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# Pricing without martingale measure

Julien Baptiste · Laurence Carassus ·  
Emmanuel Lépinette

**Abstract** For several decades, the no-arbitrage (NA) condition and the martingale measures have played a major role in the financial asset's pricing theory. A new approach is proposed in [7] to estimate the super-replication cost based on convex duality instead of martingale measures duality: Without no-arbitrage condition, the prices are expressed using Fenchel conjugate and bi-conjugate. In this paper, we propose an numerical illustration of the method on real data from the french CAC 40 index.

**Keywords** Financial market models · Super-hedging prices · No-arbitrage conditions · Conditional support · Essential supremum · CAC 40.

**Mathematics Subject Classification (2000)** 2000 MSC: 60G44 · G11-G13

## 1 Introduction

The problem of giving a fair price to a financial asset  $G$  is central in the economic and financial theory. A selling price should be an amount which is enough to initiate a hedging strategy for  $G$ , i.e. a strategy whose value at maturity is always above  $G$ . It seems also natural to ask for the infimum of such amount. This is the so called super-replication price and it has been introduced in the binomial setup for transaction costs by [5]. Characterizing and computing the super-replication price has become one of the central issue in mathematical finance theory, [37].

In the paper [7], a super-hedging or super-replicating price is the initial value of some super-hedging strategy. The set of super-hedging prices and its infimum value, called the infimum super-hedging cost, are computed without any no-arbitrage condition. Moreover, the super-hedging costs are finite if and only if the weak

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no-arbitrage condition AIP (Absence of Instantaneous Profits) holds. Indeed, the pricing formula that we obtain (see (2.6)) shows that, if the initial stock price does not belong to the convex hull of the conditional support of the stock value at the end of the period, then the super-hedging cost is equal to  $-\infty$ .

Under AIP condition, the one-step infimum super-hedging cost is the concave envelop of the payoff relative to the convex envelop of the conditional support. In the multiple-period framework, we propose a recursive scheme for the computation of the super-hedging prices of a convex option. We obtain the same computation scheme as in [6] and [8] but here it is obtained by only assuming AIP instead of the stronger NA condition. To illustrate the method, we calibrate historical data of the french index CAC 40 to our model and implement our super-hedging strategy for a call option. Our procedure is, somehow, model free and based only on statistical estimations. We actually observe that the same strategy, based on the implied volatility, provides worse results but at a lower price.

The paper is organized as follows. In Section 2, we present the one-period framework with the main results of [7] while in Section 3 the multi-period setting is deduced. Section 4 proposes some explicit pricing for a convex payoff and numerical experiments.

In the remaining of this introduction we present our framework and notations. Let  $(\Omega, (\mathcal{F}_t)_{t \in \{0, \dots, T\}}, \mathcal{F}_T, P)$  be a complete filtered probability space, where  $T$  is the time horizon. For any  $\sigma$ -algebra  $\mathcal{H}$  and any  $k \geq 1$ , we denote by  $L^0(\mathbb{R}^k, \mathcal{H})$  the set of  $\mathcal{H}$ -measurable and  $\mathbb{R}^k$ -valued random variables. We consider a non-negative process  $S := \{S_t, t \in \{0, \dots, T\}\}$  such that  $S_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)$  for all  $t \in \{0, \dots, T\}$ . The vector  $S_t$  represents the price at time  $t$  of the  $d$  risky assets in the financial market of consideration. Trading strategies are given by processes  $\theta := \{\theta_t, t \in \{0, \dots, T-1\}\}$  such that  $\theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)$  for all  $t \in \{0, \dots, T-1\}$ . The vector  $\theta_t$  represents the investor's holding in the  $d$  risky assets between times  $t$  and  $t+1$ . We assume that trading is self-financing and that the riskless asset's price is a constant equal to 1. The value at time  $t$  of a portfolio  $\theta$  starting from initial capital  $x \in \mathbb{R}$  is then given by

$$V_t^{x, \theta} = x + \sum_{u=1}^t \theta_{u-1} \Delta S_u,$$

where  $\Delta S_u = S_u - S_{u-1}$  for  $u \geq 1$  and  $xy$  is the scalar product of  $x$  and  $y$ .

## 2 The one-period framework

Let  $\mathcal{F}_t$  and  $\mathcal{F}_{t+1}$  be two complete sub- $\sigma$ -algebras of  $\mathcal{F}_T$  such that  $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$  and which represent respectively the initial and the final information. Let  $S_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)$  and  $S_{t+1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t+1})$  be two non-negative random variables. They represent the initial and the final prices of the  $d$  risky assets. We also consider a contingent claim  $Z_{t+1} \in L^0(\mathbb{R}, \mathcal{F}_{t+1})$ . We will be particularly interested by derivatives on  $S_{t+1}$  i.e.  $Z_{t+1} = g(S_{t+1})$  with  $g : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $g(S_{t+1}) : \omega \mapsto g(S_{t+1})(\omega) = g(\omega, S_{t+1}(\omega))$ .

In this section, we present a characterization of  $\mathcal{P}_t(Z_{t+1})$  the one-step set of super-hedging (or super-replicating) prices of  $Z_{t+1}$  and of its infimum value.

**Definition 2.1.** The set  $\mathcal{P}_t(Z_{t+1})$  of super-hedging prices of the contingent claim  $Z_{t+1} \in L^0(\mathbb{R}, \mathcal{F}_{t+1})$  consists in the initial values of super-hedging strategies  $\theta$

$$\mathcal{P}_t(Z_{t+1}) = \{x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists \theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t), x_t + \theta_t(S_{t+1} - S_t) \geq Z_{t+1} \text{ a.s.}\}.$$

The infimum super-hedging cost of  $Z_{t+1}$  is defined by  $p_t(Z_{t+1}) := \text{ess inf } \mathcal{P}_t(Z_{t+1})$ . When  $Z_{t+1} = g(S_{t+1})$  we write  $\mathcal{P}_t(g) = \mathcal{P}_t(Z_{t+1})$  and  $p_t(g) = p_t(Z_{t+1})$ .

The notions of conditional essential infimum  $\text{ess inf}_{\mathcal{F}_t}$  and conditional essential supremum  $\text{ess sup}_{\mathcal{F}_t}$  are recalled in the Appendix, see Proposition 5.4. We also need the notion of conditional support  $\text{supp}_{\mathcal{F}_t} S_{t+1}$  of  $S_{t+1}$ , which is introduced in Definition 5.1.

