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A non-residually finite group acting uniformly properly on a hyperbolic space

R. Coulon*, D. Osin†

Abstract

In this article we produce an example of a non-residually finite group which admits a uniformly proper action on a Gromov hyperbolic space.

1 Introduction

By default, all actions of groups on metric spaces considered in this paper are by isometries. Recall that a group is *hyperbolic* if and only if it acts properly and cocompactly on a hyperbolic metric space. It is natural to ask what kind of groups we get if we remove the requirement of cocompactness from this definition. However, it turns out that every countable group admits a proper action on a hyperbolic space, namely the parabolic action on a combinatorial horoball [14]. Thus to obtain an interesting class of groups we have to strengthen our properness assumptions.

In this paper we propose to study the class of groups that admit a uniformly proper actions on a hyperbolic length space. We denote this class of group \mathcal{P} . Recall that an action of a group G on a metric space X is *uniformly proper* if for every $r \in \mathbb{R}_+$ there exists $N \in \mathbb{N}$ such that for all $x \in X$,

$$|\{g \in G \mid d_X(x, gx) \leq r\}| \leq N.$$

It is worth noting that every countable group acts properly on a hyperbolic space (e.g., on the so-called combinatorial horoball, see [14]). On the other hand, having a uniformly proper action on a hyperbolic space is a rather restrictive condition. For instance, [18, Theorem 1.2] implies that every group $G \in \mathcal{P}$ (as well as every its subgroup) is either virtually cyclic or acylindrically hyperbolic, which imposes strong restrictions on the algebraic structure of G .

Hyperbolic groups and their subgroups obviously belong to \mathcal{P} and, in general, groups from the class \mathcal{P} have many properties similar to those of hyperbolic groups. In fact, we do not know the answer to the following question: Does \mathcal{P} coincide with the class of all

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subgroups of hyperbolic groups? Although the affirmative answer seems unlikely, we are not aware of any potential counterexamples.

This paper is inspired by the well-known open problem.

Question 1.1. *Is every hyperbolic group residually finite?*

Our main result shows that the answer to this question is negative if one replaces the class of hyperbolic groups with the class \mathcal{P} .

Theorem 1.2. *There exists a non-trivial group G acting uniformly properly on a hyperbolic graph of bounded valence such that every amenable quotient of G is trivial. In particular, $G \in \mathcal{P}$ and G is not residually finite.*

In the process of constructing such a group G , we show that a certain subclass of \mathcal{P} is closed under taking relative hyperbolicity and certain small cancellation quotients (see Section 4). These result seems to be of independent interest and can potentially be used to construct other interesting examples of groups from the class \mathcal{P} .

2 A short review of hyperbolic geometry

In this section we recall a few notations and definitions regarding hyperbolic spaces in the sense of Gromov. For more details, refer the reader to Gromov's original article [12] or [5, 11].

The four point inequality. Let (X, d) be a length space. Recall that the *Gromov product* of three points $x, y, z \in X$ is defined by

$$\langle x, y \rangle_z = \frac{1}{2} \{d(x, z) + d(y, z) - d(x, y)\}.$$

In the remainder of this section, we assume that X is δ -hyperbolic, i.e. for every $x, y, z, t \in X$,

$$\langle x, z \rangle_t \geq \min \{\langle x, y \rangle_t, \langle y, z \rangle_t\} - \delta. \tag{1}$$

We denote by ∂X the boundary at infinity of X , see [5, Chapitre 2].

Quasi-convex subsets. Let Y be a subset of X . Recall that Y is α -quasi-convex if for every $x \in X$, for every $y, y' \in Y$, we have $d(x, Y) \leq \langle y, y' \rangle_x + \alpha$. If Y is path-connected, we denote by d_Y the length pseudo-metric on Y induced by the restriction of d_X on Y . The set Y is *strongly quasi-convex* if Y is 2δ -quasi-convex and for every $y, y' \in Y$ we have

$$d_X(y, y') \leq d_Y(y, y') \leq d_X(y, y') + 8\delta.$$

We denote by $Y^{+\alpha}$, the α -neighborhood of Y , i.e. the set of points $x \in X$ such that $d(x, Y) \leq \alpha$.

