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# A SURFACE OF DEGREE 24 WITH 1440 SINGULARITIES OF TYPE $D_4$

by

CÉDRIC BONNAFÉ

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*Abstract.* — Using the invariant algebra of the reflection group denoted by  $G_{32}$  in Shephard-Todd classification, we construct three irreducible surfaces in  $\mathbf{P}^3(\mathbb{C})$  with many singularities: one of them has degree 24 and contains 1440 quotient singularities of type  $D_4$ .

Let  $\mu_{D_4}(d)$  denote the maximal number of quotient singularities of type  $D_4$  that an irreducible projective surface in  $\mathbf{P}^3(\mathbb{C})$  might have. Miyaoka [Miy] proved that

$$\mu_{D_4}(d) \leq \frac{16}{117} d(d-1)^2$$

For  $d = 8, 16$  or  $24$ , this reads

$$\mu_{D_4}(8) \leq 53, \quad \mu_{D_4}(16) \leq 492 \quad \text{and} \quad \mu_{D_4}(24) \leq 1736.$$

The main results of this note are that

$$(1) \quad \mu_{D_4}(8) \geq 44, \quad \mu_{D_4}(16) \geq 352 \quad \text{and} \quad \mu_{D_4}(24) \geq 1440$$

and that

$$(2) \quad \mu_{D_4}(8k) \geq 44k^3$$

for all  $k \geq 1$ . So, let  $\mathbb{C}[x_1, x_2, x_3, x_4]$  be the polynomial ring over  $\mathbb{C}$  in 4 indeterminates with its usual grading and let  $\mathbf{P}^3(\mathbb{C}) = \text{Proj}(\mathbb{C}[x_1, x_2, x_3, x_4])$  be the projective space of dimension 3. If  $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$  is homogeneous, we denote by  $\mathcal{Z}(f)$  the projective hypersurface it defines. If  $k$  is a natural number, we denote by  $f[k]$  the homogeneous polynomial  $f(x_1^k, x_2^k, x_3^k, x_4^k)$ . We aim to prove the following result:

**Theorem.** *There exists a polynomial  $g$  of degree 8 such that:*

- (a)  $\mathcal{Z}(g)$  is an irreducible surface of degree 8 in  $\mathbf{P}^3(\mathbb{C})$  with exactly 44 singular points which are all quotient singularities of type  $D_4$ .
- (b) If  $k \geq 1$ , then  $\mathcal{Z}(g[k])$  is an irreducible surface of degree  $8k$  with at least  $44k^3$  quotient singularities of type  $D_4$ .
- (c)  $\mathcal{Z}(g[2])$  is an irreducible surface of degree 16 with exactly 472 singular points: 24 quotient singularities of type  $A_1$ , 96 quotient singularities of type  $A_2$  and 352 quotient singularities of type  $D_4$ .
- (d)  $\mathcal{Z}(g[3])$  is an irreducible surface of degree 24 in  $\mathbf{P}^3(\mathbb{C})$  with exactly 1440 singular points which are all quotient singularities of type  $D_4$ . The automorphism group of  $\mathcal{Z}(g[3])$  contains at least 25920 elements and acts transitively on the 1440 singular points.

The proof relies on extensive MAGMA computations [Magma], and goes as follow. Let

