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## Pricing without martingale measure

Julien Baptiste · Laurence Carassus ·  
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**Abstract** For several decades, the no-arbitrage (NA) condition and the martingale measures have played a major role in the financial asset's pricing theory. We propose a new approach for estimating the super-replication cost based on convex duality instead of martingale measures duality : The prices are expressed using Fenchel conjugate and bi-conjugate. The super-hedging problem leads endogenously to a weak condition of NA called Absence of Immediate Profit (AIP). We propose several characterizations of AIP and study the relation with classical notions of no-arbitrage. We also give some promising numerical illustrations.

**Keywords** Financial market models · Super-hedging prices · No-arbitrage conditions · Conditional support · Essential supremum.

**Mathematics Subject Classification (2000)** 2000 MSC: 60G44 · G11-G13

### 1 Introduction

The problem of giving a fair price to a financial asset  $G$  is central in the economic and financial theory. A selling price should be an amount which is enough to initiate a hedging strategy for  $G$ , i.e. a strategy whose value at maturity is always above  $G$ . It seems also natural to ask for the infimum of such amount. This is the so called super-replication price and it has been introduced in the binomial setup for transaction costs by [5]. Characterizing and computing the super-replication price has become one of the central issue in mathematical finance theory, [35]. Until now it was intimately related to the no-arbitrage (NA) condition. This condition asserts that starting from a zero wealth it is not possible to reach a positive

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one (non negative almost surely and strictly positive with strictly positive probability). Characterizing the NA condition or, more generally, the No Free Lunch condition leads to the Fundamental Theorem of Asset Pricing (FTAP in short). This theorem proves the equivalence between those absence of arbitrage conditions and the existence of equivalent risk-neutral probability measures (also called martingale measures or pricing measures) which are equivalent probability measures under which the (discounted) asset price process is a martingale. This was initially formalised in [15], [16] and [23] while in [9] the FTAP is formulated in a general discrete-time setting under the NA condition. The literature on the subject is huge and we refer to [10] and [20] for a general overview. Under the NA condition, the super-replication price of  $G$  is equal to the supremum of the (discounted) expectation of  $G$  computed under the risk-neutral probability measures. This is the so called dual formulation of the super-replication price or superhedging theorem. We refer to [33] and [12] and the references therein.

In this paper, a super-hedging or super-replicating price is the initial value of some super-hedging strategy. We do not postulate any normative assumption on the financial market and we analyze from scratch the set of super-hedging prices and its infimum value which will be called the infimum super-hedging cost. We ask ourselves what the finiteness of the super-hedging cost can tell us about free lunch like opportunities in the market. To do so, we compute the super-hedging cost using convex duality instead of the usual financial duality based on martingale measures under the NA condition. We show under mild assumptions that the one-step set of super-hedging prices can be expressed using Fenchel-Legendre conjugate and the infimum super-replication cost is obtained by the Fenchel-Legendre biconjugate. In particular, we use the notion of conditional essential supremum and show, using measurable selection techniques, that the conditional essential supremum of a function of a random variable  $Y$  is equal to the usual supremum of the function evaluated on the conditional support of  $Y$  (see Proposition 2.7). The pricing formula that we obtain (see (2.10)) shows that, if the initial stock price does not belong to the convex hull of the conditional support of the stock value at the end of the period, then the super-hedging cost is equal to  $-\infty$ . To exclude this possibility we postulate the condition of Absence of Immediate Profit (AIP). The AIP is very weak : If the initial information is trivial, a one period immediate profit is a strategy which starts from 0 and leads to a deterministic strictly positive gain at time 1. The AIP is also easy to check in practice, it is sufficient to verify that the convex hull of the conditional support of the price increment contains 0. This differs from the NA condition for which it does not seem to exist any computationally feasible way of deriving whether arbitrage exist or not. It is also not easy to check the existence of equivalent martingale measure by looking directly at the dynamics of the stock price process. We propose several characterization of the AIP condition. In particular we show that AIP is equivalent to the non-negativity of the super-hedging cost of some call option. We also discuss in details the link between AIP and the others no-arbitrage conditions as the no-arbitrage of first and second type and the no-riskless arbitrage of [17] and the No Unbounded Increasing Profit and No Unbounded Profit with Bounded Risk of [22] or the No Free Lunch with Vanishing Risk of [10]. All those conditions imply AIP but the reverse does not hold true. We provide condition for the equivalence between NA and AIP and examples of pricing under AIP when NA is not satisfied. We also show that

contrarily to the no-arbitrage condition, AIP allows to obtain super-hedging cost which are AIP free.

Under AIP condition, we show that the one-step infimum super-hedging cost is the concave envelop of the payoff relatively to the convex envelop of the conditional support. Fenchel-Legendre duality have already been used to obtain a dual representation of the super-replication price thanks to deflators (see [28, Exemple 4.2] and [29, Theorem 10 and Corollary 15]). In [29, Theorem 10] the result is shown under the assumption that the set of claims that can be super-replicated from 0 is closed, which holds true under NA. Our approach is different as we do not postulate any assumption on the market and as we are not looking for a dual representation of the super-hedging price.

We then consider the multiple-period framework. We show that the global AIP condition and the local ones are equivalent. We study the link between AIP, NA and the absence of weak immediate profit (AWIP) conditions. We show that the AIP condition is the weakest-one and we also provide conditions for the equivalence between the AIP and the AWIP conditions, as well as characterization through absolutely continuous martingale measure.

We then focus on a particular, but still general setup, where we propose a recursive scheme for the computation of the super-hedging prices of a convex option. We obtain the same computation scheme as in [7] and [8] but here it is obtained by only assuming AIP instead of the stronger NA condition. We also give some numerical illustrations. We calibrate historical data of the french index CAC 40 to our model and implement our super-hedging strategy for a call option. Our procedure is, somehow, model free and based only on statistical estimations. We actually observe that the same strategy, based on the implied volatility, provides worse results but at a lower price.

The paper is organized as follows. In Section 2, we study the one-period framework while in Section 3 we study the multi-period one. Section 4 proposes some explicit pricing for a convex payoff and numerical experiments. The proofs of technical results are postponed to the appendix.

In the remaining of this introduction we present our framework and notations. Let  $(\Omega, (\mathcal{F}_t)_{t \in \{0, \dots, T\}}, \mathcal{F}_T, P)$  be a complete filtered probability space, where  $T$  is the time horizon. For any  $\sigma$ -algebra  $\mathcal{H}$  and any  $k \geq 1$ , we denote by  $L^0(\mathbb{R}^k, \mathcal{H})$  the set of  $\mathcal{H}$ -measurable and  $\mathbb{R}^k$ -valued random variables. We consider a non-negative process  $S := \{S_t, t \in \{0, \dots, T\}\}$  such that  $S_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)$  for all  $t \in \{0, \dots, T\}$ . The vector  $S_t$  represents the price at time  $t$  of the  $d$  risky assets in the financial market of consideration. Trading strategies are given by processes  $\theta := \{\theta_t, t \in \{0, \dots, T-1\}\}$  such that  $\theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)$  for all  $t \in \{0, \dots, T-1\}$ . The vector  $\theta_t$  represents the investor's holding in the  $d$  risky assets between times  $t$  and  $t+1$ . We assume that trading is self-financing and that the riskless asset's price is a constant equal to 1. The value at time  $t$  of a portfolio  $\theta$  starting from initial capital  $x \in \mathbb{R}$  is then given by

$$V_t^{x, \theta} = x + \sum_{u=1}^t \theta_{u-1} \Delta S_u,$$

where  $\Delta S_u = S_u - S_{u-1}$  for  $u \geq 1$  and  $xy$  is the scalar product of  $x$  and  $y$ .

## 2 The one-period framework

Let  $\mathcal{H}$  and  $\mathcal{F}$  be two complete sub- $\sigma$ -algebras of  $\mathcal{F}_T$  such that  $\mathcal{H} \subseteq \mathcal{F}$  and which represent respectively the initial and the final information. Let  $y \in L^0(\mathbb{R}^d, \mathcal{H})$  and  $Y \in L^0(\mathbb{R}^d, \mathcal{F})$  be two non-negative random variables. They represent the initial and the final prices of the  $d$  risky assets. We also consider a contingent claim  $Z \in L^0(\mathbb{R}, \mathcal{F})$ . We will be particularly interested by derivatives on  $Y$  i.e.  $Z = g(Y)$  with  $g : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  and  $g(Y) : \omega \mapsto g(Y)(\omega) = g(\omega, Y(\omega))$ .

The objective of the section is to obtain a characterization of  $\mathcal{P}(Z)$  the one-step set of super-hedging (or super-replicating) prices of  $Z$  and of its infimum value. The setting will be applied in Section 3 with the choices  $\mathcal{H} = \mathcal{F}_t$ ,  $\mathcal{F} = \mathcal{F}_{t+1}$ ,  $Y = S_{t+1}$  and  $y = S_t$ .

**Definition 2.1.** *The set  $\mathcal{P}(Z)$  of super-hedging prices of the contingent claim  $Z \in L^0(\mathbb{R}, \mathcal{F})$  consists in the initial values of super-hedging strategies  $\theta$*

$$\mathcal{P}(Z) = \{x \in L^0(\mathbb{R}, \mathcal{H}), \exists \theta \in L^0(\mathbb{R}^d, \mathcal{H}), x + \theta(Y - y) \geq Z \text{ a.s.}\}.$$

The infimum super-hedging cost of  $Z$  is defined by  $p(Z) := \text{ess inf}_{\mathcal{H}} \mathcal{P}(Z)$ .

When  $Z = g(Y)$  we write  $\mathcal{P}(g) = \mathcal{P}(Z)$  and  $p(g) = p(Z)$ .

The notions of conditional essential infimum  $\text{ess inf}_{\mathcal{H}}$  and conditional essential supremum  $\text{ess sup}_{\mathcal{H}}$  are at the heart of this study and are defined in Proposition 2.5 below. We also use the conditional support  $\text{supp}_{\mathcal{H}} Y$  of  $Y$  which is introduced in Definition 2.2 below. In Section 2.2 we derive the characterization of the super-hedging prices and cost from the following steps :

1. Observe that the set of super-hedging prices can be rewritten using a conditional essential supremum (see (2.5) and (2.7)).
2. Under mild conditions, show that the conditional essential supremum of a function of  $Y$  is equal to the usual supremum of the function evaluated on the random set  $\text{supp}_{\mathcal{H}} Y$  (see Proposition 2.7).
3. When  $Z = g(Y)$ , recognize that a super-hedging price can be written using a Fenchel-Legendre conjugate (see (2.8)).
4. Take the essential infimum of the set of super-hedging prices and go through the steps 2. and 3. to recognize the Fenchel-Legendre biconjugate (see (2.10)).
5. Use the classical convex biconjugate theorem to evaluate the infimum super-hedging cost.

With this pricing formula in hand (see (2.10)) the Absence of Immediate Profit (AIP) appears as the condition to postulate in order to get a finite super-hedging cost. In Section 2.3, we develop the concept of AIP and propose several characterizations of the AIP condition. In Section 2.4 we compare it with the classical no-arbitrage conditions.

### 2.1 Conditional support and conditional essential infimum

This section is the toolbox of the paper. The proofs are postponed to the appendix. We recall some results and notations that will be used without further references

in the rest of the paper. Let  $h : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ . The effective domain of  $h(\omega, \cdot)$  is defined by

$$\text{dom } h(\omega, \cdot) := \{x \in \mathbb{R}^d, h(\omega, x) < \infty\}$$

and  $h(\omega, \cdot)$  is proper if  $\text{dom } h(\omega, \cdot) \neq \emptyset$  and  $h(\omega, x) > -\infty$  for all  $x \in \mathbb{R}^d$ . Next, if  $h$  is  $\mathcal{H}$ -normal integrand (see Definition 14.27 in [32]) then  $h$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and is lower semi-continuous (l.s.c. in the sequel, see [32, Definition 1.5]) in  $x$  and the converse holds true if  $\mathcal{H}$  is complete for some measure, see [32, Corollary 14.34]. Note that if  $z \in L^0(\mathbb{R}^d, \mathcal{H})$  and  $h$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable, then  $h(z) \in L^0(\mathbb{R}^d, \mathcal{H})$ . A random set  $\mathcal{K} : \Omega \rightarrow \mathbb{R}^d$  is  $\mathcal{H}$ -measurable if for all open set  $O$  of  $\mathbb{R}^d$ , the subset  $\{\omega \in \Omega, O \cap \mathcal{K}(\omega) \neq \emptyset\} \in \mathcal{H}$ . If  $\mathcal{K}$  is a  $\mathcal{H}$ -measurable and closed-valued random set of  $\mathbb{R}^d$ , then  $\mathcal{K}$  admits a Castaing representation  $(\eta_n)_{n \in \mathbb{N}}$  (see [32, Theorem 14.5]). This means that  $\mathcal{K}(\omega) = \text{cl}\{\eta_n(\omega), n \in \mathbb{N}\}$  for all  $\omega \in \text{dom } \mathcal{K} := \{\omega \in \Omega, \mathcal{K}(\omega) \cap \mathbb{R}^d \neq \emptyset\}$  where the closure is taken in  $\mathbb{R}^d$ .

We introduce the conditional support of  $X \in L^0(\mathbb{R}^d, \mathcal{F})$  with respect to  $\mathcal{H}$ .

**Definition 2.2.** Let  $\mu$  be a  $\mathcal{H}$ -stochastic kernel i.e. for all  $\omega \in \Omega$ ,  $\mu(\cdot, \omega)$  is a probability measure on  $\mathcal{B}(\mathbb{R}^d)$  and  $\mu(A, \cdot)$  is  $\mathcal{H}$ -measurable for all  $A \in \mathcal{B}(\mathbb{R}^d)$ . We define the random set  $D_\mu : \Omega \rightarrow \mathbb{R}^d$  by

$$D_\mu(\omega) := \bigcap \left\{ A \subset \mathbb{R}^d, \text{ closed}, \mu(A, \omega) = 1 \right\}. \quad (2.1)$$

For  $\omega \in \Omega$ ,  $D_\mu(\omega) \subset \mathbb{R}^d$  is called the support of  $\mu(\cdot, \omega)$ . Let  $X \in L^0(\mathbb{R}^d, \mathcal{F})$ , we denote by  $\text{supp}_{\mathcal{H}} X$  the set defined in (2.1) when  $\mu(A, \omega) = P(X \in A | \mathcal{H})(\omega)$  is a regular version of the conditional law of  $X$  knowing  $\mathcal{H}$ . The random set  $\text{supp}_{\mathcal{H}} X$  is called the conditional support of  $X$  with respect to  $\mathcal{H}$ .

**Remark 2.3.** When  $\mathcal{H}$  is the trivial sigma-algebra,  $\text{supp}_{\mathcal{H}} X$  is just the usual support of  $X$  (see p441 of [1]). Theorems 12.7 and 12.14 of [1] show that  $P(X \in \cdot | \mathcal{H})$  admits a unique support  $\text{supp}_{\mathcal{H}} X \subset \mathbb{R}^d$  such that  $P(X \in \text{supp}_{\mathcal{H}} X | \mathcal{H}) = 1$  a.s. i.e.  $\text{supp}_{\mathcal{H}} X$  is a.s. non-empty.

For simplicity we will assume that  $Y(\omega) \in \text{supp}_{\mathcal{H}} Y(\omega)$  for all  $\omega \in \Omega$ . Moreover, as  $0 \leq Y < \infty$ ,  $\text{Dom } \text{supp}_{\mathcal{H}} Y = \Omega$ .

**Lemma 2.4.** Let  $\mu$  as in Definition 2.2.  $D_\mu$  is non-empty, closed-valued and  $\mathcal{H}$ -measurable.

It is possible to incorporate measurability in the definition of the essential supremum (see [20, Section 5.3.1] for the definition and the proof of existence of the classical essential supremum). This has been done by [3] for a single real-valued random variable and by [21] for a family of vector-valued random variables and with respect to a random partial order (see [21, Definition 3.1 and Lemma 3.9]). Proposition 2.5 is given and proved for sake of completeness and for pedagogical purpose. The authors thanks T. Jeulin who suggested this (elegant) proof.

**Proposition 2.5.** Let  $\mathcal{H}$  and  $\mathcal{F}$  be complete  $\sigma$ -algebras such that  $\mathcal{H} \subseteq \mathcal{F}$  and let  $\Gamma = (\gamma_i)_{i \in I}$  be a family of real-valued  $\mathcal{F}$ -measurable random variables. There exists a unique  $\mathcal{H}$ -measurable random variable  $\gamma_{\mathcal{H}} \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{H})$  denoted  $\text{ess sup}_{\mathcal{H}} \Gamma$  which satisfies the following properties

1. For every  $i \in I$ ,  $\gamma_{\mathcal{H}} \geq \gamma_i$  a.s.
2. If  $\zeta \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{H})$  satisfies  $\zeta \geq \gamma_i$  a.s.  $\forall i \in I$ , then  $\zeta \geq \gamma_{\mathcal{H}}$  a.s.

