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# Pricing without martingale measure

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**Abstract:** For several decades, the no-arbitrage (NA) condition and the martingale measures have played a major role in the financial asset's pricing theory. Here, we propose a new approach based on convex duality instead of martingale measures duality : our prices will be expressed using Fenchel conjugate and bi-conjugate. This is lead naturally to a weak condition of absence of arbitrage opportunity, called Absence of Immediate Profit (AIP), which asserts that the price of the zero claim should be zero. We study the link between (AIP), (NA) and the no-free lunch condition. We show in a one step model that, under (AIP), the super-hedging cost is just the payoff's concave envelop and that (AIP) is equivalent to the non-negativity of the super-hedging prices of some call option. In the multiple-period case, for a particular, but still general setup, we propose a recursive scheme for the computation of a the super-hedging cost of a convex option. We also give some numerical illustrations.

**Keywords and phrases:** Financial market models, Super-hedging prices, No-arbitrage condition, Conditional support, Essential supremum.  
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## 1. Introduction

The problem of giving a price to a financial asset  $G$  is central in the economic and financial theory. A selling price should be an amount which is enough to initiate a hedging strategy for  $G$ , i.e. a strategy whose value at maturity is always above  $G$ . It seems also natural to ask for the infimum of such amount. This is the so called super-replication price and it has been introduced in the binomial setup for transaction costs by [4]. Characterising and computing

the super-replication price has become one of the central issue in mathematical finance theory. Until now it was intimately related to the No-Arbitrage (NA) condition. This condition asserts that starting from a zero wealth it is not possible to reach a positive one (non negative almost surely and strictly positive with strictly positive probability measure). Characterizing the (NA) condition or, more generally, the No Free Lunch condition leads to the Fundamental Theorem of Asset Pricing (FTAP in short). This theorem proves the equivalence between those absence of arbitrage conditions and the existence of risk-neutral probability measures (also called martingale measures or pricing measures) which are equivalent probability measures under which the (discounted) asset price process is a martingale. This was initially formalised in [11], [12] and [16] while in [8] the FTAP is formulated in a general discrete-time setting under the (NA) condition. The literature on the subject is huge and we refer to [9] and [14] for a general overview. Under the (NA) condition, the super-replication price of  $G$  is equal to the supremum of the (discounted) expectation of  $G$  computed under the risk-neutral probability measures. This is the so called dual formulation of the super-replication price or Superhedging Theorem. We refer to [10] and the references therein.

In this paper a super-hedging or super-replicating price is the initial value of some super-hedging strategy. We propose an innovating approach: we analyse from scratch the set of super-replicating prices and its infimum value, which will be called the infimum super-replication cost. Note that this cost is not automatically a super-replicating price. Under mild assumptions, we show that the one-step set of super-replication prices can be expressed using Fenchel-Legendre conjugate and the infimum super-replication cost is obtained by the Fenchel-Legendre biconjugate. So we use here the convex duality instead of the usual financial duality based on martingale measures under the (NA) condition. We then introduce the condition of Absence of Immediate Profit (AIP). An Immediate Profit is the possibility of super-hedging 0 at a negative cost. We prove that (AIP) is equivalent to the fact that the stock value at the beginning of the period belongs to the convex envelop of the conditional (with respect to the information of the beginning of the period) support of the stock value at the end of the period. Using the notion of conditional essential supremum, it is equivalent to say that the initial stock price is between the conditional essential infimum and supremum of the stock value at the end of the period. Under (AIP) condition we show that the one-step infimum super-replication cost is the concave envelop of the payoff relatively to the convex envelop of the conditional support. We

also show that (AIP) is equivalent to the non-negativity of the super-hedging prices of any fixed call option. We then study the multiple-period framework. We show that the global (AIP) condition and the local ones are equivalent. We then focus on a particular, but still general setup, where we propose a recursive scheme for the computation of the super-hedging prices of a convex option. We obtain the same computational scheme as in [5] and [6] but here it is obtained by only assuming (AIP) instead of the stronger (NA) condition. We also give some numerical illustrations; we calibrate historical data of the french index CAC 40 to our model and implement the super-hedging strategy for a call option.

Finally, we study the link between (AIP), (NA) and the weak no-free lunch (WNFL) conditions. We show that the (AIP) condition is the weakest-one and we also provide conditions for the equivalence between the (AIP) and the (WNFL) conditions.

The paper is organized as follows. In Section 2, we study the one-period framework while in Section 3 we study the multi-period one. Section 4 is devoted to the comparison between (AIP), (NA) and (WNFL) conditions. Section 5 contains the numerical experiments. Finally, Section 6 collects the results on conditional support and conditional essential supremum.

In the remaining of this introduction we introduce our framework and recall some results that will be used without further references in the sequel. Let  $(\Omega, (\mathcal{F}_t)_{t \in \{0, \dots, T\}}, \mathcal{F}_T, P)$  be a filtered probability space where  $T$  is the time horizon. We consider a  $(\mathcal{F}_t)_{t \in \{0, \dots, T\}}$ -adapted, real-valued, non-negative process  $S := \{S_t, t \in \{0, \dots, T\}\}$ , where for  $t \in \{0, \dots, T\}$ ,  $S_t$  represents the price of some risky asset in the financial market in consideration. Trading strategies are given by  $(\mathcal{F}_t)_{t \in \{0, \dots, T\}}$ -adapted processes  $\theta := \{\theta_t, t \in \{0, \dots, T-1\}\}$  where for all  $t \in \{0, \dots, T-1\}$ ,  $\theta_t$  represents the investor's holding in the risky asset between time  $t$  and time  $t+1$ .

We assume that trading is self-financing and that the riskless asset's price is constant equal to 1. The value at time  $t$  of a portfolio  $\theta$  starting from initial capital  $x \in \mathbb{R}$  is given by

$$V_t^{x, \theta} = x + \sum_{u=1}^t \theta_{u-1} \Delta S_u.$$

For any  $\sigma$ -algebra  $\mathcal{H}$  and any  $k \geq 1$ , we denote by  $L^0(\mathbb{R}^k, \mathcal{H})$  the set of  $\mathcal{H}$ -measurable and  $\mathbb{R}^k$ -valued random variables. Let  $h : \Omega \times \mathbb{R}^k \rightarrow \mathbb{R}$ . The effective domain of  $h(\omega, \cdot)$  is  $\text{dom } h(\omega, \cdot) = \{x \in \mathbb{R}^k, h(\omega, x) < \infty\}$  and

$h(\omega, \cdot)$  is proper if  $\text{dom}h(\omega, \cdot) \neq \emptyset$  and  $h(\omega, x) > -\infty$  for all  $x \in \mathbb{R}^k$ . Then  $h$  is  $\mathcal{H}$ -normal integrand (see Definition 14.27 in [21]) if and only if  $h$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k)$ -measurable and is lower semi-continuous (l.s.c. in the sequel) in  $x$ , see [21, Corollary 14.34]. Let  $Z \in L^0(\mathbb{R}^k, \mathcal{H})$ , we will use the notation  $h(Z) : \omega \rightarrow h(Z(\omega)) = h(\omega, Z(\omega))$  and if  $h$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k)$ -measurable,  $h(Z) \in L^0(\mathbb{R}^k, \mathcal{H})$ . Let  $\mathcal{K}$  be a  $\mathcal{H}$ -measurable (see Definition 14.1 of [21]) and closed-valued random set of  $\mathbb{R}^k$  then  $\mathcal{K}$  admits a Castaing representation  $(\eta_n)_{n \in \mathbb{N}}$  (see Theorem 14.5 in [21]) :  $\mathcal{K}(\omega) = \text{cl}\{\eta_n(\omega), n \in \mathbb{N}\}$  for all  $\omega \in \text{dom } \mathcal{K} = \{\omega \in \Omega, \mathcal{K}(\omega) \cap \mathbb{R}^k \neq \emptyset\}$ , where the closure is taken in  $\mathbb{R}^k$ .

## 2. The one-period framework

For ease of notation, we consider two complete sub- $\sigma$ -algebras of  $\mathcal{F}_T$ :  $\mathcal{H} \subseteq \mathcal{F}$  and two random variables  $y \in L^0(\mathbb{R}_+, \mathcal{H})$  and  $Y \in L^0(\mathbb{R}_+, \mathcal{F})$ . The setting will be applied in Section 3 with the choices  $\mathcal{H} = \mathcal{F}_t$ ,  $\mathcal{F} = \mathcal{F}_{t+1}$ ,  $Y = S_{t+1}$ ,  $y = S_t$ .

Section's objective is to obtain a characterisation of the one-step set of superhedging or super-replicating prices of  $g(Y)$  under suitable assumptions on  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ .

In the following, the notion of conditionnal support ( $\text{supp}_{\mathcal{H}}Y$ ), conditional essential infimum ( $\text{ess inf}_{\mathcal{H}}$ ) or supremum ( $\text{ess sup}_{\mathcal{H}}$ ) will be in force, see Section 6.

**Definition 2.1.** *The set  $\mathcal{P}(g)$  of super-hedging prices of the contingent claim  $g(Y)$  consists in the initial values of super-hedging strategies  $\theta$ :*

$$\mathcal{P}(g) = \{x \in L^0(\mathbb{R}, \mathcal{H}), \exists \theta \in L^0(\mathbb{R}, \mathcal{H}), x + \theta(Y - y) \geq g(Y) \text{ a.s.}\}.$$

*The infimum super-hedging cost of  $g(Y)$  is defined as  $p(g) := \text{ess inf}_{\mathcal{H}} \mathcal{P}(g)$ .*

Notice that the infimum super-hedging cost is not a priori a price, i.e. an element of  $\mathcal{P}(g)$ , as the later may be an open interval.

**Remark 2.2.** As  $P(Y \in \text{supp}_{\mathcal{H}}Y) = 1$  (see [1, definition of support on page 441, Theorems 12.7 and 12.14]), we have that  $\text{supp}_{\mathcal{H}}Y$  is a.s. non-empty. Moreover since  $0 \leq Y < \infty$ ,  $\text{Dom } \text{supp}_{\mathcal{H}}Y = \Omega$ . We could easily include the case  $P(0 \leq Y < \infty) = 1$  by replacing  $Y$  by 0 on the complementary of  $\{0 \leq Y < \infty\}$ .

**Lemma 2.3.**

$$\mathcal{P}(g) = \left\{ \text{ess sup}_{\mathcal{H}} (g(Y) - \theta Y) + \theta y, \theta \in L^0(\mathbb{R}, \mathcal{H}) \right\} + L^0(\mathbb{R}_+, \mathcal{H}). \quad (2.1)$$

Suppose that  $g$  is a  $\mathcal{H}$ -normal integrand. Then

$$\text{ess sup}_{\mathcal{H}} (g(Y) - \theta Y) = \sup_{z \in \text{supp}_{\mathcal{H}} Y} (g(z) - \theta z) = f^*(-\theta), \quad (2.2)$$

where  $f^*$  is the Fenchel-Legendre conjugate of  $f$  i.e.

$$\begin{aligned} f^*(\omega, x) &= \sup_{z \in \mathbb{R}} (xz - f(\omega, z)), \\ f(\omega, z) &= -g(\omega, z) + \delta_{\text{supp}_{\mathcal{H}} Y}(\omega, z), \end{aligned} \quad (2.3)$$

where  $\delta_C(\omega, z) = 0$  if  $z \in C(\omega)$  and  $+\infty$  else. Both  $f^*(\omega, \cdot)$  and  $x \rightarrow f^*(\omega, -x)$  are a.s. proper, convex, l.s.c.,  $f^*$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ -measurable and  $f^*$  is a  $\mathcal{H}$ -normal integrand. Moreover, we have that

$$p(g) = -f^{**}(y),$$

where  $f^{**}$  is the Fenchel-Legendre biconjugate of  $f$  i.e.

$$f^{**}(\omega, x) = \sup_{z \in \mathbb{R}} (xz - f^*(\omega, z)).$$

*Proof.* As  $x \in \mathcal{P}(g)$  if and only if there exists  $\theta \in L^0(\mathbb{R}, \mathcal{H})$  such that  $x - \theta y \geq g(Y) - \theta Y$  a.s., we get by definition of the conditional essential supremum (see Definition 6.3) that (2.1) holds true. Then (2.2) follows from Lemma 6.8 (see Remark 2.2). Since the graph of  $\text{supp}_{\mathcal{H}} Y$  belongs to  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$  (see Lemma 6.2), we easily deduce that  $\delta_{\text{supp}_{\mathcal{H}} Y}$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ -measurable and it is clear that it is also l.s.c. As  $\text{dom } f = \text{supp}_{\mathcal{H}} Y$  is a.s. non-empty (see again Remark 2.2)  $f^*(\omega, \cdot)$  is convex and l.s.c. as the supremum of affine functions. Hence  $x \rightarrow f^*(\omega, -x)$  is also a.s. l.s.c. and convex. Moreover, using Lemma 6.6 (and Remark 2.2),  $f^*$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ -measurable.

$$\begin{aligned} p(g) &= \text{ess inf}_{\mathcal{H}} \{f^*(-\theta) + \theta y, \theta \in L^0(\mathbb{R}, \mathcal{H})\} \\ &= -\text{ess sup}_{\mathcal{H}} \{\theta y - f^*(\theta), \theta \in L^0(\mathbb{R}, \mathcal{H})\} \\ &= -\sup_{z \in \mathbb{R}} (zy - f^*(z)) = -f^{**}(y). \end{aligned}$$

The first equality is a direct consequence of (2.1), the second one is trivial. In order to obtain the third one, we want to apply Lemma 6.9. First remark that  $\text{ess sup}_{\mathcal{H}} \{\theta y - f^*(\theta), \theta \in L^0(\mathbb{R}, \mathcal{H})\} = \text{ess sup}_{\mathcal{H}} \{\theta y - f^*(\theta), \theta \in$