$$\begin{aligned}
g = & x_1^8 - 6x_1^6x_2^2 - 60x_1^6x_2x_3 - 60x_1^6x_2x_4 - 6x_1^6x_3^2 + 60x_1^6x_3x_4 - 6x_1^6x_4^2 \\
& + 2240x_1^5x_2^2x_3 - 2240x_1^5x_2^2x_4 + 2240x_1^5x_2x_3^2 - 2240x_1^5x_2x_4^2 + 2240x_1^5x_3^2x_4 \\
& - 2240x_1^5x_3x_4^2 - 14x_1^4x_2^4 + 10180x_1^4x_2^3x_3 + 10180x_1^4x_2^3x_4 + 40412x_1^4x_2^2x_3^2 \\
& - 23440x_1^4x_2^2x_3x_4 + 40412x_1^4x_2^2x_4^2 + 10180x_1^4x_2x_3^3 + 23440x_1^4x_2x_3^2x_4 \\
& + 23440x_1^4x_2x_3x_4^2 + 10180x_1^4x_2x_4^3 - 14x_1^4x_3^4 - 10180x_1^4x_3^3x_4 + 40412x_1^4x_3^2x_4^2 \\
& - 10180x_1^4x_3x_4^3 - 14x_1^4x_4^4 + 10180x_1^3x_2^4x_3 - 10180x_1^3x_2^4x_4 + 111980x_1^3x_2^3x_3^2 \\
& - 111980x_1^3x_2^3x_4^2 + 111980x_1^3x_2^2x_3^3 - 111980x_1^3x_2^2x_4^3 + 10180x_1^3x_2x_3^4 - 10180x_1^3x_2x_4^4 \\
& + 10180x_1^3x_3^4x_4 - 111980x_1^3x_3^3x_4^2 + 111980x_1^3x_2^3x_4^3 - 10180x_1^3x_3x_4^4 - 6x_1^2x_2^6 \\
& + 2240x_1^2x_2^5x_3 + 2240x_1^2x_2^5x_4 + 40412x_1^2x_2^4x_3^2 + 23440x_1^2x_2^4x_3x_4 + 40412x_1^2x_2^4x_4^2 \\
& + 111980x_1^2x_2^3x_3^3 + 111980x_1^2x_2^3x_4^3 + 40412x_1^2x_2^2x_3^4 + 154704x_1^2x_2^2x_3^2x_4^2 + 40412x_1^2x_2^2x_4^4 \\
& + 2240x_1^2x_2x_3^5 - 23440x_1^2x_2x_3^4x_4 - 23440x_1^2x_2x_3x_4^4 + 2240x_1^2x_2x_4^5 - 6x_1^2x_3^6 \\
& - 2240x_1^2x_3^5x_4 + 40412x_1^2x_3^4x_4^2 - 111980x_1^2x_3^3x_4^3 + 40412x_1^2x_3^2x_4^4 - 2240x_1^2x_3x_4^5 \\
& - 6x_1^2x_4^6 - 60x_1x_2^6x_3 + 60x_1x_2^6x_4 + 2240x_1x_2^5x_3^2 - 2240x_1x_2^5x_4^2 + 10180x_1x_2^4x_3^3 \\
& - 23440x_1x_2^4x_3^2x_4 + 23440x_1x_2^4x_3x_4^2 - 10180x_1x_2^4x_4^3 + 10180x_1x_2^3x_4^4 - 10180x_1x_2^3x_4^4 \\
& + 2240x_1x_2^2x_3^5 + 23440x_1x_2^2x_3^4x_4 - 23440x_1x_2^2x_3x_4^4 - 2240x_1x_2^2x_4^5 - 60x_1x_2x_3^6 \\
& + 23440x_1x_2x_3^4x_4^2 - 23440x_1x_2x_3^3x_4^4 + 60x_1x_2x_4^6 - 60x_1x_3^6x_4 - 2240x_1x_3^5x_4^2 \\
& + 10180x_1x_3^4x_4^3 - 10180x_1x_3^3x_4^4 + 2240x_1x_3^2x_4^5 + 60x_1x_3x_4^6 + x_2^8 - 6x_2^6x_3^2 \\
& - 60x_2^6x_3x_4 - 6x_2^6x_4^2 - 2240x_2^5x_3^2x_4 - 2240x_2^5x_3x_4^2 - 14x_2^4x_3^4 + 10180x_2^4x_3^3x_4 \\
& + 40412x_2^4x_3^2x_4^2 + 10180x_2^4x_3x_4^3 - 14x_2^4x_4^4 - 10180x_2^3x_3^4x_4 - 111980x_2^3x_3^3x_4^2 \\
& - 111980x_2^3x_3^2x_4^3 - 10180x_2^3x_3x_4^4 - 6x_2^2x_3^6 + 2240x_2^2x_3^5x_4 + 40412x_2^2x_3^4x_4^2 \\
& + 111980x_2^2x_3^3x_4^3 + 40412x_2^2x_3^2x_4^4 + 2240x_2^2x_3x_4^5 - 6x_2^2x_4^6 + 60x_2x_3^6x_4 \\
& - 2240x_2x_3^5x_4^2 - 10180x_2x_3^4x_4^3 - 10180x_2x_3^3x_4^4 - 2240x_2x_3^2x_4^5 + 60x_2x_3x_4^6 \\
& + x_3^8 - 6x_3^6x_4^2 - 14x_3^4x_4^4 - 6x_3^2x_4^6 + x_4^8.
\end{aligned}$$

This is the polynomial of the Theorem. It has coefficients in  $\mathbb{Q}$  but we will check in our computations that the singular points of  $\mathcal{Z}(g)$  and  $\mathcal{Z}(g[3])$  have coordinates in the cyclotomic field  $K = \mathbb{Q}(z)$ , where  $z$  is a primitive 12-th root of unity. We denote by  $\mathcal{Z}(g)_{\text{sing}}$  the (reduced) singular locus of  $\mathcal{Z}(g)$  respectively.

**Remark 1.** As we will see in the computations below, the polynomial  $g[3]$  is invariant under the action of the complex reflection group denoted by  $G_{32}$  in Shephard-Todd classification. In fact, we did not find  $g$  directly: we found first  $g[3]$  by looking at invariants of degree 24 of  $G_{32}$ , following ideas of Barth [Bar] (for constructing *Barth sextic* with 65 nodes) and Sarti [Sar1], [Sar2], [Sar3] (for constructing for instance the *Sarti surface* of degree 12 with 600 nodes), who used invariants of Coxeter groups of type  $H_3$  and  $H_4$ . ■

**Remark 2.** As the expert in computer algebra will notice by reading the MAGMA codes below, the author is not a specialist in programming and asks for some indulgence. ■