The conditional essential infimum  $\text{ess inf}_{\mathcal{H}} \Gamma$  is defined symmetrically.

*Proof.* Considering the homeomorphism  $\arctan$  we can restrict our-self to  $\gamma_i$  taking values in  $[0, 1]$ . We denote by  $P_{\gamma_i|\mathcal{H}}$  a regular version of the conditional law of  $\gamma_i$  knowing  $\mathcal{H}$ . Let  $\zeta \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{H})$  such that  $\zeta \geq \gamma_i$  a.s.  $\forall i \in I$ . This is equivalent to  $P_{\gamma_i|\mathcal{H}}(\cdot - \infty, x)|_{x=\zeta} = 1$  a.s. and  $\text{supp}_{\mathcal{H}} \gamma_i \subset ]-\infty, \zeta]$  a.s. follows from Definition 2.2. Let

$$A_{\gamma_i|\mathcal{H}} = \sup\{x \in [0, 1], x \in \text{supp}_{\mathcal{H}} \gamma_i\}. \quad (2.2)$$

Then  $A_{\gamma_i|\mathcal{H}} \leq \zeta$  a.s. and it is easy to see that  $A_{\gamma_i|\mathcal{H}}$  is  $\mathcal{H}$ -measurable. So taking the classical essential supremum, we get that  $\text{ess sup}_i A_{\gamma_i|\mathcal{H}} \leq \zeta$  a.s. and that  $\text{ess sup}_i A_{\gamma_i|\mathcal{H}}$  is  $\mathcal{H}$ -measurable. We conclude that  $\gamma_{\mathcal{H}} = \text{ess sup}_i A_{\gamma_i|\mathcal{H}}$  a.s. since for every  $i \in I$ ,  $P(\gamma_i \in \text{supp}_{\mathcal{H}} \gamma_i | \mathcal{H}) = 1$  (see Remark 2.3).  $\square$

**Lemma 2.6.** *Assume that  $d = 1$  and consider  $X \in L^0(\mathbb{R}, \mathcal{F})$ . Then, we have a.s. that*

$$\begin{aligned} \text{ess inf}_{\mathcal{H}} X &= \inf \text{supp}_{\mathcal{H}} X, & \text{ess sup}_{\mathcal{H}} X &= \sup \text{supp}_{\mathcal{H}} X, \\ \text{ess inf}_{\mathcal{H}} X &\in \text{supp}_{\mathcal{H}} X & \text{on the set } \{\text{ess inf}_{\mathcal{H}} X > -\infty\}, \\ \text{ess sup}_{\mathcal{H}} X &\in \text{supp}_{\mathcal{H}} X & \text{on the set } \{\text{ess sup}_{\mathcal{H}} X < \infty\}, \\ \text{convsupp}_{\mathcal{H}} X &= [\text{ess inf}_{\mathcal{H}} X, \text{ess sup}_{\mathcal{H}} X] \cap \mathbb{R}, \end{aligned} \quad (2.3)$$

where  $\text{convsupp}_{\mathcal{H}} X$  is the convex envelop of  $\text{supp}_{\mathcal{H}} X$  i.e. the smallest convex set that contains  $\text{supp}_{\mathcal{H}} X$ .

The following proposition is one of the main ingredient of the paper. It extends the fact that  $\text{ess sup}_{\mathcal{H}} X = \sup_{x \in \text{supp}_{\mathcal{H}} X} x$  a.s. (see (2.2)) and allows to compute a conditional essential supremum as a classical supremum but on a random set. A generalization is given in [25], see e.g. [26].

**Proposition 2.7.** *Let  $X \in L^0(\mathbb{R}^d, \mathcal{F})$  be such that  $\text{dom supp}_{\mathcal{H}} X = \Omega$  and let  $h : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function which is l.s.c. in  $x$ . Then,*

$$\text{ess sup}_{\mathcal{H}} h(X) = \sup_{x \in \text{supp}_{\mathcal{H}} X} h(x) \quad \text{a.s.} \quad (2.4)$$

## 2.2 Fenchel-Legendre conjugate and bi-conjugate to express super-replication prices and cost

We are now in position to perform the program announced in the beginning of the section. Let  $Z \in L^0(\mathbb{R}, \mathcal{F})$ . As  $x \in \mathcal{P}(Z)$  if and only if there exists  $\theta \in L^0(\mathbb{R}^d, \mathcal{H})$  such that  $x \geq Z - \theta(Y - y)$  a.s., we get by definition of the conditional essential supremum (see Proposition 2.5) that

$$\mathcal{P}(Z) = \left\{ \text{ess sup}_{\mathcal{H}} (Z - \theta(Y - y)), \theta \in L^0(\mathbb{R}^d, \mathcal{H}) \right\} + L^0(\mathbb{R}_+, \mathcal{H}), \quad (2.5)$$

$$p(Z) = \text{ess inf}_{\mathcal{H}} \left\{ \text{ess sup}_{\mathcal{H}} (Z - \theta(Y - y)), \theta \in L^0(\mathbb{R}^d, \mathcal{H}) \right\}. \quad (2.6)$$

In the case where  $Z = g(Y)$  we are able to perform an explicit computation of  $p(Z)$ . To do so we recall that the (upper) closure  $\bar{h}$  of  $h$  is the smallest u.s.c. function which dominates  $h$  i.e.  $\bar{h}(x) = \limsup_{y \rightarrow x} h(y)$ . The lower closure is defined symmetrically.

**Theorem 2.8.** *The set  $\mathcal{P}(g)$  of Definition 2.1 can be express as follows*

$$\mathcal{P}(g) = \left\{ \text{ess sup}_{\mathcal{H}}(g(Y) - \theta Y) + \theta y, \theta \in L^0(\mathbb{R}^d, \mathcal{H}) \right\} + L^0(\mathbb{R}_+, \mathcal{H}). \quad (2.7)$$

Suppose that  $g$  is a  $\mathcal{H}$ -normal integrand. Then, for  $\theta \in L^0(\mathbb{R}^d, \mathcal{H})$ , we get that

$$\text{ess sup}_{\mathcal{H}}(g(Y) - \theta Y) = \sup_{z \in \text{supp}_{\mathcal{H}} Y} (g(z) - \theta z) = f^*(-\theta) \quad \text{a.s.} \quad (2.8)$$

where  $f$  and  $f^*$ , its Fenchel-Legendre conjugate, are given by

$$\begin{aligned} f(\omega, z) &= -g(\omega, z) + \delta_{\text{supp}_{\mathcal{H}} Y(\omega)}(\omega, z) \\ f^*(\omega, x) &= \sup_{z \in \mathbb{R}^d} (xz - f(\omega, z)), \end{aligned} \quad (2.9)$$

and  $\delta_{C(\omega)}(\omega, z) = 0$  if  $z \in C(\omega)$  and  $+\infty$  else. Moreover suppose that  $g$  is proper and that there exists some concave function  $\varphi$  such that  $g \leq \varphi < \infty$  on  $\text{convsupp}_{\mathcal{H}} Y$ <sup>1</sup>. We have that a.s.

$$\begin{aligned} p(g) &= -f^{**}(y) = \overline{\text{conc}}(g, \text{supp}_{\mathcal{H}} Y)(y) - \delta_{\text{convsupp}_{\mathcal{H}} Y}(y) \\ &= \inf \{ \alpha x + \beta, \alpha \in \mathbb{R}^d, \beta \in \mathbb{R}, \alpha z + \beta \geq g(z), \forall z \in \text{supp}_{\mathcal{H}} Y \} - \delta_{\text{convsupp}_{\mathcal{H}} Y}(y), \end{aligned} \quad (2.10)$$

where  $f^{**}$  is the Fenchel-Legendre biconjugate of  $f$  i.e.  $f^{**}(\omega, x) = \sup_{z \in \mathbb{R}^d} (xz - f^*(\omega, z))$  and the relative concave envelop of  $g$  with respect to  $\text{supp}_{\mathcal{H}} Y$  is given by

$$\text{conc}(g, \text{supp}_{\mathcal{H}} Y)(x) = \inf \{ v(x), v \text{ is concave and } v(z) \geq g(z), \forall z \in \text{supp}_{\mathcal{H}} Y \}.$$

Notice that the infimum super-hedging cost is not a priori a price, i.e. an element of  $\mathcal{P}(g)$ , as the later may be an open interval. Note also that [7] and [4] have represented the super-hedging price as a concave envelop but this was done under the no-arbitrage condition using the dual representation of the super-replication price through martingale measures.

**Remark 2.9.** Let  $\mathcal{F} = \sigma(Y)$ . For all  $\theta \in L^0(\mathbb{R}^d, \mathcal{H})$ ,  $V = x + \theta(Y - y) \geq Z$  if and only if  $V \geq \tilde{Z}$  where  $\tilde{Z} = \text{ess sup}_{\mathcal{F}} Z$ . Thus  $Z$  and  $\tilde{Z}$  have the same super-hedging cost and as  $\tilde{Z}$  is  $\mathcal{F}$ -measurable,  $\tilde{Z} = \delta(Y)$  for some  $\delta$ . In order to apply the theorem above, one need to know  $\delta$  (and also have nice properties on  $\delta$ ). But the situation is somehow similar under the no-arbitrage condition where one need to compute  $E_Q(Z)$ , for some  $Q \sim P$  such that  $E_Q(Y|\mathcal{H}) = y$ : It is necessary to understand how  $Z$  depends on  $Y$ . In the case where  $Z = g(Y, T)$  for some random variable  $T$  which is not necessarily  $\mathcal{F}$ -measurable, we show that  $\tilde{Z} = \delta(Y)$  where  $\delta(y) = \sup_{t \in \text{supp}_{\mathcal{F}} T} g(y, t)$  is a  $\mathcal{F}$ -normal integrand under the condition that  $g$  is l.s.c. in  $(y, t)$ , see Lemma 5.3.

**Remark 2.10.** Fenchel-Legendre duality have already been used many times in financial mathematics. In particular, Pennanen obtains a dual representation of the super-replication price thanks to deflators (see [28, Example 4.2] and [29, Theorem 10 and Corollary 15]). The proof of [29, Theorem 10] is also based on the (convex) biconjugate theorem but the result is shown under the assumption that the set  $\mathcal{R}$  of claims that can be super-replicated from 0 (see (2.13)) is closed, which holds

<sup>1</sup> This is equivalent to assume that there exists  $\alpha, \beta \in \mathbb{R}$ , such that  $g(x) \leq \alpha x + \beta$  for all  $x \in \text{convsupp}_{\mathcal{H}} Y$ .



true under the no-arbitrage condition NA. In [30], the existence and the absence of duality gap in a general stochastic optimization problem is proved through dynamic programming and under a condition (that does not rely on inf-compactness) of linearity on sets constructed with recession functions. This condition in classical mathematical finance problems is equivalent to the no-arbitrage condition (see [30, Exemple 1]). Our approach is different as we do not postulate any assumption on the market and we obtain from the biconjugate representation a formula for the infimum super-hedging cost (see (2.10)). We will deduce from this the condition that should be satisfied by the market (see Proposition 2.12). Importantly, our goal is not to obtain a dual representation thanks to deflator or martingale measures.

*Proof.* First (2.7) follows from (2.5). Lemma 2.4 will be in force. Under the assumption that  $g$  is a  $\mathcal{H}$ -normal integrand, (2.8) follows from Proposition 2.7. Under the additional assumptions that  $g$  is proper and that there exists some concave function  $\varphi$  such that  $g \leq \varphi < \infty$  on  $\text{convsupp}_{\mathcal{H}}Y$ , we show first that  $\text{conv} f$  satisfies (2.11) and is proper, where  $\text{conv} f$  is the convex envelop of  $f$  i.e. the greatest convex function dominated by  $f$ . It can be written as follows (see [32, Proposition 2.31])

$$\text{conv} f(x) = \inf \left\{ \sum_{i=1}^n \lambda_i f(x_i), n \geq 1, (\lambda_i)_{i \in \{1, \dots, n\}} \in \mathbb{R}_+^n, (x_i)_{i \in \{1, \dots, n\}} \in \mathbb{R}^{d \times n}, \right. \\ \left. x = \sum_{i=1}^n \lambda_i x_i, \sum_{i=1}^n \lambda_i = 1 \right\}.$$

Let  $x = \sum_{i=1}^n \lambda_i x_i$  for some  $n \geq 1$ ,  $(\lambda_i)_{i \in \{1, \dots, n\}} \in \mathbb{R}_+^n$  such that  $\sum_{i=1}^n \lambda_i = 1$  and  $(x_i)_{i \in \{1, \dots, n\}} \in \mathbb{R}^{d \times n}$ . Assume that  $x \notin \text{convsupp}_{\mathcal{H}}Y$ . Then (see [32, Proposition 2.27, Theorem 2.29]), there exists at least one  $x_i \notin \text{supp}_{\mathcal{H}}Y$  and  $f(x_i) = +\infty$  and also  $\text{conv} f(x) = +\infty$ . If  $x \in \text{convsupp}_{\mathcal{H}}Y$ ,  $\text{conv} f(x) = -\text{conc}(g, \text{supp}_{\mathcal{H}}Y)(x)$  by definition. Moreover, for all  $x \in \text{convsupp}_{\mathcal{H}}Y$ ,  $x = \sum_{i=1}^n \lambda_i x_i$  for some  $n \geq 1$ ,  $(\lambda_i)_{i \in \{1, \dots, n\}} \in \mathbb{R}_+^n$ , and  $(x_i)_{i \in \{1, \dots, n\}} \subset \text{supp}_{\mathcal{H}}Y$  such that  $\sum_{i=1}^n \lambda_i = 1$  and

$$\text{conc}(g, \text{supp}_{\mathcal{H}}Y)(x) \geq \sum_{i=1}^n \lambda_i \text{conc}(g, \text{supp}_{\mathcal{H}}Y)(x_i) \geq \sum_{i=1}^n \lambda_i g(x_i) > -\infty.$$

Moreover,  $\text{conc}(g, \text{supp}_{\mathcal{H}}Y) \leq \varphi < \infty$  on  $\text{convsupp}_{\mathcal{H}}Y$ . Thus, for all  $x \in \text{convsupp}_{\mathcal{H}}Y$ ,  $\text{conc}(g, \text{supp}_{\mathcal{H}}Y)(x) \in \mathbb{R}$  and one may write that

$$\text{conv} f = -\text{conc}(g, \text{supp}_{\mathcal{H}}Y) + \delta_{\text{convsupp}_{\mathcal{H}}Y} \quad \text{a.s.} \quad (2.11)$$

As  $\text{convsupp}_{\mathcal{H}}Y$  is non-empty,  $\text{conv} f$  is proper and [32, Theorem 11.1] implies that  $f^*$  is proper, l.s.c and convex and that  $f^{**}(y) = \text{conv} f(y)$ . Moreover, using Lemma 5.2,  $f^*(\omega, x) = \sup_{z \in \text{supp}_{\mathcal{H}}Y(\omega)} (xz + g(\omega, z))$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. We obtain that a.s.

$$p(g) = \text{ess inf}_{\mathcal{H}} \{ f^*(-\theta) + \theta y, \theta \in L^0(\mathbb{R}^d, \mathcal{H}) \} = -\text{ess sup}_{\mathcal{H}} \{ \theta y - f^*(\theta), \theta \in L^0(\mathbb{R}^d, \mathcal{H}) \} \\ = - \sup_{z \in \mathbb{R}^d} (zy - f^*(z)) = -f^{**}(y).$$

The first equality is a direct consequence of (2.7), the second one is trivial. So to finish the proof it remains to prove the third one. Indeed (2.11) implies that

$$p(g) = -f^{**}(y) = -\text{conv} f(y) = \overline{\text{conc}}(g, \text{supp}_{\mathcal{H}}Y)(y) - \delta_{\text{convsupp}_{\mathcal{H}}Y}(y) \quad \text{a.s.}$$

and the last equality follows easily. Recall that for all  $\theta \in L^0(\mathbb{R}^d, \mathcal{H})$ , we have  $\text{ess sup}_{\mathcal{H}}\{\theta y - f^*(\theta)\} \geq \theta y - f^*(\theta)$  a.s. Thus for any  $z \in \mathbb{R}^d$ , choosing  $\theta = z$ , we get that  $\text{ess sup}_{\mathcal{H}}\{\theta y - f^*(\theta)\} \geq zy - f^*(z)$  a.s. and

$$\text{ess sup}_{\mathcal{H}}\{\theta y - f^*(\theta)\} \geq \sup_{z \in \mathbb{R}^d} (zy - f^*(z)) \quad \text{a.s.} \quad (2.12)$$

Conversely, for all  $\theta \in L^0(\mathbb{R}^d, \mathcal{H})$ , we have  $\theta y - f^*(\theta) \leq \sup_{z \in \mathbb{R}^d} (zy - f^*(z))$  a.s. If  $\sup_{z \in \mathbb{R}^d} (zy - f^*(z))$  is  $\mathcal{H}$ -measurable, we shall conclude that the equality holds true in (2.12). As  $f^*$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable,  $\text{graph dom } f^* = \{(\omega, x) \in \Omega \times \mathbb{R}^d, f^*(\omega, x) < \infty\} \in \mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$  and  $\text{dom } f^*$  is  $\mathcal{H}$ -measurable (see [32, Theorem 14.8]). As  $f^*$  is proper,  $\text{dom } f^*(\omega, \cdot) \neq \emptyset$ . By measurable selection argument, we obtain the existence of a  $\mathcal{H}$ -measurable selector  $a$  such that  $a \in \text{dom}(f^*)$  a.s. On  $\text{dom}(f^*) = \{a\}$ , we have  $\sup_{z \in \mathbb{R}^d} (zy - f^*(z)) = ay - f^*(a)$  which is  $\mathcal{H}$ -measurable. Otherwise, on  $\text{dom}(f^*) \neq \{a\}$ ,  $\text{ri dom}(f^*) \neq \emptyset$  and  $z \mapsto zy - f^*(z)$  is concave hence continuous on this set. We deduce that a.s.

$$\sup_{z \in \mathbb{R}^d} (zy - f^*(z)) = \sup_{z \in \text{ri dom } f^*} (zy - f^*(z)) = \sup_{z \in \hat{\Gamma}} (zy - f^*(z))$$

where  $\hat{\Gamma}$  is a countable  $\mathcal{H}$ -measurable dense subset of  $\mathbb{R}^d$ . The first equality is classical (see for example Lemma A.32 of [6]).