Group action. Let G be a group acting uniformly properly on X . An element $g \in G$ is either *elliptic* (it has bounded orbits, hence finite order) or *loxodromic* (it has exactly two accumulation points in ∂X) [3, Lemma 2.2]. A subgroup of G is either *elementary* (it is virtually cyclic) or contains a copy of the free group \mathbb{F}_2 [12, Paragraph 8.2]. In order to measure the action of g on X , we use the translation length defined as follows

$$\|g\|_X = \inf_{x \in X} d(gx, x).$$

If there is no ambiguity, we omit the space X in the notation. A loxodromic element $g \in G$ fixes exactly two points g^- and g^+ in ∂X . We denote by $E(g)$ the stabilizer of $\{g^-, g^+\}$. It is the maximal elementary subgroup containing g . Moreover $\langle g \rangle$ has finite index in $E(g)$ [9, Lemma 6.5].

Given a loxodromic element $g \in G$, there exists a g -invariant strongly quasi-convex subset Y_g of X which is quasi-isometric to line; its stabilizer is $E(g)$ and the quotient $Y_g/E(g)$ is bounded [8, Definition 3.12 and Lemma 3.13]. We call this set Y_g the *cylinder* of g .

We say that two elements $g, h \in G$ are *commensurable*, if there exists $n, m \in \mathbb{Z}$ and $u \in G$ such that $g^n = uh^m u^{-1}$. Every loxodromic element is contained in a unique maximal elementary subgroup [8, Lemma 3.28]. Hence two loxodromic elements g and h are commensurable if and only if there exists $u \in G$ such that g and uhu^{-1} generate an elementary subgroup.

Lemma 2.1. *Let S be a finite collection of pairwise non commensurable loxodromic elements of G . There exists $\Delta \in \mathbb{R}_+$ with the following property. For every $g, g' \in S$, for every $u \in G$, if*

$$\text{diam} \left(Y_g^{+5\delta} \cap uY_{g'}^{+5\delta} \right) > \Delta,$$

then $g = g'$ and $u \in E(g)$.

Proof. The action of G on X being uniformly proper, it is also acylindrical. According to [8, Proposition 3.44 and Lemma 6.14] there exists a constant $A, B > 0$ with the following property: if $h, h' \in G$ are two loxodromic elements generating a non-elementary subgroup, then

$$\text{diam} \left(Y_h^{+5\delta} \cap Y_{h'}^{+5\delta} \right) \leq A \max\{\|h\|, \|h'\|\} + B.$$

We now let

$$\Delta = A \max_{g \in S} \|g\| + B.$$

Let $g, g' \in S$, and $u \in G$ such that

$$\text{diam} \left(Y_g^{+5\delta} \cap uY_{g'}^{+5\delta} \right) > \Delta.$$

Recall that $uY_{g'}$ is the cylinder of $ug'u^{-1}$. It follows from our choice of Δ , that g and $ug'u^{-1}$ generate an elementary subgroup. Since the elements of S are pairwise non-commensurable it forces $g = g'$ and $u \in E(g)$. \square

3 An auxiliary class \mathcal{P}_0

To prove our main result we will make use of an auxiliary class \mathcal{P}_0 .

Definition 3.1. A subset S of a metric space X is *r -separated* if for every distinct points $s, s' \in S$, $d(s, s') \geq r$. Given a subset Y of X and $r > 0$, we define the *r -capacity* of Y , denoted by $C_r(Y)$, as the maximal number of points in an r -separated subset of Y . We say that X has *r -bounded geometry* if for every $R > 0$, there is an integer N bounding from above the r -capacity of every ball of radius R . If there exists $r > 0$ such that X has r -bounded geometry we simply say that X has *bounded geometry*.

The class \mathcal{P}_0 we are interested in consists of all groups admitting a uniformly proper action on a hyperbolic length space with bounded geometry. It is clear that $\mathcal{P}_0 \subseteq \mathcal{P}$. We will show that the class \mathcal{P}_0 is closed under relative hyperbolicity and taking certain small cancellation quotients. Before we discuss the precise statements and proofs, a few remarks are in order. First, we do not know whether \mathcal{P}_0 is indeed a proper subclass of \mathcal{P} . Second, it is possible to prove the results of the next section for the whole class \mathcal{P} . Nevertheless, the proofs become much easier for \mathcal{P}_0 . Therefore we restrict our attention to this subclass, which is sufficient for the proof of our main theorem.

We start with a few equivalent characterizations of the class \mathcal{P}_0 . A graph $\Gamma = (V, E)$ is understood here in the sense of Serre [20]. Observe that a graph Γ has bounded geometry whenever it has *uniformly bounded valence* i.e. there exists $d \in \mathbb{N}$, such that the valence of any vertex $v \in V$ is at most d .