So let us start by defining the polynomial  $g$  in MAGMA. We will define it over the rational numbers. This will not be important for the computations regarding  $\mathcal{Z}(g)$ , but this will shorten drastically computations regarding  $\mathcal{Z}(g[3])$ .

```

> Q:=RationalField();
> P3<x1,x2,x3,x4>:=ProjectiveSpace(Q,3);
>
> g:=x1^8 - 6*x1^6*x2^2 - 60*x1^6*x2*x3
> - 60*x1^6*x2*x4 - 6*x1^6*x3^2 + 60*x1^6*x3*x4
> - 6*x1^6*x4^2 + 2240*x1^5*x2^2*x3
> - 2240*x1^5*x2^2*x4 + 2240*x1^5*x2*x3^2
> - 2240*x1^5*x2*x4^2 + 2240*x1^5*x3^2*x4
> - 2240*x1^5*x3*x4^2 - 14*x1^4*x2^4
> + 10180*x1^4*x2^3*x3 + 10180*x1^4*x2^3*x4
> + 40412*x1^4*x2^2*x3^2 - 23440*x1^4*x2^2*x3*x4
> + 40412*x1^4*x2^2*x4^2 + 10180*x1^4*x2*x3^3
> + 23440*x1^4*x2*x3^2*x4 + 23440*x1^4*x2*x3*x4^2
> + 10180*x1^4*x2*x4^3 - 14*x1^4*x3^4
> - 10180*x1^4*x3^3*x4 + 40412*x1^4*x3^2*x4^2
> - 10180*x1^4*x3*x4^3 - 14*x1^4*x4^4
> + 10180*x1^3*x2^4*x3 - 10180*x1^3*x2^4*x4
> + 111980*x1^3*x2^3*x3^2 - 111980*x1^3*x2^3*x4^2
> + 111980*x1^3*x2^2*x3^3 - 111980*x1^3*x2^2*x4^3
> + 10180*x1^3*x2*x3^4 - 10180*x1^3*x2*x4^4
> + 10180*x1^3*x3^4*x4 - 111980*x1^3*x3^3*x4^2
> + 111980*x1^3*x3^2*x4^3 - 10180*x1^3*x3*x4^4
> - 6*x1^2*x2^6 + 2240*x1^2*x2^5*x3 + 2240*x1^2*x2^5*x4
> + 40412*x1^2*x2^4*x3^2 + 23440*x1^2*x2^4*x3*x4
> + 40412*x1^2*x2^4*x4^2 + 111980*x1^2*x2^3*x3^3
> + 111980*x1^2*x2^3*x4^3 + 40412*x1^2*x2^2*x3^4
> + 154704*x1^2*x2^2*x3^2*x4^2 + 40412*x1^2*x2^2*x4^4
> + 2240*x1^2*x2*x3^5 - 23440*x1^2*x2*x3^4*x4
> - 23440*x1^2*x2*x3*x4^4 + 2240*x1^2*x2*x4^5
> - 6*x1^2*x3^6 - 2240*x1^2*x3^5*x4
> + 40412*x1^2*x3^4*x4^2 - 111980*x1^2*x3^3*x4^3
> + 40412*x1^2*x3^2*x4^4 - 2240*x1^2*x3*x4^5
> - 6*x1^2*x4^6 - 60*x1*x2^6*x3 + 60*x1*x2^6*x4
> + 2240*x1*x2^5*x3^2 - 2240*x1*x2^5*x4^2
> + 10180*x1*x2^4*x3^3 - 23440*x1*x2^4*x3^2*x4
> + 23440*x1*x2^4*x3*x4^2 - 10180*x1*x2^4*x4^3
> + 10180*x1*x2^3*x3^4 - 10180*x1*x2^3*x4^4
> + 2240*x1*x2^2*x3^5 + 23440*x1*x2^2*x3^4*x4
> - 23440*x1*x2^2*x3*x4^4 - 2240*x1*x2^2*x4^5
> - 60*x1*x2*x3^6 + 23440*x1*x2*x3^4*x4^2
> - 23440*x1*x2*x3^2*x4^4 + 60*x1*x2*x4^6 - 60*x1*x3^6*x4
> - 2240*x1*x3^5*x4^2 + 10180*x1*x3^4*x4^3
> - 10180*x1*x3^3*x4^4 + 2240*x1*x3^2*x4^5
> + 60*x1*x3*x4^6 + x2^8 - 6*x2^6*x3^2 - 60*x2^6*x3*x4
> - 6*x2^6*x4^2 - 2240*x2^5*x3^2*x4 - 2240*x2^5*x3*x4^2
> - 14*x2^4*x3^4 + 10180*x2^4*x3^3*x4
> + 40412*x2^4*x3^2*x4^2 + 10180*x2^4*x3*x4^3
> - 14*x2^4*x4^4 - 10180*x2^3*x3^4*x4
> - 111980*x2^3*x3^3*x4^2 - 111980*x2^3*x3^2*x4^3
> - 10180*x2^3*x3*x4^4 - 6*x2^2*x3^6 + 2240*x2^2*x3^5*x4
> + 40412*x2^2*x3^4*x4^2 + 111980*x2^2*x3^3*x4^3
> + 40412*x2^2*x3^2*x4^4 + 2240*x2^2*x3*x4^5 - 6*x2^2*x4^6
> + 60*x2*x3^6*x4 - 2240*x2*x3^5*x4^2 - 10180*x2*x3^4*x4^3
> - 10180*x2*x3^3*x4^4 - 2240*x2*x3^2*x4^5 + 60*x2*x3*x4^6
> + x3^8 - 6*x3^6*x4^2 - 14*x3^4*x4^4 - 6*x3^2*x4^6 + x4^8;