As  $x_t \in \mathcal{P}_t(Z_{t+1})$  if and only if there exists  $\theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)$  such that  $x_t \geq Z_{t+1} - \theta_t(S_{t+1} - S_t)$  a.s., we get by definition of the conditional essential supremum (see Proposition 5.4) that

$$\mathcal{P}_t(Z_{t+1}) = \left\{ \text{ess sup}_{\mathcal{F}_t} (Z_{t+1} - \theta_t(S_{t+1} - S_t)), \theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t) \right\} + L^0(\mathbb{R}_+, \mathcal{F}_t), \quad (2.1)$$

$$p_t(Z_{t+1}) = \text{ess inf}_{\mathcal{F}_t} \left\{ \text{ess sup}_{\mathcal{F}_t} (Z_{t+1} - \theta_t(S_{t+1} - S_t)), \theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t) \right\}. \quad (2.2)$$

In the case where  $Z_{t+1} = g(S_{t+1})$  we are able to perform an explicit computation of  $p_t(Z_{t+1})$ . To do so we recall that the (upper) closure  $\bar{h}$  of  $h$  is the smallest u.s.c. function which dominates  $h$  i.e.  $\bar{h}(x) = \limsup_{y \rightarrow x} h(y)$ . The lower closure is defined symmetrically. The proof of the following theorem is given in [7].

**Theorem 2.2.** The set  $\mathcal{P}_t(g)$  of Definition 2.1 can be expressed as follows

$$\mathcal{P}_t(g) = \left\{ \text{ess sup}_{\mathcal{F}_t} (g(S_{t+1}) - \theta_t S_{t+1}) + \theta_t S_t, \theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t) \right\} + L^0(\mathbb{R}_+, \mathcal{F}_t). \quad (2.3)$$

Suppose that  $g$  is a  $\mathcal{F}_t$ -normal integrand. Then, for  $\theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ , we get that

$$\text{ess sup}_{\mathcal{F}_t} (g(S_{t+1}) - \theta_t S_{t+1}) = \sup_{z \in \text{supp}_{\mathcal{F}_t} S_{t+1}} (g(z) - \theta_t z) = f^*(-\theta_t) \text{ a.s.} \quad (2.4)$$

where  $f$  and  $f^*$ , its Fenchel-Legendre conjugate, are given by

$$\begin{aligned} f(\omega, z) &= -g(\omega, z) + \delta_{\text{supp}_{\mathcal{F}_t} S_{t+1}(\omega)}(\omega, z) \\ f^*(\omega, x) &= \sup_{z \in \mathbb{R}^d} (xz - f(\omega, z)), \end{aligned} \quad (2.5)$$

and  $\delta_{C(\omega)}(\omega, z) = 0$  if  $z \in C(\omega)$  and  $+\infty$  else. Moreover suppose that  $g$  is proper and that there exists some concave function  $\varphi$  such that  $g \leq \varphi < \infty$  on  $\text{conv supp}_{\mathcal{F}_t} S_{t+1}$ <sup>1</sup>. We have that a.s.

$$\begin{aligned} p_t(g) &= -f^{**}(S_t) = \overline{\text{conc}}(g, \text{supp}_{\mathcal{F}_t} S_{t+1})(S_t) - \delta_{\text{conv supp}_{\mathcal{F}_t} S_{t+1}}(S_t) \\ &= \inf \{ \alpha S_t + \beta, \alpha \in \mathbb{R}^d, \beta \in \mathbb{R}, \alpha z + \beta \geq g(z), \forall z \in \text{supp}_{\mathcal{F}_t} S_{t+1} \} - \delta_{\text{conv supp}_{\mathcal{F}_t} S_{t+1}}(S_t), \end{aligned} \quad (2.6)$$

where  $f^{**}$  is the Fenchel-Legendre biconjugate of  $f$  i.e.  $f^{**}(\omega, x) = \sup_{z \in \mathbb{R}^d} (xz - f^*(\omega, z))$  and the relative concave envelop of  $g$  with respect to  $\text{supp}_{\mathcal{F}_t} S_{t+1}$  is given by

$$\text{conc}(g, \text{supp}_{\mathcal{F}_t} S_{t+1})(x) = \inf \{ v(x), v \text{ is concave and } v(z) \geq g(z), \forall z \in \text{supp}_{\mathcal{F}_t} S_{t+1} \}.$$

Notice that the infimum super-hedging cost is not a priori a price, i.e. an element of  $\mathcal{P}_t(g)$ , as the later may be an open interval.

<sup>1</sup> This is equivalent to assume that there exists  $\alpha, \beta \in \mathbb{R}$ , such that  $g(x) \leq \alpha x + \beta$  for all  $x \in \text{conv supp}_{\mathcal{F}_t} Y$ .

## 2.1 The AIP condition

Theorem 2.2 shows that if  $S_t \notin \text{convsupp}_{\mathcal{F}_t} S_{t+1}$  the infimum super-hedging cost of a European claim  $p_t(g)$  equals  $-\infty$ . Avoiding this situation leads to the notion of absence of immediate profit that we present now. Let  $\mathcal{R}_{t,t+1}$  be the set of all  $\mathcal{F}_{t+1}$ -measurable claims that can be super-replicate from 0:

$$\mathcal{R}_{t,t+1} := \left\{ \theta_t(S_{t+1} - S_t) - \epsilon_{t+1}^+, \theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t), \epsilon_{t+1}^+ \in L^0(\mathbb{R}_+, \mathcal{F}_{t+1}) \right\}. \quad (2.7)$$

Then

$$\begin{aligned} \mathcal{P}_t(0) &= \{x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists \theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t), x_t + \theta_t(S_{t+1} - S_t) \geq 0 \text{ a.s.}\} \\ &= (-\mathcal{R}_{t,t+1}) \cap L^0(\mathbb{R}, \mathcal{F}_t). \end{aligned}$$

Note that  $0 \in \mathcal{P}_t(0)$ , so  $p_t(0) \leq 0$ . We say that there is an immediate profit when  $P(p_t(0) < 0) > 0$  i.e. if it is possible to super-replicate the contingent claim 0 at a negative super-hedging price.

**Definition 2.3.** *There is an immediate profit (IP) if  $P(p_t(0) < 0) > 0$ . On the contrary case if  $p_t(0) = 0$  a.s. we say that the Absence of Immediate Profit (AIP) condition holds.*

We have the following characterisation of the AIP condition, see the proof in [7].

**Proposition 2.4.** *AIP holds if and only if one of the following condition holds true.*

1.  $S_t \in \text{convsupp}_{\mathcal{F}_t} S_{t+1}$  a.s. or  $0 \in \text{convsupp}_{\mathcal{F}_t}(S_{t+1} - S_t)$  a.s.
2.  $\mathcal{P}_t(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\}$  or  $\mathcal{R}_{t,t+1} \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \{0\}$ .

**Remark 2.5.** In the case  $d = 1$ , (5.22) implies that the previous conditions are equivalent to  $S_t \in [\text{ess inf}_{\mathcal{F}_t} S_{t+1}, \text{ess sup}_{\mathcal{F}_t} S_{t+1}] \cap \mathbb{R}$  a.s.

The AIP condition is very easy to check in practice as it suffices to observe that the price of any non negative payoff is non negative.

