots, X_k\}$. The proof of Theorem 1.2 in [Du] applies to the generalized geometry $(P, \omega, \mathfrak{a})$ which must also be locally homogeneous on an open dense set. Notice that local Killing vector fields of $(P, \omega, \mathfrak{a})$ are restrictions of elements in \mathfrak{a} which commute with the vector fields X_i . It follows that local vector fields of $(P, \omega, \mathfrak{a})$ are elements in the centralizer \mathfrak{a}' of \mathfrak{a} , and \mathfrak{a}' is transitive on an open dense set. Since \mathfrak{a}' is commutative, the action is simply transitive on the open dense set (note that \mathfrak{a}' has the same dimension as X). A basis of \mathfrak{a}' is a family of commuting vector fields spanning TX at the generic point. Since the vector fields X_i commute with elements in this basis, they must be a linear combination of elements in \mathfrak{a}' with constant coefficients (on the open dense set and hence on all of X). In particular, we have $X_i \in \mathfrak{a}'$ which implies that $\mathfrak{a} = \mathfrak{a}'$ is an abelian algebra; this also implies that k is the complex dimension of X . In particular, the corresponding connected Lie group L (the connected component of the identity of the automorphism group of (P, ω)) is an abelian group of the same dimension as X acting with an open dense orbit. This open dense orbit is the complement of the vanishing set of the holomorphic section $X_1 \wedge \dots \wedge X_k$ of the anticanonical bundle.

Since L is abelian, any $\ell_0 \in L$ stabilizing some point x_0 of the open orbit stabilizes all points ℓx_0 for $\ell \in L$: $\ell_0 x_0 = x_0$, so $\ell_0 \ell x_0 = \ell \ell_0 x_0 = \ell x_0$. By density of the open dense orbit, ℓ_0 stabilizes all points of X . So the stabilizer of any point in the open orbit is trivial, i.e. X is an L -equivariant compactification of L . \square

Theorem 3.2. *Compact complex manifolds with algebraic dimension zero bearing a holomorphic Cartan geometry of algebraic type have infinite fundamental group.*

Proof. Assume, by contradiction, that a complex manifold X of algebraic dimension zero admits a holomorphic Cartan geometry (P, ω) and has finite fundamental group. Considering a finite unramified cover and the pull-back of the Cartan geometry on it, we will assume that X is simply connected. Replace the Cartan geometry by the induced effective Cartan geometry, so we can assume that it is effective.

By Lemma 3.1, the automorphism group of (P, ω) is a complex connected abelian Lie group L acting on X with an open dense orbit (the complement of an anticanonical divisor). The complement of this anticanonical divisor is biholomorphic to L (see Lemma 3.1).

We consider a compact orbit T in X for the L -action (see Lemma 2.6). Then T is biholomorphic to the quotient of L by the stabilizer of a point in T . Notice that T has positive dimension. Indeed, if T is a point then it is a fixed point in X for the L -action and, by Theorem 2.5, the group L is isomorphic as a complex Lie group to a complex algebraic group. This implies that the complement of the canonical divisor in X is a smooth complex algebraic variety and, consequently, X is of maximal algebraic dimension: a contradiction.

It follows that T is a compact complex torus and the orbital map of the L -action is a holomorphic map from L to T with fibers isomorphic to the stabilizer L_t of a point $t \in T$. By Theorem 2.5, this L_t is a complex algebraic subgroup of H . Consequently, the fibers of the quotient map $L \rightarrow T$ have a finite number of connected components.

We get a holomorphic map $L \rightarrow T$, and each fiber has a finite number of components.

Since X is an L -equivariant compactification of L , we get a meromorphic map $f : X \rightarrow T$. By resolving the singularities of its graph, the meromorphic map f extends into a holomorphic map $\bar{f} : \bar{X} \rightarrow \bar{T}$ from a modification $\pi_1 : \bar{X} \rightarrow X$ of X to a modification $\pi_2 : \bar{T} \rightarrow T$ of T . Consider then the holomorphic map $\pi_2 \circ \bar{f} : \bar{X} \rightarrow T$.

Since the fibers of $\pi_1 : \bar{X} \rightarrow X$ are covered by complex projective lines and any holomorphic map from a complex projective line to a complex torus is constant, it follows that $\pi_2 \circ \bar{f}$ descends to a holomorphic map $X \rightarrow T$.

This furnishes a holomorphic map from X to T such that the generic fibers admit a finite number of connected components. Since X is simply connected, this implies that the fundamental group of T is finite: a contradiction. \square

4. HOLOMORPHIC RIGID GEOMETRIC STRUCTURES

In this section we show that the proof of Theorem 3.2 directly adapts to the context of holomorphic rigid geometric structures in Gromov's sense [DG, Gr, Du2] and gives the following result:

Theorem 4.1. *Compact complex manifolds with algebraic dimension zero bearing a holomorphic rigid geometric structure of algebraic type have infinite fundamental group.*

Let us first recall Gromov's definition of a rigid geometric structure of algebraic type [Gr, Introduction].

