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Nonlinearly preconditioned FETI solver for substructured formulations of nonlinear problems

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Abstract

We propose a new nonlinear version of preconditioning, dedicated to nonlinear substructured and condensed formulations with dual approach – i.e. nonlinear analogues to FETI solver. By increasing the importance of local nonlinear operations, this new technique reduces communications between processors throughout the parallel solving process. More, the tangent systems produced at each step still have the exact shape of classically preconditioned linear FETI problems, which makes the tractability of the implementation barely modified. The efficiency of this new preconditioner is demonstrated on two numerical test cases, namely a water diffusion problem and a nonlinear thermal behavior.

Key words: Domain decomposition; Nonlinear mechanics; Newton solver; FETI solver; Parallel processing

1 Introduction

In the context of large linear problems and their numerical solving, parallelization techniques can be used through iterative solvers to reduce computational costs: Domain Decomposition Methods (DDM) [1], (restricted) additive Schwarz methods [2, 3], Finite Element Tearing and Interconnecting (FETI) [4] and Balancing Domain Decomposition (BDD) [5] algorithms,

are amongst the most popular of these iterative parallel methods. This article focuses on DDM solvers based on Schur condensation, namely FETI/BDD solvers, in which the global problem is substructured into small non overlapping subproblems – corresponding to subdomains of the whole structure – which can be solved independently on different processors. Nodes shared by multiple subdomains are called interface nodes, and continuity/balance conditions across subdomains have to be imposed on the corresponding unknowns in order to keep coherence with the global problem. The resulting system is called condensed interface problem, and solved in parallel by a Krylov algorithm, with the help of local equilibriums solutions at each iteration. The main difference between FETI and BDD solvers lies in the chosen type of interface unknown (dual and primal respectively). The efficiency of such solvers is now well established, and several variants have been developed. Additional constraints can be added, by augmentation in FETI-2 method [6], or by elimination of selected primal unknowns in FETI-Dual Primal (FETI-DP) [7]; a second Lagrange multiplier can also be added to create mixed boundary conditions in the FETI-2 Lagrange Multipliers (FETI-2LM) [8]. Concerning BDD algorithm, the BDD by Constraints (BDDC) [9] variant is the primal counterpart to FETI-DP.

When nonlinearities are involved, a Newton solver is often applied to the global problem, while DDM methods can be used for the resulting tangent systems: this is the framework of Newton-Krylov-Schur (NKS) methods [10, 11]. Starting from the observation that NKS methods do not take advantage of the domain decomposition to deal with nonlinearities, but only use it at the tangent (linear) level, a new class of iterative nonlinear solvers has been developed [12, 13, 14, 15, 16, 17]. Nonlinear analogues to primal, dual and mixed Schur complements are built with a nonlinear substructuring and condensation: in this formulation, the global nonlinear problem is decomposed into small nonlinear subproblems, and continuity/balance conditions are imposed on interface unknowns across subdomains, resulting in a nonlinear condensed interface problem. The latter is solved by a Newton algorithm and, depending on the chosen type of main interface unknowns, the resulting tangent systems have the exact type of BDD/FETI solvers – and can thus be solved by the dedicated algorithms. This solving procedure has been proven to reduce the main time-consuming operations of the parallel process (i.e. communications, mainly involved in the Krylov solver iterations): compared to classical NKS procedures, the building of a 'better' (nonlinear) interface condensed problem makes the required number of global Newton iterations at convergence smaller – which a fortiori implies less required tangent operations. The cost of this decrease in communications is found in the additional embarrassingly parallel local nonlinear calculations.

Whether in the case of a linear or a nonlinear study, linear/tangent solvers BDD and FETI need a preconditioner, in order to regularize the problem and improve their scalability. Classical preconditioners involve a scaled assembly of local inverses, i.e. local primal (resp. dual) Schur complements for FETI (resp. BDD) [18]: these are quasi-optimal for a given linear problem, in the sense that they lead to a condition number in $\log(H/h)^2$ – where h is the characteristic element size and H the subdomain size. Equipped with this Dirichlet (resp. Neumann) preconditioner, the numerical stability of the algorithm is thus ensured, even if in practice its parallel scalability also depends on the ability to fastly compute coarse problem [19]. Scaled assembly operators can be build from local matrices: this strategy helps to preserve the condition number of the global problem in heterogeneous cases. Although this type of preconditioner is quite easily computable during the linear solving process – since it only requires local solvings and one assembly, – cheaper and efficient variants can be build in order to increase numerical productivity, like lumped preconditioners [20, 21], or more generally *interface* preconditioners [22].

It can be observed that the nonlinear substructuring and condensation method does not take full advantage of the preconditioning step to treat nonlinearities – as it was previously found out that NKS methods did not optimize the domain decomposition in the nonlinear framework. Instead of preconditioning the system only at the tangent level, this article presents a way to build a nonlinear analogue to the scaled classical Dirichlet (or Neumann) preconditioner [18] which produces self-well-conditioned tangent operators. A similar idea can be found in the framework of Schwarz DD methods, with the development of the ASPIN preconditioner for Newton solver [23], and more recently the RASPEN solver [24] – the reader can also refer to [25] for a rather exhaustive review and comparison of achievable nonlinear algebraic solvers compositions, including left nonlinear preconditioning of Newton-Krylov methods.

Starting from an interpretation of the linear FETI preconditioned system as a linear fixed point problem [26], whose nonlinear counterpart can be easily derived, the condensed nonlinear interface problem is modified, enhanced by a more consistent preconditioner which takes into account the linear or nonlinear behavior of each subdomain. The overall solving strategy however remains the same, apart from a few local operations and assemblies: a Newton algorithm is still applied to the interface problem, while tangent systems have the exact type of FETI algorithms. Only, a decrease in global Newton iterations numbers (and thus in Krylov iterations numbers) occurs, linked to the additional nonlinear information provided in the new interface condensed problem, given by the local nonlinear preconditioning operations.

The basic principles of nonlinear substructuring and condensation method are first recalled in the following. Then, a nonlinear version of the classical linear preconditioner is build, based on a fixed point interpretation. Its performance is eventually evaluated on two numerical examples: water diffusion in soils problem, and nonlinear thermal evolution. These two test cases – involving scalar unknowns – were chosen as first easy-to-handle demonstrators of the nonlinear preconditioner efficiency, even if the ultimate goal of the nonlinear substructuring and condensation method is the solving of large structure mechanics problems. Although we tried to remain as general as possible in our formulations, physical interpretations will be given in the context of mechanics, keeping in mind that the concepts and equations can be applied to any problem satisfying sufficient regularity properties to posses a unique solution (maximal monotone coercive continous operators for example, see [27]).

2 Nonlinear substructuring and condensation

2.1 Reference problem, notations

2.1.1 Global nonlinear problem

We consider a nonlinear partial differential equation on a domain Ω, representative of a quasi-static structural mechanical or thermal problem for instance, with Dirichlet conditions on a part $\partial_u\Omega \neq \varnothing$ of its boundary, and Neumann conditions on the complementary part $\partial_f\Omega$. After discretization with the Finite Element method, the problem to solve reads:

$$f_{int}(u) + f_{ext} = 0 \tag{1}$$

The vector f_{ext} takes into account boundary conditions (Dirichlet or Neumann) and dead loads, the operator f_{int} refers to the discretization of the homogeneous partial differential equation.

Remark 1. In linear elasticity, under the small perturbations hypothesis, one has:

$$f_{int}(u) = -Ku$$

with K the stiffness matrix of the structure.

We assume that f_{int} is differentiable and such that the tangent matrices are symmetric positive semi-definite (the semi-definiteness being typically due to the existence of rigid body motions).

2.1.2 Substructuring

Classical DDM notations will be used – see figure 1: global domain Ω is partitioned into N_s subdomains $(\Omega^{(s)})$. Let $n^{(s)}$ be the number of degrees of freedom of subdomain $\Omega^{(s)}$. For each subdomain, a trace operator $t^{(s)}$ restricts local quantities $x^{(s)}$ defined on $\Omega^{(s)}$ to boundary quantities $x_b^{(s)}$ defined on $\Gamma^{(s)} \equiv \partial\Omega^{(s)} \backslash \partial\Omega$:

$$x_b^{(s)} = t^{(s)} x^{(s)} = x^{(s)}_{|\Gamma^{(s)}}$$

Quantities defined on internal nodes (belonging to $\Omega^{(s)} \backslash \Gamma^{(s)}$) are written with subscript i: $x_i^{(s)}$.