**Corollary 2.6.** *The AIP condition holds true if and only if  $p_t(g) \geq 0$  a.s. for some non-negative  $\mathcal{F}_t$ -normal integrand  $g$  such that there exists some concave function  $\varphi$  verifying that  $g \leq \varphi < \infty$ .*

In particular, the AIP condition holds true if and only if the infimum super-hedging cost of some European call option is non-negative. Note that, under AIP, the price of some non-zero payoff call option may be zero.

**Corollary 2.7.** *Suppose that AIP holds true. Let  $g$  be a proper  $\mathcal{F}_t$ -normal integrand such that there exists some concave function  $\varphi$  verifying that  $g \leq \varphi < \infty$  on  $\text{convsupp}_{\mathcal{F}_t} S_{t+1}$ . Then, a.s.*

$$\begin{aligned} p_t(g) &= \overline{\text{conc}}(g, \text{supp}_{\mathcal{F}_t} S_{t+1})(S_t) \\ &= \inf \{ \alpha S_t + \beta, \alpha \in \mathbb{R}^d, \beta \in \mathbb{R}, \alpha x + \beta \geq g(x), \forall x \in \text{supp}_{\mathcal{F}_t} S_{t+1} \}. \end{aligned} \quad (2.8)$$

If  $g$  is concave and u.s.c.,  $p_t(g) = g(S_t)$  a.s.

We finish the one-period analysis with the computation of the infimum super-hedging cost of a convex derivative when  $d = 1$ . In this case, the cost is in fact a super-hedging price and we get the super-hedging strategy explicitly.

**Corollary 2.8.** *Suppose that AIP holds true and that  $d = 1$ . Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a non-negative convex function with  $\text{dom } g = \mathbb{R}$  and  $\lim_{x \rightarrow \infty} x^{-1}g(x) = M \in [0, \infty)$ , then a.s.*

$$p_t(g) = \theta_t^* S_t + \beta^* = g(\text{ess inf}_{\mathcal{F}_t} S_{t+1}) + \theta_t^* (S_t - \text{ess inf}_{\mathcal{F}_t} S_{t+1}), \quad (2.9)$$

$$\theta_t^* = \frac{g(\text{ess sup}_{\mathcal{F}_t} S_{t+1}) - g(\text{ess inf}_{\mathcal{F}_t} S_{t+1})}{\text{ess sup}_{\mathcal{F}_t} S_{t+1} - \text{ess inf}_{\mathcal{F}_t} S_{t+1}}, \quad (2.10)$$

where we use the conventions  $\theta^* = \frac{0}{0} = 0$  in the case  $\text{ess sup}_{\mathcal{F}_t} S_{t+1} = \text{ess inf}_{\mathcal{F}_t} S_{t+1}$  a.s. and  $\theta_t^* = \frac{g(\infty)}{\infty} = M$  if  $\text{ess inf}_{\mathcal{F}_t} S_{t+1} < \text{ess sup}_{\mathcal{F}_t} S_{t+1} = +\infty$  a.s. Moreover,  $p_t(g) \in \mathcal{P}_t(g)$ .

**Example 2.9.** We compute the price of a call option under AIP in the case  $d = 1$ . Let  $G = g(S_{t+1}) = (S_{t+1} - K)_+$  for some  $K \geq 0$ .

- If  $K \geq \text{ess sup}_{\mathcal{F}_t} S_{t+1}$  then  $S_{t+1} - K \leq \text{ess sup}_{\mathcal{F}_t} S_{t+1} - K$  and  $G = 0$ . As AIP condition holds true,  $p_t(g) = p_t(0) = 0$ .
- If  $K \leq \text{ess inf}_{\mathcal{F}_t} S_{t+1}$  then  $S_{t+1} - K \geq \text{ess inf}_{\mathcal{F}_t} S_{t+1} - K$  and  $G = S_{t+1} - K$ . As  $g$  is concave and u.s.c.,  $p_t(g) = g(S_t) = S_t - K$  a.s.
- If  $\text{ess inf}_{\mathcal{F}_t} S_{t+1} \leq K \leq \text{ess sup}_{\mathcal{F}_t} S_{t+1}$ . Then, (2.10) and (2.9) imply that

$$p_t(g) = \frac{\text{ess sup}_{\mathcal{F}_t} S_{t+1} - K}{\text{ess sup}_{\mathcal{F}_t} S_{t+1} - \text{ess inf}_{\mathcal{F}_t} S_{t+1}} (S_t - \text{ess inf}_{\mathcal{F}_t} S_{t+1})$$

on  $\{\text{ess sup}_{\mathcal{F}_t} S_{t+1} \neq \text{ess inf}_{\mathcal{F}_t} S_{t+1}\}$  and 0 else. So  $p_t(g) = 0$  if and only if  $S_t = \text{ess inf}_{\mathcal{F}_t} S_{t+1}$  or  $\text{ess sup}_{\mathcal{F}_t} S_{t+1} = \text{ess inf}_{\mathcal{F}_t} S_{t+1}$ . A non-negative call option can have a zero price.

### 3 The multi-period framework

For every  $t \in \{0, \dots, T\}$ , the set  $\mathcal{R}_t^T$  of all claims that can be super-replicated from the zero initial endowment at time  $t$  is defined by

$$\mathcal{R}_t^T := \left\{ \sum_{u=t+1}^T \theta_{u-1} \Delta S_u - \epsilon_T^+, \theta_{u-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{u-1}), \epsilon_T^+ \in L^0(\mathbb{R}_+, \mathcal{F}_T) \right\}. \quad (3.11)$$

The set of (multi-period) super-hedging prices and the (multi-period) infimum super-hedging cost of some contingent claim  $g_T \in L^0(\mathbb{R}, \mathcal{F}_T)$  at time  $t$  are given for all  $t \in \{0, \dots, T\}$ , by

$$\begin{aligned} \mathcal{P}_{T,T}(g_T) &= \{g_T\} \text{ and } \pi_{T,T}(g_T) = g_T \\ \mathcal{P}_{t,T}(g_T) &= \{x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists R \in \mathcal{R}_t^T, x_t + R = g_T \text{ a.s.}\} \\ \pi_{t,T}(g_T) &= \text{ess inf}_{\mathcal{F}_t} \mathcal{P}_{t,T}(g_T). \end{aligned} \quad (3.12)$$

As in the one-period case, it is clear that the infimum super-hedging cost is not necessarily a price in the sense that  $\pi_{t,T}(g_T) \notin \mathcal{P}_{t,T}(g_T)$  when  $\mathcal{P}_{t,T}(g_T)$  is not closed.