The second equality is shown as follows. By lemma 5.4, consider a  $\mathcal{H}$ -measurable version of the affine hull  $H$  of  $\text{dom}(f^*)$ . Let  $(q_n)_{n \geq 1}$  be a dense subset of  $\mathbb{R}^d$  and let us denote by  $p_H(q_n)$  the projection of  $q_n$  onto  $H$ ,  $n \geq 1$ . By lemma 5.4, we may suppose that  $p_H(q_n)$  is  $\mathcal{H}$ -measurable for all  $n \geq 1$  and it is well known that  $(p_H(q_n))_{n \geq 1}$  is dense in  $H$ . Therefore, as  $\text{ri dom}(f^*)$  is the largest open set in  $\text{dom}(f^*)$  relatively to the affine hull  $H$  of  $\text{dom}(f^*)$ , we deduce that any  $z \in \text{ri dom}(f^*) \subseteq H$  is the limit of a countable (with random index) family  $(z_n)_{n \geq 1}$ , which is a subfamily of the larger countable family  $\hat{\Gamma}(\omega) = (p_H(q_n)(\omega))_{n \geq 1}$ . By continuity on the relative interior, we then deduce that  $\sup_{z \in \text{ri dom } f^*} (zy - f^*(z)) \leq \sup_{z \in \hat{\Gamma}} (zy - f^*(z))$  hence the equality holds by the first equality. In particular,  $\sup_{z \in \mathbb{R}^d} (zy - f^*(z))$  is  $\mathcal{H}$ -measurable, which allows to conclude.  $\square$

### 2.3 The AIP condition

Theorem 2.8 shows that if  $y \notin \text{conv sup}_{\mathcal{H}} Y$  the infimum super-hedging cost of a European claim  $p(g)$  equals  $-\infty$ . Avoiding this situation leads to the notion of absence of immediate profit that we present now. Let  $\mathcal{R}$  be the set of all  $\mathcal{F}$ -measurable claims that can be super-replicate from 0:

$$\mathcal{R} := \left\{ \theta(Y - y) - \epsilon^+, \theta \in L^0(\mathbb{R}^d, \mathcal{H}), \epsilon^+ \in L^0(\mathbb{R}_+, \mathcal{F}) \right\}. \quad (2.13)$$

Then

$$\mathcal{P}(0) = \{x \in L^0(\mathbb{R}, \mathcal{H}), \exists \theta \in L^0(\mathbb{R}^d, \mathcal{H}), x + \theta(Y - y) \geq 0 \text{ a.s.}\} = (-\mathcal{R}) \cap L^0(\mathbb{R}, \mathcal{H}).$$

Note that  $0 \in \mathcal{P}(0)$ , so  $p(0) \leq 0$ . We say that there is an immediate profit when  $P(p(0) < 0) > 0$  i.e. if it is possible to super-replicate the contingent claim 0 at a negative super-hedging price.

**Definition 2.11.** *There is an immediate profit (IP) if  $P(p(0) < 0) > 0$ . On the contrary case if  $p(0) = 0$  a.s. we say that the Absence of Immediate Profit (AIP) condition holds.*

We know propose several characterisations of the AIP condition. We will discuss in Proposition 2.18, Remark 2.20 and Lemma 2.21 the link with the no-arbitrage conditions and show that AIP is indeed very weak.

**Proposition 2.12.** *AIP holds if and only if one of the following condition holds true.*

1.  $y \in \text{convsupp}_{\mathcal{H}}Y$  a.s. or  $0 \in \text{convsupp}_{\mathcal{H}}(Y - y)$  a.s.
2.  $\sigma_{\text{supp}_{\mathcal{H}}(Y-y)} \geq 0$  a.s. where  $\sigma_D(z) = \sup_{x \in D}(-xz)$  is the support function of  $-D$
3.  $\mathcal{P}(0) \cap L^0(\mathbb{R}_-, \mathcal{H}) = \{0\}$  or  $\mathcal{R} \cap L^0(\mathbb{R}_+, \mathcal{H}) = \{0\}$ .

**Remark 2.13.** In the case  $d = 1$ , (2.3) implies that the previous conditions are equivalent to  $y \in [\text{ess inf}_{\mathcal{H}}Y, \text{ess sup}_{\mathcal{H}}Y] \cap \mathbb{R}$  a.s.

**Remark 2.14.** The AIP condition is tailor-made for pricing issues. It allows to give a super-hedging cost even in case of arbitrage opportunity (see Example 2.27 below). Note that an IP is a very strong strategy. Assume that  $\mathcal{H}$  is trivial, then an IP corresponds to some  $\theta \in \mathbb{R}^d$  such that  $\theta(Y - y)$  is deterministic and strictly positive. So excluding IP and not NA may not be enough to get existence in the problem of maximization of expected utility. Technically AIP may not insure some maximizing sequence to be bounded.

**Example 2.15.** The AIP condition is very easy to check in practice as it suffices to observe that the price of any non negative payoff is non negative. For  $d = 1$ , consider  $R = Y/y$ . To check AIP, it is enough to compute either  $\text{convsupp}_{\mathcal{H}}R$  or  $\text{ess inf}_{\mathcal{H}}R$  and  $\text{ess sup}_{\mathcal{H}}R$  and compare with 1. For example, suppose that  $R = \exp((\mu - \frac{\sigma^2}{2}) + \sigma(B_{t+1} - B_t))$  where  $B$  is a Brownian motion,  $\mathcal{H} = \sigma(\{B_u, u \leq t\})$  and  $\mathcal{F} = \sigma(\{B_u, u \leq t + 1\})$ . Then  $\text{convsupp}_{\mathcal{H}}R = [0, \infty)$  and AIP holds true.

*Proof.* The assumptions of Theorem 2.8 are satisfied for  $g = 0$  and we get that  $p(0) = -\delta_{\text{convsupp}_{\mathcal{H}}Y}(y)$  a.s. Hence, AIP holds true if and only if  $y \in \text{convsupp}_{\mathcal{H}}Y$  a.s. or equivalently  $0 \in \text{convsupp}_{\mathcal{H}}(Y - y)$  a.s. and AIP is equivalent to 1. Using Theorem 2.8, we get that

$$\mathcal{P}(0) = \left\{ \text{ess sup}_{\mathcal{H}}(-\theta(Y - y)), \theta \in L^0(\mathbb{R}^d, \mathcal{H}) \right\} + L^0(\mathbb{R}_+, \mathcal{H}).$$

Proposition 2.7 implies that for  $\theta \in L^0(\mathbb{R}^d, \mathcal{H})$ ,

$$\text{ess sup}_{\mathcal{H}}(-\theta(Y - y)) = \sup_{x \in \text{supp}_{\mathcal{H}}(Y-y)}(-\theta x) = \sigma_{\text{supp}_{\mathcal{H}}(Y-y)}(\theta).$$

So,  $\mathcal{P}(0) \cap L^0(\mathbb{R}_-, \mathcal{H}) = \{0\}$  if and only if  $\sigma_{\text{supp}_{\mathcal{H}}(Y-y)} \geq 0$  a.s. and 2. and 3. are equivalent. To finish the proof, it remains to prove that 2. is equivalent to 1. First remark that  $\sigma_{\text{supp}_{\mathcal{H}}(Y-y)} = \sigma_{\text{convsupp}_{\mathcal{H}}(Y-y)}$ . So, it remains to prove that, for any closed convex set  $D$  of  $\mathbb{R}^d$ ,  $\sigma_D \geq 0$  if and only if  $0 \in D$ . If  $0 \in D$  it is clear that  $\sigma_D \geq 0$ . Assume that  $0 \notin D$ . Then, by Hahn-Banach theorem, there exists some  $\beta > 0$  and some  $\theta_0 \in \mathbb{R}^d \setminus \{0\}$  such that  $-x\theta_0 \leq -\beta$  for all  $x \in D$  and  $\sigma_D(\theta_0) \leq -\beta < 0$  follows.  $\square$

**Corollary 2.16.** *The AIP condition holds true if and only if  $p(g) \geq 0$  a.s. for some non-negative  $\mathcal{H}$ -normal integrand  $g$  such that there exists some concave function  $\varphi$  verifying that  $g \leq \varphi < \infty$ .*

In particular, the AIP condition holds true if and only if the infimum superhedging cost of some European call option is non-negative. Note that, under AIP, the price of some non-zero payoff call option may be zero (see Example 2.27 below).

*Proof.* Assume that AIP condition holds true. Then, by Definition 2.11, we get that  $p(0) = 0$  a.s. As  $g \geq 0$ , it is clear that  $p(g) \geq p(0) = 0$  a.s. Conversely, assume that there exists an IP, Proposition 2.12 leads to  $P(y \in \text{convsupp}_{\mathcal{H}}Y) < 1$  and, since  $\overline{\text{conc}}(g, \text{supp}_{\mathcal{H}}Y)(y) \leq \varphi < \infty$ , (2.10) implies that  $P(p(g) = -\infty) > 0$ . The converse is proved.  $\square$

#### 2.4 Comparison of AIP with other no-arbitrage conditions

The no-arbitrage conditions may be quite different in discrete and continuous time. For example, going back to [18] the classical no-arbitrage (see Definition 2.17) is equivalent to the no-arbitrage with no interim losses (“weakly arbitrage-free”). We will compare AIP with the classical conditions of the literature, i.e. with NA, with No Unbounded Profit with Bounded Risk (NUPBR) and with No Free Lunch with Vanishing Risk (NFLVR). We also compare with the notions of arbitrage introduced by Ingersoll (see [17]). We will make the comparison with No Unbounded Increasing Profit (NUIP) in the multi-period part as an unbounded increasing profit is an increasing portfolio process starting from zero and strictly positive with strictly positive probability.

**Definition 2.17.** *The no-arbitrage (NA) condition holds true if  $\theta(Y - y) \geq 0$  a.s. for some  $\theta \in L^0(\mathbb{R}^d, \mathcal{H})$  implies that  $\theta(Y - y) = 0$  a.s. or equivalently  $\mathcal{R} \cap L^0(\mathbb{R}_+, \mathcal{F}) = \{0\}$ .*

We remark first that the AIP condition is strictly weaker than the NA one. Indeed it is clear from Proposition 2.12 and Definition 2.17 that NA implies AIP. Fix  $d = 1$ ,  $\mathcal{H} = \{\emptyset, \Omega\}$ ,  $y = 0$  and  $Y$  follows a uniform distribution on  $[0, 1]$ . Then  $P(Y - y > 0) = 1$  and the constant strategy equal to 1 is an arbitrage opportunity. Nevertheless  $y = 0 \in \text{convsupp}_{\mathcal{H}}Y = [0, 1]$  and AIP holds true. We now provide a necessary and sufficient condition for the equivalence between AIP and NA.

**Proposition 2.18.** *The AIP and NA conditions are equivalent if and only if*

$$0 \in \text{convsupp}_{\mathcal{H}}(Y - y) \text{ a.s.} \Leftrightarrow 0 \in \text{ri}(\text{convsupp}_{\mathcal{H}}(Y - y)) \text{ a.s.} \quad (2.14)$$

*This last condition is satisfied if*

$$P(0 \notin \text{convsupp}_{\mathcal{H}}(Y - y) \setminus \text{ri}(\text{convsupp}_{\mathcal{H}}(Y - y))) = 1. \quad (2.15)$$

**Remark 2.19.** Note that, if  $\mathcal{H} = \{\emptyset, \Omega\}$ , then (2.15) is a necessary and sufficient condition for the equivalence between the AIP and the NA conditions.

We first have a look on the case  $d = 1$ . Remark 2.13 shows that

$$\text{convsupp}_{\mathcal{H}}(Y - y) \setminus \text{ri}(\text{convsupp}_{\mathcal{H}}(Y - y)) = \{\text{ess inf}_{\mathcal{H}}Y - y, \text{ess sup}_{\mathcal{H}}Y - y\}$$

and (2.15) is equivalent to  $P(\text{ess inf}_{\mathcal{H}}Y = y) = P(\text{ess sup}_{\mathcal{H}}Y = y) = 0$ .

The preceding counter-example corresponds to  $\text{ess inf}_{\mathcal{H}}Y = 0$ ,  $\text{ess sup}_{\mathcal{H}}Y = 1$  and  $y = 0$ . The equivalence between AIP and NA will hold true in this setup if  $y \in (0, 1)$ .

The proof in the case  $d = 1$  enlightens the difference between both conditions. Assume that AIP holds true and that  $P(\text{ess inf}_{\mathcal{H}}Y = y) = P(\text{ess sup}_{\mathcal{H}}Y = y) = 0$ .

Using Remark 2.13,  $y \in [\text{ess inf}_{\mathcal{H}} Y, \text{ess sup}_{\mathcal{H}} Y] \cap \mathbb{R}$  a.s. Let  $\theta \in L^0(\mathbb{R}, \mathcal{H})$  such that  $\theta(Y - y) \geq 0$ . On the set  $\{\theta > 0\} \in \mathcal{H}$ , we have that  $Y \geq y$  hence  $\text{ess inf}_{\mathcal{H}} Y \geq y \geq \text{ess inf}_{\mathcal{H}} Y$ . We deduce that  $P(\theta > 0) = 0$ . Similarly, we get that  $P(\theta < 0) = 0$  and finally  $\theta = 0$ .

*Proof.* Proposition 2.12 shows that AIP is equivalent to  $0 \in \text{convsupp}_{\mathcal{H}}(Y - y)$  a.s. On the other hand, [18, Theorem 3g)] shows that NA is equivalent to  $0 \in \text{ri}(\text{supp}_{\mathcal{H}}(Y - y))$  a.s. So AIP is equivalent to NA if and only if (2.14) holds true. This last condition is implied by (2.15).  $\square$

**Remark 2.20.** We compare IP with other notions of arbitrage as introduced by Ingersoll (see [17]) in a one step setting with a finite set of states of the world. Arbitrage opportunity of the first type is the classical arbitrage. An arbitrage opportunity  $\theta$  of the second type is limited liability investments with a current negative commitment. As we assume the existence of a riskless asset, it means that  $\theta(Y - y)$  is not deterministic but always greater than some strictly positive deterministic number. Finally, a riskless arbitrage opportunity is a non-positive investment with a constant, positive profit. This notion is equivalent to our notion of IP (recall that there exists a riskless asset) in the context of a trivial initial filtration. If  $\mathcal{H}$  is not trivial anymore, a riskless arbitrage is an IP but the converse is not true anymore.

At last, we compare the AIP condition to the No Unbounded Profit with Bounded Risk (NUPBR) condition, which is also equivalent to the No Arbitrage of First Kind (NA1) condition, see [19]. An arbitrage opportunity of first kind is a payoff  $\xi \in L^0(\mathbb{R}_+, \mathcal{F}) \setminus \{0\}$  such that every  $x > 0$ , there exists  $\theta \in L^0(\mathbb{R}^d, \mathcal{H})$  such that  $x + \theta(Y - y) \geq \xi$  a.s. i.e. it is possible to super-hedge  $\xi$  from any arbitrarily small initial capital. We also compare with No free Lunch with Vanishing Risk (NFLVR) condition, see [10].

**Lemma 2.21.** *The NUPBR implies AIP and AIP may be strictly weaker. The same holds true for the NFLVR.*

*Proof.* Assume that there exists an immediate profit i.e. some  $\theta \in L^0(\mathbb{R}^d, \mathcal{H})$  such that  $\xi = \theta(Y - y) \in L^0(\mathbb{R}_+, \mathcal{H}) \setminus \{0\}$ . Then  $\xi$  is an unbounded profit with bounded risk i.e. an arbitrage of first kind since for all  $x > 0$   $x + \theta(Y - y) \geq \xi$  a.s. On the contrary, AIP may hold while NUPBR fails. Assume that  $y = 0$  and  $Y$  follows a uniform distribution on  $[0, 1.5]$ . As  $y = 0 \in \text{convsupp}_{\mathcal{H}} Y = [0, 1.5]$  AIP holds true. Let  $\xi = 1_{\{Y > 1\}}$ . Then  $\xi \in L^0(\mathbb{R}_+, \mathcal{F}) \setminus \{0\}$  as  $P(Y > 1) > 0$ . Moreover for all  $x > 0$   $x + (Y - y) > Y \geq \xi$  a.s. as  $P(Y \geq 0) = 1$ .