Global primal (resp. dual) interface is written $\Gamma_A = \cup_s \Gamma^{(s)}$ (resp. Γ_B). Primal assembly operators $A^{(s)}$ are defined as canonical prolongation operators from $\Gamma^{(s)}$ to Γ_A: $A^{(s)}$ is a full-ranked boolean matrix of size $n_A \times n_b^{(s)}$ - where n_A is the size of global primal interface Γ_A and $n_b^{(s)}$ the number of interface degrees of freedom of subdomain $\Omega^{(s)}$.

Remark 2 (Diamond notations). For a domain Ω substructured into N_s subdomains $(\Omega^{(s)})$, concatenated local variables are superscripted $^{\diamondplus}$, $^{\diamondminus}$ or $^{\diamondtimes}$, depending on the alignment.

$$x^{\diamondplus} = \begin{pmatrix} x^{(1)} \\ \vdots \\ x^{(N_s)} \end{pmatrix}, \qquad x^{\diamondminus} = \begin{pmatrix} x^{(1)} & \dots & x^{(N_s)} \end{pmatrix}, \qquad M^{\diamondtimes} = \begin{pmatrix} M^{(1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & M^{(N_s)} \end{pmatrix}$$

Note that in the case of a nonlinear operator, the dependence is purely local in the block notation, in particular: $f_{int}^{\diamondplus}(u^{\diamondplus}) = \left[\dots, f_{int}^{(s)^T}(u^{(s)}), \dots \right]^T$.

Any matrix $B^{\diamondminus}$ satisfying $\mathrm{Range}(B^{\diamondminus^T}) = \mathrm{Ker}(A^{\diamondminus})$ can be assigned to dual assembly operator – see figure 1 for the most classical choice. Note that in that case multiple-points lead to $B^{\diamondminus}$ being rank-deficient. The number of relations characterizing the global dual interface is written n_B.

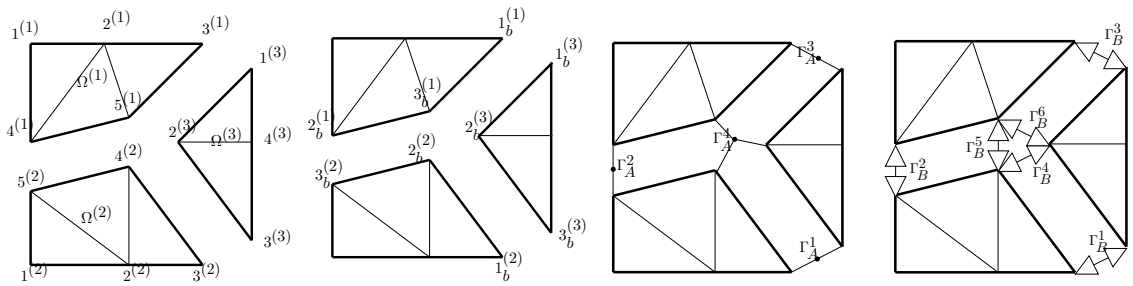
Remark 3 (Scaled assembly operators). We introduce the classical primal and dual scaled assembly operators $\tilde{A}^{\diamondminus}$ and $\tilde{B}^{\diamondminus}$ [18], they satisfy the following properties:

$$\begin{aligned} & A^{\diamondminus} \tilde{A}^{\diamondminus^T} = \tilde{A}^{\diamondminus} A^{\diamondminus^T} = I \\ & B^{\diamondminus} \tilde{B}^{\diamondminus^T} B^{\diamondminus} = B^{\diamondminus}, \quad \tilde{B}^{\diamondminus} B^{\diamondminus^T} \tilde{B}^{\diamondminus} = \tilde{B}^{\diamondminus} \end{aligned} \tag{2}$$

Remark 4. The following trivial properties are worth recalling:

$$\mathrm{Range}(B^{\diamondminus}) \cap \mathrm{Ker}(B^{\diamondminus^T}) = \{0\} \quad \text{and} \quad \mathrm{Range}(\tilde{B}^{\diamondminus}) \cap \mathrm{Ker}(B^{\diamondminus^T}) = \{0\}$$

□



(a) Subdomains (b) Local boundaries (c) Interface nodes (d) Interface connexions

$$t^{(1)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad t^{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad t^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A^{(3)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$B^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad B^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B^{(3)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

Figure 1: Local numberings, interface numberings, trace and assembly operators

Classically, the scaling operators are built as follow:

$$\begin{cases} \tilde{A}^{\diamondsuit} = \left(A^{\diamondsuit} D^{\diamondsuit} A^{\diamondsuit^T}\right)^{-1} A^{\diamondsuit} D^{\diamondsuit} \\ \tilde{B}^{\diamondsuit} = \left(B^{\diamondsuit} D^{\diamondsuit^{-1}} B^{\diamondsuit^T}\right)^{\dagger} B^{\diamondsuit} D^{\diamondsuit^{-1}} \end{cases} \quad \text{with } D^{(s)} = I^{(s)} \text{ or } \operatorname{diag}\left(K_{t_{bb}}^{(s)}\right) \tag{3}$$

Note that if the same matrix $D^{\diamondsuit}$ is chosen in definition of the primal and dual scaled assembly operators then the following property holds [28]:

$$A^{\diamondsuit^T} \tilde{A}^{\diamondsuit} + \tilde{B}^{\diamondsuit^T} B^{\diamondsuit} = I \tag{4}$$

2.2 Solving strategy

This section recalls the principle of nonlinear substructuring and condensation for dual and primal approach. More details can be found in [16].

Nonlinear problem (1) is decomposed into N_s nonlinear subproblems:

$$f_{int}^{\diamondplus}(u^{\diamondplus}) + f_{ext}^{\diamondplus} + t^{\diamondminus^T}\lambda_b^{\diamondplus} = 0^{\diamondplus} \tag{5}$$

where $\lambda_b^{(s)}$ is the unknown local interface nodal reaction, introduced to represent interactions of subdomain $\Omega^{(s)}$ with neighboring subdomains.

Transmission conditions – continuity of displacements and balance of reactions – hold:

$$\begin{cases} B^{\diamondminus} u_b^{\diamondplus} = 0 \\ A^{\diamondminus}\lambda_b^{\diamondplus} = 0 \end{cases} \tag{6}$$

In the following we make an assumption equivalent to the small strain hypothesis in mechanics: for any subdomain $\Omega^{(s)}$ lacking Dirichlet conditions (aka floating subdomain), there exists a basis $R^{(s)}$ of rigid body motions which satisfies:

$$\forall u^{(s)}, \begin{cases} f_{int}^{(s)}(u^{(s)} + R^{(s)}\alpha^{(s)}) = f_{int}^{(s)}(u^{(s)}), \ \forall \alpha^{(s)} \\ R^{(s)^T} f_{int}^{(s)}(u^{(s)}) = 0 \end{cases} \tag{7}$$

The basis of rigid body motions is directly linked to the kernel of the tangent matrix:

$$R^{(s)} = \mathrm{Ker}\left(K_t^{(s)}\right), \text{ with } K_t^{(s)} = \frac{\partial f_{int}^{(s)}}{\partial u^{(s)}}$$

2.2.1 Dual approach

Formulation of the condensed problem The dual formulation consists in defining the main interface unknown as λ_B, i.e. a balanced quantity (interface nodal reaction) across subdomains. Unknowns $u^{(s)}$ are, under this assumption, a priori not continuous between neighbors (until convergence).