$L^0(\mathbb{R}, \mathcal{H}) \cap \text{Dom } f^*$ . Now since  $f^*$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ -measurable, we deduce that  $\text{graph dom } f^* = \{(\omega, x) \in \Omega \times \mathbb{R}, f^*(\omega, x) < \infty\}$  is a  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ -measurable set and  $\text{dom } f^*$  is also  $\mathcal{H}$ -measurable (see [21, Theorem 14.8]). Since  $(\omega, z) \rightarrow zy(\omega) - f^*(\omega, z)$  is a  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ -measurable function and  $f^*(\omega, \cdot)$  is convex and thus u.s.c. on  $\text{dom } f^*(\omega)$ , we can apply Lemma 6.9 and we obtain that

$$\begin{aligned} \text{ess sup}_{\mathcal{H}} \{\theta y - f^*(\theta), \theta \in L^0(\mathbb{R}, \mathcal{H}) \cap \text{Dom } f^*\} &= \sup_{z \in \text{Dom}(f^*)} (zy - f^*(z)) \\ &= \sup_{z \in \mathbb{R}} (zy - f^*(z)). \end{aligned}$$

□

Let  $\text{conv } h$  be the convex envelop of  $h$  i.e. the greatest convex function dominated by  $h$

$$\text{conv } h(x) = \sup\{u(x), u \text{ convex and } u \leq h\}.$$

The concave envelop is defined symmetrically and denoted by  $\text{conc } h$ . We also define the (lower) closure  $\underline{h}$  of  $h$  as the greatest l.s.c. function which is dominated by  $h$  i.e.  $\underline{h} = \liminf_{y \rightarrow x} h(y)$ . The upper closure is defined symmetrically:  $\overline{h} = \limsup_{y \rightarrow x} h(y)$ . It is easy to see that

$$\underline{\text{conv}} f(y) = \sup\{\alpha y + \beta, \alpha, \beta \in \mathbb{R}, f(x) \geq \alpha x + \beta, \forall x \in \mathbb{R}\}.$$

It is well-known (see for example [21, Theorem 11.1]) that

$$f^* = (\text{conv } f)^* = (\underline{f})^* = (\underline{\text{conv}} f)^*. \quad (2.4)$$

Moreover, if  $\text{conv } f$  is proper,  $f^{**}$  is also proper, convex and l.s.c. and

$$f^{**} = \underline{\text{conv}} f. \quad (2.5)$$

So in order to compute  $p(g)$ , we need to compute  $\text{conv } f$  and  $\underline{\text{conv}} f$ . To do so, we introduce the notion of relative concave envelop of  $g$  with respect to  $\text{supp}_{\mathcal{H}} Y$  :

$$\text{conc}(g, \text{supp}_{\mathcal{H}} Y)(x) = \inf\{v(x), v \text{ is concave and } v(z) \geq g(z), \forall z \in \text{supp}_{\mathcal{H}} Y\}.$$

In the following, we use the convention  $0 \times (\pm\infty) = 0$  and  $(+\infty) \times 0 = 0$ .

**Lemma 2.4.** *Suppose that  $g$  is a  $\mathcal{H}$ -normal integrand. Then, we have:*

$$\begin{aligned} \text{conv } f(x) &= +\infty 1_{\mathbb{R} \setminus \text{convsupp}_{\mathcal{H}} Y}(x) - 1_{\text{convsupp}_{\mathcal{H}} Y}(x) \times \text{conc}(g, \text{supp}_{\mathcal{H}} Y)(x) \\ \underline{\text{conv}} f(x) &= +\infty 1_{\mathbb{R} \setminus \text{convsupp}_{\mathcal{H}} Y}(x) - 1_{\text{convsupp}_{\mathcal{H}} Y}(x) \times \overline{\text{conc}}(g, \text{supp}_{\mathcal{H}} Y)(x) \\ &= +\infty 1_{\mathbb{R} \setminus \text{convsupp}_{\mathcal{H}} Y}(x) - 1_{\text{convsupp}_{\mathcal{H}} Y}(x) \times \\ &\quad \inf \{ \alpha x + \beta, \alpha, \beta \in \mathbb{R}, \alpha z + \beta \geq g(z), \forall z \in \text{supp}_{\mathcal{H}} Y \}, \end{aligned}$$

where  $\text{convsupp}_{\mathcal{H}} Y$  is the convex envelopp of  $\text{supp}_{\mathcal{H}} Y$ , i.e. the smallest convex set that contains  $\text{supp}_{\mathcal{H}} Y$ .

**Remark 2.5.** Note that  $\text{conv } f$  is proper if and only if  $\text{conc}(g, \text{supp}_{\mathcal{H}} Y)(x) < +\infty$  for all  $x \in \text{convsupp}_{\mathcal{H}} Y$ , since  $\text{convsupp}_{\mathcal{H}} Y$  is non-empty (see Remark 2.2). So  $\text{conv } f$  is proper if there exists some concave function  $\varphi$  such that  $g \leq \varphi$  on  $\text{supp}_{\mathcal{H}} Y$  and  $\varphi < \infty$  on  $\text{convsupp}_{\mathcal{H}} Y$  (by definition,  $\text{conc}(g, \text{supp}_{\mathcal{H}} Y) \leq \varphi$ ).<sup>1</sup> As for all  $x \in \text{convsupp}_{\mathcal{H}} Y$ ,  $\text{conc}(g, \text{supp}_{\mathcal{H}} Y)(x) \geq g(x) > -\infty$ , we get that  $\text{conc}(g, \text{supp}_{\mathcal{H}} Y)(x) \in \mathbb{R}$  and also  $\overline{\text{conc}}(g, \text{supp}_{\mathcal{H}} Y)(x) \in \mathbb{R}$ , one may write that

$$\begin{aligned} \text{conv } f &= -\text{conc}(g, \text{supp}_{\mathcal{H}} Y) + \delta_{\text{convsupp}_{\mathcal{H}} Y}. \\ \underline{\text{conv}} f &= -\overline{\text{conc}}(g, \text{supp}_{\mathcal{H}} Y) + \delta_{\text{convsupp}_{\mathcal{H}} Y}. \end{aligned}$$

*Proof.* One can rewrite the convex envelop of  $f$  as follows (see [21, Proposition 2.31]):

$$\text{conv } f(x) = \inf \left\{ \sum_{i=1}^n \lambda_i f(x_i), n \geq 1, (\lambda_i)_{i \in \{1, \dots, n\}} \in \mathbb{R}_+^n, (x_i)_{i \in \{1, \dots, n\}} \in \mathbb{R}^n, \right. \\ \left. x = \sum_{i=1}^n \lambda_i x_i, \sum_{i=1}^n \lambda_i = 1 \right\}.$$

The convex envelop of  $\text{supp}_{\mathcal{H}} Y$  is given by (see [21, Proposition 2.27, Theorem 2.29]).

$$\text{convsupp}_{\mathcal{H}} Y = \left\{ \sum_{i=1}^n \lambda_i x_i, n \geq 1, (\lambda_i)_{i \in \{1, \dots, n\}} \in \mathbb{R}_+^n, \sum_{i=1}^n \lambda_i = 1, x_i \in \text{supp}_{\mathcal{H}} Y \right\}.$$

Assume that  $x \notin \text{convsupp}_{\mathcal{H}} Y$ . Then if  $x = \sum_{i=1}^n \lambda_i x_i$  for some  $n \geq 1$ ,  $(\lambda_i)_{i \in \{1, \dots, n\}} \in \mathbb{R}_+^n$ ,  $(x_i)_{i \in \{1, \dots, n\}} \in \mathbb{R}^n$  such that  $\sum_{i=1}^n \lambda_i = 1$ , at least one

<sup>1</sup>This is equivalent to assume that there exists  $\alpha, \beta \in \mathbb{R}$ , such that  $g(x) \leq \alpha x + \beta$  for all  $x \in \text{supp}_{\mathcal{H}} Y$ .



$x_i \notin \text{supp}_{\mathcal{H}}Y$  and  $f(x_i) = \infty$  and also  $\text{conv } f(x) = \infty = +\infty 1_{\mathbb{R} \setminus \text{convsupp}_{\mathcal{H}}Y}(x) - 1_{\text{convsupp}_{\mathcal{H}}Y}(x) \times \text{conc}(g, \text{supp}_{\mathcal{H}}Y)(x)$ .

If  $x \in \text{convsupp}_{\mathcal{H}}Y$ ,  $\text{conv } f(x) = -\text{conc}(g, \text{supp}_{\mathcal{H}}Y)(x)$ . One can also remark that if  $x \in \text{convsupp}_{\mathcal{H}}Y$ ,

$$\underline{\text{conv}} f(x) = -\inf \{ \alpha x + \beta, \alpha, \beta \in \mathbb{R}, g(z) \leq \alpha z + \beta, \forall z \in \text{supp}_{\mathcal{H}}Y \}.$$

□

So we have the following representation of the infimum super-hedging cost:

**Proposition 2.6.** *Suppose that  $g$  is a  $\mathcal{H}$ -normal integrand and that there exists some concave function  $\varphi$  such that  $g \leq \varphi$  on  $\text{supp}_{\mathcal{H}}Y$  and  $\varphi < \infty$  on  $\text{convsupp}_{\mathcal{H}}Y$ . Then,*

$$p(g) = -\underline{\text{conv}}(f)(y) = \overline{\text{conc}}(g, \text{supp}_{\mathcal{H}}Y)(y) - \delta_{\text{convsupp}_{\mathcal{H}}Y}(y).$$

We see that the fact that  $y$  belongs to  $\text{convsupp}_{\mathcal{H}}Y$  or not is important. In particular, in some cases, the infimum price of a European claim may be  $-\infty$ . This is related to the notion of absence of immediate profit that we present now. We say that there is an immediate profit when it is possible to super-replicate the contingent claim 0 at a negative price  $p$ . This implies that we may immediately make the positive profit  $-p$  and then start a portfolio process ending up with a non negative wealth. On the contrary case, we say that the Absence of Immediate Profit (AIP) condition holds. We will see that (AIP) is strictly weaker than (NA).

**Definition 2.7.** *There is an immediate profit (IP) if there exists a non null element of  $\mathcal{P}(0) \cap L^0(\mathbb{R}_-, \mathcal{H})$  or equivalently if  $p(0) \leq 0$  with  $P(p(0) < 0) > 0$ .*

Notice that the (AIP) condition may be seen as a particular case of the utility based No Good Deal condition introduced by Cherny, see [7, Definition 3]. In the definition above, let us explain why  $p(0) \leq 0$  with  $P(p(0) < 0) > 0$  implies the existence of an immediate profit (IP). To see it, recall that  $\mathcal{P}(0)$  is directed downward so that  $p(0) = \lim_n \downarrow p_n$  where  $p_n \in \mathcal{P}(0)$ . Since  $P(p(0) < 0) > 0$ , we deduce that there exists  $n$  such that  $P(p_n < 0) > 0$ . Let us define  $\tilde{p} = p_n 1_{p_n < 0}$ . Then,  $\tilde{p} \in \mathcal{P}(0) \cap L^0(\mathbb{R}_-, \mathcal{H})$  and  $\tilde{p} \neq 0$ , i.e.  $\tilde{p}$  generates an immediate profit.

**Proposition 2.8.** *(AIP) holds if and only if  $y \in \text{convsupp}_{\mathcal{H}}Y$  a.s.*

Notice that, from Lemma 6.10, we get that

$$\text{convsupp}_{\mathcal{H}}Y = [\text{ess inf}_{\mathcal{H}}Y, \text{ess sup}_{\mathcal{H}}Y] \cap \mathbb{R}.$$

*Proof.* The assumptions of Proposition 2.6 are satisfied for  $g = 0$  and we get that  $p(0) = -\delta_{\text{conv}\text{supp}_{\mathcal{H}}Y}(y)$ . Hence, there is no immediate profit if and only if  $y \in \text{conv}\text{supp}_{\mathcal{H}}Y$  a.s.  $\square$

**Corollary 2.9.** *The (AIP) condition holds true if and only if  $p(g) \geq 0$  a.s. for some non-negative  $\mathcal{H}$ -normal integrand  $g$  such that there exists some concave function  $\varphi$  verifying that  $g \leq \varphi < \infty$ .*

So in particular the (AIP) condition holds true if and only the infimum super-hedging cost of a european call option is non-negative.

*Proof.* Assume that (AIP) condition holds true. Then from Definition 2.7, we get that  $p(0) = 0$  a.s. As  $g \geq 0$ , it is clear that  $p(g) \geq p(0) = 0$  a.s. Conversely, assume that there exists some (IP). From Proposition 2.6, we get that

$$p(g) = \overline{\text{conc}}(g, \text{supp}_{\mathcal{H}}Y)(y) - \delta_{\text{conv}\text{supp}_{\mathcal{H}}Y}(y).$$

From Proposition 2.8, we get that  $P(y \in \text{conv}\text{supp}_{\mathcal{H}}Y) < 1$  and as  $\overline{\text{conc}}(g, \text{supp}_{\mathcal{H}}Y)(y) \leq \varphi < \infty$ ,  $P(p(g) = -\infty) > 0$  and the converse is proved.  $\square$

**Remark 2.10.** Assume that the  $\mathcal{H}$ -measurable set  $\Gamma = \{\text{ess sup}_{\mathcal{H}}Y < y\}$  has a non null probability. Then, on this set, from the zero initial capital, taking the physical position  $(y, -1)$  while keeping the zero position otherwise, one get at time 1 the terminal wealth  $y - Y \geq y - \text{ess sup}_{\mathcal{H}}Y > 0$  on  $\Gamma$  and zero otherwise, i.e. an arbitrage opportunity. Thus if  $y \notin \text{conv}\text{supp}_{\mathcal{H}}Y$  a.s., one gets an Arbitrage Opportunity and (AIP) is weaker than (NA).