```

## 1. Degree 8

As explained above, degree 8 is sufficiently small so that all computations can be performed directly over the field  $K$ .

```

> K<z>:=CyclotomicField(12);
> P3K<x>:=ProjectiveSpace(K, 3);
> time Zg:=Surface(P3K, Evaluate(g, [x[1], x[2], x[3], x[4]]));
Time: 11.040
> Zgsing:=SingularSubscheme(Zg);
> Dimension(Zgsing);
0
> unionhyp:=Scheme(P3K, x[1]*x[2]*x[3]*x[4]);
> Dimension(Intersection(Zgsing, unionhyp));
-1

```

Note that the fact that  $\dim \mathcal{Z}(g)_{\text{sing}} = 0$  implies that  $\mathcal{Z}(g)$  is geometrically irreducible. Note also that the last command implies that all the singular points are contained (for instance) in the affine chart  $\mathcal{Z}(g)^{\text{aff}}$  defined by “ $x_4 \neq 0$ ”. We will make all the remaining computations in this affine chart:

```

> Zgaff:=AffinePatch(Zg, 4);
> Zgaffsing:=SingularSubscheme(Zgaff);
> irr:=IrreducibleComponents(Zgaffsing);
> irr:=[ReducedSubscheme(i) : i in irr];
> Set([Degree(i) : i in irr]);
{ 1 }
> # irr;
44

```

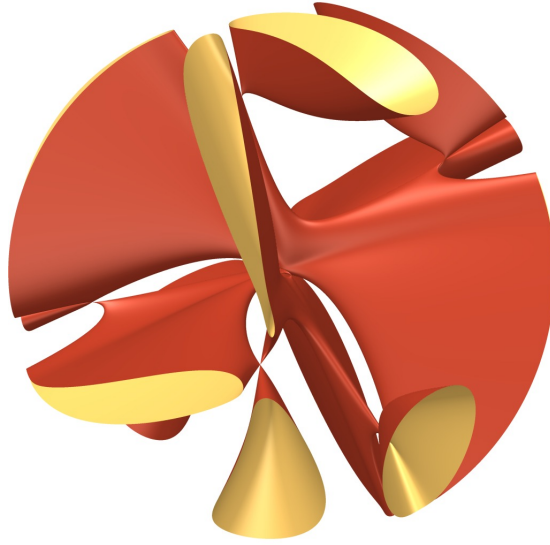
The last two commands shows that all singular points have coordinates in  $K$  and that there are 44 singular points. We now check that they are quotient singularities of type  $D_4$ :

```

> sings:=[RationalPoints(i) : i in irr];
> sings:=&cat [[Coordinates(i) : i in j] : j in sings];
> sings:=[Zgaff ! i : i in sings];
> test:=[];
> for i in sings
for>   do a,b,c:=IsSimpleSurfaceSingularity(i);
for>   test:=test cat [[* a,b,c *]];
for> end for;
> test[1];
[* true, D, 4 *]
> Set([i eq test[1] : i in test]);
{ true }

```

This concludes the proof of statement (a) of the Theorem. Note that the surface  $\mathcal{Z}(g)$  has only 8 real points: Figure 1 shows part of the real locus of  $\mathcal{Z}(g)$ .

FIGURE 1. Part of the real locus of  $\mathcal{Z}(g)$ 

## 2. Degree $8k$

Let  $U$  denote the open subset of  $\mathbf{P}^3(\mathbb{C})$  defined by  $x_1 x_2 x_3 x_4 \neq 0$  and let  $\sigma_k : \mathbf{P}^3(\mathbb{C}) \rightarrow \mathbf{P}^3(\mathbb{C})$ ,  $[x_1; x_2; x_3; x_4] \mapsto [x_1^k; x_2^k; x_3^k; x_4^k]$ . The restriction of  $\sigma_k$  to a morphism  $U \rightarrow U$  is an étale Galois covering, with group  $(\mu_k)^4 / \Delta \mu_k$  (here,  $\mu_k$  denotes the group of  $k$ -th roots of unity in  $\mathbb{C}^\times$  and  $\Delta : \mu_k \hookrightarrow (\mu_k)^4$  is the diagonal embedding). We have  $\mathcal{Z}(g[k]) = \sigma_k^*(\mathcal{Z}(g))$ .