Under hypothesis (7), if the global problem is well-posed, we can assume the existence, at least locally, of a nonlinear dual analogue $F_{nl}^{\diamondplus}$ to the dual Schur complement (i.e. a discrete Neumann-to-Dirichlet operator). The solution $u^{(s)}$ of each local equilibrium is then defined uniquely, up to a rigid body mode $R^{(s)}\alpha^{(s)}$, and interface local unknowns can be expressed as:

$$u_b^{\diamondplus} = F_{nl}^{\diamondplus}\left(\lambda_b^{\diamondplus} = B^{\diamondminus^T}\lambda_B; f_{ext}^{\diamondplus}\right) + t^{\diamondminus}R^{\diamondminus}\alpha^{\diamondplus} \tag{8}$$

An admissibility condition is also imposed on $B^{\diamondminus^T}\lambda_B$:

$$R^{\diamondminus^T} f_{int}^{\diamondplus} = 0 \quad \Rightarrow \quad R^{\diamondminus^T}\left(f_{ext}^{\diamondplus} + t^{\diamondminus^T} B^{\diamondminus^T}\lambda_B\right) = 0$$

Property 1. The tangent operator $F_t^{\diamondsuit}$ to $F_{nl}^{\diamondsuit}$ can be explicitly computed in function of the tangent stiffness $K_t^{\diamondsuit}$ [16]:

$$F_t^{\diamondsuit} = \frac{\partial F_{nl}^{\diamondsuit}}{\partial \lambda_b^{\diamondsuit}} = t^{\diamondsuit} \left(K_t^{\diamondsuit}\right)^{\dagger} t^{\diamondsuit^T} \tag{9}$$

Moreover, in the linear case, the Neumann-to-Dirichlet operator – written $F_l^{\diamondsuit}$ – is affine, with the constant term associated with external forces:

$$\begin{aligned} F_l^{\diamondsuit}\left(\lambda_b^{\diamondsuit}; f_{ext}^{\diamondsuit}\right) &= F_t^{\diamondsuit}\lambda_b^{\diamondsuit} + b_d^{\diamondsuit} \\ \text{with } b_d^{\diamondsuit} &= t^{\diamondsuit}\left(K^{\diamondsuit}\right)^{\dagger} f_{ext}^{\diamondsuit} \end{aligned} \tag{10}$$

Finally, writing $d^{(s)}$ the number of rigid body modes of each subdomain, interface condensed problem reads:

$$\text{Find } \lambda_B \in \mathbb{R}^{n_B}, \alpha^{\diamondsuit} \in \mathbb{R}^{\sum_s d^{(s)}} \text{ such that :}$$

$$\begin{cases} g_B(\lambda_B, \alpha^{\diamondsuit}) \equiv B^{\diamondsuit}\left(F_{nl}^{\diamondsuit}\left(B^{\diamondsuit^T}\lambda_B; f_{ext}^{\diamondsuit}\right) + t^{\diamondsuit}R^{\diamondsuit}\alpha^{\diamondsuit}\right) = 0 & (11) \\ R^{\diamondsuit^T}\left(f_{ext}^{\diamondsuit} + t^{\diamondsuit^T}B^{\diamondsuit^T}\lambda_B\right) = 0 & (12) \end{cases}$$

Admissibility condition In practice, the solution λ_B to the dual interface condensed problem (11) is sought iteratively in the admissible affine space defined by (12). As in classical linear FETI method, we compute an adapted initialization λ_{B_0} and look for the remaining part in $\mathrm{Ker}\left(R_b^{\diamondsuit^T}B^{\diamondsuit^T}\right)$ using a projector P_B. The unknown thus takes the following form:

$$\lambda_B = \lambda_{B_0} + P_B\tilde{\lambda}_B \text{ with: } \begin{cases} R^{\diamondsuit^T}\left(f_{ext}^{\diamondsuit} + t^{\diamondsuit^T}B^{\diamondsuit^T}\lambda_{B_0}\right) = 0, \\ R_b^{\diamondsuit^T}B^{\diamondsuit^T}P_B = 0 \end{cases}$$

In practice, if we note $R_B = B^{\diamondsuit}R_b^{\diamondsuit}$, we use the following expressions:

$$\begin{cases} P_B = I - QR_B\left(R_B^TQR_B\right)^{-1}R_B^T \\ \lambda_{B_0} = -QR_B\left(R_B^TQR_B\right)^{-1}R^Tf_{ext}^{\diamondsuit} \end{cases} \tag{13}$$

where $Q = \tilde{B}^{\diamondsuit}Q^{\diamondsuit}\tilde{B}^{\diamondsuit^T}$ is a SPD matrix homogeneous to the (linearized) preconditioner or any of its approximations (Q is homogeneous to a boundary stiffness matrix). It is crucial to note that the matrix Q does not need to be updated during the nonlinear resolution (usually it is either never updated

or updated at the beginning of the increments of the Newton solver, $Q = I$ is even a classical choice).

For any $\tilde{\lambda}_B$ (even inexact), the magnitude $\alpha^{\diamond}$ of the rigid body motions can be computed by minimizing the Q-norm of the residual g_B, which leads to:

$$\alpha^{\diamond} = (R_B^T Q R_B)^{-1} R_B^T Q B^{\diamond} F_{nl}^{\diamond} \left(B^{\diamond^T} (\lambda_{B_0} + P_B \tilde{\lambda}_B); f_{ext}^{\diamond} \right) \tag{14}$$

Using this expression for $\alpha^{\diamond}$ in equation (11), one recognizes the transposed projector. Finally the system to be solved can be written as:

$$\text{Find } \tilde{\lambda}_B \in \mathbb{R}^{n_B} \text{ such that: } g_B^P(\tilde{\lambda}_B) \equiv P_B^T B^{\diamond} F_{nl}^{\diamond} \left(B^{\diamond^T} (\lambda_{B_0} + P_B \tilde{\lambda}_B); f_{ext}^{\diamond} \right) = 0 \tag{15}$$

A Newton-Krylov algorithm The nonlinear substructuring and condensation method consists in solving interface problem (15) instead of global problem (1). The Newton method applied to this equation leads to, for $k \geqslant 0$:

$$\begin{cases} \text{Solve for } d\tilde{\lambda}_B : \left[\dfrac{\partial g_B^P}{\partial \tilde{\lambda}_B} (\tilde{\lambda}_{B_k}) \right] d\tilde{\lambda}_B = -g_B^P(\tilde{\lambda}_{B_k}) \\ \text{Update } \tilde{\lambda}_{B_{k+1}} = \tilde{\lambda}_{B_k} + d\tilde{\lambda}_B \end{cases} \tag{16}$$

Two steps are involved in the solving process:

(i) Independent solutions to subdomains nonlinear equilibriums (8) with Neumann boundary conditions are computed by applying local Newton algorithms, and assembled to build the nonlinear projected interface residual $g_B^P(\tilde{\lambda}_{B_k})$.

(ii) Solution to global interface tangent problem with the FETI method:

$$\frac{\partial g_B^P}{\partial \tilde{\lambda}_B} (\tilde{\lambda}_{B_k}) = P_B^T B^{\diamond} F_{t_k}^{\diamond} B^{\diamond^T} P_B$$

where $F_{t_k}^{\diamond}$ is the notation for the tangent operator at λ_{B_k} which is connected to the tangent stiffness matrix (9).

In the following, this solving process will be referred to as FETI-NL.

Remark 5. It is well known that the dual tangent system is only semi-definite because of the rank-deficiency of $B^{\diamond}$ in the presence of multiple points, which means that $d\tilde{\lambda}_B$ (and more generally λ_B) is defined up to corner modes [29], anyhow the mechanical quantity of interest $\lambda_b^{\diamond} = B^{\diamond^T} \lambda_B$ is uniquely defined.

FETI preconditioner The preconditioning step of FETI-NL algorithm is involved at the tangent level, when classical FETI algorithm comes into play.

The preconditioned projected FETI problem can be written as:

$$\left(\tilde{B}^{\diamondsuit} S_{t_k}^{\diamondsuit} \tilde{B}^{\diamondsuit^T}\right) P_B^T B^{\diamondsuit} F_{t_k}^{\diamondsuit} B^{\diamondsuit^T} P_B d\lambda_{B_k} = -\left(\tilde{B}^{\diamondsuit} S_{t_k}^{\diamondsuit} \tilde{B}^{\diamondsuit^T}\right) g_{B_k}^P \tag{17}$$

where $S_{t_k}^{\diamondsuit}$ is the primal Schur complement (see next subsection) of which the dual Schur complement is a pseudo-inverse: $F_{t_k}^{\diamondsuit} = S_{t_k}^{\diamondsuit^\dagger}$.

The choice of such a preconditioner is motivated by the quality of the approximation of the FETI inverse operator achieved by the scaled assembly (3) of local pseudo-inverses.

2.2.2 Primal approach

We here quickly recall the principle of primal formulation for nonlinear substructuration and condensation method.

Formulation of the nonlinearly condensed problem The primal formulation consists in defining the main interface unknown as u_A, i.e. a displacement which is continuous across subdomains. Nodal interface reactions $\lambda_b^{\diamondsuit}$ are, under this assumption, a priori not balanced between subdomains (until convergence).