We provide some examples where (AIP) holds true and is strictly weaker than (NA). This is the case if there exists  $Q_1, Q_2 \ll P$  such that  $S$  is a  $Q_1$ -super martingale (resp.  $Q_2$ -sub martingale), see Remark 6.4. This is of course true if  $\text{ess inf}_{\mathcal{H}}Y = 0$  and  $\text{ess sup}_{\mathcal{H}}Y = \infty$ . Finally, this is also the case for a model of the form  $Y = yZ$  where  $Z > 0$  is such that  $\text{supp}_{\mathcal{H}}Z = [0, 1]$  a.s. or  $\text{supp}_{\mathcal{H}}Z = [1, \infty)$  a.s. and  $y > 0$ . Indeed (recall Lemma 6.10), if  $\text{supp}_{\mathcal{H}}Z = [0, 1]$ ,  $\text{ess inf}_{\mathcal{H}}Y = y \text{ess inf}_{\mathcal{H}}Z = 0 \leq y$  and  $\text{ess sup}_{\mathcal{H}}Y = y \text{ess sup}_{\mathcal{H}}Z = y \geq y$ . The same holds if  $\text{supp}_{\mathcal{H}}Z = [1, \infty)$  a.s. Nevertheless, this kind of model does not admit a risk-neutral probability measure. Indeed, in the contrary case, there exists a density process i.e. a positive martingale  $(\rho_t)_{t=0,1}$  with  $\rho_0 = 1$  such that  $\rho S$  is a P-martingale:  $\mathbb{E}_P(\rho_1 Y | \mathcal{H}) = \rho_0 y$ . We get that  $\mathbb{E}_P(\rho_1 Z | \mathcal{H}) = \rho_0$ . Since we also have  $\rho_0 = \mathbb{E}_P(\rho_1 | \mathcal{H})$ , we deduce that

$\mathbb{E}_P(\rho_1(1 - Z)|\mathcal{H}) = 0$ . Since  $Z \leq 1$  a.s. or  $Z \geq 1$  a.s., this implies that  $\rho_1(1 - Z) = 0$  hence  $Z = 1$  which yields a contradiction.

**Corollary 2.11.** *Suppose that (AIP) holds true. Let  $g$  be a  $\mathcal{H}$ -normal integrand, such that there exists some concave function  $\varphi$  verifying that  $g \leq \varphi$  on  $\text{supp}_{\mathcal{H}}Y$  and  $\varphi < \infty$  on  $\text{conv}\text{supp}_{\mathcal{H}}Y$ . Then,*

$$\begin{aligned} p(g) &= \overline{\text{conc}}(g, \text{supp}_{\mathcal{H}}Y)(y) \\ &= \inf \{ \alpha y + \beta, \alpha, \beta \in \mathbb{R}, \alpha x + \beta \geq g(x), \forall x \in \text{supp}_{\mathcal{H}}Y \}. \end{aligned} \quad (2.6)$$

So in the case where  $g$  is concave and u.s.c., we get under (AIP) that  $p(g) = g(y)$ .

If  $g$  is convex and  $\lim_{x \rightarrow \infty} x^{-1}g(x) = M \in \mathbb{R}$ , the relative concave envelop of  $g$  with respect to  $\text{supp}_{\mathcal{H}}Y$  is the affine function that coincides with  $g$  on the extreme points of the interval  $\text{conv}\text{supp}_{\mathcal{H}}Y$  i.e.

$$p(g) = \theta^*y + \beta^* = g(\text{ess inf}_{\mathcal{H}}Y) + \theta^*(y - \text{ess inf}_{\mathcal{H}}Y), \quad (2.7)$$

$$\theta^* = \frac{g(\text{ess sup}_{\mathcal{H}}Y) - g(\text{ess inf}_{\mathcal{H}}Y)}{\text{ess sup}_{\mathcal{H}}Y - \text{ess inf}_{\mathcal{H}}Y}, \quad (2.8)$$

where we use the conventions  $\theta^* = \frac{0}{0} = 0$  in the case  $\text{ess sup}_{\mathcal{H}}Y = \text{ess inf}_{\mathcal{H}}Y$  and  $\theta^* = \frac{g(\infty)}{\infty} = M$  if  $\text{ess inf}_{\mathcal{H}}Y < \text{ess sup}_{\mathcal{H}}Y = +\infty$ . Moreover, using (2.6), we get that  $\theta^*Y + \beta^* \geq g(Y)$  a.s. (recall that  $Y \in \text{supp}_{\mathcal{H}}Y$  a.s., see Remark 2.2) and this implies using (2.7) that

$$p(g) + \theta^*(Y - y) \geq g \text{ a.s.} \quad (2.9)$$

and  $p(g) \in \mathcal{P}(g)$ .

### 3. The multi-period framework

#### 3.1. Multi-period super-hedging prices

For every  $t \in \{0, \dots, T\}$  the set  $\mathcal{R}_t^T$  of all claims that can be super-replicated from the zero initial endowment at time  $t$  is defined by

$$\mathcal{R}_t^T := \left\{ \sum_{u=t+1}^T \theta_{u-1} \Delta S_u - \epsilon_T^+, \theta_{u-1} \in L^0(\mathbb{R}, \mathcal{F}_{u-1}), \epsilon_T^+ \in L^0(\mathbb{R}_+, \mathcal{F}_T) \right\}. \quad (3.10)$$

The set of (multi-period) super-hedging prices and the (multi-period) infimum super-hedging cost of some contingent claim  $g \in L^0(\mathbb{R}, \mathcal{F}_T)$  at  $t$  are given by

$$\begin{aligned}\Pi_{T,T}(g) &:= \{g\} \\ \pi_{T,T}(g) &:= g \\ \Pi_{t,T}(g) &:= \{x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists R \in \mathcal{R}_t^T, x_t + R = g \text{ a.s.}\}, t \in \{0, \dots, T-1\} \\ \pi_{t,T}(g) &:= \text{ess inf}_{\mathcal{F}_t} \Pi_{t,T}(g).\end{aligned}\tag{3.11}$$

As in the one-period case, it is clear that the infimum super-hedging cost is not necessarily a price in the sense that  $\pi_{t,T}(g) \notin \Pi_{t,T}(g)$  when  $\Pi_{t,T}(g)$  is not closed. Alternatively, we may define sequentially

$$\begin{aligned}\mathcal{P}_{T,T}(g) &= \{g\} \\ \mathcal{P}_{t,T}(g) &= \{x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists \theta_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists p_{t+1} \in \mathcal{P}_{t+1,T}(g), x_t + \theta_t \Delta S_{t+1} \geq p_{t+1} \text{ a.s.}\}.\end{aligned}$$

The set  $\mathcal{P}_{t,T}(g)$  contains all prices at time  $t$  super-replicating some price  $p_{t+1} \in \mathcal{P}_{t+1,T}(g)$  at time  $t+1$ . First we show that for all  $t \in \{0, \dots, T\}$

$$\Pi_{t,T}(g) = \mathcal{P}_{t,T}(g).\tag{3.12}$$

It is clear at time  $T$ . Let  $x_t \in \Pi_{t,T}$ . Then there exists for all  $u \in \{t, \dots, T-1\}$ ,  $\theta_u \in L^0(\mathbb{R}, \mathcal{F}_u)$  such that  $x_t + \sum_{u=t+1}^{T-1} \theta_{u-1} \Delta S_u + \theta_{T-1} \Delta S_T \geq g$  a.s. As  $g \in \mathcal{P}_{T,T}(g)$ ,  $x_t + \sum_{u=t+1}^{T-1} \theta_{u-1} \Delta S_u \in \mathcal{P}_{T-1,T}(g)$  a.s. As  $x_t + \sum_{u=t+1}^{T-2} \theta_{u-1} \Delta S_u + \theta_{T-2} \Delta S_{T-1} = x_t + \sum_{u=t+1}^{T-1} \theta_{u-1} \Delta S_u$ , it follows that  $x_t + \sum_{u=t+1}^{T-2} \theta_{u-1} \Delta S_u \in \mathcal{P}_{T-2,T}(g)$  and recursively  $x_t \in \mathcal{P}_{t,T}$ . Conversely, let  $x_t \in \mathcal{P}_{t,T}$ , then there exists  $\theta_t \in L^0(\mathbb{R}, \mathcal{F}_t)$  and  $p_{t+1} \in \mathcal{P}_{t+1,T}(g)$ , such that  $x_t + \theta_t \Delta S_{t+1} \geq p_{t+1}$  a.s. Then as  $p_{t+1} \in \mathcal{P}_{t+1,T}(g)$ , there exists  $\theta_{t+1} \in L^0(\mathbb{R}, \mathcal{F}_{t+1})$  and  $p_{t+2} \in \mathcal{P}_{t+2,T}(g)$ , such that  $p_{t+1} + \theta_{t+1} \Delta S_{t+2} \geq p_{t+2}$  a.s. and going forward until  $T$  since  $\mathcal{P}_{T,T}(g) = \{g\}$ ,  $p_{T-1} + \theta_{T-1} \Delta S_T \geq g$  a.s., we get that  $x_t + \sum_{u=t+1}^T \theta_{u-1} \Delta S_u \geq g$  a.s. and  $x_t \in \Pi_{t,T}$  follows.

We now define a local version of super-hedging prices. Let  $g_{t+1} \in L^0(\mathbb{R}, \mathcal{F}_{t+1})$ , then the set of one-step super-hedging prices of  $g_{t+1}$  and its associated infimum super-hedging cost are given by

$$\begin{aligned}\mathcal{P}_{t,t+1}(g_{t+1}) &= \{x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists \theta_t \in L^0(\mathbb{R}, \mathcal{F}_t), x_t + \theta_t \Delta S_{t+1} \geq g_{t+1} \text{ a.s.}\} \\ \pi_{t,t+1}(g_{t+1}) &= \text{ess inf}_{\mathcal{F}_t} \mathcal{P}_{t,t+1}(g_{t+1}).\end{aligned}$$

The following lemma makes the link between local and global super-hedging under the assumption that the infimum (global) super-replication cost is a price. It provides a dynamic programming principle.

**Lemma 3.1.** *Let  $g \in L^0(\mathbb{R}, \mathcal{F}_T)$  and  $t \in \{0, \dots, T-1\}$ . Then  $\mathcal{P}_{t,T}(g) \subset \mathcal{P}_{t,t+1}(\pi_{t+1,T}(g))$  and  $\pi_{t,T}(g) \geq \pi_{t,t+1}(\pi_{t+1,T}(g))$ . Moreover if  $\pi_{t+1,T}(g) \in \Pi_{t+1,T}(g)$ , then  $\mathcal{P}_{t,T}(g) = \mathcal{P}_{t,t+1}(\pi_{t+1,T}(g))$  and  $\pi_{t,T}(g) = \pi_{t,t+1}(\pi_{t+1,T}(g))$ .*

**Remark 3.2.** So under (AIP), if at each step,  $\pi_{t+1,T}(g) \in \Pi_{t+1,T}(g)$  and if  $\pi_{t+1,T}(g) = g_{t+1}(S_{t+1})$  for some  $\mathcal{F}_t$ -normal integrand  $g_{t+1}$ , we will get from Corollary 2.6 that  $\pi_{t,T}(g) = \overline{\text{conc}}(g_{t+1}, \text{supp}_{\mathcal{F}_t} S_{t+1})(S_t)$ . We will propose in Section 3.3 a quite general setting where this holds true.

*Proof.* Let  $x_t \in \mathcal{P}_{t,T}(g)$ , then there exists  $\theta_t \in L^0(\mathbb{R}, \mathcal{F}_t)$  and  $p_{t+1} \in \mathcal{P}_{t+1,T}(g)$  such that (recall (3.12))

$$x_t + \theta_t \Delta S_{t+1} \geq p_{t+1} \geq \text{ess inf}_{\mathcal{F}_t} \Pi_{t+1,T}(g) = \pi_{t+1,T}(g) \text{ a.s.}$$

and the first statement follows. The second one follows from  $\pi_{t+1,T}(g) \in \mathcal{P}_{t+1,T}(g)$ .  $\square$

### 3.2. Multi-period (AIP)

We now define the notion of global and local immediate profit at time  $t$ . The first one says that it is possible to super-replicate at a negative cost from time  $t$  the claim 0 payed at time  $T$  and the local one the claim 0 payed at time  $t+1$ . We will see that they are equivalent.

**Definition 3.3.** *Fix some  $t \in \{0, \dots, T\}$ . A global immediate profit (IP) at time  $t$  is a non null element of  $\mathcal{P}_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t)$ .*

*A local immediate profit at time  $t$  is a non null element of  $\mathcal{P}_{t,t+1}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t)$ .*

*We say that the (AIP) condition holds if there is no global IP at any instant  $t$ , i.e. if  $\mathcal{P}_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\}$  for all  $t \in \{0, \dots, T\}$ .*

Using Proposition 2.8, we get the equivalence between the absence of local IP at time  $t$  and the fact that  $S_t \in \text{convsupp}_{\mathcal{F}_t} S_{t+1}$  a.s. So Theorem 3.4 below will show that there is an equivalence between the absence of global IP and the absence of local one.

**Theorem 3.4.** *(AIP) holds if and only if one of the the following assertions holds:*

- 1)  $S_t \in \text{convsupp}_{\mathcal{F}_t} S_{t+1}$  a.s., for all  $t \in \{0, \dots, T-1\}$ .
- 2)  $\text{ess inf}_{\mathcal{F}_t} S_{t+1} \leq S_t \leq \text{ess sup}_{\mathcal{F}_t} S_{t+1}$  a.s., for all  $t \in \{0, \dots, T-1\}$ .
- 3)  $\text{ess inf}_{\mathcal{F}_t} S_u \leq S_t \leq \text{ess sup}_{\mathcal{F}_t} S_u$  a.s. for all  $u \in \{t, \dots, T\}$ .
- 4)  $\pi_{t,T}(0) = 0$  a.s. for all  $t \leq T-1$ .

*Proof.* Let  $A_T = \Omega$  and for all  $t \in \{0, \dots, T-1\}$

$$A_t := \{\text{ess sup}_{\mathcal{F}_t} \Delta S_{t+1} \geq 0\} \cap \{\text{ess inf}_{\mathcal{F}_t} \Delta S_{t+1} \leq 0\}.$$

We show by induction that  $0 \in \mathcal{P}_{t,T}(0)$  and that under (AIP) at time  $t+1$

$$\pi_{t,T}(0) = 0 \text{ a.s.} \Leftrightarrow P(A_t) = 1 \Leftrightarrow (\text{AIP}) \text{ holds at time } t.$$

The third assertion follows from Lemma 6.5.