Remark 3.2. The converse statement is false. Indeed, consider the real line, which we think of as a graph with the vertex set \mathbb{Z} and the obvious edges; to each vertex, attach infinitely many edges of length 1. The resulting graph has 3-bounded geometry while some vertices have infinite valence.

If Γ is a graph with uniformly bounded valence, the action of a group G on Γ is uniformly proper if and only if there exists $N \in \mathbb{N}$ such that the stabilizer of any vertex contains at most N elements.

Proposition 3.3. *Let G be a group. The following assertions are equivalent.*

1. G belongs to \mathcal{P}_0 .
2. G acts uniformly properly without inversion on a hyperbolic graph Γ with uniformly bounded valence.
3. G acts on a hyperbolic graph Γ with uniformly bounded valence such that the action of G is free when restricted to the vertex set of Γ .

Proof. To show that (3) \Rightarrow (2) one simply takes the barycentric subdivision of the graph. The implication (2) \Rightarrow (1) directly follows from the definition. We now focus on (1) \Rightarrow (3). By definition there exists $r \in \mathbb{R}_+^*$ such that G acts uniformly properly on a hyperbolic length space X with r -bounded geometry. Using Zorn's Lemma we choose an r -separated subset \bar{S} of $\bar{X} = X/G$ which is maximal for this property. We denote by S the pre-image

of \bar{S} in X . We fix $S_0 \subset S$ to be a set of representatives for the action of G on S . Let $R = 2r + 1$. We now define a graph $\Gamma = (V, E)$ as follows. Its vertex set is $V = G \times S_0$. The edge set E is the set of pairs $((u, s), (u', s')) \in V \times V$ such that $d_X(us, u's') \leq R$. The initial and terminal vertices of such an edge are (u, s) and (u', s') respectively. The group G acts freely on V as follows: for every $g \in G$, for every $(u, s) \in V$, we have $g \cdot (u, s) = (gu, s)$. This action induces an action by isometries of G on Γ . Recall that $R > 2r$. This allows us to perform a variation on the Milnor-Svarč Lemma and prove that the map $V \rightarrow X$ sending (u, s) to us induces a (G -equivariant) quasi-isometry from Γ to X . In particular Γ is hyperbolic. We are left to prove that Γ has uniformly bounded valence.

Since X has r -bounded geometry, there exists $N_1 \in \mathbb{N}$ such that the r -capacity of any ball of radius R in X is at most N_1 . The group G acting uniformly properly on X , there exists $N_2 \in \mathbb{N}$ such that for every $x \in X$, the cardinality of the set

$$U(x) = \{g \in G \mid d_X(x, gx) \leq 2R\}$$

is bounded above by N_2 . We now fix a vertex $v_0 = (u_0, s_0)$ of Γ . We fix a subset S_1 of $B(u_0s_0, R)$ such that any G -orbit of S intersecting $B(u_0s_0, R)$ contains exactly one point in S_1 . It follows from our choice of S that if $s, s' \in S$ belong to distinct G -orbits, then $d_X(s, s') \geq r$. Consequently the cardinality of S_1 is bounded above by the r -capacity of this ball, i.e. N_1 . By construction for every $s \in S_1$, there exists $u_s \in G$ such that $u_s s$ belongs to S_0 . It follows from the definition of Γ combined with the triangle inequality that any neighbor of v_0 belongs to the set

$$\{(uu_s^{-1}, u_s s) \mid s \in S_1, u \in U(s)\}.$$

The cardinality of this set is bounded above by $d = N_1 N_2$, which does not depend on v_0 , hence Γ has uniformly bounded valence. \square

4 Stability of the class \mathcal{P}_0

4.1 Relative hyperbolicity

Proposition 4.1. *Let G be a group hyperbolic relative to a finite collection of subgroups $H_1, \dots, H_k \in \mathcal{P}_0$. Then $G \in \mathcal{P}_0$.*

Proof. This is a straightforward consequence of the construction of the induced action from [1]. Indeed, for every $i \in \{1, \dots, k\}$, let R_i be a hyperbolic graph of uniformly bounded valence such that H_i acts on R_i and the action is free on vertices. Let S be a finite subset of G such that G is generated by $S \cup H_1 \cup \dots \cup H_k$. Associated to these data and some additional parameters, the construction of the induced action produces an action of G on a graph Γ that is obtained from the (possibly disconnected) Cayley graph of G with respect to S by gluing copies of graphs R_i along cosets of H_i in G ; see [1, Definition 3.20]. It is straightforward from the construction that Γ has uniformly bounded valence whenever every R_i does and that the action of G on Γ is free on vertices [1, Lemma 3.19]. Moreover, the graph Γ is hyperbolic by [1, Theorem 4.9 (a)]. Thus by Proposition 3.3, $G \in \mathcal{P}_0$. \square