We now define a local version of super-hedging prices. The set of one-step super-hedging prices of the payoff  $g_{t+1} \in L^0(\mathbb{R}, \mathcal{F}_{t+1})$  and its associated infimum super-hedging cost are given by

$$\begin{aligned} \mathcal{P}_{t,t+1}(g_{t+1}) &= \left\{ x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists \theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t), x_t + \theta_t \Delta S_{t+1} \geq g_{t+1} \text{ a.s.} \right\} \\ \pi_{t,t+1}(g_{t+1}) &= \text{ess inf}_{\mathcal{F}_t} \mathcal{P}_{t,t+1}(g_{t+1}) \\ &= \text{ess inf}_{\mathcal{F}_t} \left\{ \text{ess sup}_{\mathcal{F}_t} (g_{t+1} - \theta_t \Delta S_{t+1}), \theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t) \right\}, \end{aligned} \quad (3.13)$$

see (2.2). In the following, we extend the definition of  $\mathcal{P}_{t,u}(g_u)$ ,  $u \geq t+1$ , so that the argument  $g_u$  may be a subset  $G_u \subseteq L^0(\mathbb{R}, \mathcal{F}_u)$ . Precisely,  $\mathcal{P}_{t,u}(G_u) = \cup_{g_u \in G_u} \mathcal{P}_{t,u}(g_u)$ . The following lemma makes the link between local and global super-hedging prices. It also provides a dynamic programming principle, meaning that the prices are time consistent.

**Lemma 3.1.** *Let  $g_T \in L^0(\mathbb{R}, \mathcal{F}_T)$  and  $t \in \{0, \dots, T-1\}$ . Then*

$$\mathcal{P}_{t,T}(g_T) = \mathcal{P}_{t,t+1}(\mathcal{P}_{t+1,T}(g_T)) \text{ and } \pi_{t,T}(g_T) \geq \pi_{t,t+1}(\pi_{t+1,T}(g_T)).$$

Moreover, assume that  $\pi_{t+1,T}(g_T) \in \mathcal{P}_{t+1,T}(g_T)$ . Then

$$\mathcal{P}_{t,T}(g_T) = \mathcal{P}_{t,t+1}(\pi_{t+1,T}(g_T)) \text{ and } \pi_{t,T}(g_T) = \pi_{t,t+1}(\pi_{t+1,T}(g_T)).$$

**Remark 3.2.** Under AIP, if at each step,  $\pi_{t+1,T}(g_T) \in \mathcal{P}_{t+1,T}(g_T)$  and if we have  $\pi_{t+1,T}(g_T) = g_{t+1}(S_{t+1})$  for some “nice”  $\mathcal{F}_t$ -normal integrand  $g_{t+1}$ , we will get from Corollary 2.7 that  $\pi_{t,T}(g_T) = \text{connc}(g_{t+1}, \text{supp}_{\mathcal{F}_t} S_{t+1})(S_t)$  a.s. We will propose in Section 4 a quite general setting where this holds true.

Note that the super-hedging problem is solved for general claims  $\xi_T$  through the formula (3.13). For claims of Asian type  $g((S_u)_{u \leq T})$  or of American type, what we propose for European claims could be easily adapted. Consider a general claim  $\xi_T$  and the natural filtration i.e. the one generated by the price process  $S$ . Then for any self-financing portfolio  $\theta$ ,  $V_T = x + \sum_{t=1}^T \theta_{t-1} \Delta S_t \geq \xi_T$  if and only if  $V_T \geq \tilde{\xi}_T$  where  $\tilde{\xi}_T = \text{ess sup}_{\mathcal{F}_T} \xi_T$ . Thus  $\xi_T$  and  $\tilde{\xi}_T$  have the same super-replication cost and as  $\tilde{\xi}_T$  is  $\mathcal{F}_T$ -measurable, it is of the form  $\tilde{\xi}_T = g((S_u)_{u \leq T})$ . Of course, in practice, it is necessary to have an idea about  $g$  but the same difficulty arises under the NA condition.

**Definition 3.3.** *The AIP condition holds true if for all  $t \in \{0, \dots, T\}$*

$$\mathcal{P}_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\}.$$

We now study the link between global and local immediate profit. The global (resp. local) profit means that it is possible to super-replicate from a negative cost at time  $t$  the claim 0 paid at time  $T$  (resp. time  $t+1$ ). The next proposition shows that the local and global AIP conditions are equivalent in the following sense.

**Proposition 3.4.** *The following assertions are equivalent.*

1.  $\mathcal{P}_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\}$  for all  $t \in \{0, \dots, T-1\}$  i.e. AIP.
2.  $\mathcal{P}_{t,t+1}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\}$  for all  $t \in \{0, \dots, T-1\}$ .
3.  $S_t \in \text{convsupp}_{\mathcal{F}_t} S_{t+1}$  a.s. or  $0 \in \text{convsupp}_{\mathcal{F}_t} \Delta S_{t+1}$  a.s. for all  $t \in \{0, \dots, T-1\}$ .
4.  $\pi_{t,T}(0) = 0$  a.s. for all  $t \in \{0, \dots, T-1\}$ .

#### 4 Numerical experiments

**Proposition 4.1.** *Suppose that the model is defined by  $\text{ess inf}_{\mathcal{F}_{t-1}} S_t = k_{t-1}^d S_{t-1}$  a.s. and  $\text{ess sup}_{\mathcal{F}_{t-1}} S_t = k_{t-1}^u S_{t-1}$  a.s. where  $(k_{t-1}^d)_{t \in \{1, \dots, T\}}$ ,  $(k_{t-1}^u)_{t \in \{1, \dots, T\}}$  and  $S_0$  are deterministic non-negative numbers.*

*The AIP condition holds true if and only if  $k_{t-1}^d \in [0, 1]$  and  $k_{t-1}^u \in [1, +\infty]$  for all  $t \in \{1, \dots, T\}$ .*

*Suppose that the AIP condition holds. Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a non-negative convex function with  $\text{dom } h = \mathbb{R}$  such that  $\lim_{z \rightarrow +\infty} \frac{h(z)}{z} \in [0, \infty)$ . Then the infimum super-hedging cost of the European contingent claim  $h(S_T)$  is a price and it is given by  $\pi_{t,T}(h) = h(t, S_t) \in \mathcal{P}_{t,T}(h(S_T))$  a.s. where*

$$\begin{aligned} h(T, x) &= h(x) \\ h(t-1, x) &= \lambda_{t-1} h(t, k_{t-1}^d x) + (1 - \lambda_{t-1}) h(t, k_{t-1}^u x), \end{aligned} \quad (4.14)$$

where  $\lambda_{t-1} = \frac{k_{t-1}^u - 1}{k_{t-1}^u - k_{t-1}^d} \in [0, 1]$  and  $1 - \lambda_{t-1} = \frac{1 - k_{t-1}^d}{k_{t-1}^u - k_{t-1}^d} \in [0, 1]$ , with the following conventions. When  $k_{t-1}^d = k_{t-1}^u = 1$  or  $S_{t-1} = 0$ ,  $\lambda_{t-1} = \frac{0}{0} = 0$  and  $1 - \lambda_{t-1} = 1$  and when,  $k_{t-1}^d < k_{t-1}^u = \infty$ ,  $\lambda_{t-1} = \frac{\infty}{\infty} = 1$ ,

$$(1 - \lambda_{t-1}) h(t, (+\infty)x) = (1 - k_{t-1}^d) x \frac{h(t, (+\infty)x)}{(+\infty)x} = (1 - k_{t-1}^d) x \lim_{z \rightarrow +\infty} \frac{h(z)}{z}. \quad (4.15)$$

Moreover, for every  $t \in \{1, \dots, T\}$ ,  $\lim_{z \rightarrow +\infty} \frac{h(z)}{z} = \lim_{z \rightarrow +\infty} \frac{h(t, z)}{z}$  and  $h(\cdot, x)$  is non-increasing for all  $x \geq 0$ .