We proceed by backward recursion. At time  $T$ ,  $\mathcal{P}_{T,T}(0) = \{0\}$ , thus (AIP) holds at  $T$  and  $\pi_{T,T}(0) = 0$ . Fix some  $t \in \{0, \dots, T-1\}$ , assume that the induction hypothesis holds true at  $t+1$  and that (AIP) holds at time  $t+1$ . As  $\pi_{t+1,T}(0) = 0 \in \mathcal{P}_{t+1,T}(0)$ , we can apply Lemma 3.1 and  $\mathcal{P}_{t,T}(0) = \mathcal{P}_{t,t+1}(0)$ . So we can apply Lemma 2.3 and

$$\begin{aligned} \mathcal{P}_{t,T}(0) &= \mathcal{P}_{t,t+1}(0) = \left\{ \sup_{z \in \text{supp}_{\mathcal{F}_t} S_{t+1}} (-\theta z) + \theta S_t, \theta \in L^0(\mathbb{R}, \mathcal{F}_t) \right\} + L^0(\mathbb{R}_+, \mathcal{F}_t) \\ &= \left\{ -\theta (\text{ess sup}_{\mathcal{F}_t} \Delta S_{t+1} 1_{\theta < 0} + \text{ess inf}_{\mathcal{F}_t} \Delta S_{t+1} 1_{\theta \geq 0}), \theta \in L^0(\mathbb{R}, \mathcal{F}_t) \right\} + L^0(\mathbb{R}_+, \mathcal{F}_t). \end{aligned}$$

Note that  $0 \in \mathcal{P}_{t,T}(0)$ . Moreover, (AIP) holds at time  $t$  if and only if  $P(A_t) = 1$  (this also a direct consequence of Proposition 2.8). We also obtain that  $\pi_{t,T}(0) = \text{ess inf}_{\mathcal{H}} \mathcal{P}_{t,T}(0) = (0)1_{A_t} + (-\infty)1_{\Omega \setminus A_t}$  and equivalently (AIP) holds at time  $t$  if and only if  $\pi_{t,T}(0) = 0$  a.s. In particular, under (AIP) at time  $t$ , the infimum super-hedging cost at time  $t$  is a price for 0:  $\pi_{t,T}(0) = 0 \in \mathcal{P}_{t,T}(0)$ .  $\square$

**Remark 3.5.** Fix some  $t \leq T-1$ . If  $\text{ess sup}_{\mathcal{F}_{t-1}} \Delta S_t < 0$  on a non null measure set, then as in Remark 2.10 there is an arbitrage opportunity at time  $t$ .

### 3.3. Explicit pricing of a convex payoff under (AIP)

The aim of this section is to obtain some results in a particular model where  $\text{ess inf}_{\mathcal{F}_{t-1}} S_t = k_{t-1,t}^d S_{t-1}$  and  $\text{ess sup}_{\mathcal{F}_{t-1}} S_t = k_{t-1,t}^u S_{t-1}$  for every

$t \in \{1, \dots, T\}$  where  $(k_{t-1,t}^d)_{t \in \{1, \dots, T\}}$ ,  $(k_{t-1,t}^u)_{t \in \{1, \dots, T\}}$  and  $S_0$  are deterministic non-negative numbers. We obtain the same computative scheme (see (3.13)) as in [6] but it is obtained here assuming only (AIP) and not (NA).

**Theorem 3.6.** *Suppose that the model is defined by  $\text{ess inf}_{\mathcal{F}_{t-1}} S_t = k_{t-1,t}^d S_{t-1}$  and  $\text{ess sup}_{\mathcal{F}_{t-1}} S_t = k_{t-1,t}^u S_{t-1}$  where  $(k_{t-1,t}^d)_{t \in \{1, \dots, T\}}$ ,  $(k_{t-1,t}^u)_{t \in \{1, \dots, T\}}$  and  $S_0$  are deterministic non-negative numbers.*

- *The (AIP) condition holds at every instant  $t$  if and only if the super-hedging prices of some European call option are non-negative or equivalently if  $k_{t-1,t}^d \in [0, 1]$  and  $k_{t-1,t}^u \in [1, +\infty]$  for all  $t \in \{1, \dots, T\}$ .*
- *Suppose that the (AIP) condition holds. If  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a convex function with  $\text{Dom } h = \mathbb{R}$ ,  $h(z) \geq 0$  for all  $z \geq 0$  and  $\lim_{z \rightarrow +\infty} \frac{h(z)}{z} \in [0, \infty)$ , the infimum super-hedging cost of the European contingent claim  $h(S_T)$  is a price given by  $\pi_{t,T}(h) = h(t, S_t) \in \mathcal{P}_{t,T}(h)$  where*

$$\begin{aligned} h(T, x) &= h(x) \\ h(t-1, x) &= \lambda_{t-1,t} h(t, k_{t-1,t}^d x) + (1 - \lambda_{t-1,t}) h(t, k_{t-1,t}^u x), \end{aligned} \quad (3.13)$$

where  $\lambda_{t-1,t} = \frac{k_{t-1,t}^u - 1}{k_{t-1,t}^u - k_{t-1,t}^d} \in [0, 1]$  and  $1 - \lambda_{t-1,t} = \frac{1 - k_{t-1,t}^d}{k_{t-1,t}^u - k_{t-1,t}^d} \in [0, 1]$ , with the following conventions. When  $k_{t-1,t}^d = k_{t-1,t}^u = 1$  or  $S_{t-1} = 0$ ,  $\lambda_{t-1,t} = \frac{0}{0} = 0$  and  $1 - \lambda_{t-1,t} = 1$  and when  $k_{t-1,t}^d < k_{t-1,t}^u = \infty$ ,

$$\begin{aligned} \lambda_{t-1,t} &= \frac{\infty}{\infty} = 1 \\ (1 - \lambda_{t-1,t}) h(t, (+\infty)x) &= (1 - k_{t-1,t}^d) x \frac{h(t, (+\infty)x)}{(+\infty)x} \\ &= (1 - k_{t-1,t}^d) x \lim_{z \rightarrow +\infty} \frac{h(z)}{z}. \end{aligned} \quad (3.14)$$

Moreover, for every  $t$ ,  $\lim_{z \rightarrow +\infty} \frac{h(z)}{z} = \lim_{z \rightarrow +\infty} \frac{h(t,z)}{z}$  and  $h(\cdot, x)$  is non-increasing for all  $x \geq 0$ .

In the proof, the strategy associated to the minimal price is given and, in section 5, this result is illustrated through a numerical experiment.

*Proof.* The conditions  $k_{t-1,t}^d \in [0, 1]$  and  $k_{t-1,t}^u \in [1, +\infty]$  are equivalent to the (AIP) conditions by Theorem 3.4. We denote  $M = \frac{h(\infty)}{\infty}$  and  $M_t = \lim_{z \rightarrow +\infty} \frac{h(t,z)}{z}$ . We prove the second statement. Assume that (AIP) holds true. We establish the recursive formulation  $\pi_{t,T}(h) = h(t, S_t)$  given

by (3.13), that  $h(t, \cdot) \geq h(t+1, \cdot)$  and that  $M_t = M_{t+1}$ . The case  $t = T$  is immediate. As  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a convex function with  $\text{Dom } h = \mathbb{R}$ ,  $h$  is clearly a  $\mathcal{F}_{T-1}$ -normal integrand, we can apply Proposition 2.6 and its consequence for convex functions (see (2.7) and (2.8)) and we get that

$$\begin{aligned} \pi_{T-1,T}(h) &= h(k_{T-1,T}^d S_{T-1}) + \theta_{T-1}^* (S_{T-1} - k_{T-1,T}^d S_{T-1}), \\ \theta_{T-1}^* &= \frac{h(k_{T-1,T}^u S_{T-1}) - h(k_{T-1,T}^d S_{T-1})}{k_{T-1,T}^u S_{T-1} - k_{T-1,T}^d S_{T-1}}, \end{aligned} \quad (3.15)$$

where we use the conventions  $\theta_{T-1}^* = \frac{0}{0} = 0$  if either  $S_{T-1} = 0$  or  $k_{T-1,T}^u = k_{T-1,T}^d = 1$  and  $\theta_{T-1}^* = \frac{h(\infty)}{\infty} = M$  if  $k_{T-1,T}^d < k_{T-1,T}^u = +\infty$ . Moreover, using (2.9), we obtain that  $\pi_{T-1,T}(h) + \theta_{T-1}^* \Delta S_T \geq h$  a.s. i.e.  $\pi_{T-1,T}(h) \in \mathcal{P}(h)$ . So, using Lemma 3.1, we get that  $\mathcal{P}_{T-2,T}(h) = \mathcal{P}_{T-2,T-1}(\pi_{T-1,T}(h))$  and  $\pi_{T-2,T}(h) = \pi_{T-2,T-1}(\pi_{T-1,T}(h))$  and we may continue the recursion as soon as  $\pi_{T-1,T}(h) = h(T-1, S_{T-1})$  where  $h(T-1, \cdot)$  satisfies (3.13), is convex with domain equal to  $\mathbb{R}$ , is such that  $h(T-1, z) \geq 0$  for all  $z \geq 0$  and  $M_{T-1} = M \in [0, \infty)$ . To see that we distinguish three cases. If either  $S_{T-1} = 0$  or  $k_{T-1,T}^u = k_{T-1,T}^d = 1$ ,  $\pi_{T-1,T}(h) = h(S_{T-1})$  and  $h(T-1, z) = h(z) = h(T, z)$  satisfies all the required conditions. If  $k_{T-1,T}^d < k_{T-1,T}^u = +\infty$ ,  $\pi_{T-1,T}(h) = h(k_{T-1,T}^d S_{T-1}) + M (S_{T-1} - k_{T-1,T}^d S_{T-1}) = h(T-1, S_{T-1})$  with

$$\begin{aligned} h(T-1, z) &= h(k_{T-1,T}^d z) + Mz(1 - k_{T-1,T}^d) \\ &= \lim_{k^u \rightarrow +\infty} \left( \frac{k^u - 1}{k^u - k_{T-1,T}^d} h(k_{T-1,T}^d z) + \frac{1 - k_{T-1,T}^d}{k^u - k_{T-1,T}^d} h(k^u z) \right), \end{aligned}$$

using (3.14). The term in the r.h.s. above is larger than  $h(z) = h(T, z)$  by convexity since  $\frac{k^u - 1}{k^u - k_{T-1,T}^d} k_{T-1,T}^d z + \frac{1 - k_{T-1,T}^d}{k^u - k_{T-1,T}^d} k^u z = z$ . As  $k_{T-1,T}^d \in [0, 1]$  and  $M \in [0, \infty)$ ,  $h(T-1, z) \geq 0$  for all  $z \geq 0$ , we get that  $h(T-1, \cdot)$  is convex function with domain equal to  $\mathbb{R}$  since  $h$  is so. The function  $h(T-1, \cdot)$  also satisfies (3.13) (see (3.14)). Finally  $M_{T-1} = \lim_{z \rightarrow +\infty} k_{T-1,T}^d \frac{h(k_{T-1,T}^d z)}{k_{T-1,T}^d z} + M(1 - k_{T-1,T}^d) = M$ .

The last case is when  $S_{T-1} \neq 0$  and  $k_{T-1,T}^u \neq k_{T-1,T}^d$  and  $k_{T-1,T}^u < +\infty$ . It is clear that (3.15) implies (3.13). Moreover as  $k_{T-1,T}^d \in [0, 1]$  and  $k_{T-1,T}^u \in [1, +\infty)$ ,  $\lambda_{T-1,T} = \frac{k_{T-1,T}^u - 1}{k_{T-1,T}^u - k_{T-1,T}^d} \in [0, 1]$  and  $1 - \lambda_{T-1,T} = \frac{1 - k_{T-1,T}^d}{k_{T-1,T}^u - k_{T-1,T}^d} \in [0, 1]$  and (3.13) implies that  $h(T-1, z) \geq 0$  for all  $z \geq 0$ ,  $h(T-1, \cdot)$  is convex



with domain equal to  $\mathbb{R}$  since  $h$  is so. Moreover,

$$M_{T-1} = \lambda_{T-1,T} k_{T-1,T}^d \lim_{z \rightarrow +\infty} \frac{h(k_{T-1,T}^d z)}{k_{T-1,T}^d z} + (1 - \lambda_{T-1,T}) k_{T-1,T}^u \lim_{z \rightarrow +\infty} \frac{h(k_{T-1,T}^u z)}{k_{T-1,T}^u z} = M,$$

since

$$\lambda_{T-1,T} k_{T-1,T}^d + (1 - \lambda_{T-1,T}) k_{T-1,T}^u = 1.$$

If  $h(x) = (x - K)^+$ , for some  $K \in \mathbb{R}$ ,  $h(S_T)$  is a European contingent claim and  $h : \mathbb{R} \rightarrow \mathbb{R}$  a convex function with  $\text{Dom } h = \mathbb{R}$ ,  $h \geq 0$  and  $\lim_{z \rightarrow +\infty} \frac{h(z)}{z} = 1 \in [0, \infty)$ . We have just seen that under (AIP) the infimum super-hedging cost of  $h(S_T)$  is a price  $\pi_{t,T}(h) \geq 0$ . Reversely if (AIP) does not hold true, Proposition 2.6 implies

$$\begin{aligned} \pi_{T-1,T}(h) &= \inf \{ \alpha S_{T-1} + \beta, \alpha, \beta \in \mathbb{R}, (z - K)^+ \leq \alpha z + \beta, \forall z \in \text{supp}_{\mathcal{F}_{T-1,T}} S_T \} \\ &\quad - \delta_{[k_{T-1,T}^d S_{T-1}, k_{T-1,T}^u S_{T-1}] \cap \mathbb{R}}(S_{T-1}). \end{aligned}$$

As (AIP) does not hold true, either  $k_{T-1,T}^d > 1$  or  $k_{T-1,T}^u < 1$  and in both cases,  $S_{T-1} \notin [k_{T-1,T}^d S_{T-1}, k_{T-1,T}^u S_{T-1}] \cap \mathbb{R}$  and  $\pi_{T-1,T}(h) = -\infty$  since  $\inf \{ \alpha S_{T-1} + \beta, \alpha, \beta \in \mathbb{R}, (z - K)^+ \leq \alpha z + \beta, \forall z \in \text{supp}_{\mathcal{F}_{T-1,T}} S_T \} \leq S_{T-1}$ . Thus the convex subset  $\mathcal{P}_{T-1,T}(h)$  is equal to  $L^0(\mathbb{R}, \mathcal{F}_{T-1})$ . Similarly  $\pi_{t,T}(h) = -\infty$  for all  $t \in \{0, \dots, T-3\}$ . This allows to conclude about the first statement.  $\square$

**Remark 3.7.** *The infimum price of the European contingent claim  $h(S_T)$  in our model is a price, precisely the same than the price we get in a binomial model  $S_t \in \{k_{t-1,t}^d S_{t-1}, k_{t-1,t}^u S_{t-1}\}$  a.s.,  $t = 1, \dots, T$ .*

#### 4. Comparison between the (AIP) condition and classical no-arbitrage conditions

Examples have already show that (AIP) condition can be weaker than the classical absence of arbitrage opportunity (NA) characterized by the fundamental theorem of asset pricing (FTAP), see the Dalang-Morton-Willinger theorem in [8]. The goal of this section is compare the (AIP) condition with a weaker form of the classical No Free Lunch condition.