4.2 Small cancellation quotient

We now explain how \mathcal{P}_0 behaves under small cancellation. To that end we first review the geometric theory of small cancellation as it has been introduced by M. Gromov [13] and further developed in [10, 6, 9]. For a detailed exposition we refer the reader to [7, Sections 4-6].

Settings. Let X be a δ -hyperbolic length space and G a group acting on X . Let \mathcal{Q} be a family of pairs (H, Y) such that Y is a strongly quasi-convex subset of X and H a subgroup of $\text{Stab}(Y)$. We assume that \mathcal{Q} is closed under the following action of G . For every $(H, Y) \in \mathcal{Q}$, for every $g \in G$, $g(H, Y) = (gHg^{-1}, gY)$. In addition we require that \mathcal{Q}/G is finite. We denote by K the (normal) subgroup generated by the subgroups H where $(H, Y) \in \mathcal{Q}$. The goal is to study the quotient $\bar{G} = G/K$ and the corresponding projection $\pi: G \rightarrow \bar{G}$. To that end we define the following two small cancellation parameters

$$\begin{aligned} \Delta(\mathcal{Q}, X) &= \sup \left\{ \text{diam} \left(Y_1^{+5\delta} \cap Y_2^{+5\delta} \right) \mid (H_1, Y_1) \neq (H_2, Y_2) \in \mathcal{Q} \right\}, \\ \text{inj}(\mathcal{Q}, X) &= \inf \{ \|h\| \mid h \in H \setminus \{1\}, (H, Y) \in \mathcal{Q} \}. \end{aligned}$$

They play the role of the length of the longest piece and the shortest relation respectively. We now fix a number $\rho > 0$. Its value will be made precise later. It should be thought of as a very large parameter.

Cones. Let $(H, Y) \in \mathcal{Q}$. The *cone of radius ρ over Y* , denoted by $Z(Y)$, is the quotient of $Y \times [0, \rho]$ by the equivalence relation that identifies all the points of the form $(y, 0)$. The equivalence class of $(y, 0)$, denoted by a , is called the *apex* of the cone. By abuse of notation, we still write (y, r) for the equivalence class of (y, r) . The map $\iota: Y \rightarrow Z(Y)$ that send y to (y, ρ) provides a natural embedding from Y to $Z(Y)$. This space can be endowed with a metric as described below. For the geometric interpretation of the distance see [7, Section 4.1].

Proposition 4.2. [4, Chapter I.5, Proposition 5.9] *The cone $Z(Y)$ is endowed with a metric characterized in the following way. Let $x = (y, r)$ and $x' = (y', r')$ be two points of $Z(Y)$ then*

$$\cosh d_{Z(Y)}(x, x') = \cosh r \cosh r' - \sinh r \sinh r' \cos \theta(y, y'),$$

where $\theta(y, y')$ is the angle at the apex defined by $\theta(y, y') = \min \{ \pi, d_Y(y, y') / \sinh \rho \}$.

Cone-off over a metric space. The *cone-off of radius ρ over X relative to \mathcal{Y}* denoted by $\dot{X}_\rho(\mathcal{Q})$ (or simply \dot{X}) is obtained by attaching for every $(H, Y) \in \mathcal{Q}$, the cone $Z(Y)$ on X along Y according to ι . We endow \dot{X} with the largest pseudo-metric $d_{\dot{X}}$ for which all the maps $X \rightarrow \dot{X}$ and $Z(Y) \rightarrow \dot{X}$ – when (H, Y) runs over \mathcal{Q} – are 1-Lipschitz. It turns out that this pseudo-distance is actually a distance on \dot{X} [7, Proposition 5.10]. Actually $(\dot{X}, d_{\dot{X}})$ is a length space.