As NFLVR condition is equivalent to the NA condition and the NA1 condition, NFLVR implies AIP but the reverse does not hold true.  $\square$

We finish this section with the following interesting result: Contrary to the no-arbitrage condition, AIP makes it possible to obtain super-hedging cost which are AIP free. We adopt the definition and concepts of [13, Section 1]. Precisely, the goal is to understand whether the cost we have defined in the last sections under AIP is compatible in the sense that it does not create immediate profits in the extended market where it is possible to trade the additional asset whose initial value is  $p(Z)$ , the infimum super-hedging cost of the terminal claim  $Z \in L^0(\mathbb{R}, \mathcal{F})$ . In that case, such cost is said immediate profit-free, in the spirit of [13, Definition 1.29] where the concept was introduced for the no-arbitrage condition and the super-replication price. This problem naturally arises in the models under NA but

also, more generally, for other types of no-arbitrage condition, as in [34]. In the following,  $Z \in L^0(\mathbb{R}, \mathcal{F})$  is fixed and  $p(Z)$  is defined in Definition 2.1.

As in Definition 2.11, AIP in the extended market is equivalent to  $p^{Y,Z}(0) = 0$  where  $p^{Y,Z}(0)$  is the super-hedging cost of 0 in the extended market i.e. the conditional essential infimum of the  $x \in L^0(\mathbb{R}, \mathcal{H})$  such that  $x + \alpha(Z - p(Z)) + \theta(Y - y) \geq 0$  a.s. where  $\alpha \in L^0(\mathbb{R}, \mathcal{H})$  and  $\theta \in L^0(\mathbb{R}^d, \mathcal{H})$ .

**Theorem 2.22.** *The AIP condition holds if and only if it holds true in the extended market, i.e. the market with the additional asset  $(p(Z), Z)$ .*

*Proof.* If the extended market satisfies AIP, then Proposition 2.12 implies that  $0_{\mathbb{R}^2} \in \text{convsupp}_{\mathcal{H}}(\hat{Y} - \hat{y})$  a.s. where  $\hat{y} = (y, p(Z))$  and  $\hat{Y} = (Y, Z)$ . As

$$\text{supp}_{\mathcal{H}}(\hat{Y} - \hat{y}) \subset \text{supp}_{\mathcal{H}}(Y - y) \times \text{supp}_{\mathcal{H}}(Z - p(Z)), \quad (2.16)$$

$0 \in \text{convsupp}_{\mathcal{H}}(Y - y)$  a.s. and the initial market satisfies AIP as well. Reciprocally, suppose that the initial market satisfies AIP. Consider a super-hedging price  $x \in L^0(\mathbb{R}, \mathcal{H})$  in the extended market for the zero claim at time 1 i.e. such that  $x + \alpha(Z - p(Z)) + \theta(Y - y) \geq 0$  a.s. where  $\alpha \in L^0(\mathbb{R}, \mathcal{H})$  and  $\theta \in L^0(\mathbb{R}^d, \mathcal{H})$ . We show below that  $x \geq 0$  a.s. which will imply that  $p^{Y,Z}(0) = 0$  and thus AIP in the extended market. Let  $A^1 = \{\alpha < 0\} \in \mathcal{H}$ . Then,

$$\left(\frac{x}{-\alpha} + p(Z)\right) 1_{A^1} + \frac{\theta 1_{A^1}}{-\alpha} (Y - y) \geq Z 1_{A^1}.$$

Let  $\bar{x} \in \mathcal{P}(Z)$  and  $\bar{\theta} \in L^0(\mathbb{R}^d, \mathcal{H})$  such that  $\bar{x} + \bar{\theta}(Y - y) \geq Z$  a.s. Thus

$$\left(\bar{x} 1_{\Omega \setminus A^1} + \left(\frac{x}{-\alpha} + p(Z)\right) 1_{A^1}\right) + \left(\bar{\theta} 1_{\Omega \setminus A^1} + \frac{\theta 1_{A^1}}{-\alpha}\right) (Y - y) \geq Z$$

and  $\bar{x} 1_{\Omega \setminus A^1} + \left(\frac{x}{-\alpha} + p(Z)\right) 1_{A^1} \geq p(Z)$  and we conclude that  $x \geq 0$  on  $\{\alpha < 0\}$ .

On the set  $A^2 = \{\alpha = 0\} \in \mathcal{H}$ , we have  $x 1_{A^2} + \theta 1_{A^2} (Y - y) \geq 0$ . Therefore,  $x 1_{A^2} \geq p(0) = 0$  since AIP holds true for the initial market defined only by  $(y, Y)$ . We deduce that  $x \geq 0$  on  $\{\alpha = 0\}$ .

At last, before analyzing the problem on the set  $A^3 = \{\alpha > 0\} \in \mathcal{H}$ , let us recall that as  $\mathcal{P}(Z)$  is downward-directed there exists  $z_n \in \mathcal{P}(Z)$ , for all  $n$  such that  $p(Z) = \lim_n \downarrow z^n$ . Fix  $n > 0$ . Let  $k_n = \inf\{k, z^k < p(Z) + n^{-1}\}$ . Then, by a measurable selection argument one may assume that  $k_n \in L^0(\mathbb{R}, \mathcal{H})$ . Let  $r^n = \sum_{l \geq 0} z^l 1_{\{k_n = l\}}$ . As  $\{k_n = l\} \in \mathcal{H}$  and  $z^l \in \mathcal{P}(Z)$ ,  $r^n \in \mathcal{P}(Z)$  and  $r^n + \theta^n (Y - y) \geq Z$  for some  $\theta^n \in L^0(\mathbb{R}^d, \mathcal{H})$ . Hence  $p(Z) \leq r^n \leq p(Z) + n^{-1}$  and  $Z - p(Z) \leq \theta^n (Y - y) + n^{-1}$ . Therefore, on the set  $A^3$ , we have that

$$\frac{x}{\alpha} + n^{-1} + \left(\theta^n + \frac{\theta}{\alpha}\right) (Y - y) \geq \frac{x}{\alpha} + (Z - p(Z)) + \frac{\theta}{\alpha} (Y - y) \geq 0$$

and  $\left(\frac{x}{\alpha} + n^{-1}\right) 1_{A^3} \geq p(0) = 0$  as AIP holds for the initial market defined by  $(y, Y)$ . Therefore, as  $n \rightarrow \infty$ , we deduce that  $x \geq 0$  on  $\{\alpha > 0\}$ . The conclusion follows.  $\square$

**Corollary 2.23.** *Suppose that AIP holds. Then,  $p(Z) \in [\text{ess inf}_{\mathcal{H}} Z, \text{ess sup}_{\mathcal{H}} Z] \cap \mathbb{R}$  a.s.*

*Proof.* Suppose that AIP holds. Then the extended market i.e. the market with the additional asset  $(p(Z), Z)$  satisfies the AIP condition by the theorem above. Thus (2.16) shows that  $0 \in \text{convsupp}_{\mathcal{H}}(Z - p(Z))$  a.s. and Remark 2.13 implies that  $p(Z) \in [\text{ess inf}_{\mathcal{H}} Z, \text{ess sup}_{\mathcal{H}} Z] \cap \mathbb{R}$  a.s.  $\square$

**Remark 2.24.** Remark that Theorem 2.22 does not hold true in an incomplete market for the super-replication price under the no-arbitrage condition NA. Indeed, let us consider a one step incomplete market. Let  $Z \in L^0(\mathbb{R}, \mathcal{F})$  be a non replicable contingent claim. Then [13, Theorem 1.32] implies that there exists  $\theta \in \mathbb{R}^d$  such that  $z + \theta(Y - y) \geq Z$  a.s. where  $z = \inf\{x \in \mathbb{R}, \exists \theta \in \mathbb{R}^d, x + \theta(Y - y) \geq Z \text{ a.s.}\}$  is the super-replication price of  $Z$ . As  $Z$  is not replicable  $P(z + \theta(Y - y) > Z) > 0$  and  $\theta(Y - y) - (Z - z)$  is a.s. non negative and strictly positive with strictly positive probability. There exists an arbitrage opportunity in the extended market  $((y, Y), (z, Z))$ .

## 2.5 Super-hedging cost under AIP

We now provide the characterization of the infimum super-hedging cost under the AIP condition.

**Corollary 2.25.** *Suppose that AIP holds true. Let  $g$  be a proper  $\mathcal{H}$ -normal integrand such that there exists some concave function  $\varphi$  verifying that  $g \leq \varphi < \infty$  on  $\text{convsupp}_{\mathcal{H}}Y$ . Then, a.s.*

$$\begin{aligned} p(g) &= \overline{\text{conc}}(g, \text{supp}_{\mathcal{H}}Y)(y) \\ &= \inf \{ \alpha y + \beta, \alpha \in \mathbb{R}^d, \beta \in \mathbb{R}, \alpha x + \beta \geq g(x), \forall x \in \text{supp}_{\mathcal{H}}Y \}. \end{aligned} \quad (2.17)$$

If  $g$  is concave and u.s.c.,  $p(g) = g(y)$  a.s.

*Proof.* The two equalities are direct consequence of Theorem 2.8. If  $g$  is concave and u.s.c., the result is trivial.  $\square$

We finish the one-period analysis with the computation of the infimum super-hedging cost of a convex derivative when  $d = 1$ . In this case, the cost is in fact a super-hedging price and we get the super-hedging strategy explicitly.

**Corollary 2.26.** *Suppose that AIP holds true and that  $d = 1$ . Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a non-negative convex function with  $\text{dom } g = \mathbb{R}$  and  $\lim_{x \rightarrow \infty} x^{-1}g(x) = M \in [0, \infty)$ , then a.s.*

$$p(g) = \theta^* y + \beta^* = g(\text{ess inf}_{\mathcal{H}}Y) + \theta^* (y - \text{ess inf}_{\mathcal{H}}Y), \quad (2.18)$$

$$\theta^* = \frac{g(\text{ess sup}_{\mathcal{H}}Y) - g(\text{ess inf}_{\mathcal{H}}Y)}{\text{ess sup}_{\mathcal{H}}Y - \text{ess inf}_{\mathcal{H}}Y}, \quad (2.19)$$

where we use the conventions  $\theta^* = \frac{0}{0} = 0$  in the case  $\text{ess sup}_{\mathcal{H}}Y = \text{ess inf}_{\mathcal{H}}Y$  a.s. and  $\theta^* = \frac{g(\infty)}{\infty} = M$  if  $\text{ess inf}_{\mathcal{H}}Y < \text{ess sup}_{\mathcal{H}}Y = +\infty$  a.s. Moreover,  $p(g) \in \mathcal{P}(g)$ .

*Proof.* As  $g$  is convex, the relative concave envelop of  $g$  with respect to  $\text{supp}_{\mathcal{H}}Y$  is the affine function that coincides with  $g$  on the extreme points of the interval  $\text{convsupp}_{\mathcal{H}}Y$  and (2.18) and (2.19) follow from (2.17) and Remark 2.13. Then using (2.17) and (2.18), we get that (recall that  $Y \in \text{supp}_{\mathcal{H}}Y$ )

$$p(g) + \theta^*(Y - y) = \theta^*Y + \beta^* \geq g(Y) \text{ a.s.} \quad (2.20)$$

and  $p(g) \in \mathcal{P}(g)$  follows.  $\square$

**Example 2.27.** We compute the price of a call option under AIP in the case  $d = 1$ . Let  $G = g(Y) = (Y - K)_+$  for some  $K \geq 0$ .

- If  $K \geq \text{ess sup}_{\mathcal{H}} Y$  then  $Y - K \leq \text{ess sup}_{\mathcal{H}} Y - K$  and  $G = 0$ . As AIP condition holds true,  $p(g) = p(0) = 0$ .
- If  $K \leq \text{ess inf}_{\mathcal{H}} Y$  then  $Y - K \geq \text{ess inf}_{\mathcal{H}} Y - K$  and  $G = Y - K$ . As  $g$  is concave and u.s.c.,  $p(g) = g(y) = y - K$  a.s.
- If  $\text{ess inf}_{\mathcal{H}} Y \leq K \leq \text{ess sup}_{\mathcal{H}} Y$ . Then, (2.19) and (2.18) imply that

$$p(g) = \frac{\text{ess sup}_{\mathcal{H}} Y - K}{\text{ess sup}_{\mathcal{H}} Y - \text{ess inf}_{\mathcal{H}} Y} (y - \text{ess inf}_{\mathcal{H}} Y)$$

on  $\{\text{ess sup}_{\mathcal{H}} Y \neq \text{ess inf}_{\mathcal{H}} Y\}$  and 0 else. So  $p(g) = 0$  if and only if  $y = \text{ess inf}_{\mathcal{H}} Y$  or  $\text{ess sup}_{\mathcal{H}} Y = \text{ess inf}_{\mathcal{H}} Y$ . A non-negative call option can have a zero price.

We finish with an example of computation for a call price under AIP but when there is some arbitrage opportunity. We choose a simple model that will be studied in Section 4. We assume a.s. that for  $y > 0$ ,  $\text{ess inf}_{\mathcal{H}} Y = dy$  and  $\text{ess sup}_{\mathcal{H}} Y = uy$  for two constants  $u$  and  $d$ . From Remark 2.13, AIP is equivalent to  $d \leq 1 \leq u$ . If ( $d = 1$  and  $u > 1$ ) or ( $u = 1$  and  $d < 1$ ), AIP holds but the NA condition does not hold true. Suppose that  $d = 1$  and  $u > 1$ . If  $K \geq y$ , the super-replication price under AIP is zero and if  $K \leq y$ , it is  $y - K$ . The same holds true if  $u = 1$  and  $d < 1$ .

### 3 The multi-period framework

#### 3.1 Multi-period super-hedging prices

For every  $t \in \{0, \dots, T\}$ , the set  $\mathcal{R}_t^T$  of all claims that can be super-replicated from the zero initial endowment at time  $t$  is defined by

$$\mathcal{R}_t^T := \left\{ \sum_{u=t+1}^T \theta_{u-1} \Delta S_u - \epsilon_T^+, \theta_{u-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{u-1}), \epsilon_T^+ \in L^0(\mathbb{R}_+, \mathcal{F}_T) \right\}. \quad (3.21)$$

The set of (multi-period) super-hedging prices and the (multi-period) infimum super-hedging cost of some contingent claim  $g_T \in L^0(\mathbb{R}, \mathcal{F}_T)$  at time  $t$  are given for all  $t \in \{0, \dots, T\}$ , by

$$\begin{aligned} \mathcal{P}_{T,T}(g_T) &= \{g_T\} \text{ and } \pi_{T,T}(g_T) = g_T \\ \mathcal{P}_{t,T}(g_T) &= \{x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists R \in \mathcal{R}_t^T, x_t + R = g_T \text{ a.s.}\} \\ \pi_{t,T}(g_T) &= \text{ess inf}_{\mathcal{F}_t} \mathcal{P}_{t,T}(g_T). \end{aligned} \quad (3.22)$$

As in the one-period case, it is clear that the infimum super-hedging cost is not necessarily a price in the sense that  $\pi_{t,T}(g_T) \notin \mathcal{P}_{t,T}(g_T)$  when  $\mathcal{P}_{t,T}(g_T)$  is not closed.

We now define a local version of super-hedging prices. The set of one-step super-hedging prices of the payoff  $g_{t+1} \in L^0(\mathbb{R}, \mathcal{F}_{t+1})$  and its associated infimum super-hedging cost are given by

$$\begin{aligned} \mathcal{P}_{t,t+1}(g_{t+1}) &= \left\{ x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists \theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t), x_t + \theta_t \Delta S_{t+1} \geq g_{t+1} \text{ a.s.} \right\} \\ \pi_{t,t+1}(g_{t+1}) &= \text{ess inf}_{\mathcal{F}_t} \mathcal{P}_{t,t+1}(g_{t+1}) \\ &= \text{ess inf}_{\mathcal{F}_t} \left\{ \text{ess sup}_{\mathcal{F}_t} (g_{t+1} - \theta_t \Delta S_{t+1}), \theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t) \right\}, \end{aligned} \quad (3.23)$$



see (2.6). In the following, we extend the definition of  $\mathcal{P}_{t,u}(g_u)$ ,  $u \geq t+1$ , so that the argument  $g_u$  may be a subset  $G_u \subseteq L^0(\mathbb{R}, \mathcal{F}_u)$ . Precisely,  $\mathcal{P}_{t,u}(G_u) = \cup_{g_u \in G_u} \mathcal{P}_{t,u}(g_u)$ . The following lemma makes the link between local and global super-hedging prices. It also provides a dynamic programming principle, meaning that the prices are time consistent.