Recall that the set of all prices for the zero claim at time  $t$  is given by  $\mathcal{P}_{t,T}(0) = (-\mathcal{R}_t^T) \cap L^0(\mathbb{R}, \mathcal{F}_t)$  (see (3.10), (3.11) and (3.12)). It follows that (AIP) reads as  $\mathcal{R}_t^T \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \{0\}$ . Recall that the (NA) condition is

$\overline{\mathcal{R}_t^T} \cap L^0(\mathbb{R}_+, \mathcal{F}_T) = \{0\}$ . We also study a stronger condition than (AIP), i.e.  $\overline{\mathcal{R}_t^T} \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \{0\}$  for all  $t \in \{0, \dots, T\}$ , where the closure of  $\mathcal{R}_t^T$  is taken with respect to the convergence in probability. Note that this condition is a weak form of the classical No Free Lunch condition  $\overline{\mathcal{R}_t^T} \cap L^0(\mathbb{R}_+, \mathcal{F}_T) = \{0\}$  for all  $t \in \{0, \dots, T\}$ ; we call it (WNFL) for Weak No Free Lunch. The following result implies that (WNFL) may be equivalent to (AIP) condition under an extra closedness condition. It also provides a characterization through (absolutely continuous) martingale measures.

**Theorem 4.1.** *The following statements are equivalent:*

- (WNFL) holds.
- For every  $t \in \{0, \dots, T\}$ , there exists  $Q \ll P$  with  $\mathbb{E}(dQ/dP|\mathcal{F}_t) = 1$  such that  $(S_u)_{u \in \{t, \dots, T\}}$  is a  $Q$ -martingale.
- (AIP) holds and  $\overline{\mathcal{R}_t^T} \cap L^0(\mathbb{R}, \mathcal{F}_t) = \mathcal{R}_t^T \cap L^0(\mathbb{R}, \mathcal{F}_t)$  for every  $t \in \{0, \dots, T\}$ .

*Proof.* Suppose that (WNFL) holds and fix some  $t \in \{0, \dots, T\}$ . We may suppose without loss of generality that the process  $S$  is integrable under  $P$ . Under (WNFL), we then have  $\overline{\mathcal{R}_t^T} \cap L^1(\mathbb{R}_+, \mathcal{F}_t) = \{0\}$  where the closure is taken in  $L^1$ . Therefore, for every nonzero  $x \in L^1(\mathbb{R}_+, \mathcal{F}_t)$ , there exists by the Hahn-Banach theorem a non-zero  $Z_x \in L^\infty(\mathbb{R}_+, \mathcal{F}_T)$  such that (recall that  $\mathcal{R}_t^T$  is a cone)  $\mathbb{E}Z_x x > 0$  and  $\mathbb{E}Z_x \xi \leq 0$  for every  $\xi \in \mathcal{R}_t^T$ . Since  $-L^1(\mathbb{R}_+, \mathcal{F}_T) \subseteq \mathcal{R}_t^T$ , we deduce that  $Z_x \geq 0$  and we may renormalise  $Z_x$  so that  $\|Z_x\|_\infty = 1$ . Let us consider the family  $\mathcal{G} = \{\mathbb{E}(Z_x|\mathcal{F}_t) > 0\}$ ,  $x \in L^1(\mathbb{R}_+, \mathcal{F}_t) \setminus \{0\}$ . Consider any non null set  $\Gamma \in \mathcal{F}_t$ . Taking  $x = 1_\Gamma \in L^1(\mathbb{R}_+, \mathcal{F}_t) \setminus \{0\}$ , since  $\mathbb{E}(Z_x 1_\Gamma) > 0$ , we deduce that  $\Gamma$  has a non null intersection with  $\{\mathbb{E}(Z_x|\mathcal{F}_t) > 0\}$ . By [14, Lemma 2.1.3], we deduce an at most countable subfamily  $(x_i)_{i \geq 1}$  such that the union  $\bigcup_i \{\mathbb{E}(Z_{x_i}|\mathcal{F}_t) > 0\}$  is of full measure. Therefore,  $Z = \sum_{i=1}^\infty 2^{-i} Z_{x_i} \geq 0$  is such that  $\mathbb{E}(Z|\mathcal{F}_t) > 0$  and we define  $Q \ll P$  such that  $dQ = (Z/\mathbb{E}(Z|\mathcal{F}_t))dP$ . As the subset  $\{\sum_{u=t+1}^T \theta_{u-1} \Delta S_u, \theta_{u-1} \in L(\mathbb{R}, \mathcal{F}_{u-1})\}$  is a linear vector space contained in  $\mathcal{R}_t^T$ , we deduce that  $(S_u)_{u \in \{t, \dots, T\}}$  is a  $Q$ -martingale.

Suppose that for every  $t \in \{0, \dots, T\}$ , there exists  $Q \ll P$  such that  $(S_u)_{u \in \{t, \dots, T\}}$  is a  $Q$ -martingale with  $\mathbb{E}(dQ/dP|\mathcal{F}_t) = 1$ . Let us define for  $u \in \{t, \dots, T\}$ ,  $\rho_u = \mathbb{E}_P(dQ/dP|\mathcal{F}_u)$  then  $\rho_u \geq 0$  and  $\rho_t = 1$ . Consider  $\gamma_t \in \mathcal{R}_t^T \cap L^0(\mathbb{R}_+, \mathcal{F}_t)$ , i.e.  $\gamma_t$  is  $\mathcal{F}_t$ -measurable and is of the form  $\gamma_t = \sum_{u=t}^{T-1} \theta_u \Delta S_{u+1} - \epsilon_T^+$ . Since  $\theta_u$  is  $\mathcal{F}_u$ -measurable,  $\theta_u \Delta S_{u+1}$  admits a generalized conditional expectation under  $Q$  knowing  $\mathcal{F}_u$  and, by assumption, we

have  $\mathbb{E}_Q(\theta_u \Delta S_{u+1} | \mathcal{F}_u) = 0$ . We deduce by the tower law that

$$\gamma_t = \mathbb{E}_Q(\gamma_t | \mathcal{F}_t) = \sum_{u=t}^{T-1} \mathbb{E}_Q(\mathbb{E}_Q(\theta_u \Delta S_{u+1} | \mathcal{F}_u) | \mathcal{F}_t) - \mathbb{E}_Q(\epsilon_T^+ | \mathcal{F}_t) = -\mathbb{E}_Q(\epsilon_T^+ | \mathcal{F}_t).$$

Hence  $\gamma_t = 0$ , i.e. (AIP) holds. It remains to show that  $\overline{\mathcal{R}_t^T} \cap L^0(\mathbb{R}, \mathcal{F}_t) \subseteq \mathcal{R}_t^T \cap L^0(\mathbb{R}, \mathcal{F}_t)$ .

Consider first a one step model, where  $(S_u)_{u \in \{T-1, T\}}$  is a  $Q$ -martingale with  $\rho_T \geq 0$  and  $\rho_{T-1} = 1$ . Suppose that  $\gamma^n = \theta_{T-1}^n \Delta S_T - \epsilon_T^{n+} \in L^0(\mathbb{R}, \mathcal{F}_{T-1})$  converges in probability to  $\gamma^\infty \in L^0(\mathbb{R}, \mathcal{F}_{T-1})$ . We need to show that  $\gamma^\infty \in \mathcal{R}_{T-1}^T$ . On the  $\mathcal{F}_{T-1}$ -measurable set  $\Lambda_{T-1} := \{\liminf_n |\theta_{T-1}^n| < \infty\}$ , by [14, Lemma 2.1.2], we may assume w.l.o.g. that  $\theta_{T-1}^n$  is convergent to some  $\theta_{T-1}^\infty$  hence  $\epsilon_T^{n+}$  is also convergent and we can conclude. Otherwise, on  $\Omega \setminus \Lambda_{T-1}$ , we use the normalized sequences  $\tilde{\theta}_{T-1}^n := \theta_{T-1}^n / (|\theta_{T-1}^n| + 1)$ ,  $\tilde{\epsilon}_T^{n+} := \epsilon_T^{n+} / (|\theta_{T-1}^n| + 1)$ . By [14, Lemma 2.1.2], we may assume that  $\tilde{\theta}_{T-1}^n \rightarrow \tilde{\theta}_{T-1}^\infty$ ,  $\tilde{\epsilon}_T^{n+} \rightarrow \tilde{\epsilon}_T^{\infty+}$  and  $\tilde{\theta}_{T-1}^\infty \Delta S_T - \tilde{\epsilon}_T^{\infty+} = 0$ . As  $|\tilde{\theta}_{T-1}^\infty| = 1$  a.s., first consider the subset  $\Lambda_{T-1}^2 := \{\tilde{\theta}_{T-1}^\infty = 1\} \in \mathcal{F}_{T-1}$ . We then have  $\Delta S_T \geq 0$  on  $\Lambda_{T-1}^2$ . Since  $\mathbb{E}_Q(\Delta S_T 1_{\Lambda_{T-1}^2} | \mathcal{F}_{T-1}) = 0$ , we get that  $\rho_T \Delta S_T 1_{\Lambda_{T-1}^2} = 0$  a.s. Hence  $\rho_T \gamma^n 1_{\Lambda_{T-1}^2} = -\rho_T \epsilon_T^{n+} 1_{\Lambda_{T-1}^2} \leq 0$ . Taking the limit, we get that  $\rho_T \gamma^\infty 1_{\Lambda_{T-1}^2} \leq 0$  and, since  $\gamma^\infty \in L^0(\mathbb{R}, \mathcal{F}_{T-1})$ , we deduce that  $\rho_{T-1} \gamma^\infty 1_{\Lambda_{T-1}^2} \leq 0$ . Recall that  $\rho_{T-1} = 1$  hence  $\gamma^\infty 1_{\Lambda_{T-1}^2} \leq 0$  and  $\gamma^\infty 1_{\Lambda_{T-1}^2} \in \mathcal{R}_{T-1}^T$ . On the subset  $\{\tilde{\theta}_{T-1}^\infty = -1\}$  we may argue similarly and the conclusion follows in the one step model.

Fix some  $s \in \{t, \dots, T-1\}$ . We show that  $\overline{\mathcal{R}_{s+1}^T} \cap L^0(\mathbb{R}, \mathcal{F}_{s+1}) \subseteq \mathcal{R}_{s+1}^T \cap L^0(\mathbb{R}, \mathcal{F}_{s+1})$  implies the same property for  $s$  instead of  $s+1$ . By assumption  $(S_u)_{u \in \{s, \dots, T\}}$  is a  $Q$ -martingale with  $\rho_u \geq 0$  for  $u \in \{s, \dots, T\}$  and  $\rho_s = 1$ . Suppose that  $\gamma^n = \sum_{u=s}^{T-1} \theta_u^n \Delta S_{u+1} - \epsilon_T^{n+} \in L^0(\mathbb{R}, \mathcal{F}_s)$  converges to  $\gamma^\infty \in L^0(\mathbb{R}, \mathcal{F}_s)$ . If  $\gamma^\infty = 0$  there is nothing to prove. On the  $\mathcal{F}_s$ -measurable set  $\Lambda_s := \{\liminf_n |\theta_s^n| < \infty\}$ , by [14, Lemma 2.1.2], we may assume w.l.o.g. that  $\theta_s^n$  converges to  $\theta_s^\infty$ . Therefore, by the induction hypothesis,  $\sum_{u=s+1}^{T-1} \theta_u^n \Delta S_{u+1} - \epsilon_T^{n+}$  is also convergent to an element of  $\mathcal{R}_{s+1}^T \cap L^0(\mathbb{R}, \mathcal{F}_{s+1})$  and we conclude that  $\gamma^\infty \in \mathcal{R}_s^T$ . On  $\Omega \setminus \Lambda_{s-1}$ , we use the normalisation procedure, and deduce the equality  $\sum_{u=s}^{T-1} \tilde{\theta}_u^\infty \Delta S_{u+1} - \tilde{\epsilon}_T^{\infty+} = 0$  for some  $\tilde{\theta}_u^\infty \in L^0(\mathbb{R}, \mathcal{F}_u)$ ,  $u \in \{s, \dots, T-1\}$  and  $\tilde{\epsilon}_T^{\infty+} \geq 0$  such that  $|\tilde{\theta}_s^\infty| = 1$  a.s. We then argue as in the one step model on  $\Lambda_s^2 := \{\tilde{\theta}_s^\infty = 1\} \in \mathcal{F}_s$  and  $\Lambda_s^3 := \{\tilde{\theta}_s^\infty = -1\} \in \mathcal{F}_s$  respectively. When  $\tilde{\theta}_s^\infty = 1$ , we deduce that  $\Delta S_{s+1} + \sum_{u=s+1}^{T-1} \theta_u^\infty \Delta S_{u+1} - \tilde{\epsilon}_T^{\infty+} =$

0, i.e.  $\Delta S_{s+1} \in \mathcal{P}_{s+1,T}(0)$  hence  $\Delta S_{s+1} \geq 0$  under (AIP), see Theorem 3.4. Since  $\mathbb{E}_Q(\Delta S_{s+1} 1_{\Lambda_s^2} | \mathcal{F}_s) = 0$ ,  $\rho_{s+1} \Delta S_{s+1} 1_{\Lambda_s^2} = 0$  a.s. So,  $\rho_{s+1} \gamma^n 1_{\Lambda_s^2} \in \mathcal{R}_{s+1}^T \cap L^0(\mathbb{R}, \mathcal{F}_{s+1})$  hence  $\rho_{s+1} \gamma^\infty 1_{\Lambda_s^2} \in \mathcal{R}_{s+1}^T \cap L^0(\mathbb{R}, \mathcal{F}_{s+1})$  by induction. As  $\rho_{s+1} \gamma^\infty 1_{\Lambda_s^2}$  admits a generalized conditional expectation knowing  $\mathcal{F}_s$ , we deduce from (AIP) that  $\mathbb{E}_Q(\rho_{s+1} \gamma^\infty 1_{\Lambda_s^2} | \mathcal{F}_s) \leq 0$  hence  $\rho_s \gamma^\infty 1_{\Lambda_s^2} \leq 0$ . Recall that  $\rho_s = 1$  hence  $\gamma^\infty 1_{\Lambda_s^2} \leq 0$  so that  $\gamma^\infty 1_{\Lambda_s^2} \in \mathcal{R}_s^T \cap L^0(\mathbb{R}, \mathcal{F}_s)$ .