The action of G on X naturally extends to an action on \dot{X} as follows. Let $(H, Y) \in \mathcal{Q}$. For every $x = (y, r) \in Z(Y)$, for every $g \in G$, gx is the point of $Z(gY)$ defined by

$gx = (gy, r)$. The space $\bar{X}_\rho(\mathcal{Q})$ (or simply \bar{X}) is the quotient $\bar{X} = \dot{X}/K$. The metric on \dot{X} induces a pseudo-metric on \bar{X} . We write $\zeta: \dot{X} \rightarrow \bar{X}$ for the canonical projection from \dot{X} to \bar{X} . The quotient \bar{G} naturally acts by isometries on \bar{X} .

Proposition 4.3. *Assume that for every $(H, Y) \in \mathcal{Q}$, the space Y/H is bounded. Then the spaces \bar{X} and X/K are quasi-isometric.*

Proof. Recall that the embedding $X \rightarrow \dot{X}$ is 1-Lipschitz. Hence it induces a 1-Lipschitz embedding $X/K \rightarrow \bar{X}$. We claim that the map $X/K \rightarrow \bar{X}$ is actually bi-Lipschitz. For simplicity, we implicitly identify X/K with its image in \bar{X} . Recall that \mathcal{Q}/G is finite. It follows from our assumption that there exists $D \in \mathbb{R}_+$ such that for every $(H, Y) \in \mathcal{Q}$, the image of Y in X/K has diameter at most D .

Let $\bar{x}, \bar{x}' \in X/K$. Let $\eta \in \mathbb{R}_+^*$. There exist $x, x' \in X$, respective pre-images of \bar{x} and \bar{x}' , such that $d_{\dot{X}}(x, x') < d_{\bar{X}}(\bar{x}, \bar{x}') + \eta$. Following the construction of the metric on \dot{X} – see for instance [7, Section 5.1] – we observe that there exists a sequence of points $(x_0, y_0, x_1, y_1, \dots, x_m, y_m)$ which approximates the distance between x and x' in the following sense:

1. $x_0 = x$ and $y_m = x'$;
2. For every $i \in \{0, \dots, m-1\}$, there exists $(H_i, Y_i) \in \mathcal{Q}$ such that $y_i, x_{i+1} \in Y_i$;
- 3.

$$\sum_{i=0}^m d_X(x_i, y_i) + \sum_{i=0}^{m-1} d_{Z(Y_i)}(y_i, x_{i+1}) < d_{\dot{X}}(x, x') + \eta. \quad (2)$$

For every $i \in \{0, \dots, m\}$, we write \bar{x}_i and \bar{y}_i for the images in X/K of x_i and y_i respectively. It follows from the triangle inequality that

$$d_{X/K}(\bar{x}, \bar{x}') \leq \sum_{i=0}^m d_{X/K}(\bar{x}_i, \bar{y}_i) + \sum_{i=0}^{m-1} d_{X/K}(\bar{y}_i, \bar{x}_{i+1}) \quad (3)$$

We are going to compare the terms of the latter inequality with the ones of (2). Note first that for every $i \in \{0, \dots, m\}$, we have

$$d_{X/K}(\bar{x}_i, \bar{y}_i) \leq d_X(x_i, y_i). \quad (4)$$

Let $i \in \{0, \dots, m-1\}$. In order to estimate $d_{X/K}(\bar{y}_i, \bar{x}_{i+1})$, we distinguish two cases. Assume first that $d_{Y_i}(y_i, x_{i+1}) \leq \pi \sinh \rho$. It follows from the definition of the metric on $Z(Y_i)$ that

$$d_{X/K}(\bar{y}_i, \bar{x}_{i+1}) \leq d_X(y_i, x_{i+1}) \leq d_{Y_i}(y_i, x_{i+1}) \leq \frac{\pi \sinh \rho}{2\rho} d_{Z(Y_i)}(y_i, x_{i+1}). \quad (5)$$

Assume now that $d_{Y_i}(y_i, x_{i+1}) > \pi \sinh \rho$. In particular $d_{Z(Y_i)}(y_i, x_{i+1}) = 2\rho$. Recall that the diameter of the image of Y_i in X/K is at most D . Hence

$$d_{X/K}(\bar{y}_i, \bar{x}_{i+1}) \leq \frac{D}{2\rho} d_{Z(Y_i)}(y_i, x_{i+1}). \quad (6)$$

Combining (2) - (6) we get that

$$d_{X/K}(\bar{x}, \bar{x}') \leq \lambda (d_{\dot{X}}(x, x') + \eta) \leq \lambda (d_{\bar{X}}(\bar{x}, \bar{x}') + 2\eta),$$

where

$$\lambda = \max \left\{ 1, \frac{\pi \sinh \rho}{2\rho}, \frac{D}{2\rho} \right\}.$$