We assume the well-posedness of local Dirichlet problems, so that we can define a nonlinear primal analogue $S_{nl}^{\diamondsuit}$ to the Schur primal complement (i.e. a discrete Dirichlet-to-Neumann operator). The solution $\lambda_b^{(s)}$ of each local equilibrium is then defined uniquely, and local interface nodal reactions can be expressed as:

$$\lambda_b^{\diamondsuit} = S_{nl}^{\diamondsuit}\left(u_b^{\diamondsuit} = A^{\diamondsuit^T} u_A; f_{ext}^{\diamondsuit}\right) \tag{18}$$

Property 2. The tangent operator $S_t^{\diamondsuit}$ to $S_{nl}^{\diamondsuit}$ can be explicitly computed in function of the tangent stiffness $K_t^{\diamondsuit}$:

$$\forall s \in \{0, \ldots, N_s\}, \quad S_t^{(s)} = \frac{\partial S_{nl}^{(s)}}{\partial u_b^{(s)}} = K_{t_{bb}}^{(s)} - K_{t_{bi}}^{(s)} K_{t_{ii}}^{(s)^{-1}} K_{t_{ib}}^{(s)} \tag{19}$$

Moreover, in the linear case, the discrete Dirichlet-to-Neumann operator, written $S_l^{\diamondsuit}$, is affine, with the constant term associated with external forces:

$$\begin{aligned} S_l^{\diamondsuit}\left(u_b^{\diamondsuit}; f_{ext}^{\diamondsuit}\right) &= S_t^{\diamondsuit} u_b^{\diamondsuit} - b_p^{\diamondsuit} \\ \text{with } b_p^{\diamondsuit} &= f_{ext_b}^{\diamondsuit} - K_{t_{bi}}^{\diamondsuit} K_{t_{ii}}^{\diamondsuit^{-1}} f_{ext_i}^{\diamondsuit} \end{aligned} \tag{20}$$

Note the link between primal and dual right-hand sides: $b_d^{\diamondsuit} = F_t^{\diamondsuit} b_p^{\diamondsuit}$.

Finally, interface condensed problem reads:

$$\begin{aligned} &\text{Find } u_A \in \mathbb{R}^{n_A} \text{ such that :} \\ &g_A(u_A) \equiv A^{\diamondsuit} S_{nl}^{\diamondsuit}\left(A^{\diamondsuit^T} u_A; f_{ext}^{\diamondsuit}\right) = 0 \end{aligned} \tag{21}$$

Newton-Krylov algorithm The strategy defined at section 2.2.1 still holds with primal approach. The tangent problem of global Newton algorithm becomes, at each iteration k:

$$\left(A^{\diamondsuit} S_{t_k}^{\diamondsuit} A^{\diamondsuit^T}\right) du_A = -A^{\diamondsuit} S_{nl}^{\diamondsuit}\left(A^{\diamondsuit^T} u_{A_k}; f_{ext}^{\diamondsuit}\right) \tag{22}$$

with $S_{t_k}^{\diamondsuit}$ the tangent primal Schur complement defined in (19). The right-hand side evaluation involves the assembly of the nodal reactions associated with the solution of local equilibriums with imposed interface displacements $A^{\diamondsuit^T} u_{A_k}$. Internal local displacements $u_{i_k}^{\diamondsuit}$ and interface nodal reactions $\lambda_{b_k}^{\diamondsuit}$ are computed in parallel, and the lack of balance of $\lambda_{b_k}^{\diamondsuit}$ is the interface nonlinear condensed residual $g_A(u_{A_k}) = A^{\diamondsuit}\lambda_{b_k}^{\diamondsuit} = A^{\diamondsuit} S_{nl}^{\diamondsuit}\left(A^{\diamondsuit^T} u_{A_k}; f_{ext}^{\diamondsuit}\right)$.

The left-hand side corresponds to the assembly of local primal Schur complements, i.e. the operator of a classical primal Schur domain decomposition method. Tangent problems are thus solved with a classical BDD algorithm.

In the following, this solving process will be referred to as BDD-NL.

BDD preconditioner The classical preconditioner makes use of a scaled assembly of local inverses, which involves solving local Neumann problems, here again an initialization/projection procedure is used:

$$\begin{aligned} &du_A = du_A^0 + P_A d\tilde{u}_A \\ &\left(\tilde{A}^{\diamondsuit} F_{t_k}^{\diamondsuit} \tilde{A}^{\diamondsuit^T}\right) P_A^T \left(A^{\diamondsuit} S_{t_k}^{\diamondsuit} A^{\diamondsuit^T}\right) P_A d\tilde{u}_A = -\left(\tilde{A}^{\diamondsuit} F_{t_k}^{\diamondsuit} \tilde{A}^{\diamondsuit^T}\right) P_A^T g_A(u_{A_k}) \end{aligned} \tag{23}$$

where:

$$\left.\begin{aligned} du_A^0 &= -\tilde{R}_A\left(\tilde{R}_A^T S_A \tilde{R}_A\right)^{-1} \tilde{R}_A^T g_A(u_{A_k}) \\ P_A &= I - \tilde{R}_A\left(\tilde{R}_A^T S_A \tilde{R}_A\right)^{-1} \tilde{R}_A^T S_A \end{aligned}\right\} \text{ with } \left\{\begin{aligned} \tilde{R}_A &= \tilde{A}^{\diamondsuit} R_b^{\diamondsuit} \\ S_A &= A^{\diamondsuit} S_{t_k}^{\diamondsuit} A^{\diamondsuit^T} \end{aligned}\right.$$

is introduced to satisfy the optimality condition required when rigid body modes exist within substructures. It is important to note that contrary to the dual case, the projector needs to be updated at each iteration because the image space of the projector, $\mathrm{Ker}\left(\tilde{R}_A^T S_A\right)$, depends on the current state of the system.

2.3 Typical algorithm

Algorithm 1 sums up the main steps of the method with the dual nonlinear local problems, and FETI tangent solver. For simplicity reasons, only one load increment was considered. A similar algorithm can be written for the primal approach.

Algorithm 1 FETI-NL

define:
$r_{nl}^{d\,\diamondplus}\left(u^{\diamondplus},\, \lambda_b^{\diamondplus}\right) = f_{int}^{\diamondplus}\left(u^{\diamondplus}\right) + f_{ext}^{\diamondplus} + t^{\diamondminus T}\lambda_b^{\diamondplus}$
initialization: $u_{00}^{\diamondplus},\, \lambda_{b00}^{\diamondplus}$ given
$K_{t_{00}}^{\diamondminus} = f'^{\diamondminus}_{int}\left(u_{00}^{\diamondplus}\right),\, P_B = I - QR_B\left(R_B^T QR_B\right)^{-1} R_B^T$
admissibility condition:
$\lambda_{B_0} = -QR_B\left(R_B^T QR_B\right)^{-1} R^T r_{nl}^{d\,\diamondplus}\left(u_{00}^{\diamondplus},\, \lambda_{b00}^{\diamondplus}\right)$
local linear initialization step:
$u_0^{\diamondplus} = -K_{t_{00}}^{\diamondminus\;-1} r_{nl}^{d\,\diamondplus}\left(u_{00}^{\diamondplus},\, B^{\diamondminus T}\lambda_{B_0}^{\diamondplus}\right)$
assemble $b_{B_0}^d = -B^{\diamondminus} t^{\diamondminus} u_0^{\diamondplus}$
set $k = 0$
while $\|b_{B_k}^d\| > \varepsilon_{NG}$ **do**
  tangent problem:
  set $m = 0$ and $d\lambda_{B_{k,m}} = 0$
  while $\|P_B^T b_{B_k}^d - P_B^T\left(B^{\diamondminus} F_{t_{k,m}}^{\diamondminus} B^{\diamondminus T}\right) P_B d\lambda_{B_{k,m}}\| > \varepsilon_K$ **do**
    FETI iterations with index m
  compute $u_{k+1}^{\diamondplus} = u_k^{\diamondplus} - K_{t_{k,m}}^{\diamondminus\;-1} t^{\diamondminus T} B^{\diamondminus T} P_B d\lambda_{B_{k,m}}$
  update $\lambda_{B_{k+1}} = \lambda_{B_k} + P_B d\lambda_{B_{k,m}}$
  increment $k = k + 1$
  local nonlinear step:
  set $j = 0$ and $u_{k,j}^{\diamondplus} = u_k^{\diamondplus}$
  while $\|r_{nl}^{d\,\diamondplus}\left(u_{k,j}^{\diamondplus},\, \lambda_{b_k}^{\diamondplus}\right)\| > \varepsilon_{NL}^{\diamondplus}$ **do**
    $u_{k,j+1}^{\diamondplus} = u_{k,j}^{\diamondplus} - K_{t_{k,j}}^{\diamondminus\;-1} r_{nl}^{d\,\diamondplus}\left(u_{k,j}^{\diamondplus},\, \lambda_{b_k}^{\diamondplus}\right)$
    increment $j = j + 1$
  assemble $b_{B_k}^d = B^{\diamondminus} t^{\diamondminus} u_{k,j}^{\diamondplus}$

3 Nonlinear preconditioner for the FETI-NL solver

The aim of this section is to define and develop a new preconditioning strategy for the nonlinear interface condensed problem. The purpose is to have a formulation where the tangent operator is directly well conditioned (for regular problems) without resorting to a linear preconditioner.