Finally, notice that the (AIP) condition implies (WNFL) as soon as the equality  $\overline{\mathcal{R}_t^T} \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \mathcal{R}_t^T \cap L^0(\mathbb{R}_+, \mathcal{F}_t)$  holds for every  $t \in \{0 \dots, T-1\}$ .  $\square$

**Proposition 4.2.** *Suppose that  $P(\text{ess inf}_{\mathcal{F}_t} S_{t+1} = S_t) = P(\text{ess sup}_{\mathcal{F}_t} S_{t+1} = S_t) = 0$  for all  $t \in \{0 \dots, T-1\}$ . Then, (WNFL) is equivalent to (AIP) and, under these equivalent conditions,  $\mathcal{R}_t^T$  is closed in probability for every  $t \in \{0 \dots, T-1\}$ .*

*Proof.* It suffices to show that  $\mathcal{R}_t^T$  is closed in probability for every  $t \in \{0 \dots, T-1\}$  under (AIP). Consider first the one step model, i.e. suppose that  $\gamma^n = \theta_{T-1}^n \Delta S_T - \epsilon_T^{n+} \in \mathcal{R}_{T-1}^T$  is a convergent sequence to  $\gamma^\infty \in L^0(\mathbb{R}, \mathcal{F}_T)$ . It is then sufficient to show that the  $\mathcal{F}_{T-1}$ -measurable set  $\Lambda_{T-1} := \{\liminf_n |\theta_{T-1}^n| < \infty\}$  satisfies  $P(\Lambda_{T-1}) = 1$ . Following the normalization procedure of proof of Theorem 4.1 on  $\Omega \setminus \Lambda_{T-1}$ , we get that  $\tilde{\theta}_{T-1}^\infty \Delta S_T$  where  $|\tilde{\theta}_{T-1}^\infty| = 1$  a.s. First consider the subset  $\Lambda_{T-1}^2 := \{\tilde{\theta}_{T-1}^\infty = 1\} \in \mathcal{F}_{T-1}$ . We have  $\Delta S_T \geq 0$  and hence  $\text{ess inf}_{\mathcal{F}_{T-1}} S_T \geq S_{T-1}$  on  $\Lambda_{T-1}^2$ . By (AIP) (see Theorem 3.4), we deduce that  $\text{ess inf}_{\mathcal{F}_{T-1}} S_T = S_{T-1}$  on  $\Lambda_{T-1}^2$ . The assumption implies that  $P(\Lambda_{T-1}^2) = 0$ . On the remaining subset  $\Lambda_{T-1}^3 := \{\tilde{\theta}_{T-1}^\infty = -1\} \in \mathcal{F}_{T-1}$ , we obtain similarly that  $\text{ess sup}_{\mathcal{F}_{T-1}} S_T = S_{T-1}$  and thus that  $P(\Lambda_{T-1}^3) = 0$ .

By induction, assume that  $\mathcal{R}_{t+1}^T$  is closed in probability and let us show that  $\mathcal{R}_t^T$  is also closed in probability. To do so, suppose that  $\gamma^n = \sum_{u=t+1}^T \theta_{u-1}^n \Delta S_u - \epsilon_T^{n+} \in \mathcal{R}_t^T$  converges to  $\gamma^\infty \in L^0(\mathbb{R}, \mathcal{F}_T)$ . On the  $\mathcal{F}_t$ -measurable set  $\Lambda_t := \{\liminf_n |\theta_t^n| < \infty\}$ , by [14, Lemma 2.1.2], we may assume w.l.o.g. that  $\theta_t^n$  is convergent to  $\theta_t^\infty$ . Therefore, by the induction hypothesis,  $\sum_{u=t+2}^T \theta_{u-1}^n \Delta S_u - \epsilon_T^{n+}$  is also convergent to an element of  $\mathcal{R}_{t+1}^T$  and we conclude that  $\gamma^\infty \in \mathcal{R}_t^T$ . On  $\Omega \setminus \Lambda_{t-1}$ , we use the normalization procedure, and deduce an equality  $\sum_{u=t+1}^T \tilde{\theta}_{u-1}^\infty \Delta S_u - \tilde{\epsilon}_T^{\infty+} = 0$  where  $\tilde{\theta}_{u-1}^\infty \in L(\mathbb{R}, \mathcal{F}_{u-1})$ ,  $u \in \{t, \dots, T-1\}$  and  $\tilde{\epsilon}_T^{\infty+} \geq 0$  such that  $|\tilde{\theta}_t^\infty| = 1$  a.s. We then argue on  $\Lambda_t^2 := \{\tilde{\theta}_t^\infty = 1\} \in \mathcal{F}_t$  and  $\Lambda_t^3 := \{\tilde{\theta}_t^\infty = -1\} \in \mathcal{F}_t$  respectively. On  $\Lambda_t^2$ , we obtain that  $\Delta S_{t+1} \in \mathcal{P}_{t+1,T}(0)$  hence under (AIP), with Theorem 3.4,

we obtain that  $\Delta S_{t+1} \geq 0$  and  $\text{ess inf}_{\mathcal{F}_t} S_{t+1} = S_t$  on  $\Lambda_t^2$ . This implies that  $P(\Lambda_t^2) = 0$  and similarly  $P(\Lambda_t^3) = 0$ . The conclusion follows.  $\square$

**Remark 4.3.** *Under the assumption of Proposition 4.2, the infimum superhedging cost is a price.*

**Lemma 4.4.** *The (AIP) condition is not necessarily equivalent to (WNFL).*

*Proof.* Let us consider a positive process  $(\tilde{S}_t)_{t \in \{0, \dots, T\}}$  which is a  $P$ -martingale. We suppose that  $\text{ess inf}_{\mathcal{F}_0} \tilde{S}_1 < \tilde{S}_1$  a.s., which holds in particular if  $\tilde{S}$  a geometric Brownian motion as  $\text{ess inf}_{\mathcal{F}_0} \tilde{S}_1 = 0$  a.s. Let us define  $S_t := \tilde{S}_t$  for  $t \in \{1, \dots, T\}$  and  $S_0 := \text{ess inf}_{\mathcal{F}_0} S_1$ . We have  $\text{ess inf}_{\mathcal{F}_0} S_1 \leq S_0$  and  $\text{ess sup}_{\mathcal{F}_0} S_1 \geq S_1 \geq \text{ess inf}_{\mathcal{F}_0} S_1 = S_0$  hence (AIP) holds at time 0 (see Theorem 3.4). Moreover, by the martingale property, (AIP) also holds at any time  $t \in \{1, \dots, T\}$  (see Remark 6.4). Let us suppose that (WNFL) holds. Then, there exists  $\rho_T \geq 0$  with  $\mathbb{E}(\rho_T) = 1$  such that  $S$  is a  $Q$ -martingale where  $dQ = \rho_T dP$ . Therefore,  $\mathbb{E}(\rho_T \Delta S_1) = 0$ . Since  $\Delta S_1 > 0$  by assumption, we deduce that  $\rho_T = 0$  hence a contradiction.  $\square$

## 5. Numerical experiments

### 5.1. Calibration

In this section, we suppose that the discrete dates are given by  $t_i^n = \frac{iT}{n}$ ,  $i \in \{0, \dots, n\}$  where  $n \geq 1$ . We assume that  $k_{t_{i-1}^n, t_i^n}^u = 1 + \sigma_{t_{i-1}^n} \sqrt{\Delta t_i^n}$  and  $k_{t_{i-1}^n, t_i^n}^d = 1 - \sigma_{t_{i-1}^n} \sqrt{\Delta t_i^n} \geq 0$  where  $t \mapsto \sigma_t$  is a positive Lipschitz-continuous function on  $[0, T]$ . This model implies that  $\text{ess inf}_{\mathcal{F}_{t_{j-1}^n}} S_{t_i^n} = k_{t_{j-1}^n, t_i^n}^d S_{t_{j-1}^n}$  and  $\text{ess sup}_{\mathcal{F}_{t_{j-1}^n}} S_{t_i^n} = k_{t_{j-1}^n, t_i^n}^u S_{t_{j-1}^n}$ , where for all  $j \leq i$ ,

$$k_{t_{j-1}^n, t_i^n}^u = \prod_{r=j}^i k_{t_{r-1}^n, t_r^n}^u, \quad k_{t_{j-1}^n, t_i^n}^d = \prod_{r=j}^i k_{t_{r-1}^n, t_r^n}^d.$$

By Theorem 3.6, we deduce that the (minimal) price of the European Call option  $(S_T - K)^+$  is given by  $h^n(t, S_t)$  defined by (3.13) with terminal condition  $h^n(T, x) = h(x) := (x - K)^+$ . We extend the function  $h^n$  on  $[0, T]$  in such a way that  $h^n$  is constant on each interval  $[t_i^n, t_{i+1}^n[$ ,  $i \in \{0, \dots, n\}$ . Such a scheme is proposed by Milstein [19] where a convergence theorem is proven when the terminal condition, i.e. the payoff function, is smooth. Precisely, the sequence of functions  $h^n$  converges uniformly to  $h(t, x)$ , solution to the

diffusion equation:

$$\partial_t h(t, x) + \sigma_t^2 \frac{x^2}{2} \partial_{xx} h(t, x) = 0, \quad h(T, x) = h(x).$$

In [19], it is supposed that the successive derivatives of the P.D.E.'s solution  $h$  are uniformly bounded. This is not the case for the Call payoff function  $g(x) = (x - K)^+$ . On the contrary the successive derivatives of the P.D.E.'s solution explode at the horizon date, see [18]. In [2], it is proven that the uniform convergence still holds when the payoff function is not smooth provided that the successive derivatives of the P.D.E. solution do not explode *too much*.

Supposing that  $\Delta t_i^n$  is closed to 0, we identify the observed prices of the call option with the limit theoretical prices  $h(t, S_t)$  at any instant  $t$  to deduce an evaluation of the deterministic function  $t \mapsto \sigma_t$ . Note that the assumptions on the multipliers  $k_{t_{i-1}^n}^u$  and  $k_{t_{i-1}^n}^d$  mean that

$$\left| \frac{S_{t_{i+1}^n}}{S_{t_i^n}} - 1 \right| \leq \sigma_{t_i} \sqrt{\Delta t_i^n}, \text{ a.s.} \quad (5.16)$$

We propose to verify (5.16) on real data. The data set is composed of historical values of the french index CAC 40 and European call option prices of maturity 3 months from the 23rd of October 2017 to the 19th of January 2018. For several strikes, matching the observed prices to the theoretical ones derived from the Black an Scholes formula with time-dependent volatility, we deduce the associated *implied* volatility  $t \mapsto \sigma_t$  and we compute the proportion of observations satisfying (5.16):

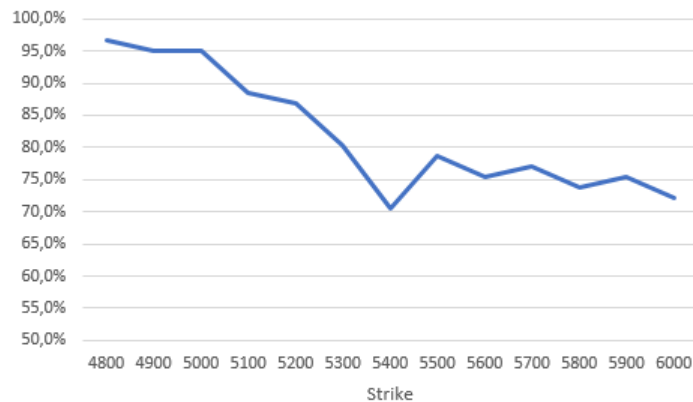


FIG 1. Ratio of observations satisfying (5.16) as a function of the strike.

Strike	4800	4900	5000	5100	5200	5300	5400	5500	5600	5700	5800	5900	6000
Ratio	96,7%	95,1%	95,1%	88,5%	86,9%	80,3%	70,5%	78,7%	75,4%	77,0%	73,8%	75,4%	72,1%

## 5.2. Super-hedging

We test the infimum super-hedging cost deduced for Theorem 3.6 on some data set composed of historical daily closing values of the french index CAC 40 from the 5th of January 2015 to the 12th of March 2018. The interval  $[0, T]$  we choose corresponds to one week composed of 5 days so that the number of discrete dates is  $n = 5$ . We first evaluate  $\sigma_{t_i}^2$ ,  $i = 0, \dots, 3$ , as

$$\sigma_{t_i} = \overline{\max} \left( \left| \frac{S_{t_{i+1}}}{S_{t_i}} - 1 \right| / \sqrt{\Delta t_{i+1}^n} \right) \quad i = 0, \dots, 3,$$

where  $\overline{\max}$  is the empirical maximum taken over a one year sliding sample window of 52 weeks. We then implement the super-hedging strategy on each of the 112 weeks following the sliding samples, i.e. every week from the 11th of January 2016 to the 5th of March 2018. We observe the empirical average  $\mathbb{E}(S_{t_0}) = 4044$ . The payoff function is  $h(x) = (x - K)^+$ .