The previous inequality holds for every $\eta \in \mathbb{R}_+^*$, hence $X/K \rightarrow \bar{X}$ is bi-Lipschitz, which completes the proof of our claim. Note that the diameter of the cones attached to X to form the cone-off space \dot{X} have diameter at most 2ρ . Hence any point of \bar{X} is a distance at most 2ρ from a point of X/K . Consequently the map $X/K \rightarrow \bar{X}$ is a quasi-isometry. \square

Small cancellation theorem. The small cancellation theorem recalled below is a compilation of Proposition 6.7, Corollary 3.12, and Proposition 6.12 from [7]

Theorem 4.4. *There exist positive constants $\delta_0, \delta_1, \Delta_0$ and ρ_0 satisfying the following property. Let X be a δ -hyperbolic length space and G a group acting by isometries on X . Let \mathcal{Q} be a G -invariant family of pairs (H, Y) where Y is a strongly quasi-convex subset of X and H a subgroup of G stabilizing Y . We assume that \mathcal{Q}/G is finite. Let $\rho \geq \rho_0$. If $\delta \leq \delta_0$, $\Delta(\mathcal{Q}, X) \leq \Delta_0$ and $\text{inj}(\mathcal{Q}, X) \geq 2\pi \sinh \rho$ then the following holds.*

1. *The space $\bar{X} = \bar{X}_\rho(\mathcal{Q})$ is a δ_1 -hyperbolic length space.*
2. *Let $(H, Y) \in \mathcal{Q}$. Let a be the apex of $Z(Y)$ and \bar{a} its image in \bar{X} . The projection $\pi: G \rightarrow \bar{G}$ induces an isomorphism from $\text{Stab}(Y)/H$ onto $\text{Stab}(\bar{a})$.*
3. *For every $x \in X$, the projection $\pi: G \rightarrow \bar{G}$ induces a bijection from the set $\{g \in G \mid d(gx, x) \leq \rho/100\}$ onto its image.*
4. *Let \bar{F} be an elliptic subgroup of \bar{G} . Either there exists an elliptic subgroup F of G such that the projection $\pi: G \rightarrow \bar{G}$ induces an isomorphism from F onto \bar{F} , or there exists $(H, Y) \in \mathcal{Q}$ such that \bar{F} is contained in $\text{Stab}(\bar{a})$, where \bar{a} stands for the image in \bar{X} of the apex a of the cone $Z(Y)$.*

We are now in position to prove the following statement.

Proposition 4.5. *Let G be a group acting uniformly properly without inversion on a hyperbolic graph Γ with uniformly bounded valence. Let $\{g_1, \dots, g_m\}$ be a finite subset of G whose elements are loxodromic (with respect to the action of G on Γ) and pairwise non-commensurable. In addition, we assume that for every $i \in \{1, \dots, m\}$, the group $\langle g_i \rangle$ is normal in $E(g_i)$. Then for every finite subset $U \subseteq G$ there exists $N \in \mathbb{N}$ with the following property. Let $n_1, \dots, n_m \in \mathbb{N}$, all bounded below by N . Let K be the normal closure of $\{g_1^{n_1}, \dots, g_m^{n_m}\}$ in G . Then the quotient $\bar{G} = G/K$ belongs to \mathcal{P}_0 . Moreover, we have the following.*

1. *For every $i \in \{1, \dots, m\}$, the natural homomorphism $\pi: G \rightarrow \bar{G}$ induces an embedding of $E(g_i)/\langle g_i \rangle$ into \bar{G} .*

2. The projection π is injective when restricted to U .

3. Let \bar{F} be a finite subgroup of \bar{G} . Then either there exists a finite subgroup F of G such that $\pi(F) = \bar{F}$ or \bar{F} is conjugate to a subgroup of $\pi(E(g_i))$ for some $i \in \{1, \dots, m\}$.

Proof. The constant δ_0 , δ_1 , Δ_0 , and ρ_0 are the one given by Theorem 4.4. We choose an arbitrary $\rho \geq \rho_0$. We write δ for the hyperbolicity constant of Γ . According to Lemma 2.1 there exists a constant Δ such that for every $u \in G$, for every $i \neq j$ in $\{1, \dots, m\}$, if

$$\text{diam} \left(Y_{g_i}^{+5\delta} \cap uY_{g_j}^{+5\delta} \right) > \Delta,$$

then $i = j$ and u belongs to $E(g_i)$. Up to replacing Γ by a rescaled version of Γ , that we denote X , we may assume that the following holds

- $\delta \leq \delta_0$ and $\Delta \leq \Delta_0$,
- there exists $x \in X$, such that for every $u \in U$ we have $d_X(ux, x) \leq \rho/100$.