**Lemma 3.1.** *Let  $g_T \in L^0(\mathbb{R}, \mathcal{F}_T)$  and  $t \in \{0, \dots, T-1\}$ . Then*

$$\mathcal{P}_{t,T}(g_T) = \mathcal{P}_{t,t+1}(\mathcal{P}_{t+1,T}(g_T)) \text{ and } \pi_{t,T}(g_T) \geq \pi_{t,t+1}(\pi_{t+1,T}(g_T)).$$

Moreover, assume that  $\pi_{t+1,T}(g_T) \in \mathcal{P}_{t+1,T}(g_T)$ . Then

$$\mathcal{P}_{t,T}(g_T) = \mathcal{P}_{t,t+1}(\pi_{t+1,T}(g_T)) \text{ and } \pi_{t,T}(g_T) = \pi_{t,t+1}(\pi_{t+1,T}(g_T)).$$

*Proof.* Consider  $x_t \in \mathcal{P}_{t,T}(g_T)$ . Then, for all  $u \in \{t, \dots, T-1\}$ , there exist  $\theta_u \in L^0(\mathbb{R}^d, \mathcal{F}_u)$  such that

$$x_t + \theta_t \Delta S_{t+1} + \sum_{u=t+2}^T \theta_{u-1} \Delta S_u \geq g_T \text{ a.s.}$$

We deduce that  $x_{t+1} := x_t + \theta_t \Delta S_{t+1} \in \mathcal{P}_{t+1,T}(g_T)$ . Moreover  $x_t \in \mathcal{P}_{t,t+1}(x_{t+1})$  and  $x_t \in \mathcal{P}_{t,t+1}(\mathcal{P}_{t+1,T}(g_T))$ . Reciprocally, suppose that  $x_t \in \mathcal{P}_{t,t+1}(\mathcal{P}_{t+1,T}(g_T))$ , i.e.  $x_t \in \mathcal{P}_{t,t+1}(x_{t+1})$  for some  $x_{t+1} \in \mathcal{P}_{t+1,T}(g_T)$ . Then,  $x_t + \theta_t \Delta S_{t+1} \geq x_{t+1}$  for some  $\theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)$  and  $x_{t+1} + \sum_{u=t+2}^T \theta_{u-1} \Delta S_u \geq g_T$  where  $\theta_u \in L^0(\mathbb{R}^d, \mathcal{F}_u)$  for all  $u \geq t+1$ . It follows that  $x_t + \sum_{u=t+1}^T \theta_{u-1} \Delta S_u \geq g_T$  a.s. and  $x_t \in \mathcal{P}_{t,T}(g_T)$ . Let  $x_t \in \mathcal{P}_{t,T}(g_T) = \mathcal{P}_{t,t+1}(\mathcal{P}_{t+1,T}(g_T))$ , then there exists  $\theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)$  and  $x_{t+1} \in \mathcal{P}_{t+1,T}(g_T)$  such that

$$x_t + \theta_t \Delta S_{t+1} \geq x_{t+1} \geq \text{ess inf}_{\mathcal{F}_{t+1}} \mathcal{P}_{t+1,T}(g_T) = \pi_{t+1,T}(g_T) \text{ a.s.}$$

Thus  $x_t \in \mathcal{P}_{t,t+1}(\pi_{t+1,T}(g_T))$  and  $\mathcal{P}_{t,T}(g_T) \subset \mathcal{P}_{t,t+1}(\pi_{t+1,T}(g_T))$ . Moreover  $x_t \geq \pi_{t,t+1}(\pi_{t+1,T}(g_T))$  and the first statement follows. If  $\pi_{t+1,T}(g_T) \in \mathcal{P}_{t+1,T}(g_T)$ , then  $\mathcal{P}_{t,t+1}(\pi_{t+1,T}(g_T)) \subset \mathcal{P}_{t,t+1}(\mathcal{P}_{t+1,T}(g_T)) = \mathcal{P}_{t,T}(g_T)$ .  $\square$

**Remark 3.2.** Under AIP, if at each step,  $\pi_{t+1,T}(g_T) \in \mathcal{P}_{t+1,T}(g_T)$  and if we have  $\pi_{t+1,T}(g_T) = g_{t+1}(S_{t+1})$  for some “nice”  $\mathcal{F}_t$ -normal integrand  $g_{t+1}$ , we will get from Corollary 2.25 that  $\pi_{t,T}(g_T) = \overline{\text{conc}}(g_{t+1}, \text{supp}_{\mathcal{F}_t} S_{t+1})(S_t)$  a.s. We will propose in Section 4 a quite general setting where this holds true.

Note that the super-hedging problem is solved for general claims  $\xi_T$  through the formula (3.23). For claims of Asian type  $g((S_u)_{u \leq T})$  or of American type, what we propose for European claims could be easily adapted. Consider a general claim  $\xi_T$  and the natural filtration i.e. the one generated by the price process  $S$ . Then for any self-financing portfolio  $\theta$ ,  $V_T = x + \sum_{t=1}^T \theta_{t-1} \Delta S_t \geq \xi_T$  if and only if  $V_T \geq \tilde{\xi}_T$  where  $\tilde{\xi}_T = \text{ess sup}_{\mathcal{F}_T} \xi_T$ . Thus  $\xi_T$  and  $\tilde{\xi}_T$  have the same super-replication cost and as  $\tilde{\xi}_T$  is  $\mathcal{F}_T$ -measurable, it is of the form  $\tilde{\xi}_T = g((S_u)_{u \leq T})$ . Of course, in practice, it is necessary to have an idea about  $g$  but the same difficulty arises under the NA condition.

### 3.2 Multi-period AIP

**Definition 3.3.** *The AIP condition holds true if for all  $t \in \{0, \dots, T\}$*

$$\mathcal{P}_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\}.$$

We now study the link between global and local immediate profit. The global (resp. local) profit means that it is possible to super-replicate from a negative cost at time  $t$  the claim 0 payed at time  $T$  (resp. time  $t+1$ ). The next proposition shows that the local and global AIP conditions are equivalent in the following sense.

**Proposition 3.4.** *The following assertions are equivalent.*

1.  $\mathcal{P}_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\}$  for all  $t \in \{0, \dots, T-1\}$  i.e. AIP.
2.  $\mathcal{P}_{t,t+1}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\}$  for all  $t \in \{0, \dots, T-1\}$ .
3.  $S_t \in \text{convsupp}_{\mathcal{F}_t} S_{t+1}$  a.s. or  $0 \in \text{convsupp}_{\mathcal{F}_t} \Delta S_{t+1}$  a.s. for all  $t \in \{0, \dots, T-1\}$ .
4.  $\sigma_{\text{supp}_{\mathcal{F}_t} \Delta S_{t+1}} \geq 0$  a.s. for all  $t \in \{0, \dots, T-1\}$ .
5.  $\pi_{t,T}(0) = 0$  a.s. for all  $t \in \{0, \dots, T-1\}$ .

*Proof.* For some fix  $t \in \{0, \dots, T-1\}$  we show that

$$\mathcal{P}_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\} \iff \pi_{t,T}(0) = 0. \quad (3.24)$$

For the implication, note that  $\pi_{t,T}(0) \leq 0$  is always true. Let  $x_t \in \mathcal{P}_{t,T}(0)$ . Then there exist  $\theta_u \in L^0(\mathbb{R}^d, \mathcal{F}_u)$  for  $u \geq t$  such that  $x_t + \sum_{u=t+1}^T \theta_{u-1} \Delta S_u \geq 0$  a.s. Thus  $x_t 1_{x_t < 0} + \sum_{u=t+1}^T \theta_{u-1} 1_{x_t < 0} \Delta S_u \geq 0$  a.s. If  $P(x_t < 0) > 0$  then  $x_t 1_{x_t < 0} \in \mathcal{P}_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t)$ , a contradiction. Thus  $x_t \geq 0$  a.s. and  $\pi_{t,T}(0) = \text{ess inf}_{\mathcal{F}_t} \mathcal{P}_{t,T}(0) \geq 0$ . For the reverse implication let  $x_t \in \mathcal{P}_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t)$ . If  $P(x_t < 0) > 0$  then  $x_t 1_{x_t < 0} \in \mathcal{P}_{t,T}(0)$  and  $x_t 1_{x_t < 0} \geq \pi_{t,T}(0) = 0$  a.s., a contradiction. Thus  $x_t \geq 0$  a.s. and  $x_t = 0$  a.s. follows.

It is clear that (3.24) implies that 1. is equivalent to 5. Now we show that 1. is equivalent to 2. Suppose that 1. holds. Then,  $\pi_{t+1,T}(0) = 0 \in \mathcal{P}_{t+1,T}(0)$  and Lemma 3.1 implies that  $\mathcal{P}_{t,T}(0) = \mathcal{P}_{t,t+1}(\pi_{t+1,T}(0)) = \mathcal{P}_{t,t+1}(0)$ . and 2. holds. Reciprocally suppose that 2. holds. Then,  $\mathcal{P}_{T-1,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_{T-1}) = \{0\}$ . From (3.24) with  $t = T-1$  we get that  $\pi_{T-1,T}(0) = 0 \in \mathcal{P}_{T-1,T}(0)$  and Lemma 3.1 implies that

$$\mathcal{P}_{T-2,T}(0) = \mathcal{P}_{T-2,T-1}(\pi_{T-1,T}(0)) = \mathcal{P}_{T-2,T-1}(0) = L^0([0, \infty), \mathcal{F}_{T-2}).$$

It is trivial  $L^0([0, \infty), \mathcal{F}_{T-2}) \subset \mathcal{P}_{T-2,T-1}(0)$ . As  $\mathcal{P}_{T-2,T-1}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_{T-2}) = \{0\}$ , the same reasoning as in the proof of the implication in (3.24) proves the reverse inclusion. It follows that  $\mathcal{P}_{T-2,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_{T-2}) = \{0\}$ . Using backward induction, 1. holds true. By Proposition 2.12 and Definition 2.11, we conclude that 3. and 4. are equivalent to 2.  $\square$

### 3.3 Comparison with other no-arbitrage conditions

In this section we compare again with others no-arbitrage conditions. In particular, we study a condition stronger than AIP in the spirit of the No free Lunch condition, i.e., by considering the closure of the set  $\mathcal{R}_t^T$ . Before that, we first recall the classical multiperiod no-arbitrage NA condition.

**Definition 3.5.** *The no-arbitrage NA condition holds if for all  $t \in \{0, \dots, T\}$ ,*

$$\mathcal{R}_t^T \cap L^0(\mathbb{R}_+, \mathcal{F}_T) = \{0\}.$$

It is easy to see that the NA condition can also be formulated as follows :  $V_T^{0,\theta} \geq 0$  a.s. implies that  $V_T^{0,\theta} = 0$  a.s. Recall that the set of all super-hedging prices for the zero claim at time  $t$  is given by  $\mathcal{P}_{t,T}(0) = (-\mathcal{R}_t^T) \cap L^0(\mathbb{R}, \mathcal{F}_t)$  (see (3.21) and (3.22)). It follows that (see Definition 3.3)

$$\text{AIP reads as } \mathcal{R}_t^T \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \{0\} \text{ for all } t \in \{0, \dots, T\}.$$

It is clear that the NA condition implies the AIP one and, as already mentioned, the equivalence does not hold true: The AIP condition is strictly weaker than the NA one. As in the one period case, this is also true for the NUPBR and NFLVR conditions (see Lemma 2.21). We now consider the No Unbounded Increasing Profit (NUIP) no-arbitrage condition of [22], see also [14], that we compare to the AIP condition. The NUIP condition means the absence of non decreasing portfolio process  $V$  starting from zero and such that  $P(V_T > 0) > 0$ .

**Lemma 3.6.** *The condition NUIP implies AIP and AIP may be strictly weaker.*

*Proof.* Assume that AIP fails. By Proposition 3.4, there exists  $t \in \{0, \dots, T-1\}$  such that  $\mathcal{P}_{t,t+1}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) \neq \{0\}$ . Let  $p_t \in L^0(\mathbb{R}_-, \mathcal{F}_t) \setminus \{0\}$  such that  $p_t + \theta_t \Delta S_{t+1} \geq 0$  for some  $\theta_t \in L^0(\mathbb{R}, \mathcal{F}_t)$ . Let us define  $V_u = 0$  if  $u \leq t$  and  $V_u = \theta_t \Delta S_{t+1}$  if  $u \geq t+1$ . Since  $p_t$  is a.s. non positive and strictly negative with strictly positive probability,  $V$  is non decreasing and  $P(V_T > 0) > 0$  hence an unbounded increasing profit.

Consider  $T = 2$  and the model where for  $i \in \{0, 1\}$ ,  $S_{i+1} \in \{S_i, 2S_i\}$  with  $S_0 > 0$  and  $P(S_{i+1} = 2S_i) \in (0, 1)$ . Then for  $i \in \{0, 1\}$ ,  $\text{ess inf}_{\mathcal{F}_i} S_{i+1} = S_i \leq S_i \leq \text{ess sup}_{\mathcal{F}_i} S_{i+1} = 2S_i$  and AIP holds true. However, the constant strategy equal to one and starting from zero generates an unbounded increasing profit since  $V_1 = \Delta S_1 \in \{0, S_0\}$  and  $V_2 - V_1 = \Delta S_2 \in \{0, S_1\}$  are a.s. non negative and strictly positive with strictly positive probability. Thus AIP does not necessarily implies NUIP.  $\square$

As in the one period case, we able to prove that the super-hedging cost is immediate profit-free i.e. does not create immediate profits in the extended market where it is possible to trade the additional asset. In the following,  $C_T \in L^0(\mathbb{R}, \mathcal{F}_T)$  is fixed and  $C_t = \pi_{t,t+1}(C_{t+1})$  is defined recursively for  $t \leq T-1$  by (3.23).

**Theorem 3.7.** *The AIP condition holds if and only if the extended market, i.e. the market with the additional asset  $(C_t)_{t \in \{0, \dots, T\}}$  satisfies AIP.*

*Suppose that AIP holds. Then,  $\text{ess inf}_{\mathcal{F}_t} C_{t+1} \leq C_t \leq \text{ess sup}_{\mathcal{F}_t} C_{t+1}$  a.s. for all  $t \in \{0, \dots, T-1\}$ .*

*Proof.* Proposition 3.4 implies that AIP holds true if and only if  $\mathcal{P}_{t,t+1}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\}$ , for all  $t \in \{0, \dots, T-1\}$  and Theorem 2.22 shows that this last condition is equivalent to  $\pi_{t,t+1}^{S,C}(0) = 0$  where  $\pi_{t,t+1}^{S,C}(0)$  is the super-hedging cost in the extended market of 0 i.e. the conditional essential infimum of the  $p_t \in L^0(\mathbb{R}, \mathcal{F}_t)$  such that  $p_t + \alpha_t \Delta C_{t+1} + \theta_t \Delta S_{t+1} \geq 0$  a.s. where  $\alpha_t \in L^0(\mathbb{R}, \mathcal{F}_t)$  and  $\theta_t \in L^0(\mathbb{R}^d, \mathcal{F}_t)$ . Using again Proposition 3.4 (in the extended market) this is equivalent to AIP in the extended market. The last assertion follows from Corollary 2.23.  $\square$

We now introduce a weaker form of IP.

**Definition 3.8.** *The absence of weak immediate profit (AWIP) condition holds true if for all  $t \in \{0, \dots, T\}$*

$$\overline{\mathcal{R}_t^T} \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \{0\},$$

where the closure of  $\mathcal{R}_t^T$  is taken with respect to the convergence in probability.

We shall see in Lemma 3.11 that the AIP condition is not necessarily equivalent to AWIP. Before, in the case  $d = 1$ , we show that AWIP may be equivalent to AIP condition under an extra closeness condition. It also provides a characterization through (absolutely continuous) martingale measures.

**Theorem 3.9.** *Assume that  $d = 1$ . The following statements are equivalent.*

1. AWIP holds.
2. For every  $t \in \{0, \dots, T\}$ , there exists  $Q \ll P$  with  $\mathbb{E}(dQ/dP|\mathcal{F}_t) = 1$  such that  $(S_u)_{u \in \{t, \dots, T\}}$  is a  $Q$ -martingale.
3. AIP holds and  $\overline{\mathcal{R}_t^T} \cap L^0(\mathbb{R}, \mathcal{F}_t) = \mathcal{R}_t^T \cap L^0(\mathbb{R}, \mathcal{F}_t)$  for every  $t \in \{0, \dots, T\}$ .

The proof, which is postponed to the appendix, is based on classical Hahn-Banach Theorem arguments, see for example the textbooks of [10] and [20].

**Remark 3.10.** From above, it is clear that AIP and AWIP are equivalent if  $\mathcal{R}_t^T$  is closed. Therefore, Lemma 3.11 implies that  $\mathcal{R}_t^T$  is not necessarily closed under AIP.