Let us first prove that  $\mathcal{Z}(g[k])$  is irreducible. We may assume that  $k \geq 2$ , as the result has been proved for  $k = 1$  in the previous section. Note the following fact: if  $i \neq j$ , then the scheme  $\mathcal{Z}_{i,j}$  defined by the ideal  $\langle g, \frac{\partial g}{\partial x_i}, \frac{\partial g}{\partial x_j} \rangle$  has dimension 0:

```

> dg := [Derivative(g, i) : i in [1..4]];
> Dimension(Scheme(P3, [g, dg[1], dg[2]]));
0
> Dimension(Scheme(P3, [g, dg[1], dg[3]]));
0
> Dimension(Scheme(P3, [g, dg[1], dg[4]]));
0
> Dimension(Scheme(P3, [g, dg[2], dg[3]]));
0
> Dimension(Scheme(P3, [g, dg[2], dg[4]]));
0
> Dimension(Scheme(P3, [g, dg[3], dg[4]]));
0

```

Recall that

$$\frac{\partial g[k]}{\partial x_i} = k x_i^{k-1} \left( \frac{\partial g}{\partial x_i} \circ \sigma_k \right),$$

so the singular locus of  $\mathcal{Z}(g[k])$  is contained in

$$\{p_1, p_2, p_3, p_4\} \cup \left( \bigcup_{i \neq j} \sigma_k^*(\mathcal{Z}_{i,j}) \right),$$

where  $p_i = [\delta_{i1}; \delta_{i2}; \delta_{i3}; \delta_{i4}]$  (and  $\delta_{ij}$  is the Kronecker symbol). Since  $\sigma_k$  is finite, this implies that  $\mathcal{Z}(g[k])_{\text{sing}}$  has dimension 0, so  $\mathcal{Z}(g[k])$  is irreducible.

Now, as  $\sigma_k : U \rightarrow U$  is étale and the singular locus of  $\mathcal{X}(g)$  is contained in  $U$  (see the command `Dimension(Intersection(Zgsing, unionhyp))` of the previous section), the 44 singularities of  $\mathcal{X}(g)$  give  $44k^3$  singularities in  $\mathcal{X}(g[k])$  of the same type, i.e. quotient singularities of type  $D_4$ . This proves the statement (b) of the Theorem.

Note that, for  $k = 2, 3$  and 4 (and maybe for bigger  $k$ ) we will prove in the next sections that  $\mathcal{X}(g[k])$  contains singular points outside of  $U$ .

### 3. Degree 16

Using the morphism  $\sigma_2$  defined in the previous section and the fact that it is étale over  $U$ , we get that  $\mathcal{X}(g[2]) \cap U$  has exactly 352 singular points, which are all quotient singularities of type  $D_4$ . We now need to determine the singularities which are not contained in  $U$ . So let  $H$  be the complement of  $U$  in  $\mathbf{P}^3(\mathbb{C})$ .

```
> g2:=Evaluate(g, [x1^2, x2^2, x3^2, x4^2]);
> Z2:=Surface(P3, g2);
> Z2sing:=SingularSubscheme(Z2);
> time Dimension(Z2sing);
0
Time: 20.920
> unionhyp:=Scheme(P3, x1*x2*x3*x4);
> Z2singhyp:=Intersection(Z2sing, unionhyp);
> Z2singhyp:=ReducedSubscheme(Z2singhyp);
> irrsinghyp:=IrreducibleComponents(Z2singhyp);
> # irrsinghyp;
18
> Set([Degree(i) : i in irrsinghyp]);
{ 4, 8 }
> &+ [Degree(i) : i in irrsinghyp];
120
```

This shows that  $\mathcal{X}(g[2])_{\text{sing}} \cap H$  contains 120 points, none of which being defined over  $\mathbb{Q}$ . We denote by  $a$  an 8-th root of unity and by  $K_4$  the group the cyclotomic field  $\mathbb{Q}(a)$ . We denote by  $K_8$  the splitting field of the polynomial

$$T^{16} + 3248T^{12} + 23100000T^8 + 20300000000T^4 + 3906250000000,$$

and by  $u$  a root of this polynomial (one can check that  $K_8$  is a Galois extension of  $\mathbb{Q}$ ).

It turns out that every singular point which is defined over a field extension of degree 4 (resp. 8) has coordinates in  $K_4$  (resp.  $K_8$ ). We separate the treatment of these two families.

**Singular points defined over  $K_4$ .** We will check that all of them are quotient singularities of type  $A_1$ :

```
> singhyp4:=[i : i in irrsinghyp | Degree(i) eq 4];
> K4<a>:=CyclotomicField(8);
> Z2K4:=ChangeRing(Z2, K4);
> P3K4<x>:=AmbientSpace(Z2K4);
> singhyp4:=[ChangeRing(i, K4) : i in singhyp4];
> singhyp4:=[* [j : j in RationalPoints(i)]
> : i in singhyp4 *];
> singhyp4:=[Coordinates(i[1]) : i in singhyp4];
> singhyp4:=[Z2K4 ! i : i in singhyp4];
```

```

> types:=[];
> for i in singhyp4 do
for>   te,ty,ra:=IsSimpleSurfaceSingularity(i);
for>   types:=types cat [[* te,ty,ra *]];
for> end for;
> types;
[ [* true, A, 1 *], [* true, A, 1 *],
[* true, A, 1 *], [* true, A, 1 *],
[* true, A, 1 *], [* true, A, 1 *] ]

```

We have chosen one point in each irreducible components over  $\mathbb{Q}$ , so this proves that  $\mathcal{Z}(g)$  contains at least 24 quotient singularities of tpe  $A_1$ .