The strategy associated to the infimum super-hedging price is given by:

$$\theta_t^* = \frac{h(t+1, k_t^u S_t) - h(t+1, k_t^d S_t)}{(k_t^u - k_t^d) S_t}.$$

*Proof.* The conditions  $k_{t-1}^d \in [0, 1]$  and  $k_{t-1}^u \in [1, +\infty]$  for all  $t \in \{1, \dots, T\}$  are equivalent to the AIP condition (see Remark 2.5). Let  $M = \frac{h(\infty)}{\infty}$  and  $M_t = \lim_{z \rightarrow +\infty} \frac{h(t, z)}{z}$ . We prove the second statement. Assume that AIP holds true. We establish (i) the recursive formulation  $\pi_{t,T}(h(S_T)) = h(t, S_t)$  given by (4.14), (ii)  $h(t, \cdot) \geq h(t+1, \cdot)$  and (iii)  $M_t = M_{t+1}$ . The case  $t = T$  is immediate. As  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a convex function with  $\text{dom } h = \mathbb{R}$ ,  $h$  is clearly a  $\mathcal{F}_{T-1}$ -normal integrand, we can apply Corollary 2.8 (see (2.9) and (2.10)) and we get that a.s.

$$\begin{aligned} \pi_{T-1,T}(h(S_T)) &= h(k_{T-1}^d S_{T-1}) + \theta_{T-1}^* (S_{T-1} - k_{T-1}^d S_{T-1}), \\ \theta_{T-1}^* &= \frac{h(k_{T-1}^u S_{T-1}) - h(k_{T-1}^d S_{T-1})}{k_{T-1}^u S_{T-1} - k_{T-1}^d S_{T-1}}, \end{aligned} \quad (4.16)$$

with the conventions  $\theta_{T-1}^* = \frac{0}{0} = 0$  if either  $S_{T-1} = 0$  or  $k_{T-1}^u = k_{T-1}^d = 1$  and  $\theta_{T-1}^* = \frac{h(\infty)}{\infty} = M$  if  $k_{T-1}^d < k_{T-1}^u = +\infty$ . Moreover, using (2.8) and (2.9), we obtain that

$$\pi_{T-1,T}(h(S_T)) + \theta_{T-1}^* \Delta S_T \geq h(S_T) \text{ a.s. i.e. } \pi_{T-1,T}(h(S_T)) \in \mathcal{P}_{T-1,T}(h(S_T)).$$



So, using Lemma 3.1, we get that  $\mathcal{P}_{T-2,T}(h(S_T)) = \mathcal{P}_{T-2,T-1}(\pi_{T-1,T}(h(S_T)))$ ,

$$\pi_{T-2,T}(h(S_T)) = \pi_{T-2,T-1}(\pi_{T-1,T}(h(S_T)))$$

and we may continue the recursion as soon as  $\pi_{T-1,T}(h(S_T)) = h(T-1, S_{T-1})$  where  $h(T-1, \cdot)$  satisfies (4.14), is convex with domain equal to  $\mathbb{R}$ , is such that  $h(T-1, z) \geq 0$  for all  $z \geq 0$  and  $M_{T-1} = M \in [0, \infty)$ . To see that, we distinguish three cases. If either  $S_{T-1} = 0$  or  $k_{T-1}^u = k_{T-1}^d = 1$ ,  $\pi_{T-1,T}(h(S_T)) = h(S_{T-1})$  and  $h(T-1, z) = h(z) = h(T, z)$  satisfies all the required conditions. If  $k_{T-1}^d < k_{T-1}^u = +\infty$ ,  $\theta_{T-1}^* = M$  and  $\pi_{T-1,T}(h(S_T)) = h(T-1, S_{T-1})$ , where

$$\begin{aligned} h(T-1, z) &= h(k_{T-1}^d z) + Mz \left(1 - k_{T-1}^d\right) \\ &= \lim_{k^u \rightarrow +\infty} \left( \frac{k^u - 1}{k^u - k_{T-1}^d} h(k_{T-1}^d z) + \frac{1 - k_{T-1}^d}{k^u - k_{T-1}^d} h(k^u z) \right), \end{aligned}$$

using (4.15). The term in the r.h.s. above is larger than  $h(z) = h(T, z)$  by convexity. As  $k_{T-1}^d \in [0, 1]$  and  $M \in [0, \infty)$ ,  $h(T-1, z) \geq 0$  for all  $z \geq 0$ , we get that  $h(T-1, \cdot)$  is convex function with domain equal to  $\mathbb{R}$  since  $h$  is so. The function  $h(T-1, \cdot)$  also satisfies (4.14) (see (4.15)). Finally

$$M_{T-1} = \lim_{z \rightarrow +\infty} k_{T-1}^d \frac{h(k_{T-1}^d z)}{k_{T-1}^d z} + M \left(1 - k_{T-1}^d\right) = M.$$

The last case is when  $S_{T-1} \neq 0$  and  $k_{T-1}^u \neq k_{T-1}^d$  and  $k_{T-1}^u < +\infty$ . It is clear that (4.16) implies (4.14). Moreover as  $k_{T-1}^d \in [0, 1]$  and  $k_{T-1}^u \in [1, +\infty)$ ,  $\lambda_{T-1} \in [0, 1]$ ,  $1 - \lambda_{T-1} \in [0, 1]$  and (4.14) implies that  $h(T-1, z) \geq 0$  for all  $z \geq 0$ ,  $h(T-1, \cdot)$  is convex with domain equal to  $\mathbb{R}$  since  $h$  is so. Moreover, by convexity

$$M_{T-1} = \lambda_{T-1} k_{T-1}^d \lim_{z \rightarrow +\infty} \frac{h(k_{T-1}^d z)}{k_{T-1}^d z} + (1 - \lambda_{T-1}) k_{T-1}^u \lim_{z \rightarrow +\infty} \frac{h(k_{T-1}^u z)}{k_{T-1}^u z} = M. \square$$

**Remark 4.2.** The infimum super-hedging cost of the European contingent claim  $h(S_T)$  in our model is a price, precisely the same than the price we get in a binomial model  $S_t \in \{k_{t-1}^d S_{t-1}, k_{t-1}^u S_{t-1}\}$  a.s.,  $t \in \{1, \dots, T\}$ . Moreover, as in Corollary 2.6, one can prove that the AIP condition holds at every instant  $t$  if and only if the super-hedging price of some European call option at  $t$  is non-negative.