Since the g_i 's are loxodromic, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, for every $i \in \{1, \dots, m\}$, we have $\|g_i^n\|_X \geq 2\pi \sinh \rho$. Let $n_1, \dots, n_m \in \mathbb{N}$, all bounded below by N . Let K be the normal closure of $\{g_1^{n_1}, \dots, g_m^{n_m}\}$ and \bar{G} be the quotient $\bar{G} = G/K$.

Since G acts without inversion on Γ , the quotient of $\bar{\Gamma} = \Gamma/K$ is a graph endowed with an action without inversion of \bar{G} . According to our assumptions there exist $d, M \in \mathbb{N}$ such that given any vertex v of Γ , its valence is at most d and the cardinality of its stabilizer is bounded above by M . Observe that the same holds for the vertices of $\bar{\Gamma}$. To prove that \bar{G} belongs to \mathcal{P}_0 , it suffices to show that $\bar{\Gamma}$ is hyperbolic. To that end, we use small cancellation theory. Let \mathcal{Q} be the following collection

$$\mathcal{Q} = \{ (\langle ug_i^{n_i}u^{-1}, uY_g \rangle \mid u \in G, 1 \leq i \leq m) \}.$$

By construction $\Delta(\mathcal{Q}, X) \leq \Delta_0$ and $\text{inj}(\mathcal{Q}, X) \geq 2\pi \sinh \rho$. The cone-off space $\dot{X} = \dot{X}_\rho(\mathcal{Q})$ and the quotient $\bar{X} = \dot{X}/K$ are built as above. The parameters have been chosen in such a way so that the family \mathcal{Q} satisfies the assumptions of Theorem 4.4. It follows that \bar{X} is a hyperbolic length space. Note that for every $(H, Y) \in \mathcal{Q}$, the quotient Y/H is bounded, hence \bar{X} is quasi-isometric to X/K (Proposition 4.3). Nevertheless X/K is just a rescaled copy of $\bar{\Gamma}$. Thus $\bar{\Gamma}$ is quasi-isometric to \bar{X} , and therefore hyperbolic. Points (1)-(3) directly follows from Theorem 4.4. \square

5 Proof of the main theorem

Lemma 5.1. *Let Q be a non-residually finite finitely presented group and let H be a torsion free hyperbolic group splitting as*

$$1 \rightarrow N \rightarrow H \rightarrow Q \rightarrow 1,$$

where the subgroup N is finitely generated and non-trivial. Then there exist $a, b \in N \setminus \{1\}$ which are not commensurable in N , but are conjugate in every finite quotient of N .

Proof. Arguing by contradiction, suppose that the conclusion fails. Let $q \in Q$ be a non-trivial element of Q that maps to 1 in every finite quotient of Q and let $h \in H$ be a preimage of q in H .

We claim that for every $g \in N$, the elements g and $h^{-1}gh$ are conjugate in N . Assume on the contrary that is not the case. As H is a torsion free hyperbolic group, g and $h^{-1}gh$ are not commensurable in N . Thus there exists a finite index normal subgroup K of N such that the images of g and $h^{-1}gh$ are not conjugate in N/K . Since N is finitely generated, there are only finitely many subgroups of any finite index in N . Replacing K with the intersection of all subgroups of N of index $[N : K]$ if necessary, we can assume that K is normal in H . Thus the group H acts on the (finite) set Ω of conjugacy classes of N/K . As every element of N acts on Ω trivially, the action of H on Ω gives rise to a homomorphism $\epsilon: Q \rightarrow \text{Sym}(\Omega)$. Since g and $h^{-1}gh$ are not conjugate in N/K , we have $\epsilon(q) \neq 1$, which contradicts the choice of q .