**Singular points defined over  $K_8$ .** We will check that all of them are quotient singularities of type  $A_2$ . So let us first pick one point in each irreducible component over  $\mathbb{Q}$ :

```

> singhyp8:=[i : i in irrsinghyp | Degree(i) eq 8];
> POL<T>:=PolynomialRing(Q);
> K8<u>:=NumberField(T^16 + 3248*T^12
> + 23100000*T^8 + 20300000000*T^4
> + 39062500000000);
> Z2K8:=ChangeRing(Z2,K8);
> P3K8<[x]>:=AmbientSpace(Z2K8);
> singhyp8:=[ChangeRing(i,K8) : i in singhyp8];
> singhyp8:=[* [j : j in RationalPoints(i)]
> : i in singhyp8 *];
> singhyp8:=[Coordinates(i[1]) : i in singhyp8];
> singhyp8:=[Z2K8 ! i : i in singhyp8];

```

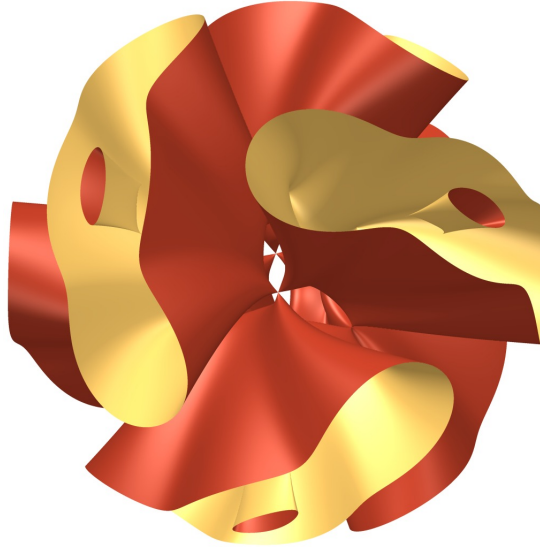
The Magma command `IsSimpleSurfaceSingularity` does not answer in a reasonable amount of time. Let us explain how we proceed for one of the points in the list `singhyp8`. So let  $p$  be the first point in the list. It lives in the affine chart defined by " $x_3 \neq 0$ ", and we denote by  $(x, y, z)$  the variables  $(x_1/x_3, x_2/x_3, x_4/x_3)$ . Let  $f(x, y, z)$  the defining polynomial of  $\mathcal{Z}(g[2])$  in this affine chart, after translating  $p$  at the origin  $(0, 0, 0)$ . We denote by  $f_2$  its homogeneous component of degree 2:

```

> A3K8<x, y, z>:=AffineSpace(K8, 3);
> p:=singhyp8[1];
> cop:=Coordinates(p);
> f:=Evaluate(g2,[x+cop[1],y+cop[2],1,z+cop[4]]);
> cof:=Coefficients(f);
> mof:=Monomials(f);
> l:=# mof;
> f2:=&+ [cof[i]*mof[i] : i in [1..l] |
> Degree(mof[i]) eq 2];
> f2:=f2/Reverse(Coefficients(f2))[1];
> Factorization(f2-z^2);
[
  <x + 1/264000000000000*(-57*u^14 - 5500*u^12
  - 372636*u^10 - 17864000*u^8
  - 1456950000*u^6 - 92675000000*u^4
  - 3075225000000*u^2 -
  55825000000000)*y, 2>
]

```



FIGURE 2. Part of the real locus of  $\mathcal{Z}(g[2])$ 

The last command shows that, up to a change of coordinates,  $f_2$  might be transformed to  $y^2 + z^2$ . By standard arguments, this proves that  $p$  is a quotient singularity of type  $A_k$ , for some  $k \geq 2$ , which can be obtained as the Milnor number of  $\mathcal{Z}(g[2])$  at  $p$ : however, Magma cannot compute this Milnor number in a reasonable amount of time and we need to copy the polynomial  $f$  in the software Singular [DGPS] to compute this Milnor number (!): we obtain 2. So  $p$  is a quotient singularity of type  $A_2$ .

Repeating 12 times the same method (we omit details), we get that  $\mathcal{Z}(g[2])$  has 96 quotient singularities of type  $A_2$ . This proves statement (c) of the Theorem.

Figure 2 shows part of the real locus of  $\mathcal{Z}(g)$ .