#### 4.0.1 Calibration

In this sub-section, we suppose that the discrete dates are given by  $t_i^n = \frac{iT}{n}$ ,  $i \in \{0, \dots, n\}$  where  $n \geq 1$ . We assume that for  $i \in \{1, \dots, n\}$

$$k_{i-1}^u = 1 + \sigma_{t_{i-1}^n} \sqrt{\Delta t_i^n} \text{ and } k_{i-1}^d = 1 - \sigma_{t_{i-1}^n} \sqrt{\Delta t_i^n} \geq 0, \quad (4.17)$$

where  $t \mapsto \sigma_t$  is a positive Lipschitz-continuous function on  $[0, T]$ . In this sub-section our aim is to calibrate  $\sigma$ . Note that the assumptions on the multipliers  $k_{i-1}^u$  and  $k_{i-1}^d$  imply that

$$\left| \frac{S_{i+1}^n}{S_i^n} - 1 \right| \leq \sigma_{t_i^n} \sqrt{\Delta t_{i+1}^n}, \text{ a.s.} \quad (4.18)$$

By Proposition 4.1, the infimum super-hedging cost of the European Call option  $(S_T - K)^+$  is given by  $h^n(t_i^n, S_{t_i^n})$  where  $h^n$  is defined by (4.14) with terminal condition  $h^n(T, x) = g(x) = (x - K)^+$ . We extend the function  $h^n$  on  $[0, T]$  in such a way that  $h^n$  is constant on each interval  $[t_i^n, t_{i+1}^n[$ ,  $i \in \{0, \dots, n\}$ . Such a scheme is proposed by Milstein [29] where a convergence theorem is proved when the terminal condition, i.e. the payoff function, is smooth. Precisely, the sequence of functions  $(h^n(t, x))_{n \geq 1}$  converges uniformly to  $h(t, x)$ , solution of the Black and Scholes formula with time-dependent volatility:

$$\partial_t h(t, x) + \sigma_t^2 \frac{x^2}{2} \partial_{xx} h(t, x) = 0, \quad h(T, x) = g(x). \quad (4.19)$$

In [29], it is supposed that the successive derivatives of the solution of the P.D.E. solution  $h$  are uniformly bounded. This is not the case for the Call payoff function  $g$ . On the contrary the successive derivatives of the solution of the P.D.E. explode at the horizon date, see [26]. In [2], it is proven that the uniform convergence still holds when the payoff function is not smooth provided that the successive derivatives of the solution of the P.D.E. do not explode *too much*.

Supposing that  $\Delta t_i^n$  is closed to 0, we can identify the observed prices of the Call option with the theoretical limit prices  $h(t, S_t)$  at any instant  $t$  given by (4.19). For several strikes, matching the observed prices to the theoretical ones allows to deduce the associated *implied* volatility  $t \mapsto \sigma_t$ , see [11] for a study of the implied volatility.

The data set is composed of historical values of the french index CAC 40 and European call option prices of maturity 3 months from the 23rd of October 2017 to the 19th of January 2018. The values of  $S$  are distributed as in Figure 1.

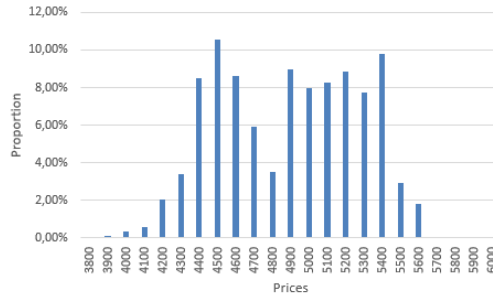


Fig. 1: Distribution of the observed prices.

We compute the proportion of observations satisfying (4.18). The results are satisfactory for strikes lower than 5000, see Fig. 2. Note that, when the strikes are too large with respect to the current price  $S$ , price observations are less available for the calibration, see Figure 1. This could explain the degradation of our results.

#### 4.0.2 Super-hedging prices

We now propose an approach where the coefficients  $k^u$  and  $k^d$  of Proposition 4.1 are not strike dependent as before and which is in this sense model free. The

Strike	4800	4900	5000	5100	5200	5300
Ratio	96,7%	95,1%	95,1%	88,5%	86,9%	80,3%
Strike	5400	5500	5600	5700	5800	5900
Ratio	70,5%	78,7%	75,4%	77,0%	73,8%	75,4%

Fig. 2

data set is composed of historical daily closing values of the french index CAC 40 from the 5th of January 2015 to the 12th of March 2018. The chosen interval  $[0, T]$  corresponds to one week of 5 working days so that the discrete dates are  $t_i^4$ ,  $i \in \{0, \dots, 4\}$  and  $n = 4$ . We estimate the parameters of the model  $k_{t_{i-1}^n}^d$  and  $k_{t_{i-1}^n}^u$ . The first approach is called symmetric :  $\sigma_{t_i}$  is estimated as an upper bound in (4.18) and then the  $k^d$  and  $k^u$  are given in (4.17). The asymmetric approach is the intuitive approach where the empirical minimum and maximum are taken. For  $i \in \{0, \dots, 3\}$

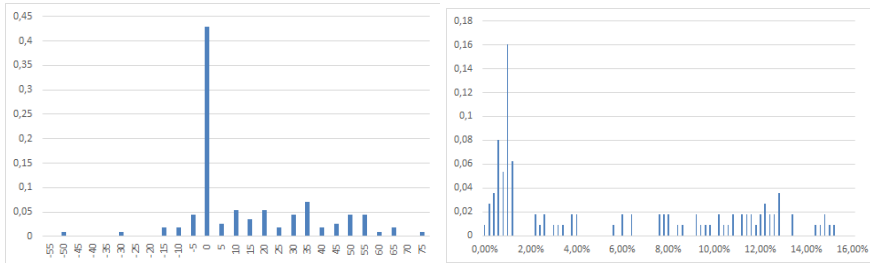
$$\sigma_{t_i} = \overline{\max} \left( \left| \frac{S_{t_{i+1}}}{S_{t_i}} - 1 \right| / \sqrt{\Delta t_{i+1}^4} \right) \quad (4.20)$$

$$k_{t_{i-1}^n}^d = \overline{\min} \frac{S_{t_i^n}}{S_{t_{i-1}^n}} \text{ and } k_{t_{i-1}^n}^u = \overline{\max} \frac{S_{t_i^n}}{S_{t_{i-1}^n}}.$$

Note that  $\overline{\max}$  (resp.  $\overline{\min}$ ), the empirical maximum (resp. min), is taken over a one year sliding sample window of 52 weeks. We estimate the parameters on 52 weeks and we implement our hedging strategy on the fifty third one. We then repeat the procedure by sliding the window of one week.