Suppose now that  $P(\text{ess inf}_{\mathcal{F}_t} S_{t+1} = S_t) = P(\text{ess sup}_{\mathcal{F}_t} S_{t+1} = S_t) = 0$  for all  $t \in \{0, \dots, T-1\}$ . Then, using Proposition 2.18, AIP is equivalent to NA. Under NA, the set  $\mathcal{R}_t^T$  is closed in probability for every  $t \in \{0, \dots, T-1\}$  and Theorem 3.9 implies that AWIP, AIP and NA are equivalent conditions.

**Lemma 3.11.** *The AIP condition is not necessarily equivalent to AWIP.*

*Proof.* Assume that  $d = 1$ . Let us consider a positive process  $(\tilde{S}_t)_{t \in \{0, \dots, T\}}$  which is a  $P$ -martingale. We suppose that  $\text{ess inf}_{\mathcal{F}_0} \tilde{S}_1 < \tilde{S}_1$  a.s., which holds in particular if  $\tilde{S}$  a geometric Brownian motion as  $\text{ess inf}_{\mathcal{F}_0} \tilde{S}_1 = 0$  a.s. Let us define  $S_t := \tilde{S}_t$  for  $t \in \{1, \dots, T\}$  and  $S_0 := \text{ess inf}_{\mathcal{F}_0} S_1$ . We have  $\text{ess inf}_{\mathcal{F}_0} S_1 \leq S_0$  and  $\text{ess sup}_{\mathcal{F}_0} S_1 \geq \text{ess inf}_{\mathcal{F}_0} S_1 = S_0$  hence AIP holds at time 0 (see Remark 2.13). Moreover, by the martingale property (see Theorem 3.9), AIP and also AWIP hold at any time  $t \in \{1, \dots, T\}$ . Let us suppose that AWIP holds true at  $t = 0$ . Using Theorem 3.9, there exists  $\rho_T \geq 0$  with  $\mathbb{E}(\rho_T) = 1$  such that  $S$  is a  $Q$ -martingale where  $dQ = \rho_T dP$ . Therefore,  $\mathbb{E}(\rho_T \Delta S_1) = 0$ . Since  $\Delta S_1 > 0$  by assumption, we deduce that  $\rho_T = 0$  hence a contradiction.  $\square$

#### 4 Explicit pricing of a convex payoff under AIP

In the first section, we obtain the same computative scheme (see (4.25)) as in [8] but assuming only AIP and not NA. Then this result is illustrated through a numerical experiment in Section 4.2.

## 4.1 The algorithm

**Proposition 4.1.** *Suppose that the model is defined by  $\text{ess inf}_{\mathcal{F}_{t-1}} S_t = k_{t-1}^d S_{t-1}$  a.s. and  $\text{ess sup}_{\mathcal{F}_{t-1}} S_t = k_{t-1}^u S_{t-1}$  a.s. where  $(k_{t-1}^d)_{t \in \{1, \dots, T\}}$ ,  $(k_{t-1}^u)_{t \in \{1, \dots, T\}}$  and  $S_0$  are deterministic non-negative numbers.*

*The AIP condition holds true if and only if  $k_{t-1}^d \in [0, 1]$  and  $k_{t-1}^u \in [1, +\infty]$  for all  $t \in \{1, \dots, T\}$ .*

*Suppose that the AIP condition holds. Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a non-negative convex function with  $\text{dom } h = \mathbb{R}$  such that  $\lim_{z \rightarrow +\infty} \frac{h(z)}{z} \in [0, \infty)$ . Then the infimum super-hedging cost of the European contingent claim  $h(S_T)$  is a price and it is given by  $\pi_{t,T}(h) = h(t, S_t) \in \mathcal{P}_{t,T}(h(S_T))$  a.s. where*

$$\begin{aligned} h(T, x) &= h(x) \\ h(t-1, x) &= \lambda_{t-1} h(t, k_{t-1}^d x) + (1 - \lambda_{t-1}) h(t, k_{t-1}^u x), \end{aligned} \quad (4.25)$$

where  $\lambda_{t-1} = \frac{k_{t-1}^u - 1}{k_{t-1}^u - k_{t-1}^d} \in [0, 1]$  and  $1 - \lambda_{t-1} = \frac{1 - k_{t-1}^d}{k_{t-1}^u - k_{t-1}^d} \in [0, 1]$ , with the following conventions. When  $k_{t-1}^d = k_{t-1}^u = 1$  or  $S_{t-1} = 0$ ,  $\lambda_{t-1} = \frac{0}{0} = 0$  and  $1 - \lambda_{t-1} = 1$  and when,  $k_{t-1}^d < k_{t-1}^u = \infty$ ,  $\lambda_{t-1} = \frac{\infty}{\infty} = 1$ ,

$$(1 - \lambda_{t-1}) h(t, (+\infty)x) = (1 - k_{t-1}^d) x \frac{h(t, (+\infty)x)}{(+\infty)x} = (1 - k_{t-1}^d) x \lim_{z \rightarrow +\infty} \frac{h(z)}{z}. \quad (4.26)$$

Moreover, for every  $t \in \{1, \dots, T\}$ ,  $\lim_{z \rightarrow +\infty} \frac{h(z)}{z} = \lim_{z \rightarrow +\infty} \frac{h(t, z)}{z}$  and  $h(\cdot, x)$  is non-increasing for all  $x \geq 0$ .

In the proof, which is postponed to the appendix, the strategy associated to the infimum super-hedging price is given.

**Remark 4.2.** The infimum super-hedging cost of the European contingent claim  $h(S_T)$  in our model is a price, precisely the same than the price we get in a binomial model  $S_t \in \{k_{t-1}^d S_{t-1}, k_{t-1}^u S_{t-1}\}$  a.s.,  $t \in \{1, \dots, T\}$ . Moreover, as in Corollary 2.16, one can prove that the AIP condition holds at every instant  $t$  if and only if the super-hedging price of some European call option at  $t$  is non-negative.

## 4.2 Numerical experiments

### 4.2.1 Calibration

In this sub-section, we suppose that the discrete dates are given by  $t_i^n = \frac{iT}{n}$ ,  $i \in \{0, \dots, n\}$  where  $n \geq 1$ . We assume that for  $i \in \{1, \dots, n\}$

$$k_{i-1}^u = 1 + \sigma_{t_{i-1}^n} \sqrt{\Delta t_i^n} \quad \text{and} \quad k_{i-1}^d = 1 - \sigma_{t_{i-1}^n} \sqrt{\Delta t_i^n} \geq 0, \quad (4.27)$$

where  $t \mapsto \sigma_t$  is a positive Lipschitz-continuous function on  $[0, T]$ . In this sub-section our aim is to calibrate  $\sigma$ . Note that the assumptions on the multipliers  $k_{i-1}^u$  and  $k_{i-1}^d$  imply that

$$\left| \frac{S_{i+1}^n}{S_i^n} - 1 \right| \leq \sigma_{t_i^n} \sqrt{\Delta t_{i+1}^n}, \text{ a.s.} \quad (4.28)$$

By Proposition 4.1, the infimum super-hedging cost of the European Call option  $(S_T - K)^+$  is given by  $h^n(t_i^n, S_{t_i^n})$  where  $h^n$  is defined by (4.25) with terminal condition  $h^n(T, x) = g(x) = (x - K)^+$ . We extend the function  $h^n$  on  $[0, T]$  in such a way that  $h^n$  is constant on each interval  $[t_i^n, t_{i+1}^n[$ ,  $i \in \{0, \dots, n\}$ . Such a scheme is proposed by Milstein [27] where a convergence theorem is proved when the terminal condition, i.e. the payoff function, is smooth. Precisely, the sequence of functions  $(h^n(t, x))_{n \geq 1}$  converges uniformly to  $h(t, x)$ , solution of the Black and Scholes formula with time-dependent volatility:

$$\partial_t h(t, x) + \sigma_t^2 \frac{x^2}{2} \partial_{xx} h(t, x) = 0, \quad h(T, x) = g(x). \quad (4.29)$$

In [27], it is supposed that the successive derivatives of the solution of the P.D.E. solution  $h$  are uniformly bounded. This is not the case for the Call payoff function  $g$ . On the contrary the successive derivatives of the solution of the P.D.E. explode at the horizon date, see [24]. In [2], it is proven that the uniform convergence still holds when the payoff function is not smooth provided that the successive derivatives of the solution of the P.D.E. do not explode *too much*.

Supposing that  $\Delta t_i^n$  is closed to 0, we can identify the observed prices of the Call option with the theoretical limit prices  $h(t, S_t)$  at any instant  $t$  given by (4.29). For several strikes, matching the observed prices to the theoretical ones allows to deduce the associated *implied* volatility  $t \mapsto \sigma_t$ , see [11] for a study of the implied volatility.

The data set is composed of historical values of the french index CAC 40 and European call option prices of maturity 3 months from the 23rd of October 2017 to the 19th of January 2018. The values of  $S$  are distributed as in Figure 1.



Fig. 1: Distribution of the observed prices.

We compute the proportion of observations satisfying (4.28). The results are satisfactory for strikes lower than 5000, see Fig. 2. Note that, when the strikes are too large with respect to the current price  $S$ , price observations are less available for the calibration, see Figure 1. This could explain the degradation of our results.

#### 4.2.2 Super-hedging prices

We now propose an approach where the  $k^u$  and the  $k^d$  of Proposition 4.1 are not strike dependent as before and which is in this sense model free. The data set is

Strike	4800	4900	5000	5100	5200	5300	5400	5500	5600	5700	5800	5900	6000
Ratio	96,7%	95,1%	95,1%	88,5%	86,9%	80,3%	70,5%	78,7%	75,4%	77,0%	73,8%	75,4%	72,1%

Fig. 2

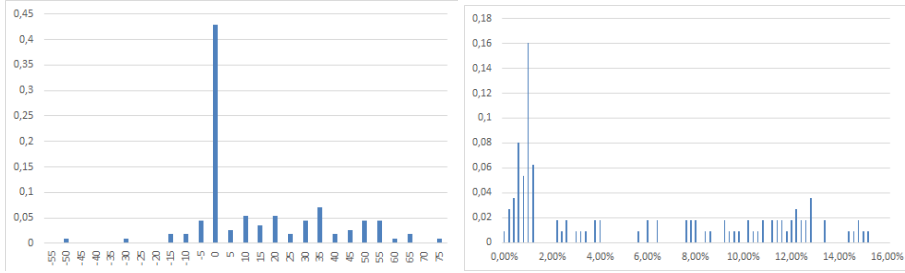
composed of historical daily closing values of the french index CAC 40 from the 5th of January 2015 to the 12th of March 2018. The chosen interval  $[0, T]$  corresponds to one week of 5 working days so that the discrete dates are  $t_i^4$ ,  $i \in \{0, \dots, 4\}$  and  $n = 4$ . We estimate the parameters of the model  $k_{t_{i-1}}^d$  and  $k_{t_{i-1}}^u$ . The first approach is called symmetric :  $\sigma_{t_i}$  is estimated as an upper bound in (4.28) and then the  $k^d$  and  $k^u$  are given in (4.27). The asymmetric approach is the intuitive approach where the empirical minimum and maximum are taken. For  $i \in \{0, \dots, 3\}$

$$\begin{aligned} \sigma_{t_i} &= \overline{\max} \left( \left| \frac{S_{t_{i+1}}}{S_{t_i}} - 1 \right| / \sqrt{\Delta t_{i+1}^4} \right) \\ k_{t_{i-1}}^d &= \overline{\min} \frac{S_{t_i}^n}{S_{t_{i-1}}^n} \text{ and } k_{t_{i-1}}^u = \overline{\max} \frac{S_{t_i}^n}{S_{t_{i-1}}^n}. \end{aligned} \quad (4.30)$$

Note that  $\overline{\max}$  (resp.  $\overline{\min}$ ), the empirical maximum (resp. min), is taken over a one year sliding sample window of 52 weeks. We estimate the parameters on 52 weeks and we implement our hedging strategy on the fifty third one. We then repeat the procedure by sliding the window of one week.

For a payoff function  $g(x) = (x - K)^+$ , we implement the super-hedging strategies  $(\theta_{t_i^4}^*)_{i \in \{0, \dots, 3\}}$  associated to the super-hedging cost given by Proposition 4.1 and (5.32). We denote by  $V_T$  the terminal value of our strategy starting from the super-hedging cost  $V_0 = \pi_{0,T} = h(0, S_0)$  i.e.  $V_T = V_0 + \sum_{i=0}^3 \theta_{t_i^4}^* \Delta S_{t_{i+1}^4}$ .

We study below the super-hedging error  $\varepsilon_T = V_T - (S_T - K)^+$  for different strikes. In the symmetric case, we present in Figures 3a and 3b the distribution of the super-hedging error  $\varepsilon_T$  and of  $V_0/S_0$  for  $K = 4700$ . The graphs in the asymmetric case are similar.



(a) Distribution of the super-hedging error  $\varepsilon_T = V_T - (S_T - K)^+$  for  $K = 4700$ . (b) Distribution of the ratio  $V_0/S_0$  for  $K = 4700$ .

Fig. 3: Symmetric case.

We now compare the result of both methods in the table below. In the symmetric (resp. asymmetric) case the empirical average of the error  $\varepsilon_T$  is 12.76 (resp.

	Mean of $V_0/S_0$	Variance of $V_0/S_0$	Mean of $\varepsilon_T$	Variance of $\varepsilon_T$	$P(\varepsilon_T < 0)$	VaR 95 %
Symmetric	5.61%	5.14 %	12.76	21.65	14.29%	-10.33
Asymmetric	5.52%	5.22%	9.47	14.20	8.04%	-1.81

Fig. 4: Comparison of the two methods of estimation for  $K = 4700$ . The mean of  $S_0$  is 4844,93 and the mean of  $(S_0 - K)^+$  is 278.73.

9.47) and its standard deviation is 21.65 (resp. 14.20). This result is rather satisfactory in comparison to the large value of the empirical mean of  $S_0$  which is equal to 4044. Notice that we observe  $E(S_T - K)^+ \simeq 282.69$ . This empirically confirms the efficiency of our suggested method. The empirical probability of  $\{\varepsilon_T < 0\}$  is equal to 14.29% (resp. 8.04%) but the Value at Risk at 95 % is  $-10.33$  (resp.  $-1.81$ ) which shows that our strategy is conservative.

Now we estimate the cost of our strategy in comparison with  $S_0$ . The empirical average of  $V_0/S_0$  is 5.61% (resp. 5.52%) and its standard deviation is 5.14% (resp. 5.22%). This means that  $V_0$  is much smaller than  $S_0$  which is the theoretical super-hedging price in some incomplete markets (this is for example the case when  $k^d = 0$  and  $k^u = \infty$ , in particular when the dynamics of  $S$  is modeled by a (discrete) geometric Brownian motion, see [7]). Note that the huge loss (50 in the symmetric case) is linked to so-called *black friday* week that occurs the 24th of June 2016. Large falls of risky assets were observed in European markets, mainly explained by the Brexit vote. In particular, the CAC 40 felt from 4340 to 4106, with a loss of  $-8\%$  on Friday.

We also present the “at the money” case  $K = S_0$ , see Fig. 5. We see that the results are better than for  $K = 4700$ :  $V_0$  is smaller with respect to  $S_0$  and the probability of loss is smaller.

	Mean of $V_0/S_0$	Variance of $V_0/S_0$	Mean of $\varepsilon_T$	Variance of $\varepsilon_T$	$P(\varepsilon_T < 0)$	VaR 95 %
Symmetric	1.51%	0.47 %	35.69	34.11	9.82 %	-11.41%
Asymmetric	1.47%	0.49%	33.37	32.78	12.50%	-9.29

Fig. 5: Comparison of the two methods of estimation for  $K = S_0$ . The mean of  $S_0$  is 4844.93.



Fig. 6

**Remark 4.3.** We have compared our results, i.e. when  $\sigma_{t_i}$  is given by (4.30), to the ones where  $\sigma_{t_i}$  is the implied volatility. The comparison is made for  $K = 5400$ , see Figure 6. The implementation of the classical approach gives worse results as we may expect. Indeed, as observed in the Section 4.2.1, the implied volatility does not capture the coefficients as well as the empirical observations do. Therefore, from the implied volatility, we get larger coefficients  $k^d$  and smaller coefficients  $k^u$ . Thus, the minimal super-hedging prices are smaller, as observed in Figure 6, but the super-hedging error is negative in most of the cases, see Figure 7. Our approach is definitively more conservative.