The method we derive is inspired from the analysis of the role of scaling operators in [26] which lead to techniques for the parallel recovery of admissible fields in [30]. The idea is to reinterpret FETI as a (nonlinear) fixed point method on the interface, which acts as a nonlinear preconditioner for the FETI-NL method.

For sake of clarity, the dependency of nonlinear operators $F_{nl}^{\diamondplus}$ and $S_{nl}^{\diamondplus}$ on external load $f_{ext}^{\diamondplus}$ will be made implicit in the following. Reference to the current iteration number k will not be reminded either.

3.1 A nonlinear fixed point

Let $\lambda_B = \lambda_{B_0} + P_B \tilde{\lambda}_B$ be an interface reaction field balanced with respect to rigid body motions. We define the displacement resulting from the solving of local Neumann problems: We introduce the following notation for nonlinear local solutions with dual formulation, i.e. discontinuous displacements:

$$u_b^{\diamondplus} = F_{nl}^{\diamondplus}\left(B^{\diamondminus^T}\lambda_B\right) + R_b^{\diamondminus}\alpha^{\diamondplus}$$

$u_b^{\diamondplus}$ is not continuous ($B^{\diamondminus}u_b^{\diamondplus} \neq 0$), except if λ_B is the solution to the FETI-NL system. A continuous displacement can be build from $u_b^{\diamondplus}$ by substracting the scaled interface jump:

$$\hat{u}_b^{\diamondplus} \equiv \left[I - \tilde{B}^{\diamondminus^T} B^{\diamondminus}\right] u_b^{\diamondplus} \qquad \Rightarrow \qquad B^{\diamondminus}\hat{u}_b^{\diamondplus} = 0 \tag{24}$$

Note that using the expression of $\alpha^{\diamondplus}$ given in (14) we have:

$$\begin{aligned} B^{\diamondminus}F_{nl}^{\diamondplus}\left(B^{\diamondminus^T}\lambda_B\right) + B^{\diamondminus}R_b^{\diamondminus}\alpha^{\diamondplus} &= P_B^T B^{\diamondminus} F_{nl}^{\diamondplus}\left(B^{\diamondminus^T}\lambda_B\right) \\ \Rightarrow \hat{u}_b^{\diamondplus} &= \left[I - \tilde{B}^{\diamondminus^T} P_B^T B^{\diamondminus}\right] F_{nl}^{\diamondplus}\left(B^{\diamondminus^T}\lambda_B\right) + R_b^{\diamondminus}\alpha^{\diamondplus} \end{aligned} \tag{25}$$

The continuous displacement $\hat{u}_b^{\diamondplus}$ can be used as an input to local Dirichlet problems, themselves leading to non-balanced reactions which can be balanced by substracting the scaled interface lack of balance:

$$\hat{\lambda}_b^{\diamondplus} \equiv \left[I - \tilde{A}^{\diamondminus^T} A^{\diamondminus}\right] S_{nl}^{\diamondplus}\left(\hat{u}_b^{\diamondplus}\right) \qquad \Rightarrow \qquad A^{\diamondminus}\hat{\lambda}_b^{\diamondplus} = 0 \tag{26}$$

If λ_B is converged, $u_b^{\diamondsuit}$ is continuous; we then have $\hat{u}_b^{\diamondsuit} = u_b^{\diamondsuit}$ and:

$$S_{nl}^{\diamondsuit}\left(\hat{u}_b^{\diamondsuit}\right) = S_{nl}^{\diamondsuit}\left(F_{nl}^{\diamondsuit}\left(B^{\diamondsuit^T}\lambda_B\right) + R_b^{\diamondsuit}\alpha^{\diamondsuit}\right) = S_{nl}^{\diamondsuit}\left(F_{nl}^{\diamondsuit}\left(B^{\diamondsuit^T}\lambda_B\right)\right) = B^{\diamondsuit^T}\lambda_B$$

We here use the hypothesis formulated in (7) (equivalent to the small strain hypothesis in mechanics) to remove rigid body motions, and the fact that primal and dual condensed operator are pseudo-inverse of each other.

Thus, at convergence, and using (4) in order to get rid of the primal assembly operators, one has:

$$\hat{\lambda}_b^{\diamondsuit} = B^{\diamondsuit^T}\tilde{B}^{\diamondsuit}S_{nl}^{\diamondsuit}\left(u_b^{\diamondsuit}\right) = B^{\diamondsuit^T}\tilde{B}^{\diamondsuit}B^{\diamondsuit^T}\lambda_B = B^{\diamondsuit^T}\lambda_B$$

We consequently define the following nonlinear operator:

$$H_{nl,d}^{\diamondsuit}\left(B^{\diamondsuit^T}\lambda_B\right) \equiv B^{\diamondsuit^T}\tilde{B}^{\diamondsuit}S_{nl}^{\diamondsuit}\left(\left[I - \tilde{B}^{\diamondsuit^T}P_B^T B^{\diamondsuit}\right]F_{nl}\left(B^{\diamondsuit^T}\lambda_B\right)\right) - B^{\diamondsuit^T}\lambda_B \quad (27)$$

and a new interface condensed problem can be defined in the shape of a fixed point system:

$$\text{Find } \tilde{\lambda}_B \text{ such that} \qquad H_{nl,d}^{\diamondsuit}\left(B^{\diamondsuit^T}(\lambda_{B_0} + P_B\tilde{\lambda}_B)\right) = 0 \qquad (28)$$

Remark 6. In the linear case, a fixed point system can also be derived for the primal approach in term of a continuous displacement u_A. In the nonlinear case, the difficulty is due to the handling of rigid body motions: in order to lead to well-posed Neumann problems, the displacement should be such that $\tilde{R}_A^T A^{\diamondsuit} S_{nl}^{\diamondsuit}\left(A^{\diamondsuit^T}u_A\right) = 0$ (nonlinear equivalent of the BDD-optimality condition). This equation characterizes a manifold, which is much more complex to handle than the constant affine space of the dual approach.

Linear case We simplify the fixed point in the case of a linear system, using the explicit expressions of the Dirichlet-to-Neumann and Neumann-to-Dirichlet operators given in equations (20,10), as well as the link between the primal and dual right-hand sides.

$$\begin{aligned}
H_{l,d}^{\diamondsuit}\left(B^{\diamondsuit^T}\lambda_B\right) &= B^{\diamondsuit^T}\tilde{B}^{\diamondsuit}S_l^{\diamondsuit}\left(\left[I - \tilde{B}^{\diamondsuit^T}P_B^T B^{\diamondsuit}\right]F_l^{\diamondsuit}\left(B^{\diamondsuit^T}\lambda_B\right)\right) - B^{\diamondsuit^T}\lambda_B \\
&= B^{\diamondsuit^T}\tilde{B}^{\diamondsuit}\left(S_t^{\diamondsuit}\left[I - \tilde{B}^{\diamondsuit^T}P_B^T B^{\diamondsuit}\right]\left(F_t^{\diamondsuit}B^{\diamondsuit^T}\lambda_B + b_d^{\diamondsuit}\right) - b_p^{\diamondsuit}\right) - B^{\diamondsuit^T}\lambda_B \\
&= -B^{\diamondsuit^T}\tilde{B}^{\diamondsuit}S_t^{\diamondsuit}\tilde{B}^{\diamondsuit^T}P_B^T B^{\diamondsuit}\left(F_t^{\diamondsuit}B^{\diamondsuit^T}\lambda_B + b_d^{\diamondsuit}\right) \\
&\qquad + B^{\diamondsuit^T}\left(\tilde{B}^{\diamondsuit}\left(S_t^{\diamondsuit}F_t^{\diamondsuit}(B^{\diamondsuit^T}\lambda_B + b_p^{\diamondsuit}) - b_p^{\diamondsuit}\right) - \lambda_B\right) \\
&= -B^{\diamondsuit^T}\tilde{B}^{\diamondsuit}S_t^{\diamondsuit}\tilde{B}^{\diamondsuit^T}P_B^T B^{\diamondsuit}\left(F_t^{\diamondsuit}B^{\diamondsuit^T}\lambda_B + b_d^{\diamondsuit}\right)
\end{aligned}$$

where we used the duality between the Schur complements and the definition of the scaled assembly matrices.