For a payoff function  $g(x) = (x - K)^+$ , we implement the super-hedging strategies  $(\theta_{t_i^4}^*)_{i \in \{0, \dots, 3\}}$  associated to the super-hedging cost given by Proposition 4.1 and (4.16). We denote by  $V_T$  the terminal value of our strategy starting from the super-hedging cost  $V_0 = \pi_{0,T} = h(0, S_0)$  i.e.  $V_T = V_0 + \sum_{i=0}^3 \theta_{t_i^4}^* \Delta S_{t_{i+1}^4}$ .

We study below the super-hedging error  $\varepsilon_T = V_T - (S_T - K)^+$  for different strikes. In the symmetric case, we present in Figures 3a and 3b the distribution of the super-hedging error  $\varepsilon_T$  and of  $V_0/S_0$  for  $K = 4700$ . The graphs in the asymmetric case are similar.



(a) Distribution of the super-hedging error  $\varepsilon_T = V_T - (S_T - K)^+$  for  $K = 4700$ . (b) Distribution of the ratio  $V_0/S_0$  for  $K = 4700$ .

Fig. 3: Symmetric case.

We now compare the result of both methods in the table below. In the sym-

	Mean of $V_0/S_0$	Variance of $V_0/S_0$	Mean of $\varepsilon_T$	Variance of $\varepsilon_T$	$P(\varepsilon_T < 0)$	VaR 95 %
Symmetric	5.61%	5.14 %	12.76	21.65	14.29%	-10.33
Asymmetric	5.52%	5.22%	9.47	14.20	8.04%	-1.81

Fig. 4: Comparison of the two methods of estimation for  $K = 4700$ . The mean of  $S_0$  is 4844,93 and the mean of  $(S_0 - K)^+$  is 278.73.

metric (resp. asymmetric) case the empirical average of the error  $\varepsilon_T$  is 12.76 (resp. 9.47) and its standard deviation is 21.65 (resp. 14.20). This result is rather satisfactory in comparison to the large value of the empirical mean of  $S_0$  which is equal to 4044. Notice that we observe  $E(S_T - K)^+ \simeq 282.69$ . This empirically confirms the efficiency of our suggested method. The empirical probability of  $\{\varepsilon_T < 0\}$  is equal to 14.29% (resp. 8.04%) but the Value at Risk at 95 % is  $-10.33$  (resp.  $-1.81$ ) which shows that our strategy is conservative.

Now we estimate the cost of our strategy in comparison with  $S_0$ . The empirical average of  $V_0/S_0$  is 5.61% (resp. 5.52%) and its standard deviation is 5.14% (resp. 5.22%). This means that  $V_0$  is much smaller than  $S_0$  which is the theoretical super-hedging price in some incomplete markets (this is for example the case when  $k^d = 0$  and  $k^u = \infty$ , in particular when the dynamics of  $S$  is modeled by a (discrete) geometric Brownian motion, see [6]). Note that the huge loss (50 in the symmetric case) is linked to so-called *black friday* week that occurs the 24th of June 2016. Large falls of risky assets were observed in European markets, mainly explained by the Brexit vote. In particular, the CAC 40 felt from 4340 to 4106, with a loss of  $-8\%$  on Friday.

We also present the ‘‘at the money’’ case  $K = S_0$ , see Fig. 5. We see that the results are better than for  $K = 4700$ :  $V_0$  is small with respect to  $S_0$  and the probability of loss is small as well.

	Mean of $V_0/S_0$	Variance of $V_0/S_0$	Mean of $\varepsilon_T$	Variance of $\varepsilon_T$	$P(\varepsilon_T < 0)$	VaR 95 %
Symmetric	1.51%	0.47 %	35.69	34.11	9.82 %	-11.41%
Asymmetric	1.47%	0.49%	33.37	32.78	12.50%	-9.29

Fig. 5: Comparison of the two methods of estimation for  $K = S_0$ . The mean of  $S_0$  is 4844.93.

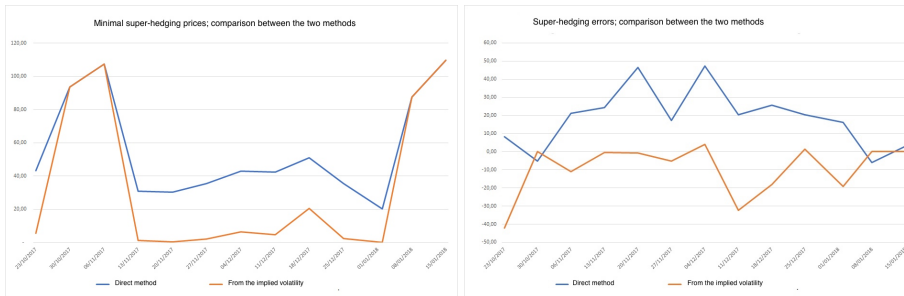


Fig. 6

**Remark 4.3.** We have compared our results, i.e. when  $\sigma_{t_i}$  is given by (4.20), to the ones where  $\sigma_{t_i}$  is the implied volatility. The comparison is made for  $K = 5400$ , see Figure 6. The implementation of the classical approach gives worse results as we may expect. Indeed, as observed in the Section 4.0.1, the implied volatility does

not capture the coefficients as well as the empirical observations do. Therefore, from the implied volatility, we get larger coefficients  $k^d$  and smaller coefficients  $k^u$ . Thus, the minimal super-hedging prices are smaller, as observed in Figure 6, but the super-hedging error is negative in most of the cases, see Figure 7. Our approach is definitively more conservative.

## 5 Appendix

We recall some results and notations that will be used without further references in the rest of the paper. The proofs are given in [7]. Let  $h : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ . The effective domain of  $h(\omega, \cdot)$  is defined by

$$\text{dom } h(\omega, \cdot) := \{x \in \mathbb{R}^d, h(\omega, x) < \infty\}$$

and  $h(\omega, \cdot)$  is proper if  $\text{dom } h(\omega, \cdot) \neq \emptyset$  and  $h(\omega, x) > -\infty$  for all  $x \in \mathbb{R}^d$ . Next, if  $h$  is  $\mathcal{F}_t$ -normal integrand (see Definition 14.27 in [34]) then  $h$  is  $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and is lower semi-continuous (l.s.c. in the sequel, see [34, Definition 1.5]) in  $x$  and the converse holds true if  $\mathcal{F}_t$  is complete for some measure, see [34, Corollary 14.34]. Note that if  $z_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)$  and  $h$  is  $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable, then  $h(z_t) \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ .

A random set  $\mathcal{K} : \Omega \rightarrow \mathbb{R}^d$  is  $\mathcal{F}_t$ -measurable if for all open set  $O$  of  $\mathbb{R}^d$ , the subset  $\{\omega \in \Omega, O \cap \mathcal{K}(\omega) \neq \emptyset\} \in \mathcal{F}_t$ . If  $\mathcal{K}$  is a  $\mathcal{F}_t$ -measurable and closed-valued random set of  $\mathbb{R}^d$ , then  $\mathcal{K}$  admits a Castaing representation  $(\eta_n)_{n \in \mathbb{N}}$  (see [34, Theorem 14.5]). This means that  $\mathcal{K}(\omega) = \text{cl}\{\eta_n(\omega), n \in \mathbb{N}\}$  for all  $\omega \in \text{dom } \mathcal{K} := \{\omega \in \Omega, \mathcal{K}(\omega) \cap \mathbb{R}^d \neq \emptyset\}$  where the closure is taken in  $\mathbb{R}^d$ .