## 5 Appendix

*Proof of Lemma 2.4.* It is clear from (2.1) that for all  $\omega \in \Omega$ ,  $D_\mu(\omega)$  is a non-empty and closed subset of  $\mathbb{R}^d$ . We show that  $D_\mu$  is  $\mathcal{H}$ -measurable. Let  $O$  be a fixed open set in  $\mathbb{R}^d$  and  $\mu_O : \omega \in \Omega \mapsto \mu_O(\omega) := \mu(O, \omega)$ . As  $\mu$  is a stochastic kernel,  $\mu_O$  is  $\mathcal{H}$ -measurable. By definition of  $D_\mu(\omega)$  we get that  $\{\omega \in \Omega, D_\mu(\omega) \cap O \neq \emptyset\} = \{\omega \in \Omega, \mu_O(\omega) > 0\} \in \mathcal{H}$  and  $D_\mu$  is  $\mathcal{H}$ -measurable.  $\square$

*Proof of Lemma 2.6.* The two first statements follow from the construction of  $\text{ess sup}_{\mathcal{H}} X$  in Proposition 2.5 (see (2.2)). Suppose that  $\text{ess inf}_{\mathcal{H}} X \notin \text{supp}_{\mathcal{H}} X$  on some non-null measure subset  $\Lambda \in \mathcal{H}$  of  $\{\text{ess inf}_{\mathcal{H}} X > -\infty\}$ . As  $\text{supp}_{\mathcal{H}} X$  is  $\mathcal{H}$ -measurable and closed-valued, by a measurable selection argument, we deduce the existence of  $r \in L^0(\mathbb{R}_+, \mathcal{H})$  such that  $r > 0$  a.s. and  $(\text{ess inf}_{\mathcal{H}} X - r, \text{ess inf}_{\mathcal{H}} X + r) \subseteq \mathbb{R} \setminus \text{supp}_{\mathcal{H}} X$  on  $\Lambda$ . As  $X \in \text{supp}_{\mathcal{H}} X$  a.s. (see Remark 2.3) and  $X \geq \text{ess inf}_{\mathcal{H}} X$  a.s., we deduce that  $X \geq \text{ess inf}_{\mathcal{H}} X + r$  on  $\Lambda$ , which contradicts the definition of  $\text{ess inf}_{\mathcal{H}} X$ . The next statement is similarly shown and the last one follows.  $\square$

The proof of Proposition 2.7 is based on the two following useful lemmata.

**Lemma 5.1.** *Let  $\mathcal{K} : \Omega \rightarrow \mathbb{R}^d$  be a  $\mathcal{H}$ -measurable and closed-valued random set such that  $\text{dom } \mathcal{K} = \Omega$  and let  $h : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  be l.s.c. in  $x$ . Then,*

$$\sup_{x \in \mathcal{K}} h(x) = \sup_{n \in \mathbf{N}} h(\eta_n), \quad (5.31)$$

where  $(\eta_n)_{n \in \mathbf{N}}$  is a Castaing representation of  $\mathcal{K}$ .

*Proof.* Let  $\omega \in \Omega$ . As  $(\eta_n(\omega))_{n \in \mathbf{N}} \subset \mathcal{K}(\omega)$ ,  $h(\omega, \eta_n(\omega)) \leq \sup_{x \in \mathcal{K}(\omega)} h(\omega, x)$  and thus  $\sup_n h(\eta_n) \leq \sup_{x \in \mathcal{K}} h(x)$ . Let  $x \in \mathcal{K}(\omega) = \text{cl}\{\eta_n(\omega), n \in \mathbf{N}\}$ , by lower semicontinuity of  $h$ , we get that  $h(\omega, x) \leq \liminf_n h(\omega, \eta_n(\omega)) \leq \sup_n h(\omega, \eta_n(\omega))$ . We conclude that  $\sup_{x \in \mathcal{K}} h(x) \leq \sup_n h(\eta_n)$  and (5.31) is proved.  $\square$

**Lemma 5.2.** *Let  $\mathcal{K} : \Omega \rightarrow \mathbb{R}^d$  be a  $\mathcal{H}$ -measurable and closed-valued random set such that  $\text{dom } \mathcal{K} = \Omega$  and let  $h : \Omega \times \mathbb{R}^k \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function such that  $h(\omega, x, \cdot)$  is l.s.c. for all  $(\omega, x) \in \Omega \times \mathbb{R}^k$ . Then  $(\omega, x) \in \Omega \times \mathbb{R}^k \mapsto s(\omega, x) = \sup_{z \in \mathcal{K}(\omega)} h(\omega, x, z)$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k)$ -measurable.*

*Proof.* Lemma 5.1 implies that  $s(\omega, x) = \sup_n h(\omega, x, \eta_n(\omega))$ , where  $(\eta_n)_{n \in \mathbf{N}}$  is a Castaing representation of  $\mathcal{K}$ . This implies that for any fixed  $c \in \mathbb{R}$

$$\{(\omega, x) \in \Omega \times \mathbb{R}^d, s(\omega, x) \leq c\} = \cap_n \{(\omega, x) \in \Omega \times \mathbb{R}^d, h(\omega, x, \eta_n(\omega)) \leq c\}.$$

As  $h$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and  $\eta_n$  is  $\mathcal{H}$ -measurable,  $(\omega, x) \mapsto h(\omega, x, \eta_n(\omega))$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k)$ -measurable and so is  $s$ .  $\square$

*Proof of Proposition 2.7.* As  $P(X \in \text{supp}_{\mathcal{H}} X | \mathcal{H}) = 1$  (see Remark 2.3) we have that  $\sup_{x \in \text{supp}_{\mathcal{H}} X} h(x) \geq h(X)$  a.s. and the definition of  $\text{ess sup}_{\mathcal{H}} h(X)$  implies that  $\sup_{x \in \text{supp}_{\mathcal{H}} X} h(x) \geq \text{ess sup}_{\mathcal{H}} h(X)$  a.s. since  $\sup_{x \in \text{supp}_{\mathcal{H}} X} h(x)$  is  $\mathcal{H}$ -measurable by Lemmata 2.4 and 5.2.

Let  $(\gamma_n)_{n \in \mathbf{N}}$  be a Castaing representation of  $\text{supp}_{\mathcal{H}} X$ . Lemmata 2.4 and 5.1 imply that  $\sup_{x \in \text{supp}_{\mathcal{H}} X} h(x) = \sup_n h(\gamma_n)$ . Fix some rational number  $\varepsilon > 0$  and some integer  $n > 0$  and set  $Z_{\varepsilon, n} = 1_{B(\gamma_n, \varepsilon)}(X)$ , where  $B(\gamma_n, \varepsilon)$  is the closed ball of center  $\gamma_n$  and radius  $\varepsilon$ . Let  $\Omega_{\varepsilon, n} = \{E(Z_{\varepsilon, n} | \mathcal{H}) > 0\}$ . Then  $P(\Omega_{\varepsilon, n}) = 1$ . Otherwise,

$P(\Omega \setminus \Omega_{\varepsilon,n}) > 0$  and on  $\Omega \setminus \Omega_{\varepsilon,n} \in \mathcal{H}$ ,  $P(X \in \mathbb{R}^d \setminus B(\gamma_n, \varepsilon) | \mathcal{H}) = 1$  and by definition 2.2,  $\text{supp}_{\mathcal{H}} X \subset \mathbb{R}^d \setminus B(\gamma_n, \varepsilon)$ , which contradicts  $\gamma_n \in \text{supp}_{\mathcal{H}} X$ . By definition of the essential supremum, we have that  $\text{ess sup}_{\mathcal{H}} h(X) \geq h(X)$  a.s. and that  $\text{ess sup}_{\mathcal{H}} h(X)$  is  $\mathcal{H}$ -measurable. This implies that, for all fixed  $\omega \in \Omega_{\varepsilon,n}$ ,

$$\begin{aligned} \text{ess sup}_{\mathcal{H}} h(X)(\omega) &\geq \frac{\mathbb{E}(Z_{\varepsilon,n} h(X) | \mathcal{H})(\omega)}{\mathbb{E}(Z_{\varepsilon,n} | \mathcal{H})(\omega)} = \frac{\int 1_{B(\gamma_n(\omega), \varepsilon)}(x) h(\omega, x) P_{X|\mathcal{H}}(dx; \omega)}{\mathbb{E}(Z_{\varepsilon,n} | \mathcal{H})(\omega)} \\ &\geq \frac{\int (\inf_{y \in B(\gamma_n(\omega), \varepsilon)} h(\omega, y)) 1_{B(\gamma_n(\omega), \varepsilon)}(x) P_{X|\mathcal{H}}(dx; \omega)}{\mathbb{E}(Z_{\varepsilon,n} | \mathcal{H})(\omega)} \\ &\geq \inf_{y \in B(\gamma_n(\omega), \varepsilon)} h(\omega, y). \end{aligned}$$

As  $h$  is l.s.c. (recall [32, Definition 1.5, equation 1(2)]), we have that

$$\lim_{\varepsilon \rightarrow 0} \inf_{y \in B(\gamma_n, \varepsilon)} h(y) = \liminf_{x \rightarrow \gamma_n} h(x) = h(\gamma_n).$$

So on the full measure set  $\cap_{\varepsilon \in \mathbb{Q}, \varepsilon > 0, n \in \mathbb{N}} \Omega_{\varepsilon,n}$ ,  $\text{ess sup}_{\mathcal{H}} h(X) \geq h(\gamma_n)$ . Taking the supremum over all  $n$ , we get that

$$\text{ess sup}_{\mathcal{H}} h(X) \geq \sup_n h(\gamma_n) = \sup_{x \in \text{supp}_{\mathcal{H}} X} h(x) \geq \text{ess sup}_{\mathcal{H}} h(X) \text{ a.s. } \square$$

**Lemma 5.3.** *Let  $\mathcal{F} = \sigma(Y)$  and  $T$  be a  $\mathbb{R}^d$ -valued random variable, which is not necessarily  $\mathcal{F}$ -measurable, such that  $\text{dom supp}_{\mathcal{F}} T = \Omega$ . Let  $g : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function such that  $(y, t) \mapsto g(\omega, y, t)$  is l.s.c. for all  $\omega \in \Omega$ . Let  $Z = g(Y, T)$ . Then  $\tilde{Z} = \text{ess sup}_{\mathcal{F}} Z = \delta(Y)$  where  $\delta(y) = \sup_{t \in \text{supp}_{\mathcal{F}} T} g(y, t)$  is l.s.c. in  $y$  and  $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable w.r.t.  $(\omega, y)$ .*

*Proof.* Proposition 2.7 implies that  $\delta(y) = \text{ess sup}_{\mathcal{F}} g(y, T)$ . Lemmata 2.4 and 5.2 prove that  $\delta$  is  $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable. In particular,  $\delta(Y)$  is  $\mathcal{F}$ -measurable. As  $g(Y, T) \leq \delta(Y)$ , we deduce that  $\tilde{Z} \leq \delta(Y)$  a.s. Moreover, by Lemma 5.1,  $\delta(y) = \sup_n g(y, \tau_n)$  where  $(\tau_n)_{n \geq 1}$  is a Castaing representation of  $\text{supp}_{\mathcal{F}} T$ . Moreover,  $T(\omega) \in \Lambda(\omega)$  a.s. where  $\Lambda$  is the closed  $\mathcal{F}$ -measurable random set defined as  $\Lambda(\omega) = \{t \in \mathbb{R}^d, g(\omega, Y(\omega), t) \leq \tilde{Z}(\omega)\}$ . Therefore,  $\text{supp}_{\mathcal{F}}(T) \subseteq \Lambda$  and we deduce that  $g(Y, \tau_n) \leq \tilde{Z}$  a.s. hence  $\delta(Y) \leq \tilde{Z}$  a.s. Therefore,  $\delta(Y) = \tilde{Z}$  a.s.,  $\delta$  is jointly measurable w.r.t.  $(\omega, y)$  and is clearly l.s.c. in  $y$ .  $\square$

*Proof of Theorem 3.9.* First we prove that 1. implies 2. Suppose that AWIP holds and fix some  $t \in \{0, \dots, T\}$ . We may suppose without loss of generality that the process  $S$  is integrable under  $P$ . Under AWIP, we then have  $\overline{\mathcal{R}_t^T} \cap L^1(\mathbb{R}_+, \mathcal{F}_t) = \{0\}$  where the closure is taken in  $L^1$ . Therefore, for every nonzero  $x \in L^1(\mathbb{R}_+, \mathcal{F}_t)$ , there exists by the Hahn-Banach theorem a non-zero  $Z_x \in L^\infty(\mathbb{R}_+, \mathcal{F}_t)$  such that (recall that  $\mathcal{R}_t^T$  is a cone)  $\mathbb{E} Z_x x > 0$  and  $\mathbb{E} Z_x \xi \leq 0$  for every  $\xi \in \mathcal{R}_t^T$ . Since  $-L^1(\mathbb{R}_+, \mathcal{F}_t) \subseteq \mathcal{R}_t^T$ , we deduce that  $Z_x \geq 0$  and we may renormalise  $Z_x$  so that  $\|Z_x\|_\infty = 1$ . Let us consider the family

$$\mathcal{G} = \{\{\mathbb{E}(Z_x | \mathcal{F}_t) > 0\}, x \in L^1(\mathbb{R}_+, \mathcal{F}_t) \setminus \{0\}\}.$$

Consider any non-null set  $\Gamma \in \mathcal{F}_t$ . Taking  $x = 1_\Gamma \in L^1(\mathbb{R}_+, \mathcal{F}_t) \setminus \{0\}$ , since  $\mathbb{E}(Z_x 1_\Gamma) > 0$ , we deduce that  $\Gamma$  has a non-null intersection with  $\{\mathbb{E}(Z_x | \mathcal{F}_t) > 0\}$ . By [20, Lemma 2.1.3], we deduce an at most countable subfamily  $(x_i)_{i \geq 1}$  such that the union  $\bigcup_i \{\mathbb{E}(Z_{x_i} | \mathcal{F}_t) > 0\}$  is of full measure. Therefore,  $Z = \sum_{i=1}^\infty 2^{-i} Z_{x_i} \geq 0$

is such that  $\mathbb{E}(Z|\mathcal{F}_t) > 0$  and we define  $Q \ll P$  such that  $dQ = (Z/\mathbb{E}(Z|\mathcal{F}_t))dP$ . As the subset  $\{\sum_{u=t+1}^T \theta_{u-1} \Delta S_u, \theta_{u-1} \in L(\mathbb{R}, \mathcal{F}_{u-1})\}$  is a linear vector space contained in  $\mathcal{R}_t^T$ , we deduce that  $(S_u)_{u \in \{t, \dots, T\}}$  is a  $Q$ -martingale.

We now prove that 2. implies 3. Suppose that for every  $t \in \{0, \dots, T\}$ , there exists  $Q \ll P$  such that  $(S_u)_{u \in \{t, \dots, T\}}$  is a  $Q$ -martingale with  $\mathbb{E}(dQ/dP|\mathcal{F}_t) = 1$ . Let us define, for  $u \in \{t, \dots, T\}$ ,  $\rho_u = \mathbb{E}_P(dQ/dP|\mathcal{F}_u)$ . Then,  $\rho_u \geq 0$  and  $\rho_t = 1$ . Consider  $\gamma_t \in \mathcal{R}_t^T \cap L^0(\mathbb{R}_+, \mathcal{F}_t)$ , i.e.  $\gamma_t$  is  $\mathcal{F}_t$ -measurable and is of the form  $\gamma_t = \sum_{u=t}^{T-1} \theta_u \Delta S_{u+1} - \epsilon_T^+$ . Since  $\theta_u$  is  $\mathcal{F}_u$ -measurable,  $\theta_u \Delta S_{u+1}$  admits a generalized conditional expectation under  $Q$ , knowing  $\mathcal{F}_u$ , and we have, by assumption, that  $\mathbb{E}_Q(\theta_u \Delta S_{u+1}|\mathcal{F}_u) = 0$ . The tower law implies that a.s.

$$\gamma_t = \mathbb{E}_Q(\gamma_t|\mathcal{F}_t) = \sum_{u=t}^{T-1} \mathbb{E}_Q(\mathbb{E}_Q(\theta_u \Delta S_{u+1}|\mathcal{F}_u)|\mathcal{F}_t) - \mathbb{E}_Q(\epsilon_T^+|\mathcal{F}_t) = -\mathbb{E}_Q(\epsilon_T^+|\mathcal{F}_t).$$

Hence  $\gamma_t = 0$  a.s., i.e. AIP holds. Let us show that  $\overline{\mathcal{R}_t^T} \cap L^0(\mathbb{R}, \mathcal{F}_t) \subseteq \mathcal{R}_t^T \cap L^0(\mathbb{R}, \mathcal{F}_t)$ . Consider first a one step model, where  $(S_u)_{u \in \{T-1, T\}}$  is a  $Q$ -martingale with  $\rho_T \geq 0$  and  $\rho_{T-1} = 1$ . Suppose that  $\gamma^n = \theta_{T-1}^n \Delta S_T - \epsilon_T^{n+} \in L^0(\mathbb{R}, \mathcal{F}_T)$  converges in probability to  $\gamma^\infty \in L^0(\mathbb{R}, \mathcal{F}_{T-1})$ . We need to show that  $\gamma^\infty \in \mathcal{R}_{T-1}^T \cap L^0(\mathbb{R}, \mathcal{F}_{T-1})$ .