For any integer $k \geq 1$, we consider the principal bundle of k -frames $R^k(X) \rightarrow X$, which is the bundle of k -jets of local biholomorphisms from \mathbb{C}^d to X . The corresponding structure group D^k is the group of k -jets of local biholomorphisms of \mathbb{C}^d fixing 0. The group D^k is a complex algebraic group.

Definition 4.2. A holomorphic geometric structure (of order k) of algebraic type ϕ on the complex manifold X is a holomorphic D^k -equivariant map $\phi : R^k(X) \rightarrow Z$, with Z a complex algebraic manifold Z endowed with an algebraic action of D^k .

In this framework, the natural concept of symmetry is the following. A (local) biholomorphism of X is a (local) automorphism of ϕ if its canonical lift to $R^k(X)$ preserves the fibers of ϕ .

The geometric structure ϕ is called *rigid* (at order $k + r$, for some integer $r \geq 0$), if any local automorphism of ϕ is completely determined by its $k + r$ -jet at any given point [Gr, Introduction] and [DG, Section 5].

Holomorphic affine connections, holomorphic projective connections and holomorphic conformal structures (in complex dimension ≥ 3) are holomorphic rigid geometric structures of algebraic type [DG, Gr].

Let us give the proof of Theorem 4.1.

Proof. Assume, by contradiction, that the complex manifold X admits a holomorphic rigid geometric structure (of order k) of algebraic type ϕ and has finite fundamental group. Considering the pull-back of the rigid geometric structure on the universal cover of X we will consider that X is simply connected.

By [DM, Proposition 2], the connected component of the automorphism group of (X, ϕ) is a complex (connected) abelian Lie group L acting on X with an open dense orbit. As in the proof of Lemma 3.1, the stabilizer of any point in the open orbit is trivial, i.e. X is an L -equivariant compactification of L .

For any point $x_0 \in X$ in any L -orbit, its stabilizer $L_{x_0} \subset L$ is isomorphic to a complex algebraic subgroup in D^{k+r} , for some integer $r \geq 0$. Indeed, fix an integer $r \geq 0$ and consider the lifted L -action on the frame bundles $R^{k+r}(X)$. This action preserves the fibers of the r -jet of ϕ , which is a D^{k+r} -equivariant map ϕ^r from $R^{k+r}(X)$ to some complex manifold $J^r(Z)$ [DG, Section 5].

If $f \in L_{x_0}$, then for any lift $\hat{x}_0 \in R^{k+r}(X)$ of x_0 , there exists a unique element $i_f \in D^{k+r}$ such that $f(\hat{x}_0) = \hat{x}_0 \cdot i_f^{-1}$. This defines a group homomorphism $i: f \mapsto i_f$ from L_{x_0} to D^{k+r} .

For $r \geq 0$, let I^r denote the stabilizer in D^{k+r} of $\phi^r(\hat{x}_0) \in J^r(Z)$. Our homomorphism i takes L_{x_0} to I^r , as L_{x_0} preserves the fibers of ϕ^r . By rigidity, the homomorphism i is injective for all r sufficiently large. For r large enough, the previous homomorphism defines an isomorphism from the identity component of L_{x_0} onto the identity component of I^r (see [DG, 5.15.C] and [Gr, 1.7]). Since the natural D^{k+r} -action on $J^r(Z)$ is algebraic [DG, Section 5], it follows that I^r is a complex algebraic subgroup in D^{k+r} . In particular, I^r has finitely many connected components. As in the proof of Lemma 3.1, I^r has finitely many path components, and $L_{x_0} \rightarrow I^r$ is onto at least on one of them, so L_{x_0} is a finite index subgroup of I^r , so a union of finitely many components of I^r , so a complex algebraic group.

Moreover, if $x_0 \in X$ is a global fixed point for the L -action, this implies that $L = L_{x_0}$ is a complex algebraic group and, consequently, X has maximal algebraic dimension: a contradiction.

Using Gromov's stratification result for the orbits of local Killing fields [Gr, Section 3.4], the arguments in the proof of Lemma 2.6 show that the L -action admits a compact orbit T in X .

We have seen in the proof of Theorem 3.2 that T must have positive dimension: it is not reduced to a (fixed) point. It follows that T is a complex torus and we finish the proof as in Theorem 3.2. \square

Theorem 4.1 corrects a flawed statement in [DM, Proposition 1] asserting that toroidal actions (e.g. holomorphic actions of $(\mathbb{C}^*)^d$ with an open dense orbit) are rigid geometric structures of algebraic type. Interesting geometric examples of compact complex simply connected manifolds of algebraic dimension zero bearing toroidal actions were constructed in [BM, Mer, PUV]. Theorem 3.2 and Theorem 4.1 show that those manifolds do not admit neither holomorphic Cartan geometries, nor holomorphic rigid geometric structures of algebraic type.

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