We introduce the conditional support of  $X \in L^0(\mathbb{R}^d, \mathcal{F}_{t+1})$  with respect to  $\mathcal{F}_t$ .

**Definition 5.1.** Let  $\mu$  be a  $\mathcal{F}_t$ -stochastic kernel i.e. for all  $\omega \in \Omega$ ,  $\mu(\cdot, \omega)$  is a probability measure on  $\mathcal{B}(\mathbb{R}^d)$  and  $\mu(A, \cdot)$  is  $\mathcal{F}_t$ -measurable for all  $A \in \mathcal{B}(\mathbb{R}^d)$ . We define the random set  $D_\mu : \Omega \rightarrow \mathbb{R}^d$  by

$$D_\mu(\omega) := \bigcap \left\{ A \subset \mathbb{R}^d, \text{ closed}, \mu(A, \omega) = 1 \right\}. \quad (5.21)$$

For  $\omega \in \Omega$ ,  $D_\mu(\omega) \subset \mathbb{R}^d$  is called the support of  $\mu(\cdot, \omega)$ . Let  $X_{t+1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t+1})$ , we denote by  $\text{supp}_{\mathcal{F}_t} X_{t+1}$  the set defined in (5.21) when  $\mu(A, \omega) = P(X_{t+1} \in A | \mathcal{F}_t)(\omega)$  is a regular version of the conditional law of  $X_{t+1}$  knowing  $\mathcal{F}_t$ . The random set  $\text{supp}_{\mathcal{F}_t} X_{t+1}$  is called the conditional support of  $X_{t+1}$  with respect to  $\mathcal{F}_t$ .

**Remark 5.2.** When  $\mathcal{F}_t$  is the trivial sigma-algebra,  $\text{supp}_{\mathcal{F}_t} X_{t+1}$  is just the usual support of  $X_{t+1}$  (see p441 of [1]). Theorems 12.7 and 12.14 of [1] show that  $P(X_{t+1} \in \cdot | \mathcal{F}_t)$  admits a unique support  $\text{supp}_{\mathcal{F}_t} X_{t+1} \subset \mathbb{R}^d$  such that  $P(X_{t+1} \in \text{supp}_{\mathcal{F}_t} X_{t+1} | \mathcal{F}_t) = 1$  a.s. i.e.  $\text{supp}_{\mathcal{F}_t} X_{t+1}$  is a.s. non-empty.

For simplicity we will assume that  $X_{t+1}(\omega) \in \text{supp}_{\mathcal{F}_t} X_{t+1}(\omega)$  for all  $\omega \in \Omega$ . Moreover, if  $0 \leq X_{t+1} < \infty$ ,  $\text{Dom } \text{supp}_{\mathcal{F}_t} X_{t+1} = \Omega$ .

**Lemma 5.3.** Let  $\mu$  as in Definition 5.1.  $D_\mu$  is non-empty, closed-valued and  $\mathcal{F}_t$ -measurable.

It is possible to incorporate measurability in the definition of the essential supremum (see [22, Section 5.3.1] for the definition and the proof of existence of the classical essential supremum). This has been done by [3] for a single real-valued

random variable and by [23] for a family of vector-valued random variables and with respect to a random partial order (see [23, Definition 3.1 and Lemma 3.9]). Let us recall the following:

**Proposition 5.4.** *Let  $\mathcal{F}_t$  and  $\mathcal{F}_{t+1}$  be complete  $\sigma$ -algebras such that  $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$  and let  $\Gamma = (\gamma_i)_{i \in I}$  be a family of real-valued  $\mathcal{F}_T$ -measurable random variables. There exists a unique  $\mathcal{F}_t$ -measurable random variable  $\gamma_t \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{F}_t)$  denoted  $\text{ess sup}_{\mathcal{F}_t} \Gamma$  which satisfies the following properties*

1. For every  $i \in I$ ,  $\gamma_t \geq \gamma_i$  a.s.
2. If  $\zeta_t \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{F}_t)$  satisfies  $\zeta_t \geq \gamma_i$  a.s.  $\forall i \in I$ , then  $\zeta_t \geq \gamma_t$  a.s.

The conditional essential infimum  $\text{ess inf}_{\mathcal{F}_t} \Gamma$  is defined symmetrically.

**Lemma 5.5.** *Assume that  $d = 1$  and consider  $X_{t+1} \in L^0(\mathbb{R}, \mathcal{F}_{t+1})$ . Then, we have a.s. that*

$$\begin{aligned} \text{ess inf}_{\mathcal{F}_t} X_{t+1} &= \inf \text{supp}_{\mathcal{F}_t} X_{t+1}, & \text{ess sup}_{\mathcal{F}_t} X_{t+1} &= \sup \text{supp}_{\mathcal{F}_t} X_{t+1}, \\ \text{ess inf}_{\mathcal{F}_t} X_{t+1} &\in \text{supp}_{\mathcal{F}_t} X_{t+1} & \text{on the set } \{\text{ess inf}_{\mathcal{F}_t} X_{t+1} > -\infty\}, \\ \text{ess sup}_{\mathcal{F}_t} X_{t+1} &\in \text{supp}_{\mathcal{F}_t} X_{t+1} & \text{on the set } \{\text{ess sup}_{\mathcal{F}_t} X_{t+1} < \infty\}, \\ \text{convsupp}_{\mathcal{F}_t} X_{t+1} &= [\text{ess inf}_{\mathcal{F}_t} X_{t+1}, \text{ess sup}_{\mathcal{F}_t} X_{t+1}] \cap \mathbb{R}, \end{aligned} \quad (5.22)$$

where  $\text{convsupp}_{\mathcal{F}_t} X_{t+1}$  is the convex envelop of  $\text{supp}_{\mathcal{F}_t} X_{t+1}$  i.e. the smallest convex set that contains  $\text{supp}_{\mathcal{F}_t} X_{t+1}$ .

The following proposition is one of the main ingredient of the paper. It allows to compute a conditional essential supremum as a classical supremum but on a random set. A generalization is given in [27], see e.g. [28].

**Proposition 5.6.** *Let  $X \in L^0(\mathbb{R}^d, \mathcal{F}_T)$  be such that  $\text{dom supp}_{\mathcal{F}_t} X = \Omega$  and let  $h : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function which is l.s.c. in  $x$ . Then,*

$$\text{ess sup}_{\mathcal{F}_t} h(X) = \sup_{x \in \text{supp}_{\mathcal{F}_t} X} h(x) \quad \text{a.s.} \quad (5.23)$$

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