Thus g and $h^{-1}gh$ are conjugate in N for every $g \in N$, i.e. h defines a *pointwise inner* automorphism of N . By [15, Corollary 5.4], this automorphism must actually be inner. (Note that our assumption that N is non-trivial and torsion free ensures that all assumptions of [15, Corollary 5.4] hold.) Thus there exists $n \in N$ such that $h^{-1}n$ belongs to the centralizer $C_H(N)$ of N in H . Since H is torsion free hyperbolic, we have $C_H(N) = \{1\}$. Therefore $h = n$ belongs to N . This implies $q = 1$, which again contradicts our choice of q . \square

Let G be a relatively hyperbolic group. Recall that a subgroup H of G is *suitable* if it contains two non-commensurable loxodromic elements h_1 and h_2 such that $E(h_1) \cap E(h_2)$ is trivial [17, Definition 2.2]. Equivalently it means that H is not virtually cyclic and no nontrivial finite subgroup of G is normalized by H (see [2, Proposition 3.4]). We are now ready to prove our main result.

Theorem 5.2. *There exists a finitely generated group $G \in \mathcal{P}_0$ such that every amenable quotient of G is trivial.*

Proof. The first step is to produce a non-residually finite group in \mathcal{P}_0 . Let

$$1 \rightarrow N \rightarrow H \rightarrow Q \rightarrow 1$$

be a short exact sequence satisfying all assumptions of Lemma 5.1. For example, one can use the Rips construction [19]. According to the previous lemma, there exist two non-commensurable elements $a, b \in N \setminus \{1\}$ which are conjugate in every finite quotient of N . Note that $N \in \mathcal{P}_0$ as it acts freely on the Cayley graph of H with respect to a finite generating set. By Proposition 4.5, there exist distinct primes p and q such that $\bar{N} = N/\langle\langle a^p, b^q \rangle\rangle$ belongs to \mathcal{P}_0 , and the images of a and b in \bar{N} have orders p and q , respectively. In addition we can assume that the image \bar{g} of ab in \bar{N} is not trivial. It follows that \bar{N} is not residually finite. Indeed \bar{g} maps to 1 in every finite quotient of \bar{N} .

Let G_0 be any infinite hyperbolic group with Kazhdan's property (T). Let N_1 and N_2 be two distinct copies of \bar{N} . By [2, Theorem 1.1] there exists a quotient group G_1 of G_0

such that the group $P = N_1 * N_2$ embeds in G_1 and G_1 is hyperbolic relative to P . Since P is hyperbolic relative to $\{N_1, N_2\}$, so is G_1 .

Let g_1 and g_2 denote the copies of $\bar{g} \in \bar{N}$ in N_1 and N_2 respectively. We claim that the normal closure P_0 of g_1g_2 in P is a suitable subgroup of G_1 with respect to the peripheral structure $\{N_1, N_2\}$. Indeed it is clear that P_0 is non-cyclic. Thus it suffices to check that P_0 contains a loxodromic element with respect to the peripheral structure $\{N_1, N_2\}$ and has no non-trivial finite normal subgroups. To prove the first fact, we note that g_1g_2 has infinite order; it remains to check that g_1g_2 is not conjugate to an element of N_1 or N_2 in G_1 . Assume that $u^{-1}g_1g_2u \in P$ for some $u \in G_1$. Since P is almost malnormal (see [16, Theorem 1.4]) and $u^{-1}Pu \cap P$ contains $\langle g_1g_2 \rangle$, we obtain $u \in P$. However it is clear that g_1g_2 is not conjugate to an element of N_1 or N_2 in the free product $P = N_1 * N_2$. Thus g_1g_2 is loxodromic with respect to the peripheral structure $\{N_1, N_2\}$. To prove the second fact, we first notice that if F is a finite normal subgroup of G_1 , then a finite index subgroup of G_1 must centralize F . Using almost malnormality of P as in the previous paragraph, we obtain F is contained in $P = N_1 * N_2$ and consequently $F = 1$.

Thus P_0 is suitable in G_1 and we can apply [17, Theorem 2.4] to this subgroup and a finite set of generators $\{t_1, \dots, t_n\}$ of G_1 . This allows us to produce a quotient group G_2 of G_1 such that the restriction of the natural homomorphism $\varepsilon: G_1 \rightarrow G_2$ is injective on N_1 and N_2 , G_2 is hyperbolic relative to $\{N_1, N_2\}$, and $\varepsilon(t_i) \in \varepsilon(P_0)$ for all $i = 1, \dots, n$. In particular, G_2 is normally generated by the image of g_1g_2 in G_2 . Since g_1g_2 is mapped to 1 in every finite quotient of G_1 it follows that G_2 has no nontrivial finite quotient. Since G_2 has property (T), it has no non-trivial amenable quotient as well. Finally we note that $G_2 \in \mathcal{P}_0$ by Proposition 4.1. Hence $G = G_2$ satisfies the conclusion of the theorem. \square

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