On the  $\mathcal{F}_{T-1}$ -measurable set  $A_{T-1} := \{\liminf_n |\theta_{T-1}^n| < \infty\}$ , by [20, Lemma 2.1.2], we may assume w.l.o.g. that  $\theta_{T-1}^n$  is convergent to some  $\theta_{T-1}^\infty$  hence  $\epsilon_T^{n+}$  is also convergent and we can conclude that  $\gamma^\infty 1_{A_{T-1}} \in \mathcal{R}_{T-1}^T \cap L^0(\mathbb{R}, \mathcal{F}_{T-1})$ . Otherwise, on  $\Omega \setminus A_{T-1}$ , we use the normalized sequences for  $i \in \{1, \dots, d\}$

$$\tilde{\theta}_{T-1}^{n,i} := \theta_{T-1}^{n,i} / (|\theta_{T-1}^n| + 1), \quad \tilde{\epsilon}_T^{n+} := \epsilon_T^{n+} / (|\theta_{T-1}^n| + 1).$$

By [20, Lemma 2.1.2] again, we may assume, taking  $d+1$  sub-sequences, that a.s.  $\tilde{\theta}_{T-1}^n \rightarrow \tilde{\theta}_{T-1}^\infty$ ,  $\tilde{\epsilon}_T^{n+} \rightarrow \tilde{\epsilon}_T^{\infty+}$  and  $\tilde{\theta}_{T-1}^\infty \Delta S_T - \tilde{\epsilon}_T^{\infty+} = 0$  a.s. Remark that  $|\tilde{\theta}_{T-1}^\infty| = 1$  a.s. First consider the subset  $A_{T-1}^2 := (\Omega \setminus A_{T-1}) \cap \{|\tilde{\theta}_{T-1}^\infty| = 1\} \in \mathcal{F}_{T-1}$  on which  $\Delta S_T \geq 0$  a.s. Since  $\mathbb{E}_Q(\Delta S_T 1_{A_{T-1}^2}|\mathcal{F}_{T-1}) = 0$  a.s., we get that  $\rho_T \Delta S_T 1_{A_{T-1}^2} = 0$  a.s. Hence  $\rho_T \gamma^n 1_{A_{T-1}^2} = -\rho_T \epsilon_T^{n+} 1_{A_{T-1}^2} \leq 0$  a.s. Taking the limit, we get that  $\rho_T \gamma^\infty 1_{A_{T-1}^2} \leq 0$  a.s. and, since  $\gamma^\infty \in L^0(\mathbb{R}, \mathcal{F}_{T-1})$ , we deduce that  $\rho_{T-1} \gamma^\infty 1_{A_{T-1}^2} \leq 0$  a.s. Recall that  $\rho_{T-1} = 1$  hence  $\gamma^\infty 1_{A_{T-1}^2} \leq 0$  a.s. and  $\gamma^\infty 1_{A_{T-1}^2} \in \mathcal{R}_{T-1}^T \cap L^0(\mathbb{R}, \mathcal{F}_{T-1})$ . On the subset  $(\Omega \setminus A_{T-1}) \cap \{|\tilde{\theta}_{T-1}^\infty| = -1\}$  we may argue similarly and the conclusion follows in the one step model.

We now show the result in multi-step models by recursion. Fix some  $s \in \{t, \dots, T-1\}$ . We show that  $\overline{\mathcal{R}_{s+1}^T} \cap L^0(\mathbb{R}, \mathcal{F}_{s+1}) \subseteq \mathcal{R}_{s+1}^T \cap L^0(\mathbb{R}, \mathcal{F}_{s+1})$  implies the same property for  $s$  instead of  $s+1$ . By assumption  $(S_u)_{u \in \{s, \dots, T\}}$  is a  $Q$ -martingale with  $\mathbb{E}_P(dQ/dP|\mathcal{F}_u) = \rho_u \geq 0$  for  $u \in \{s, \dots, T\}$  and  $\rho_s = 1$ . Suppose that  $\gamma^n = \sum_{u=s+1}^T \theta_{u-1}^n \Delta S_u - \epsilon_T^{n+} \in \mathcal{R}_s^T \cap L^0(\mathbb{R}, \mathcal{F}_T)$  converges to  $\gamma^\infty \in L^0(\mathbb{R}, \mathcal{F}_s)$ . If  $\gamma^\infty = 0$  there is nothing to prove. As before on the  $\mathcal{F}_s$ -measurable set  $A_s := \{\liminf_n |\theta_s^n| < \infty\}$ , we may assume w.l.o.g. that  $\theta_s^n$  converges to  $\theta_s^\infty$ . Therefore on  $A_s$

$$\sum_{u=s+2}^T \theta_{u-1}^n \Delta S_u - \epsilon_T^{n+} = \gamma^n - \theta_s^n \Delta S_{s+1} \rightarrow \gamma^\infty - \theta_s^\infty \Delta S_{s+1}$$

and, by the induction hypothesis,  $\sum_{u=s+2}^T \theta_{u-1}^n \Delta S_u - \epsilon_T^{n+}$  also converges to an element of  $\mathcal{R}_{s+1}^T \cap L^0(\mathbb{R}, \mathcal{F}_{s+1})$  and we conclude that  $\gamma^\infty 1_{A_s} \in \mathcal{R}_s^T \cap L^0(\mathbb{R}, \mathcal{F}_s)$ .

On  $\Omega \setminus A_{s-1}$ , we use the normalisation procedure as before, and deduce the equality

$\sum_{u=s+1}^T \tilde{\theta}_{u-1}^\infty \Delta S_u - \tilde{\epsilon}_T^{\infty+} = 0$  a.s. for some  $\tilde{\theta}_u^\infty \in L^0(\mathbb{R}, \mathcal{F}_u)$ ,  $u \in \{s, \dots, T-1\}$  and  $\tilde{\epsilon}_T^{\infty+} \geq 0$  such that  $|\tilde{\theta}_s^\infty| = 1$  a.s. We then argue on  $A_s^2 := (\Omega \setminus A_{s-1}) \cap \{\tilde{\theta}_s^\infty = 1\} \in \mathcal{F}_s$  and  $A_s^3 := (\Omega \setminus A_{s-1}) \cap \{\tilde{\theta}_s^\infty = -1\} \in \mathcal{F}_s$  respectively. When  $\tilde{\theta}_s^\infty = 1$ , we deduce that

$$\Delta S_{s+1} + \sum_{u=s+2}^T \tilde{\theta}_{u-1}^\infty \Delta S_u - \tilde{\epsilon}_T^{\infty+} = 0 \text{ a.s., i.e. } \Delta S_{s+1} \in \mathcal{P}_{s+1, T}(0)$$

hence  $\Delta S_{s+1} \geq \pi_{s+1, T}(0) = 0$  a.s. under AIP, see Proposition 3.4. Since  $\mathbb{E}_Q(\Delta S_{s+1} 1_{A_s^2} | \mathcal{F}_s) = 0$  a.s.,  $\rho_{s+1} \Delta S_{s+1} 1_{A_s^2} = 0$  a.s. So,

$$\rho_{s+1} \gamma^n 1_{A_s^2} = \sum_{u=s+2}^T \theta_{u-1}^n \rho_{s+1} 1_{A_s^2} \Delta S_u - \epsilon_T^{n+} \rho_{s+1} 1_{A_s^2} \in \mathcal{R}_{s+1}^T \cap L^0(\mathbb{R}, \mathcal{F}_{s+1}).$$

Hence,  $\rho_{s+1} \gamma^\infty 1_{A_s^2} \in \mathcal{R}_{s+1}^T \cap L^0(\mathbb{R}, \mathcal{F}_{s+1})$  by induction. As  $\rho_{s+1} \gamma^\infty 1_{A_s^2}$  admits a generalized conditional expectation knowing  $\mathcal{F}_s$ , the tower property implies a.s.

$$\begin{aligned} 1_{A_s^2} \mathbb{E}(\rho_s \gamma^\infty | \mathcal{F}_s) &= \mathbb{E}(\rho_{s+1} \gamma^\infty 1_{A_s^2} | \mathcal{F}_s) = \sum_{u=s+2}^T 1_{A_s^2} \mathbb{E} \left( \theta_{u-1}^\infty \mathbb{E} \left( \frac{dQ}{dP} \Delta S_u | \mathcal{F}_{u-1} \right) | \mathcal{F}_s \right) \\ &\quad - 1_{A_s^2} \mathbb{E}(\epsilon_T^{\infty+} \rho_{s+1} | \mathcal{F}_s) \leq 0, \end{aligned}$$

since  $(S_u)_{u \in \{s, \dots, T\}}$  is a  $Q$ -martingale. Hence,  $\rho_s \gamma^\infty 1_{A_s^2} \leq 0$  a.s. As  $\rho_s = 1$ ,  $\gamma^\infty 1_{A_s^2} \leq 0$  a.s. so that  $\gamma^\infty 1_{A_s^2} \in \mathcal{R}_s^T \cap L^0(\mathbb{R}, \mathcal{F}_s)$ .

Finally, notice that the AIP condition implies AWIP as soon as the equality  $\overline{\mathcal{R}_t^T} \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \mathcal{R}_t^T \cap L^0(\mathbb{R}_+, \mathcal{F}_t)$  holds for every  $t \in \{0, \dots, T\}$ .  $\square$

*Proof of Proposition 4.1.* The conditions  $k_{t-1}^d \in [0, 1]$  and  $k_{t-1}^u \in [1, +\infty]$  for all  $t \in \{1, \dots, T\}$  are equivalent to the AIP condition (see Remark 2.13). Let  $M = \frac{h(\infty)}{\infty}$  and  $M_t = \lim_{z \rightarrow +\infty} \frac{h(t, z)}{z}$ . We prove the second statement. Assume that AIP holds true. We establish (i) the recursive formulation  $\pi_{t, T}(h(S_T)) = h(t, S_t)$  given by (4.25), (ii)  $h(t, \cdot) \geq h(t+1, \cdot)$  and (iii)  $M_t = M_{t+1}$ . The case  $t = T$  is immediate. As  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a convex function with  $\text{dom } h = \mathbb{R}$ ,  $h$  is clearly a  $\mathcal{F}_{T-1}$ -normal integrand, we can apply Corollary 2.26 (see (2.18) and (2.19)) and we get that a.s.

$$\begin{aligned} \pi_{T-1, T}(h(S_T)) &= h(k_{T-1}^d S_{T-1}) + \theta_{T-1}^* \left( S_{T-1} - k_{T-1}^d S_{T-1} \right), \\ \theta_{T-1}^* &= \frac{h(k_{T-1}^u S_{T-1}) - h(k_{T-1}^d S_{T-1})}{k_{T-1}^u S_{T-1} - k_{T-1}^d S_{T-1}}, \end{aligned} \tag{5.32}$$

with the conventions  $\theta_{T-1}^* = \frac{0}{0} = 0$  if either  $S_{T-1} = 0$  or  $k_{T-1}^u = k_{T-1}^d = 1$  and  $\theta_{T-1}^* = \frac{h(\infty)}{\infty} = M$  if  $k_{T-1}^d < k_{T-1}^u = +\infty$ . Moreover, using (2.20), we obtain that

$$\pi_{T-1, T}(h(S_T)) + \theta_{T-1}^* \Delta S_T \geq h(S_T) \text{ a.s. i.e. } \pi_{T-1, T}(h(S_T)) \in \mathcal{P}_{T-1, T}(h(S_T)).$$

So, using Lemma 3.1, we get that  $\mathcal{P}_{T-2, T}(h(S_T)) = \mathcal{P}_{T-2, T-1}(\pi_{T-1, T}(h(S_T)))$ ,

$$\pi_{T-2, T}(h(S_T)) = \pi_{T-2, T-1}(\pi_{T-1, T}(h(S_T)))$$

and we may continue the recursion as soon as  $\pi_{T-1, T}(h(S_T)) = h(T-1, S_{T-1})$  where  $h(T-1, \cdot)$  satisfies (4.25), is convex with domain equal to  $\mathbb{R}$ , is such that

$h(T-1, z) \geq 0$  for all  $z \geq 0$  and  $M_{T-1} = M \in [0, \infty)$ . To see that, we distinguish three cases. If either  $S_{T-1} = 0$  or  $k_{T-1}^u = k_{T-1}^d = 1$ ,  $\pi_{T-1, T}(h(S_T)) = h(S_{T-1})$  and  $h(T-1, z) = h(z) = h(T, z)$  satisfies all the required conditions. If  $k_{T-1}^d < k_{T-1}^u = +\infty$ ,  $\theta_{T-1}^* = M$  and  $\pi_{T-1, T}(h(S_T)) = h(T-1, S_{T-1})$ , where

$$\begin{aligned} h(T-1, z) &= h(k_{T-1}^d z) + Mz \left(1 - k_{T-1}^d\right) \\ &= \lim_{k^u \rightarrow +\infty} \left( \frac{k^u - 1}{k^u - k_{T-1}^d} h(k_{T-1}^d z) + \frac{1 - k_{T-1}^d}{k^u - k_{T-1}^d} h(k^u z) \right), \end{aligned}$$

using (4.26). The term in the r.h.s. above is larger than  $h(z) = h(T, z)$  by convexity. As  $k_{T-1}^d \in [0, 1]$  and  $M \in [0, \infty)$ ,  $h(T-1, z) \geq 0$  for all  $z \geq 0$ , we get that  $h(T-1, \cdot)$  is convex function with domain equal to  $\mathbb{R}$  since  $h$  is so. The function  $h(T-1, \cdot)$  also satisfies (4.25) (see (4.26)). Finally

$$M_{T-1} = \lim_{z \rightarrow +\infty} k_{T-1}^d \frac{h(k_{T-1}^d z)}{k_{T-1}^d z} + M \left(1 - k_{T-1}^d\right) = M.$$

The last case is when  $S_{T-1} \neq 0$  and  $k_{T-1}^u \neq k_{T-1}^d$  and  $k_{T-1}^u < +\infty$ . It is clear that (5.32) implies (4.25). Moreover as  $k_{T-1}^d \in [0, 1]$  and  $k_{T-1}^u \in [1, +\infty)$ ,  $\lambda_{T-1} \in [0, 1]$ ,  $1 - \lambda_{T-1} \in [0, 1]$  and (4.25) implies that  $h(T-1, z) \geq 0$  for all  $z \geq 0$ ,  $h(T-1, \cdot)$  is convex with domain equal to  $\mathbb{R}$  since  $h$  is so. Moreover, by convexity

$$M_{T-1} = \lambda_{T-1} k_{T-1}^d \lim_{z \rightarrow +\infty} \frac{h(k_{T-1}^d z)}{k_{T-1}^d z} + (1 - \lambda_{T-1}) k_{T-1}^u \lim_{z \rightarrow +\infty} \frac{h(k_{T-1}^u z)}{k_{T-1}^u z} = M. \square$$

**Lemma 5.4.** *Let  $\Gamma$  be a  $\mathcal{F}$ -measurable random set in  $\mathbb{R}^d$ . Suppose that  $\Gamma$  is non empty almost surely. There exists a  $\mathcal{F}$ -measurable version  $H$  of the pointwise affine hull  $\bar{H}(\omega)$  of  $\Gamma(\omega)$ ,  $\omega \in \Omega$ , such that  $L^0(\bar{H}, \mathcal{F}) = L^0(H, \mathcal{F})$ . Moreover, for any  $x \in L^0(\mathbb{R}^d, \mathcal{F})$ , there exists a  $\mathcal{F}$ -measurable version of the orthogonal projection  $p_H(x)$  of  $x$  onto  $H$ .*

*Proof.* Recall that the affine hull of  $\Gamma$  is the smallest affine subspace  $H$  which contains  $\Gamma$ . As  $H$  is also the affine hull of the closed convex hull of  $\Gamma$ , we may suppose without loss of generality that  $\Gamma$  is convex almost surely as there exists a measurable version of the closed convex hull by [26]. As  $\Gamma \neq 0$  a.s., consider a measurable selection  $\hat{\gamma} \in \Gamma$  a.s. Then,  $\tilde{\Gamma} = \Gamma - \hat{\gamma}$  is a convex set containing zero. Therefore, we may assume without loss of generality that  $\hat{\gamma} = 0$ .

Consider the pointwise affine hull  $\bar{H}(\omega)$  of  $\Gamma(\omega)$ . By [26, Proposition 2.8], there exists a measurable version  $H$  of  $\bar{H}$  such that  $L^0(\bar{H}, \mathcal{F}) = L^0(H, \mathcal{F})$ . As  $H$  contains 0, it is a convex cone by [26, Proposition 2.8]. Moreover, the same holds for  $-H$  hence  $H$  is a linear space.

As  $H$  is closed, consider a Castaing representation of  $H$ , i.e. a countable family  $(h_n)_{n \geq 1}$  of  $L^0(H, \mathcal{F})$  such that  $(h_n(\omega))_{n \geq 1}$  is almost surely dense in  $H(\omega)$ ,  $\omega \in \Omega$ . Then,  $d(H, x) = \inf_n \|x - h_n\|$  is  $\mathcal{F}$ -measurable and the random set  $\Gamma(\omega) = \{y \in H : \|x(\omega) - y\| \leq d(H, x)\}$  is  $\mathcal{F}$ -measurable and non empty almost surely as it contains the pointwise projection  $p_H(x)$ . Therefore, by measurable selection argument, it is possible to choose a  $\mathcal{F}$ -measurable version of  $p_H(x)$ .  $\square$

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