One recognizes the projected preconditioned FETI system, multiplied on the left by $B^{\diamondsuit^T}$ – we recall that this operation converts the partially undefined traction between subdomains (because of redundancy at multiple points in Γ_B) into the well defined mechanical effort applied to subdomains (in $(\Gamma^{(s)})$).

Remark 7. Even if the preconditioned FETI system takes the form of a fixed point method, it is impossible to apply a stationary iteration. It is indeed well known that the operator is not a contraction; more precisely the spectrum is bounded from below by 1 [31].

3.2 Newton method applied to the fixed point system

Since a stationary iteration is not expected to converge, we propose to use a Newton solver. The method is motivated by the following relation obtained using the chain rule:

$$\frac{\partial H^{\diamondsuit}_{nl,d}\left(B^{\diamondsuit^T}(\lambda_{B_0}+P_B\tilde{\lambda}_B)\right)}{\partial\tilde{\lambda}_B} = B^{\diamondsuit^T}\left(\tilde{B}^{\diamondsuit}S^{\diamondsuit}_t\left[I-\tilde{B}^{\diamondsuit^T}P_B^TB^{\diamondsuit}\right]F^{\diamondsuit}_tB^{\diamondsuit^T}-I\right)P_B \tag{29}$$

Note that $F^{\diamondsuit}_t$ is computed from the subdomains' tangent matrix evaluated at $\lambda^{\diamondsuit}_b = B^{\diamondsuit^T}\lambda_B$ whereas $S^{\diamondsuit}_t$ is computed from the subdomains' tangent matrix evaluated at $[I-\tilde{B}^{\diamondsuit^T}P_B^TB^{\diamondsuit}]F_{nl}(B^{\diamondsuit^T}\lambda_B)$.

It can be convenient to adopt a modified Newton strategy where $F^{\diamondsuit}_t$ and $S^{\diamondsuit}_t$ are computed from the same configuration. In that case we have $F^{\diamondsuit}_t = S^{\diamondsuit^\dagger}_t$ and the following simplification holds:

$$\frac{\partial H^{\diamondsuit}_{nl,d}\left(B^{\diamondsuit^T}(\lambda_{B_0}+P_B\tilde{\lambda}_B)\right)}{\partial\tilde{\lambda}_B} \simeq -B^{\diamondsuit^T}\left(\tilde{B}^{\diamondsuit}S^{\diamondsuit}_t\tilde{B}^{\diamondsuit^T}P_B^TB^{\diamondsuit}F^{\diamondsuit}_tB^{\diamondsuit^T}P_B\right) \tag{30}$$

where we recognize a preconditioned FETI system which is well conditioned – at least for sufficiently regular problems and decompositions; coupling with robustification techniques [32, 33] will be considered in future studies.

If we consider the kth iteration, $\lambda_{B_k} = \lambda_{B_0} + P_B\tilde{\lambda}_{B_k}$, of a Newton method applied to (28), we have:

$$\frac{\partial H^{\diamondsuit}_{nl,d}\left(B^{\diamondsuit^T}\lambda_{B_k}\right)}{\partial\tilde{\lambda}_B}d\tilde{\lambda}_B = -H^{\diamondsuit}_{nl,d}\left(B^{\diamondsuit^T}\lambda_{B_k}\right) \text{ and } \lambda_{B_{k+1}} = \lambda_{B_k} + P_Bd\tilde{\lambda}_B$$

From the expression (30) of the tangent operator and the definition (27) of nonlinear operator $H^{\diamondsuit}_{nl}$, one can notice that $B^{\diamondsuit^T}$ can be put in factor on the

left of the equation above:

$$B^{\diamondsuit^T}\underbrace{\left(\tilde{B}^{\diamondsuit} S_t^{\diamondsuit} \tilde{B}^{\diamondsuit^T} P_B^T B^{\diamondsuit} F_t^{\diamondsuit} B^{\diamondsuit^T} P_B\right)}_{-H_{t,B}} d\tilde{\lambda}_B = \dots$$

$$\dots B^{\diamondsuit^T}\underbrace{\left[\tilde{B}^{\diamondsuit} S_{nl}^{\diamondsuit}\left(\left[I - \tilde{B}^{\diamondsuit^T} P_B^T B^{\diamondsuit}\right] F_{nl}\left(B^{\diamondsuit^T}\lambda_{B_k}\right)\right) - \lambda_{B_k}\right]}_{H_{nl,B}}$$

Thanks to Remark 4, the remaining terms $H_{t,B}$ and $H_{nl,B}$ do not belong to $\mathrm{Ker}(B^{\diamondsuit^T})$, we can thus directly consider the system:

$$H_{t,B}\, d\tilde{\lambda}_B = -H_{nl,B}\left(\lambda_{B_k}\right)$$

The left-hand side is a typical preconditioned FETI operator, it possesses solutions and the mechanical quantity $B^{\diamondsuit^T}\lambda_{B_{k+1}}$ is uniquely defined.

The Newton system has the following properties:

- The right-hand side is not classical in the sense that it does not take the form of a (linearly) preconditioned residual. It is thus not easy to fully benefit from the symmetry of the operators, in particular the system can not be solved by a projected preconditioned conjugate gradient. Even if smarter strategies might have been applied, we chose to use a GMRes solver applied to the whole preconditioned system.

- The right-hand side is the composition of two nonlinear local operators: all subdomain first solve independent nonlinear Neumann systems, then there is one all-neighbor communication (application of $B^{\diamondsuit}$) and a coarse projection, then all subdomains solve independent nonlinear Dirichlet problems and there is another all-neighbor communication (application of $\tilde{B}^{\diamondsuit}$).

Compared to the FETI-NL strategy, the tangent operator is naturally well conditioned, and the right-hand side is computed by two nonlinear solves instead of one (and two assemblies instead of one). This solving process will be refered to as FETI-precNL in the following.

3.3 Equivalence between classical and nonlinearly preconditioned problems

By construction, the fixed point is attained when the dual system is solved:

$$g_B^P(\tilde{\lambda}_B) = 0 \qquad \Rightarrow \qquad H_{nl,d}^{\diamondsuit}(B^{\diamondsuit^T}(\lambda_{B_0} + P_B\tilde{\lambda}_B)) = 0$$

Conversely, we can prove that if $\tilde{\lambda}_B$ is such that $H^{\diamondplus}_{nl,d}(B^{\diamondminus^T}(\lambda_{B_0}+P_B\tilde{\lambda}_B)) = 0$ then, at least locally, $\tilde{\lambda}_B$ is the unique solution to $g^P_B(\tilde{\lambda}_B) = 0$ (up to a term in $\mathrm{Ker}(R^T_B)$). Indeed, let us define:

$$u^{\diamondplus}_b = F^{\diamondplus}_{nl}\left(B^{\diamondminus^T}\lambda_B\right)$$

The fixed point can be written as – note that potential rigid body motions cancel out in the following computations:

$$B^{\diamondminus^T}\tilde{B}^{\diamondminus}S^{\diamondplus}_{nl}\left(u^{\diamondplus}_b - \tilde{B}^{\diamondminus^T}P^T_B B^{\diamondminus}u^{\diamondplus}_b\right) = S^{\diamondplus}_{nl}\left(u^{\diamondplus}_b\right)$$

Premultiplying by $\tilde{B}^{\diamondminus}$ and using (2), we have:

$$\tilde{B}^{\diamondminus}\left(S^{\diamondplus}_{nl}\left(u^{\diamondplus}_b - \tilde{B}^{\diamondminus^T}P^T_B B^{\diamondminus}u^{\diamondplus}_b\right) - S^{\diamondplus}_{nl}\left(u^{\diamondplus}_b\right)\right) = 0 \tag{31}$$

Using a first order Taylor series, we have:

$$\tilde{B}^{\diamondminus}S^{\diamondminus}_t\tilde{B}^{\diamondminus^T}P^T_B B^{\diamondminus}u^{\diamondplus}_b + o\left(\tilde{B}^{\diamondminus^T}P^T_B B^{\diamondminus}u^{\diamondplus}_b\right) = 0$$

Being given the symmetry, positiveness and semi-definiteness properties satisfied by matrix $\left(\tilde{B}^{\diamondminus}S^{\diamondminus}_t\tilde{B}^{\diamondminus^T}\right)$, the following implication holds:

$$H^{\diamondplus}_{nl,d}\left(B^{\diamondminus^T}\lambda_B\right) = 0 \Rightarrow P^T_B B^{\diamondminus}F^{\diamondplus}_{nl}\left(B^{\diamondminus^T}\lambda_B\right) = 0$$

4 Assessments

4.1 Water diffusion in soils

This test-case is inspired from the standard Polmann case [34, 35]: the problem to solve is the diffusion in two directions of water in a column of soil (see figure 2a). Geometrical parameters are given in table 1. Initial pressure field is taken homogeneous:

$$h(x, t < 0) = h_{D_0}$$

At time $t = 0$, a pressure $h = h_{D_1}$ is imposed on the edge defined by $y = L$, and $h = h_{D_2}$ on the edge defined by $(x = 0,\ y \in [3L/8; L/2])$. Remaining outer walls are chosen to be impermeable (null flow). Load parameters are given on table 1.