### 5.2.1. Case where $K = 4700$ .

We implement the strategy associated to the super-hedging cost given by Theorem 3.6. We deduce the distribution of the super-hedging error  $\varepsilon_T := V_T - (S_T - K)^+$ , see Figure 4:

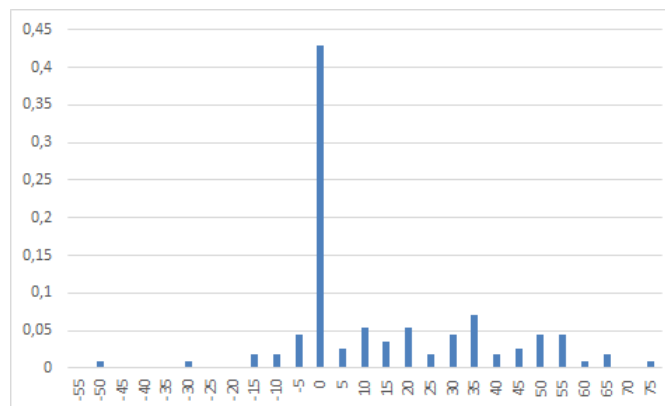


FIG 2. Distribution of the super-hedging error  $\varepsilon_T = V_T - (S_T - K)^+$ .

The empirical average of the error  $\varepsilon_T$  is 12.63 and its standard deviation is 21.65. This result is rather satisfactory in comparison to the large value  $\mathbb{E}(S_{t_0}) = 4044$ . This empirically confirms the efficiency of our suggested method.

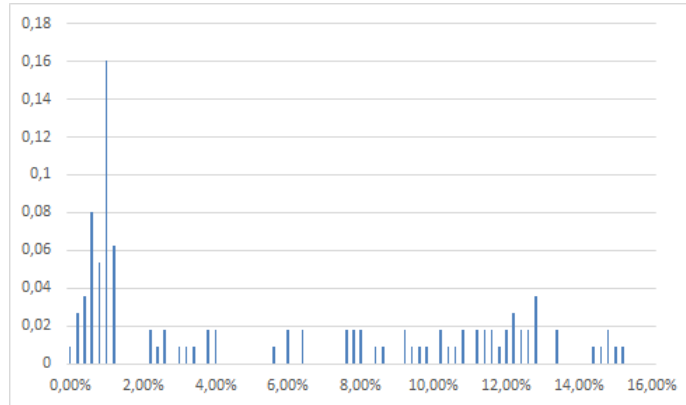


FIG 3. *Distribution of the ratio  $V_{t_0}/S_{t_0}$ .*

The empirical average of  $V_{t_0}/S_{t_0}$  is 5.63% and its standard deviation is 5.14%. Notice that, in the discrete case with  $k^d = 0$  and  $k^u = \infty$ , in particular when the dynamics of  $S$  is modeled by a (discrete) geometric Brownian motion, then the theoretical minimal initial price is  $V_{t_0} = S_{t_0}$ .

### 5.2.2. Case where $K = S_{t_0}$ .

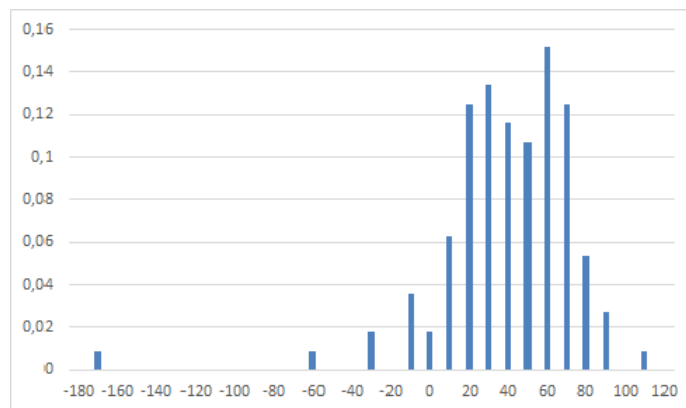


FIG 4. *Distribution of the super-hedging error  $\varepsilon_T$ .*



The empirical average of  $\varepsilon_T = V_T - (S_T - K)^+$  is 8.1 and its standard deviation is 30.78. Once again, this is rather satisfactory despite the possible loss of 170 which represents 4.2% of  $\mathbb{E}(S_{t_0}) = 4044$ .

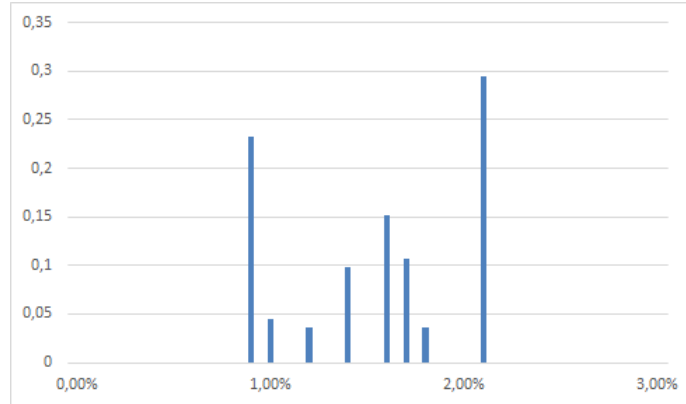


FIG 5. *Distribution of the ratio  $V_{t_0}/S_{t_0}$ .*

The empirical average of  $V_0/S_0$  is 2.51% and its standard deviation is 0.53%.

## 6. Appendix

### 6.1. Conditional support of a vector-valued random variable

We consider a random variable  $X$  defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with values in  $\mathbb{R}^d$ ,  $d \geq 1$ , endowed with the Borel  $\sigma$ -algebra. The goal of this section is to define the conditional support of  $X$  with respect to a sub  $\sigma$ -algebra  $\mathcal{H} \subseteq \mathcal{F}$ . This notion is very well known in the case where  $\mathcal{H}$  is the trivial sigma-algebra. Precisely, this is the usual support of  $X$ , i.e. the intersection of all closed deterministic subsets  $F$  of  $\mathbb{R}^d$  such that  $\mathbb{P}(X \in F) = 1$ .

**Definition 6.1.** *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and  $\mathcal{H}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$ . Let  $\mu$  be a  $\mathcal{H}$ -stochastic kernel (i.e. for all  $\omega \in \Omega$ ,  $\mu(\cdot, \omega)$  is a probability on  $\mathcal{B}(\mathbb{R}^d)$  and  $\mu(A, \cdot)$  is  $\mathcal{H}$ -measurable, for all  $A \in \mathcal{B}(\mathbb{R}^d)$ ). We define the random set  $D_\mu : \Omega \rightarrow \mathbb{R}^d$  :*

$$D_\mu(\omega) := \bigcap \{A \subset \mathbb{R}^d, \text{ closed, } \mu(A, \omega) = 1\}. \quad (6.17)$$

For  $\omega \in \Omega$ ,  $D_\mu(\omega) \subset \mathbb{R}^d$  is called the support of  $\mu(\cdot, \omega)$ . We will also call  $\text{supp}_{\mathcal{H}} X$  the set defined in (6.17) when  $\mu(A, \omega) = P(X \in A | \mathcal{H})(\omega)$  is the regular version of the conditional law of  $X$  knowing  $\mathcal{H}$  and call it the conditional support of  $X$  with respect to  $\mathcal{H}$ .

Using Theorems 12.7 and 12.14 of [1], we have that  $\mu(\cdot, \omega)$  admits a unique support  $D_\mu(\omega) \subset \mathbb{R}^d$  and that  $\mu(D_\mu(\omega), \omega) = 1$  (see also the definition of support in [1] on page 441).

**Lemma 6.2.**  $D_\mu$  is non-empty, closed-valued,  $\mathcal{H}$ -measurable and graph-measurable random set (i.e.  $\text{Graph}(D_\mu) \in \mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$ ).

*Proof.* It is clear from the definition (6.17) that for all  $\omega \in \Omega$ ,  $D_\mu(\omega)$  is a non-empty and closed subset of  $\mathbb{R}^d$ . We now show that  $D_\mu$  is  $\mathcal{H}$ -measurable. Let  $O$  be a fixed open set in  $\mathbb{R}^d$  and  $\mu_O : \omega \in \Omega \rightarrow \mu_O(\omega) := \mu(O, \omega)$ . As  $\mu$  is a stochastic kernel,  $\mu_O$  is  $\mathcal{H}$ -measurable. By definition of  $D_\mu(\omega)$  we get that  $\{\omega \in \Omega, D_\mu(\omega) \cap O \neq \emptyset\} = \{\omega \in \Omega, \mu_O(\omega) > 0\} \in \mathcal{H}$ , and  $D_\mu$  is  $\mathcal{H}$ -measurable. Now using Theorem 14.8 of [21],  $\text{Graph}(D_\mu) \in \mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$  (recall that  $D_\mu$  is closed-valued) and  $D_\mu$  is  $\mathcal{H}$ -graph-measurable.  $\square$

## 6.2. Conditional essential supremum

A very general concept of conditional essential supremum of a family of vector-valued random variables is defined in [15] with respect to a random partial order. In the real case, a generalization of the definition of essential supremum (see [14, Section 5.3.1] for the definition and the proof of existence of the classical essential supremum and Definition 3.1 and Lemma 3.9 in [15] for its conditional generalization as well as the existence, see also [3] where the conditional supremum is defined in the case where  $I$  is a singleton) is given by the following result:

**Proposition 6.3.** Let  $\mathcal{H} \subseteq \mathcal{F}$  be two  $\sigma$ -algebras on a probability space. Let  $\Gamma = (\gamma_i)_{i \in I}$  be a family of real-valued  $\mathcal{F}$ -measurable random variables. There exists a unique  $\mathcal{H}$ -measurable random variable  $\gamma_{\mathcal{H}} \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{H})$  denoted  $\text{ess sup}_{\mathcal{H}} \Gamma$  which satisfies the following properties:

1. For every  $i \in I$ ,  $\gamma_{\mathcal{H}} \geq \gamma_i$  a.s.
2. If  $\zeta \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{H})$  satisfies  $\zeta \geq \gamma_i$  a.s.  $\forall i \in I$ , then  $\zeta \geq \gamma_{\mathcal{H}}$  a.s.

*Proof.* The proof is given for sake of completeness and pedagogical purpose. The authors thanks T. Jeulin who suggested this (elegant) proof. Considering

the homeomorphism  $\arctan$  we can restrict ourself to  $\gamma_i$  taking values in  $[0, 1]$ . We denote by  $P_{\gamma_i|\mathcal{H}}$  a regular version of the conditional law of  $\gamma_i$  knowing  $\mathcal{H}$ . Let  $\zeta \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{H})$  such that  $\zeta \geq \gamma_i$  a.s.  $\forall i \in I$ . We have that

$$\begin{aligned} \zeta \geq \gamma_i \text{ a.s.} &\Leftrightarrow E(P(\zeta < \gamma_i|\mathcal{H})) = 0 \Leftrightarrow P(\zeta < \gamma_i|\mathcal{H}) = 0 \text{ a.s.} \\ &\Leftrightarrow P(\zeta \geq \gamma_i|\mathcal{H}) = P_{\gamma_i|\mathcal{H}}(\cdot - \infty, x)|_{x=\zeta} = 1 \text{ a.s.} \end{aligned}$$

From Definition 6.1,  $\text{supp}_{\mathcal{H}}\gamma_i \subset ]-\infty, \zeta]$  a.s. Let  $\Lambda_{\gamma_i|\mathcal{H}} = \sup\{x \in [0, 1], x \in \text{supp}_{\mathcal{H}}\gamma_i\}$  then  $\Lambda_{\gamma_i|\mathcal{H}} \leq \zeta$  a.s. For any  $c \in \mathbb{R}$ ,  $\{\Lambda_{\gamma_i|\mathcal{H}} \leq c\} = \{P_{\gamma_i|\mathcal{H}}(\cdot - \infty, c) = 1\} \in \mathcal{H}$  since we have chosen for  $P_{\gamma_i|\mathcal{H}}$  a regular version of the conditional law of  $\gamma_i$  knowing  $\mathcal{H}$ . It follows that  $\Lambda_{\gamma_i|\mathcal{H}}$  is  $\mathcal{H}$ -measurable. So taking the classical essential supremum, we get that  $\text{ess sup}_i \Lambda_{\gamma_i|\mathcal{H}} \leq \zeta$  a.s. and that  $\text{ess sup}_i \Lambda_{\gamma_i|\mathcal{H}}$  is  $\mathcal{H}$ -measurable. We conclude that  $\gamma_{\mathcal{H}} = \text{ess sup}_i \Lambda_{\gamma_i|\mathcal{H}}$  a.s. since for every  $i \in I$ ,  $P(\gamma_i \in \text{supp}_{\mathcal{H}}\gamma_i|\mathcal{H}) = 1$  and thus  $\text{ess sup}_i \Lambda_{\gamma_i|\mathcal{H}} \geq \gamma_i$  a.s.  $\square$

**Remark 6.4.** Let  $Q$  be an absolutely continuous probability measure with respect to  $P$ . Let  $Z = dQ/dP$  and  $\mathbb{E}_Q$  be the expectation under  $Q$ . As for every  $i \in I$ ,  $\text{ess sup}_{\mathcal{H}}\Gamma \geq \gamma_i$  a.s. and  $\text{ess sup}_{\mathcal{H}}\Gamma$  is  $\mathcal{H}$ -measurable,

$$\text{ess sup}_{\mathcal{H}}\Gamma \geq \frac{E(Z\gamma_i|\mathcal{H})}{E(Z|\mathcal{H})} = \mathbb{E}_Q(\gamma_i|\mathcal{H}).$$

Inspired by Theorem 2.8 in [3], we may easily show the following tower property:

**Lemma 6.5.** *Let  $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \mathcal{F}$  be  $\sigma$ -algebras on a probability space and let  $\Gamma = (\gamma_i)_{i \in I}$  be a family of real-valued  $\mathcal{F}$ -measurable random variables. Then,*

$$\text{ess sup}_{\mathcal{H}_1} (\text{ess sup}_{\mathcal{H}_2} \Gamma) = \text{ess sup}_{\mathcal{H}_1} \Gamma.$$

### 6.3. Link between two notions

Our goal is to extend the the fact that (see the proof of Proposition 6.3)

$$\text{ess sup}_{\mathcal{H}} X = \sup_{x \in \text{supp}_{\mathcal{H}} X} x \text{ a.s.}$$

First we show two useful lemmata on the measurability of the supremum and infimum.