#### 4. Degree 24

We now come to the surface  $\mathcal{Z}(g[3])$ . For shortening the first computations, we will define it over  $\mathbb{Q}$ .

```
> g3:=Evaluate(g, [x1^3, x2^3, x3^3, x4^3]);
> Zg3:=Surface(P3, g3);
> Zg3sing:=SingularSubscheme(Zg3);
> time Dimension(Zg3sing);
0
Time: 13.210
```

Again, the fact that the  $\dim \mathcal{Z}(g[3])_{\text{sing}} = 0$  implies that  $\mathcal{Z}(g[3])$  is geometrically irreducible. We will compute the singular locus of  $\mathcal{Z}(g[3])$  over  $\mathbb{Q}$ , by separating its intersection with the affine chart " $x_1 \neq 0$ " and its intersection with the hyperplane " $x_1 = 0$ ".

```
> Zg3aff:=AffinePatch(Zg3, 1);
> Zg3affsing:=SingularSubscheme(Zg3aff);
> time irr3aff:=IrreducibleComponents(Zg3affsing);
Time: 48.370
> irr3aff:=[ReducedSubscheme(i) : i in irr3aff];
> hyp:=Intersection(Zg3sing, Scheme(P3, x1));
> irrhyp:=IrreducibleComponents(hyp);
> irrhyp:=[ReducedSubscheme(i) : i in irrhyp];
```

We now check that all the singular points are rational over the cyclotomic field  $K = \mathbb{Q}(z)$ :

```
> K<z>:=CyclotomicField(12);
> irr3aff:=[ChangeRing(i,K) : i in irr3aff];
> irr3aff:=&cat [IrreducibleComponents(i) : i in irr3aff];
> irrhyp:=[ChangeRing(i,K) : i in irrhyp];
> irrhyp:=&cat [IrreducibleComponents(i) : i in irrhyp];
> Set([Degree(i) : i in irr3aff]);
{ 1 }
> Set([Degree(i) : i in irrhyp]);
{ 1 }
> (# irr3aff) + (# irrhyp);
1440
```

The above last command shows that  $\mathcal{Z}(g[3])$  has exactly 1440 singular points. We now pick a singular point  $p$  in  $\mathcal{Z}(g[3])$ , and show that it is a quotient singularity of type  $D_4$ :

```
> Zg3K:=ChangeRing(Zg3,K);
> p:= Zg3K ! [0,-(2*z^3-z^2-z+1), (z^3+z^2+z), 2];
> IsSingular(Zg3K,p);
true
> IsSimpleSurfaceSingularity(p);
true D 4
```

We could prove that all other singular points are quotient singularities of type  $D_4$  (by making 1440 computations!), but we prefer to do it using group theory. So let  $W$  be the complex reflection group denoted by  $G_{32}$  in Shephard-Todd classification (see for instance [LeTa]). We could obtain it by the MAGMA command  $W := \text{ShephardTodd}(32)$ , but it turns out that the group implemented in this way in MAGMA does not stabilize  $g[3]$ : with the version of  $G_{32}$  implemented in MAGMA, we would need to replace  $g[3]$  by a much more complicated polynomial with huge coefficients lying in the cyclotomic field  $\mathbb{Q}(z^4)$ . This would have led to very long computations (we could even not check that the 1440 singular points are the only ones: MAGMA computations do not conclude, at least after few hours). We prefer to use the model implemented by Jean Michel [Mic] in the CHEVIE package of GAP3.

```
> a:=z^4;
> s1:=Matrix(K,4,4,
> [ [ 1, 0, 0, 0 ],
> [ 0, 1, 0, 0 ],
> [ 0, 0, a, 0 ],
> [ 0, 0, 0, 1 ] ]);
>
> s2:=Matrix(K,4,4,
> [ [-1/3*a-2/3*a^2, 2/3*a+1/3*a^2, 2/3*a+1/3*a^2, 0 ],
> [ 2/3*a+1/3*a^2, -1/3*a-2/3*a^2, 2/3*a+1/3*a^2, 0 ],
> [ 2/3*a+1/3*a^2, 2/3*a+1/3*a^2, -1/3*a-2/3*a^2, 0 ],
> [ 0, 0, 0, 1 ] ]);
>
> s3:=Matrix(K,4,4,
> [ [ 1, 0, 0, 0 ],
> [ 0, a, 0, 0 ],
> [ 0, 0, 1, 0 ],
> [ 0, 0, 0, 1 ] ]);
>
> s4:=Matrix(K,4,4,
```

```

> [ [-1/3*a-2/3*a^2, -2/3*a-1/3*a^2, 0, -2/3*a-1/3*a^2 ],
> [-2/3*a-1/3*a^2, -1/3*a-2/3*a^2, 0, 2/3*a+1/3*a^2 ],
> [ 0, 0, 1, 0 ],
> [-2/3*a-1/3*a^2, 2/3*a+1/3*a^2, 0, -1/3*a-2/3*a^2 ] ] );
>
> W:=MatrixGroup<4,K | [s1,s2,s3,s4]>;
> Order(W);
155520
> Order(W/Centre(W));
25920
> test:=IsIsomorphic(W,ShephardTodd(32));
> test;
true

```

We then check that  $g[3]$  is invariant under the action of  $W$ :

```

> R:=InvariantRing(W);
> P<X1,X2,X3,X4>:=PolynomialRing(R);
> g3copy:=Evaluate(g3,[X1,X2,X3,X4]);
> [g3copy^w eq g3copy : w in Generators(W)];
[ true, true, true, true ]