The Richards equation for water diffusion is considered here in its classical formulation:

Find scalar pressure field $h(x,t)$ such that:

$$\frac{\partial\theta(h)}{\partial t} - \nabla.\left(K(h)\nabla(h-z)\right)$$

where:

Geometrical parameters	
L	1
W	0.2
Van Genuchten parameters	
θ_r	0.368
θ_s	0.102
K_s	9.22×10^{-5}
n	2
m	0.5
Load parameters	
h_{D_0}	-10
h_{D_1}	-1.5
h_{D_2}	-1.5

Table 1: Geometrical, Van Genuchten and load parameters for the soil column

- $\theta(h)$ is the volumic water content:

$$\theta(h) = \theta_r + (\theta_s - \theta_r) S(h)$$

  with θ_r, θ_s the residual and maximal water contents, and S the saturation degree.

- $K(h)$ is the soil hydraulic conductivity:

$$K(h) = K_s K_r(h)$$

  with K_s, K_r the intrinsic and relative conductivities.

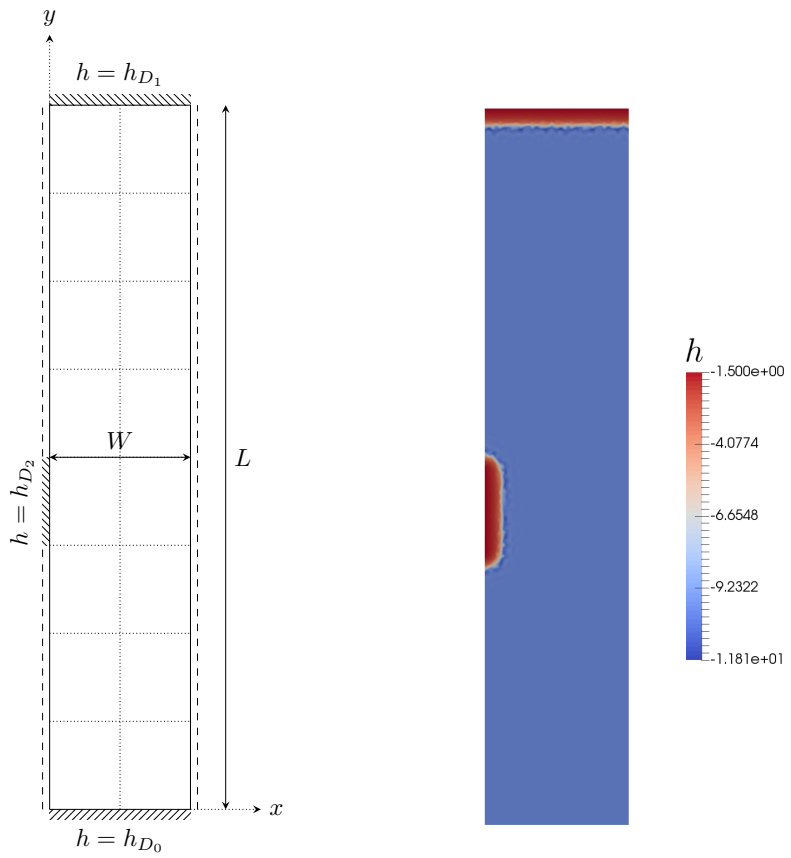
Van Genuchten model is used for the saturation degree and the relative conductivity:

$$\begin{cases} S(h) = \left(1 + \varepsilon|h|^n\right)^{-m} \\ K_r(h) = \left(1 - (\varepsilon|h|)^{n-1}\left(1 + (\varepsilon|h|)^n\right)^{-m}\right)^2 (S(h))^{1/2} \end{cases}$$

Material parameters are also given in table 1.

Remark 8. Richards equation does not involve rigid body modes within subdomains. This issue is adressed in next assessment.

For the time integration, a simple discrete implicit Euler scheme is used. Time step is taken equal to: $\Delta t = 20s$. Spatial discretization involves 5244 degrees of freedom. Substructuration involves 16 subdomains and 333 interface degrees of freedom (see figure 2a.



(a) Soil column

(b) Soil column with water diffusion at $t = 1000s$

Figure 2: Richards equation: soil column with water diffusion in two directions

Relative precision of each solver is set to $\varepsilon_{rel} = 10^{-6}$, and absolute precision to $\varepsilon_{abs} = 10^{-20}$.

The three following methods are compared:

- BDD-NL
- FETI-NL
- FETI-precNL

Comparison is performed on the three different imbricated solvers iterations numbers: global Newton iterations (cumulated over time steps), Krylov it-

erations (cumulated over global Newton iterations and time steps) and local Newton iterations (being given the parallelism of these solves, at each global Newton loop, the maximum over subdomains of local Newton iterations is stored; this value is then cumulated over global Newton iterations and time steps).

Results are given in tables 2, 3 and 4, in terms of global Newton, Krylov and local Newton iterations numbers respectively. Figure 2b shows the state of the column after 1000 seconds.

On this problem, BDD-NL and FETI-NL method were quite equivalent (see tables 2 and 3: at last time step, a total of 1535 cumulated Krylov iterations was recorded for both methods, while global Newton cumulated iterations numbers reached 311 and 325 for primal and dual approach respectively), with a smaller local cost for primal approach (at last time step, only 2559 cumulated local iterations were needed, versus 3173 for dual approach). FETI-precNL method was however way more efficient than these two approaches, with a total of only 179 cumulated global Newton iterations at last time step (see table 2). The corresponding gains, in terms of global Newton iterations, compared to BDD-NL (resp. FETI-NL) method, are comprised between 42 and 57% (resp. 45 and 62%) on the whole resolution. For Krylov cumulated iterations (see table 3), gains of FETI-precNL solver are comprised between 46 and 59% versus primal approach, and vary from 46 to 62% versus dual approach. The gain of FETI-precNL solver, compared to BDD-NL and FETI-NL methods, depends on the intensity of the nonlinearity during each time step: as time increases, the speed of the diffusion decreases, resulting in a slow decrease of the FETI-precNL gain.

The cost of nonlinear preconditioning, i.e. additional local nonlinear iterations (in parallel), is evaluated in table 4 by the ratios of local Newton iterations numbers for FETI-precNL solver over other methods: a maximum of 1.41 ratio is reached at the end of the resolution for BDD-NL method (1.14 for FETI-NL method). This additional cost is expected to be much less expensive than the decrease of about 50% in cumulated Krylov iterations (i.e., communications between processors).

Remark 9. Despite the relative smallness of the test cases presented here, we expect them to be representative of computations on larger structures. Unfortunately our code did not allow large scale computations, and thus meaningful time measurements. Limiting communications as we try to do should be even more appreciable on computations involving many processors: the number of Krylov iterations seems here to be the fairest and most reliable performance measurement.

A comparison is also made with classical NKS solving process with BDD

# Global Newton cumulated iterations					
Time	BDD-NL	FETI-NL	FETI-precNL	Gains of FETI-precNL (%)	
				vs. BDD-NL	vs. FETI-NL
20	8	9	4	50	56
40	16	17	7	56	59
60	25	27	11	56	59
80	35	39	15	57	62
100	45	51	20	56	61
200	83	94	42	49	55
300	115	129	58	50	55
400	147	159	73	50	54
500	174	184	88	49	52
600	201	211	103	49	51
700	231	241	121	48	50
800	261	271	141	46	48
900	286	300	161	44	46
1000	311	325	179	42	45

Table 2: Richards equation - Global iterations

algorithm. Results are given in table 5 for the three involved algorithms (global Newton, Krylov and local Newton). For each time step, local Newton iterations numbers are equal, for classical NKS method, to global Newton iterations numbers plus one (the last one is required to initialize the global solver). Gains are comprised, for global Newton and Krylov solver, between 51 and 82%, over the whole resolution. The ratio between local iterations is close to 10 at the beginning of the resolution and then slowly decreases, a cost which should be largely balanced by the 80% gain in Krylov iterations numbers (i.e. communications between processors).