**Lemma 6.6.** *Let  $\mathcal{K} : \Omega \rightarrow \mathbb{R}^d$  be a  $\mathcal{H}$ -measurable and closed random set such that  $\text{dom } \mathcal{K} = \{\omega \in \Omega, \mathcal{K}(\omega) \cap \mathbb{R}^d \neq \emptyset\} = \Omega$  and let  $h : \Omega \times \mathbb{R}^k \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable function, such that  $h(\omega, x, \cdot)$  is either l.s.c. or u.s.c., for all  $(\omega, x) \in \Omega \times \mathbb{R}^k$ . Let for all  $(\omega, x) \in \Omega \times \mathbb{R}^k$*

$$s(\omega, x) = \sup_{z \in \mathcal{K}(\omega)} h(\omega, x, z) \text{ and } i(\omega, x) = \inf_{z \in \mathcal{K}(\omega)} h(\omega, x, z).$$

*Then  $i$  and  $s$  are  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k)$ -measurable.*

*Proof.* Let  $(\eta_n)_{n \in \mathbb{N}}$  be a Castaing representation of  $\mathcal{K} : \mathcal{K}(\omega) = \text{cl}\{\eta_n(\omega), n \in \mathbb{N}\}$  where the closure is taken in  $\mathbb{R}^d$  and  $\eta_n(\omega) \in \mathcal{K}(\omega)$  for all  $n$ . Note that  $\eta_n$  is defined in the whole space  $\Omega$  since  $\text{dom } \mathcal{K} = \Omega$ . Fix some  $c \in \mathbb{R}$ . Then, we get that

$$\{(\omega, x) \in \Omega \times \mathbb{R}^d, s(\omega, x) \leq c\} = \bigcap_n \{(\omega, x) \in \Omega \times \mathbb{R}^d, h(\omega, x, \eta_n(\omega)) \leq c\}.$$

Indeed the first inclusion follows from the fact that  $\eta_n(\omega) \in \mathcal{K}(\omega)$  for all  $n$  and all  $\omega$ . For the reverse inclusion, fix some  $(\omega, x) \in \bigcap_n \{(\omega, x), h(\omega, x, \eta_n(\omega)) \leq c\}$ . For any  $z \in \mathcal{K}(\omega)$  one gets that  $z = \lim_n \eta_n(\omega)$ . Then from  $h(\omega, x, \eta_n(\omega)) \leq c$  we get that  $h(\omega, x, z) = \liminf h(\omega, x, \eta_n(\omega)) \leq c$  in the case where  $h(\omega, x, \cdot)$  is l.s.c. and  $h(\omega, x, z) = \limsup h(\omega, x, \eta_n(\omega)) \leq c$  in the case where  $h(\omega, x, \cdot)$  is u.s.c. Now recalling that  $h$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable and that  $\eta_n$  is  $\mathcal{H}$ -measurable,  $(\omega, x) \rightarrow h(\omega, x, \eta_n(\omega))$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k)$ -measurable,  $\{(\omega, x), h(\omega, x, \eta_n(\omega)) \leq c\} \in \mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k)$  and we deduce that  $s$  is  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^k)$ -measurable. Then we apply the same arguments for  $i$  replacing  $\leq c$  by  $\geq c$ .  $\square$

**Lemma 6.7.** *Let  $\mathcal{K} : \Omega \rightarrow \mathbb{R}^d$  be a  $\mathcal{H}$ -measurable and closed random set such that  $\text{dom } \mathcal{K} = \Omega$  and  $h : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  be l.s.c. in  $x$ . Then,*

$$\sup_{x \in \mathcal{K}} h(x) = \sup_n h(\eta_n), \tag{6.18}$$

*where  $(\eta_n)_n$  be a Castaing representation of  $\mathcal{K}$ .*

*Proof.* As  $(\eta_n)_n \subset \mathcal{K}$ ,  $h(\eta_n) \leq \sup_{x \in \mathcal{K}} h(x)$  and thus  $\sup_n h(\eta_n) \leq \sup_{x \in \mathcal{K}} h(x)$ . Let  $x \in \mathcal{K}$  and  $\eta_n \rightarrow x$ . By lower semicontinuity of  $h$ ,  $h(x) = \liminf_n h(\eta_n) \leq \sup_n h(\eta_n)$  and  $\sup_{x \in \mathcal{K}} h(x) \leq \sup_n h(\eta_n)$  and (6.18) is proved.  $\square$

**Lemma 6.8.** *Let  $X \in L^0(\mathbb{R}^d, \mathcal{F})$  such that  $\text{dom supp}_{\mathcal{H}} X = \Omega$  and  $h : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $\mathcal{H}$ -normal integrand. Then,*

$$\text{ess sup}_{\mathcal{H}} h(X) = \sup_{x \in \text{supp}_{\mathcal{H}} X} h(x) = \sup_n h(\gamma_n) \text{ a.s.}, \quad (6.19)$$

where  $(\gamma_n)_{n \in \mathbb{N}}$  is a Castaing representation of  $\text{supp}_{\mathcal{H}} X$ .

*Proof.* As  $P(X \in \text{supp}_{\mathcal{H}} X | \mathcal{H}) = 1$  we have that  $\sup_{x \in \text{supp}_{\mathcal{H}} X} h(x) \geq h(X)$  a.s. and by definition of  $\text{ess sup}_{\mathcal{H}} h(X)$ , we get that  $\sup_{x \in \text{supp}_{\mathcal{H}} X} h(x) \geq \text{ess sup}_{\mathcal{H}} h(X)$  since  $\sup_{x \in \text{supp}_{\mathcal{H}} X} h(x)$  is  $\mathcal{H}$ -measurable by Lemma 6.6 (recall that  $\text{supp}_{\mathcal{H}} X$  is  $\mathcal{H}$ -measurable and closed, see lemma 6.2).

By definition of the essential supremum we also get that  $\text{ess sup}_{\mathcal{H}} h(X) \geq h(X)$  a.s. Let  $(\gamma_n)_n$  the Castaing representation of  $\text{supp}_{\mathcal{H}} X(\omega)$ , Lemma 6.7 implies that  $\sup_{x \in \text{supp}_{\mathcal{H}} X} h(x) = \sup_n h(\gamma_n)$  a.s.. Fix some  $\varepsilon > 0$  and set  $Z_\varepsilon = 1_{B(\gamma_n, \varepsilon)}(X)$ , where  $B(\gamma_n, \varepsilon)$  is the closed ball of center  $\gamma_n$  and radius  $\varepsilon$ . Note that  $E(Z_\varepsilon | \mathcal{H}) = P(X \in B(\gamma_n, \varepsilon) | \mathcal{H}) > 0$ . Indeed if it does not hold true  $P(X \in \mathbb{R}^d \setminus B(\gamma_n, \varepsilon) | \mathcal{H}) = 1$  on some  $H \in \mathcal{H}$  such that  $P(H) > 0$  and by definition 6.1,  $\text{supp}_{\mathcal{H}} X \subset \mathbb{R}^d \setminus B(\gamma_n, \varepsilon)$  on  $H$ , which contradicts  $\gamma_n \in \text{supp}_{\mathcal{H}} X$  a.s. As  $\text{ess sup}_{\mathcal{H}} h(X)$  is  $\mathcal{H}$ -measurable we get that

$$\begin{aligned} \text{ess sup}_{\mathcal{H}} h(X) &\geq \frac{\mathbb{E}(Z_\varepsilon h(X) | \mathcal{H})}{E(Z_\varepsilon | \mathcal{H})} = \frac{\int 1_{B(\gamma_n, \varepsilon)}(x) h(x) P_{X| \mathcal{H}}(dx)}{E(Z_\varepsilon | \mathcal{H})} \\ &\geq \frac{\int (\inf_{y \in B(\gamma_n, \varepsilon)} h(y) 1_{B(\gamma_n, \varepsilon)}(x)) P_{X| \mathcal{H}}(dx)}{E(Z_\varepsilon | \mathcal{H})} \\ &\geq \inf_{y \in B(\gamma_n, \varepsilon)} h(y), \end{aligned}$$

since  $\inf_{y \in B(\gamma_n, \varepsilon)} h(y)$  is  $\mathcal{H}$ -measurable (see Lemma 6.6). Since  $h$  is l.s.c., we have that  $\lim_{\varepsilon \rightarrow 0} \inf_{x \in B(\gamma_n, \varepsilon)} h(x) = \lim_{x \rightarrow \gamma_n} h(x) = h(\gamma_n)$  and it follows that  $\text{ess sup}_{\mathcal{H}} h(X) \geq h(\gamma_n)$ . Taking the supremum over all  $n$ , we get that  $\text{ess sup}_{\mathcal{H}} h(X) \geq \sup_n h(\gamma_n) = \sup_{x \in \text{supp}_{\mathcal{H}} X} h(x)$ .

□

We have the easy extension

**Lemma 6.9.** *Let  $\mathcal{X} \subset L^0(\mathbb{R}^d, \mathcal{F})$  such that  $\text{dom supp}_{\mathcal{H}} X = \Omega$  for all  $X \in \mathcal{X}$  and  $\cup_{X \in \mathcal{X}} \text{supp}_{\mathcal{H}} X$  is  $\mathcal{H}$ -measurable and closed valued. Let  $h : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a  $\mathcal{H}$ -normal integrand. Then,*

$$\text{ess sup}_{\mathcal{H}} \{h(X), X \in \mathcal{X}\} = \sup_{x \in \cup_{X \in \mathcal{X}} \text{supp}_{\mathcal{H}} X} h(x), \text{ a.s.} \quad (6.20)$$

Note that if  $\mathcal{X}$  is countable,  $\cup_{X \in \mathcal{X}} \text{supp}_{\mathcal{H}} X$  is clearly  $\mathcal{H}$ -measurable. If  $\mathcal{X} = L^0(\mathbb{R}^d, \mathcal{F})$ , then  $\cup_{X \in \mathcal{X}} \text{supp}_{\mathcal{H}} X = \mathbb{R}^d$ , which is clearly  $\mathcal{H}$ -measurable and closed valued.

*Proof.* For all  $X \in \mathcal{X}$ ,  $\text{ess sup}_{\mathcal{H}}\{h(X), X \in \mathcal{X}\} \geq h(X)$  and  $\text{ess sup}_{\mathcal{H}}\{h(X), X \in \mathcal{X}\}$  is  $\mathcal{H}$ -measurable, so we get that by definition of  $\text{ess sup}_{\mathcal{H}} h(X)$  that  $\text{ess sup}_{\mathcal{H}}\{h(X), X \in \mathcal{X}\} \geq \text{ess sup}_{\mathcal{H}} h(X)$  and also  $\text{ess sup}_{\mathcal{H}}\{h(X), X \in \mathcal{X}\} \geq \sup_{X \in \mathcal{X}} \text{ess sup}_{\mathcal{H}} h(X)$ . Conversely, for all  $X \in \mathcal{X}$ ,  $\sup_{X \in \mathcal{X}} \text{ess sup}_{\mathcal{H}} h(X) \geq \text{ess sup}_{\mathcal{H}} h(X) \geq h(X)$  and if  $\sup_{X \in \mathcal{X}} \text{ess sup}_{\mathcal{H}} h(X)$  is  $\mathcal{H}$ -measurable, we obtain by definition of  $\text{ess sup}_{\mathcal{H}}\{h(X), X \in \mathcal{X}\}$  that  $\sup_{X \in \mathcal{X}} \text{ess sup}_{\mathcal{H}} h(X) \geq \text{ess sup}_{\mathcal{H}}\{h(X), X \in \mathcal{X}\}$ . Using Lemma 6.8, we get that

$$\sup_{X \in \mathcal{X}} \text{ess sup}_{\mathcal{H}} h(X) = \sup_{X \in \mathcal{X}} \sup_{x \in \text{supp}_{\mathcal{H}} X} h(x) = \sup_{x \in \cup_{X \in \mathcal{X}} \text{supp}_{\mathcal{H}} X} h(x)$$

Since  $\cup_{X \in \mathcal{X}} \text{supp}_{\mathcal{H}} X$  is  $\mathcal{H}$ -measurable and closed valued, Lemma 6.6 implies that  $\sup_{x \in \cup_{X \in \mathcal{X}} \text{supp}_{\mathcal{H}} X} h(x)$  is  $\mathcal{H}$ -measurable and the proof is complete.  $\square$

**Lemma 6.10.** *Consider  $X \in L^0(\mathbb{R}_+, \mathcal{F})$ . Then, we have a.s. that*

$$\begin{aligned} \text{ess inf}_{\mathcal{H}} X &= \inf \text{supp}_{\mathcal{H}} X, & \text{ess sup}_{\mathcal{H}} X &= \sup \text{supp}_{\mathcal{H}} X, \\ \text{ess inf}_{\mathcal{H}} X &\in \text{supp}_{\mathcal{H}} X, & & \text{on the set } \{\text{ess inf}_{\mathcal{H}} X > -\infty\}, \\ \text{ess sup}_{\mathcal{H}} X &\in \text{supp}_{\mathcal{H}} X, & & \text{on the set } \{\text{ess sup}_{\mathcal{H}} X < \infty\}. \end{aligned}$$

*Proof.* The two first statements are deduced from the construction of  $\text{ess sup}_{\mathcal{H}} X$  in Proposition 6.3. Suppose that  $\text{ess inf}_{\mathcal{H}} X \notin \text{supp}_{\mathcal{H}} X$  on some non null measure subset  $\Lambda \in \mathcal{H}$  of  $\{\text{ess inf}_{\mathcal{H}} X > -\infty\}$ . By a measurable selection argument, we deduce the existence of  $r \in L^0(\mathbb{R}_+, \mathcal{H})$  such that  $r > 0$  and  $[\text{ess inf}_{\mathcal{H}} X - r, \text{ess inf}_{\mathcal{H}} X + r] \subseteq \mathbb{R} \setminus \text{supp}_{\mathcal{H}} X$  on  $\Lambda$ . As  $X \in \text{supp}_{\mathcal{H}} X$  a.s. and  $X \geq \text{ess inf}_{\mathcal{H}} X$  a.s., we deduce that  $X \geq \text{ess inf}_{\mathcal{H}} X + r$  on  $\Lambda$ , which contradicts the definition of  $\text{ess inf}_{\mathcal{H}} X$ . The last statement is similarly shown.  $\square$

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