```

This shows that the group  $W$  defined as above, which is a complex reflection group of type  $G_{32}$  in Shephard-Todd classification, acts on  $\mathcal{Z}$ . Its center  $Z(W)$  has order 6, so  $W/Z(W)$  has order 25920 and acts faithfully on  $\mathcal{Z}$ . We conclude by computing the  $W$ -orbit of the point  $p$ . We do this by first computing in the vector space  $V \simeq \mathbb{C}^4$  and then projecting on  $\mathbf{P}(V) \simeq \mathbf{P}^3(\mathbb{C})$ :

```

> P3K:=ProjectiveSpace(K,3);
> ZW:=Centre(W);
> rep:=Transversal(W,ZW);
> V:=VectorSpace(W);
> v:=V ! Coordinates(p);
> orbv:=[v^(Transpose(w)) : w in rep];
> orbp:=[P3K ! Coordinates(V,i) : i in orbv];
> orbp:=Set(orbp);
> # orbp;
1440

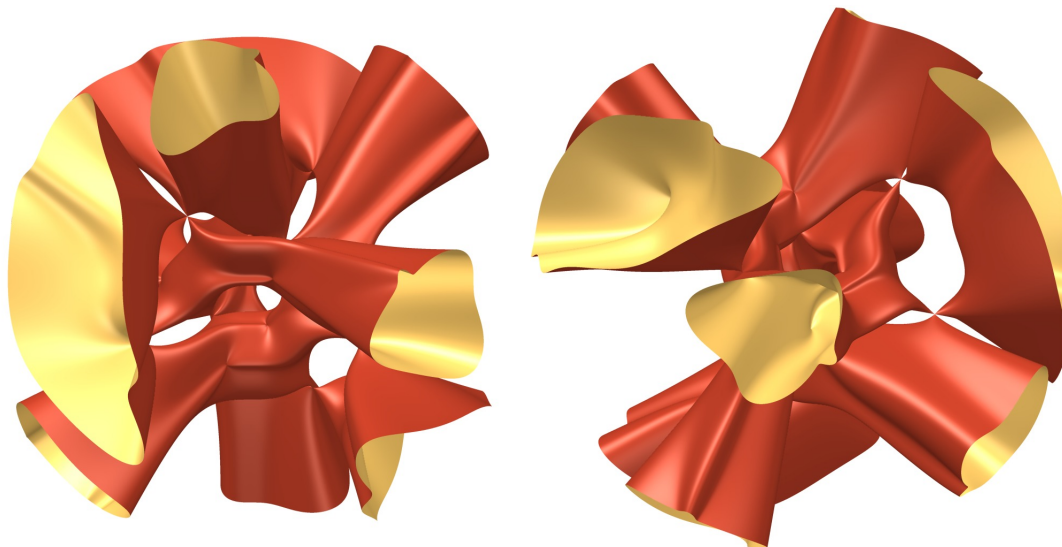
```

As the  $W$ -orbit of  $p$  has 1440 elements and as  $\mathcal{Z}(g[3])$  has 1440 singular points, the proof of statement (b) of the Theorem is complete. Note also that, in the given model, the surface  $\mathcal{Z}(g[3])$  has only 32 real points: Figure 3 gives partial views of its real locus.

**Remark 3.** From Section 2, we deduce that  $\mathcal{Z}(g[4])_{\text{sing}}$  has 2816 quotient singularities of type  $D_4$  in the open subset  $U$  and it can be checked that it has 480 other singular points outside of  $U$ , for which we did not determine the type. ■

**Remark 4.** After investigations in the invariant rings of several complex reflection groups, we have also been able to construct curves with many singularities. For example:

- Using the reflection group  $W = G_{24}$ , we have obtained a cuspidal curve of degree 14 in  $\mathbf{P}^2(\mathbb{C})$  with exactly 42 cusps (all lying in a single  $W$ -orbit). Note that  $W/Z(W)$  has order 168 and is isomorphic to  $\mathbf{GL}_3(\mathbb{F}_2)$ .
- Using the reflection group  $W = G_{26}$ , we have obtained a curve of degree 18 in  $\mathbf{P}^2(\mathbb{C})$  with 72 cusps and 12 nodes (these are the two  $W$ -orbits of singular points). Note that  $W/Z(W)$  has order 216.

FIGURE 3. Part of the real locus of  $\mathcal{Z}(g[3])$ 

Also, other singular surfaces have been obtained. For example:

- Using the reflection group  $W = G_{29}$  (note that  $W/Z(W)$  has order 1920), we have obtained:
  - a surface of degree 8 in  $\mathbf{P}^3(\mathbb{C})$  with 160 nodes, all belonging to the same  $W$ -orbit. Recall that *Endraß octic surface* [End] has degree 8 and 168 nodes and its automorphism group has order 16.
  - a surface of degree 8 in  $\mathbf{P}^3(\mathbb{C})$  with 20 singular points of multiplicity 3 and Milnor number 11, all belonging to the same  $W$ -orbit.
- Using the reflection group  $W = G_{31}$  (note that  $W/Z(W)$  has order 11520), we have obtained a surface of degree 20 in  $\mathbf{P}^3(\mathbb{C})$  with 1920 nodes, all lying in the same  $W$ -orbit. Recall that *Chmutov surface* [Chm] of degree 20 has 2926 nodes.

Details will appear in a forthcoming paper. ■

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