4.2 Nonlinear thermal problem

A second numerical test involves a symmetrized stationary nonlinear thermal behavior, described by the following partial differential equation in a domain Ω:

$$r_{vol} - \nabla \cdot \left(-K\left(T\right)\nabla T\right) = 0$$

with T the temperature, r_{vol} a volumic source of heat, and $K\left(T\right)$ the thermal conductivity. Nonlinearity comes from the dependance of K to temperature

# Krylov cumulated iterations					
Time	BDD-NL	FETI-NL	FETI-precNL	Gains of FETI-precNL (%)	
				vs. BDD-NL	vs. FETI-NL
20	32	37	16	50	57
40	64	69	29	55	58
60	107	114	47	56	59
80	157	169	65	59	62
100	207	225	89	57	60
200	397	417	189	52	55
300	557	569	263	53	54
400	715	714	338	53	53
500	850	839	413	51	51
600	985	974	488	50	50
700	1135	1123	573	50	49
800	1285	1269	663	48	48
900	1410	1410	753	47	47
1000	1535	1535	835	46	46

Table 3: Richards equation - Krylov iterations

T, which is usually set to a linear or a power law:

$$K(T) = 1 + \alpha T \quad \text{or} \quad K(T) = (1 + T)^{\alpha} \tag{32}$$

where α is a real number, parameter of the conductivity law. In the context of the nonlinear preconditioner for FETI-NL solver, in order to satisfy property (7), we chose a slightly different expression, where K depends on the gradient of T, in the spirit of linear elasticity:

$$K(T) = (1 + \nabla T \cdot \nabla T)^{\alpha}$$

Domain Ω is rectangular with Dirichlet boundary conditions (see figure 3). Geometry and load parameters are given in Table 6. Different levels of non-linearity are considered, via an incremental variation of the parameter α, which is sampled between $\alpha = 0$ (linear case) and $\alpha_{max} = 0.8$.

The spatial discretization involves 5244 degrees of freedom. Substructuration involves 16 subdomains and 333 interface degrees of freedom (see figure 3).

The numbers of iterations of the three imbricated solvers are compared for BDD-NL, FETI-NL and FETI-precNL methods: global Newton iterations (for each value of parameter α), Krylov iterations (cumulated over global

# Local Newton cumulated iterations					
Time	BDD-NL	FETI-NL	FETI-precNL	Ratios of FETI-precNL	
				over BDD-NL	over FETI-NL
20	224	249	241	1,08	0,97
40	366	393	364	0,99	0,93
60	477	529	485	1,02	0,92
80	595	714	618	1,04	0,87
100	700	899	769	1,10	0,86
200	1085	1429	1354	1,25	0,95
300	1359	1783	1706	1,26	0,96
400	1576	2020	1981	1,26	0,98
500	1763	2211	2256	1,28	1,02
600	1939	2402	2505	1,29	1,04
700	2113	2604	2779	1,32	1,07
800	2283	2815	3079	1,35	1,09
900	2423	3011	3366	1,39	1,12
1000	2559	3173	3608	1,41	1,14

Table 4: Richards equation - Local iterations

Newton loops) and local Newton iterations (cumulated over global Newton loops). Results are given in tables 7 and 8. Figure 4 shows the state of the structure for $\alpha = 0.8$.

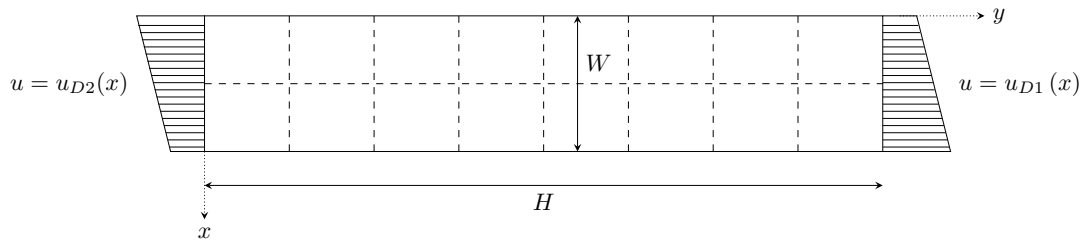


Figure 3: Nonlinear thermal problem

A first observation can be made by comparing BDD-NL and FETI-NL methods. On this test case, the primal approach is (moderately) more performant than the dual one: indeed, except for the case where the highest value of parameter α is considered ($\alpha_{max} = 0.8$), the total number of Krylov

# Global Newton cumulated iterations			
Time	NKS	FETI-precNL	Gains of FETI-precNL (%)
20	21	4	81
100	72	20	72
200	117	42	64
400	189	73	61
700	279	121	57
1000	368	179	51

# Krylov cumulated iterations			
Time	NKS	FETI-precNL	Gains of FETI-precNL (%)
20	84	16	81
100	321	89	72
200	546	189	65
400	906	338	63
700	1356	573	58
1000	1801	835	54

# Local Newton cumulated iterations			
Time	NKS	FETI-precNL	Ratio of FETI-precNL over NKS
20	22	241	10.95
100	77	769	9.99
200	127	1354	10.66
400	209	1981	9.48
700	314	2779	8.85
1000	418	3608	8.63

Table 5: Richards equation - Comparison with NKS method

iterations required at convergence is on average 20% inferior for primal approach.

Despite the (moderate) advantage of primal formulation which seems to prevail for this nonlinear thermal behavior, FETI-precNL solver is, as for previous test case, more performant than BDD-NL and FETI-NL methods: gains, in terms of global Newton iterations, are comprised between 25 and 40%, depending on the rate of nonlinearity (parameter α) inside subdomains. In terms of Krylov iterations numbers, gains vary from 13 to 35%, while the cost of nonlinear preconditioning (except from the linear case) gives a ratio lower than 1.73 (resp. 2.71) between local Newton iterations of FETI-precNL and FETI-NL (resp. BDD-NL) methods.

Figure 4: Nonlinear thermal problem: $\alpha = 0.8$

Geometrical parameters	
H	1
W	0.2
Material parameters	
α	$\in \{0, 0.2, 0.4, 0.6, 0.8\}$
Load parameters	
$u_{D1}(x)$	$10x + 2$
$u_{D2}(x)$	$-10x + 4$
r_{vol}	0

Table 6: Geometrical, material and load parameters: nonlinear thermal behavior

5 Conclusion

This article investigates a new technique of preconditioning for FETI solver, in the context of the nonlinear substructuration and condensation method. A nonlinear version of the classical scaled Dirichlet preconditioner is build under the form of a nonlinear fixed point condensed system, in place of the classical FETI-NL nonlinear interface condensed problem. A global Newton algorithm is used to solve this new nonlinearly preconditioned interface problem, and

# Global Newton iterations					
α	BDD-NL	FETI-NL	FETI-precNL	Gains of FETI-precNL (%)	
				vs. BDD-NL	vs. FETI-NL
0	1	1	1	0	0
0.2	3	4	3	0	25
0.4	4	5	3	25	40
0.6	4	5	3	25	40
0.8	5	6	3	40	40

# Krylov iterations					
α	BDD-NL	FETI-NL	FETI-precNL	Gains of FETI-precNL (%)	
				vs. BDD-NL	vs. FETI-NL
0	8	8	8	0	0
0.2	24	32	25	-4	22
0.4	32	39	28	13	28
0.6	34	40	28	18	30
0.8	43	40	28	35	30

Table 7: Nonlinear thermal behavior: global Newton and Krylov iterations

tangent operators have the exact form of classical preconditioned projected FETI operators, thus ensuring a quality condition number. The so-called solving process is called FETI-precNL.

Results show that great perfomance can be achieved with FETI-precNL method, when compared to BDD-NL and FETI-NL solvers on two test cases. The first test case is a water diffusion problem in a soil column (which does not produce rigid body modes), and the second one a symmetrized nonlinear thermal case. For both of them, FETI-precNL method clearly performs better than BDD-NL and FETI-NL methods (gains are about 50% for the first test case, and 30% for the second one). These first results are promising, and large scale implementation for mechanical problems is in progress.

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# Local Newton iterations					
α	BDD-NL	FETI-NL	FETI-precNL	Ratios of FETI-precNL (%)	
				over BDD-NL	over FETI-NL
0	1	1	3	3.0	3.0
0.2	7	11	19	2.71	1.73
0.4	10	12	22	2.20	1.83
0.6	11	16	26	2.36	1.63
0.8	12	17	32	2.67	1.88

Table 8: Nonlinear thermal behavior: Local iterations

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